Self-similar families of groups

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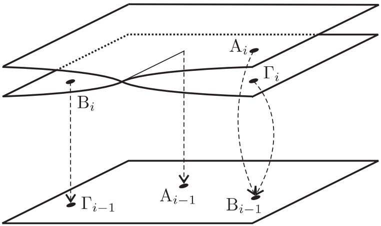
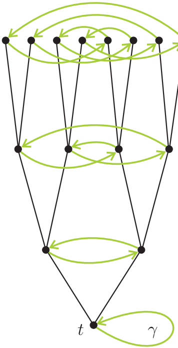
March 14, 2008,
Les Diablerets

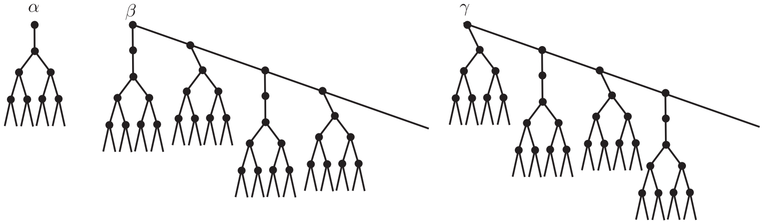
$\mathcal{D}_w$ as Iterated Monodromy Groups

Let C_i be planes and let $\mathrm{A}_i, \mathrm{B}_i, \Gamma_i \in C_i$. Let $f_i : C_i \to C_{i-1}$ by 2-fold branched coverings such that



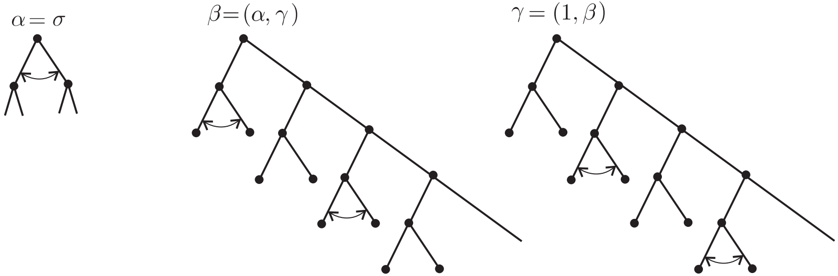


Let α, β and γ be loops around A_0, B_0 and Γ_0, respectively.
Then the trees of orbits of the corresponding automorphisms of the rooted tree are



Hence they are conjugate to $\alpha = \sigma$, $\beta = (\alpha, \gamma)$ and $\gamma = (1, \beta)$.

α, β, γ



The Family $\mathcal{D}_w$

Let $w \in \{0,1\}^\infty$ and $w = x\overline{w}$.
Define

$$\alpha_w = \sigma,$$
$$\beta_w = \left(\alpha_{\overline{w}}, \gamma_{\overline{w}}\right),$$
$$\gamma_w = \begin{cases} (\beta_{\overline{w}}, 1) & \text{if } x = 0, \\ (1, \beta_{\overline{w}}) & \text{if } x = 1. \end{cases}$$

Let $\mathcal{D}_w = \langle \alpha_w, \beta_w, \gamma_w \rangle$.
$\mathcal{D}_{00\ldots} = \mathrm{IMG}\left(z^2 + i\right)$
$\mathcal{D}_{11\ldots} = G_{0101\ldots}$ (a Grigorchuk group).

Proposition

Suppose that h_0, h_1, h_2 *are conjugate to* α, β, γ *in* $\mathrm{Aut}(X^*)$*. Then there exists a unique* $w \in \{0,1\}^\infty$ *such that* h_0, h_1, h_2 *are simultaneously conjugate to* $\alpha_w, \beta_w, