The self-avoiding walk on the hexagonal lattice

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Self-Avoiding Walks on the hexagonal lattice $\mathbb{H}$:

Conjecture (Flory, 1948; Nienhuis, 1982)

Precise asymptotics for the mean-square displacement and for the number $c_n$ of SAWs of length $n$:

$$\langle |\omega(n)|^2 \rangle \sim D_n^{2\nu} \text{ as } n \to \infty,$$

$$c_n \sim A_n^{\gamma - 1} \mu n \text{ as } n \to \infty,$$

where $\nu := \frac{3}{4}$ and $\mu_c := \sqrt{2 + \sqrt{2}}$.

$\gamma$ and $\nu$ are universal; $\mu_c$ is lattice-dependent.
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where $\nu := 3/4$ and $\mu_c := \sqrt{2 + \sqrt{2}}$, $\gamma := 43/32$.

$\gamma$ and $\nu$ are universal; $\mu_c$ is lattice-dependent.
The connective constant satisfies $\mu_c := \lim_{n \to \infty} c_n^{\frac{1}{n}} = \sqrt{2 + \sqrt{2}}$.

- Easy observations:
  
  $$c_{n+m} < c_n \cdot c_m \Rightarrow \exists \mu_c := \lim_{n \to \infty} c_n^{\frac{1}{n}},$$
  
  $$2^{n/2} \leq c_n \leq 3 \cdot 2^{n-1} \Rightarrow \sqrt{2} \leq \mu_c \leq 2.$$
Theorem (H. Duminil-Copin, S. Smirnov, 2010)

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• The generating function (diverges \( \mu < \mu_c \), converges \( \mu > \mu_c \)):

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G(\mu) := \sum_{\omega} \mu^{-\ell(\omega)} = \sum_{n} c_n \cdot \mu^{-n}.
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- The generating function (diverges $\mu < \mu_c$, converges $\mu > \mu_c$):

  $$G_{a \to z}(\mu) := \sum_{\omega \subset \Omega: a \to z} \mu^{-\ell(\omega)} = \sum_n c_{n,a \to z} \cdot \mu^{-n} .$$

  It is expected that $G(\mu) \sim (\mu_c - \mu)^{-\gamma}$.

Try to count simpler objects, **bridges**: Walks that never go below the first step and above the last one.

The number of bridges grows at the same (exponential) speed as walks.
**Definition**

A **self-avoiding bridge** is a SAW $\omega$ such that the first site is of minimal second coordinate and the last one of maximal second coordinate. Let $b_n$ be the number of self-avoiding bridges of length $n$. 

![Diagram of self-avoiding bridge](image-url)
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Proposition (Hammersley 1961)

$\mu_c$ is the same for bottom-top bridges, bottom-bottom bridges, loops.
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$\bullet$ $b_n \leq c_n$ for obvious reasons.
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$b_n \leq c_n$ for obvious reasons. Moreover, $c_n \leq r_n^2 b_n$ where $r_n$ is the number of **partitions** of $n$ into increasing positive integers. Since $r_n \leq Ce^{c\sqrt{n}}$, we obtain that $b_n$ and $c_n$ are logarithmically equivalent.
Definition

The **winding** $W_\omega(a, b)$ of a curve $\omega$ between $a$ and $b$ is the rotation (in radians) of the curve between $a$ and $b$.

![Diagram](image)

$W_\gamma(a, b) = 0$

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W_\gamma(a, b) = 0 \quad \text{and} \quad W_\gamma(a, b) = 2\pi
\]

With this definition, we can define the **parafermionic operator** for \( a \in \partial \Omega \) and \( z \in \Omega \):

\[
F(z) = F(a, z, \mu, \sigma) := \sum_{\omega \subset \Omega: \ a \rightarrow z} e^{-i\sigma W_\omega(a, z)} \mu^{-\ell(\omega)}.
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![Diagram of winding](image)

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**Lemma (Discrete integrals on elementary contours vanish)**

If $\mu = \mu_* = \sqrt{2 + \sqrt{2}}$ and $\sigma = \frac{5}{8}$, then $F$ satisfies the following relation for every vertex $v \in V(\Omega)$,

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0$$

where $p, q, r$ are the mid-edges of the three edges adjacent to $v$.  

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**Hugo Duminil-Copin & Stanislav Smirnov**

The self-avoiding walk on the hexagonal lattice
We write \( c(\omega) \) for the contribution of the walk \( \omega \) to the sum.

One can partition the set of walks \( \omega \) finishing at \( p \), \( q \) or \( r \) into pairs and triplets of walks:

In the first case,

\[
\begin{align*}
  c(\omega_1) + c(\omega_2) &= (q - v) e^{-i\sigma W_{\omega_1}(a, q)} - \ell(\omega_1) \\
  (r - v) e^{-i\sigma W_{\omega_2}(a, r)} - \ell(\omega_2)
\end{align*}
\]

In the second case,

\[
\begin{align*}
  c(\omega_1) + c(\omega_2) + c(\omega_3) &= (p - v) e^{-i\sigma W_{\omega_1}(a, p)} - \ell(\omega_1) \\
  (1 + \mu - 1) e^{i\frac{2\pi}{3}} e^{-i\frac{5\pi}{8}} - \pi e^{-i\frac{2\pi}{3}} e^{-i\frac{5\pi}{8}}
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In the second case,

$$c(\omega_1) + c(\omega_2) + c(\omega_3) = (p - v) e^{-i \sigma W \omega_1(a, p)} \mu - \ell(\omega_1) + \left(1 + \mu - \frac{1}{e^{i 2 \pi/3}} e^{-i \frac{5\pi}{3}} \cdot -\frac{\pi}{3} \right).$$
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$$= (p - \nu)e^{-\frac{5}{8}W_{\omega_1}(a,p)}\mu^{-\ell(\omega_1)} \left( e^{i\frac{2\pi}{3}} e^{-\frac{i5}{8} - \frac{4\pi}{3}} + e^{i\frac{2\pi}{3}} e^{-\frac{i5}{8} \cdot \frac{4\pi}{3}} \right) = 0$$

In the second case,

$$c(\omega_1) + c(\omega_2) + c(\omega_3)$$

$$= (p - \nu)e^{-i\sigma W_{\omega_1}(a,p)}\mu^{-\ell(\omega_1)} \left( 1 + \mu^{-1}e^{i\frac{2\pi}{3}} e^{-\frac{i5}{8} \cdot \frac{\pi}{3}} + \mu^{-1}e^{-i\frac{2\pi}{3}} e^{-\frac{i5}{8} \cdot \frac{\pi}{3}} \right).$$
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In the second case, providing \( \mu = \mu_* := \sqrt{2 + \sqrt{2}} \),

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\]
If $\mu = \mu_*$ then $\oint F(z) dz = 0$ along an elementary contour.

**Proposition ((partial) Discrete holomorphicity)**

*If $\Omega$ is simply connected, then $\oint_{\Gamma} F(z) dz = 0$ for any discrete contour $\Gamma$.***

Will be used to show $\mu_c = \mu_*$. Take a trapezoid contour $S_{T,L}$:
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$$0 = -\sum_{z \in \alpha} F(z) + \sum_{z \in \beta} F(z) + e^{i \frac{2\pi}{3}} \sum_{z \in \varepsilon} F(z) + e^{-i \frac{2\pi}{3}} \sum_{z \in \bar{\varepsilon}} F(z)$$
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1 = \cos\left(\frac{3\pi}{8}\right) \sum_{\omega: a \to \alpha} \mu_{\omega}^{-\ell(\omega)} + \sum_{\omega: a \to \beta} \mu_{\omega}^{-\ell(\omega)} + \cos\left(\frac{\pi}{4}\right) \sum_{\omega: a \to \epsilon \cup \bar{\epsilon}} \mu_{\omega}^{-\ell(\omega)}.

💡 We know the winding on the boundary!
So we can replace F by the sum of Boltzmann weights.

1 = \frac{\sqrt{2 - \sqrt{2}}}{2} A(T, L, \mu_{\omega}) + B(T, L, \mu_{\omega}) + \frac{1}{\sqrt{2}} E(T, L, \mu_{\omega}).
An upper bound on $\mu_c$:

$$1 = \frac{\sqrt{2} - \sqrt{2}}{2} A(T, L, \mu^*) + B(T, L, \mu^*) + \frac{1}{\sqrt{2}} E(T, L, \mu^*),$$
An upper bound on $\mu_c$:

$$1 = \frac{\sqrt{2} - \sqrt{2}}{2} A(T, L, \mu_*) + B(T, L, \mu_*) + \frac{1}{\sqrt{2}} E(T, L, \mu_*),$$

implies

$$\frac{2}{\sqrt{2} - \sqrt{2}} \geq A(T, L, \mu_*) .$$
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Send $T, L \to \infty$

$$\infty > \frac{2}{\sqrt{2 - \sqrt{2}}} \geq G_{\text{bottom-bottom bridges}}(\mu_*),$$
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Send $T, L \to \infty$

$$\infty > \frac{2}{\sqrt{2} - \sqrt{2}} \geq G_{\text{bottom-bottom bridges}}(\mu_*),$$

hence $\mu_c \leq \mu_*$. 

A lower bound on $\mu_c$:

$$1 = \frac{\sqrt{2} - \sqrt{2}}{2} A(T, L, \mu_*) + B(T, L, \mu_*) + \frac{1}{\sqrt{2}} E(T, L, \mu_*)$$
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As $L \to \infty$, $A$ and $B$ increase to their limits $A(T, \mu_*)$ and $B(T, \mu_*)$. Hence $E$ decreases to its limit $E(T, \mu_*)$. 
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💡 If $E(T, \mu_*) > 0$ for some $T$, then

$$G(\mu_*) \geq \sum_L E(T, L, \mu_*) = \infty.$$

Therefore $\mu_c \geq \mu_*$. 

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💡 If $E(T, \mu_*) > 0$ for some $T$, then

$$G(\mu_*) \geq \sum_L E(T, L, \mu_*) = \infty .$$

Therefore $\mu_c \geq \mu_*$. 

💡 If $E(T, \mu_*) = 0$ for all $T$, then

$$1 = \frac{\sqrt{2 - \sqrt{2}}}{2} A(T, \mu_*) + B(T, \mu_*) .$$
A lower bound on $\mu_c$ (continued):

\[ 1 = \frac{\sqrt{2} - \sqrt{2}}{2} A(T, \mu_*) + B(T, \mu_*) . \]
A lower bound on $\mu_c$ (continued):

$$1 = \frac{\sqrt{2} - \sqrt{2}}{2} A(T, \mu_*) + B(T, \mu_*) .$$

Also clearly

$$A(T + 1, \mu_*) \leq A(T, \mu_*) + B(T, \mu_*)^2 .$$
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We conclude that

$$B(T + 1, \mu_*) \geq B(T, \mu_*) - \frac{\sqrt{2} - \sqrt{2}}{2} \cdot B(T, \mu_*)^2 .$$
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hence

$$B(T, \mu_*) \geq \frac{\text{const}}{\text{const} + T} ,$$

Thus $G(\mu_*) \geq \sum T B(T, \mu_*) = \infty$ and $\mu_c \geq \mu_*$. 

Hugo Duminil-Copin & Stanislav Smirnov
The self-avoiding walk on the hexagonal lattice
Determined the connective constant.

Introduced a discrete holomorphic parafermion.
DONE

- Determined the connective constant.
- Introduced a discrete holomorphic parafermion.

TO DO

- What to do next?
- What not to do next?
What to do next? The case of the self-avoiding walk.

Conjecture (Nienhuis, 1982; Flory, 1948)

- **Combinatorial question:** Up to $n^{o(1)}$ (up to a multiplicative constant?) we have:

  $$c_n \sim n^{\gamma - 1} \left( \sqrt{2 + \sqrt{2}} \right)^n$$

  as $n \to \infty$

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Would follow from the following conjecture
Conjecture (Lawler, Schramm, Werner, 2001)

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For \( \delta > 0 \), we define a probability measure on self-avoiding paths from \( a_\delta \) to \( b_\delta \) by assigning a weight proportional to \( \mu^{-\ell(\omega)} \). When \( \delta \to 0 \), the sequence converges to a random continuous curve.

A strategy to tackle this problem?

1. Precompactness of the family of curves
2. Conformally invariant martingales which are given by the ratio of two parafermionic observables: \( F(a, z, \Omega)/F(a, b, \Omega) \).

Main missing point: show that \( F \) is fully discrete holomorphic.
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What to do next? \( O(n) \) models (1).

The \( O(n) \) model is a model on \textbf{closed loops} lying on a finite subgraph of the hexagonal lattice. The probability of a configuration is equal to

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$$z = \frac{1}{\sqrt{2 + \sqrt{2 - n}}}$$

Critical phase 2: $\text{SLE} \left( \frac{4\pi}{\arccos \left( -\frac{n}{2} \right)} \right)$

Critical phase 1: $\text{SLE} \left( \frac{4\pi}{2\pi - \arccos \left( -\frac{n}{2} \right)} \right)$

Sub-critical phase

Hugo Duminil-Copin & Stanislav Smirnov

The self-avoiding walk on the hexagonal lattice
What to do next? \( O(n) \) models (2).

💡 In the case \( n = 1 \) of the **Ising model**, a similar fermionic observable \( F \) is discrete holomorphic at criticality:

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F(a, z, x) = \sum_{\omega} \text{with a curve } \omega \text{ from } a \text{ to } z \cdot e^{-i \frac{W(\omega)(a, z)}{2}} x^{\text{#edges}}.
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For \( O(n) \) models, the **parafermionic observable**

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F(a, z, x, \sigma) := \sum_{\omega} \text{with a curve } \omega \text{ from } a \text{ to } z \cdot e^{-i \sigma W(\omega)(a, z)} x^{\text{#edges}} n^{\text{#loops}}.
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should be discrete holomorphic for \( x = x_c \) and

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2 \cos\left(\frac{4 \sigma \pi}{3}\right) = -n.
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So far only partial discrete holomorphicity observed.
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**Conjecture**

For $n \in [0, 2]$ and $x = x_c(n)$, the interface between two points $a$ and $b$ (on the boundary) converges, as the lattice step goes to zero, to SLE($\kappa$) where

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Determined the connective constant.

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⚠️ Do not work with the square lattice self-avoiding walk!

Consider a more general model on the square lattice, with the following weights

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \]

There are only two families of solutions: one possesses negative weights, the other is exactly equivalent to the hexagonal $O(n)$ model at criticality. The solutions correspond to integrable points of the model (when the Yang-Baxter condition applies).
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![Graphical representation of weights]

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Conclusion

We can introduce parafermionic observables for a wide variety of models: $O(n)$-models, random-cluster models, self-avoiding walks...

We can extract information from these operators in order to study the critical phase (example of the connective constant of the hexagonal lattice).

In some cases, the information is total – universality class of the Ising model – and we can derive conformal invariance.

Question: Can we do the same for other models?
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Thank you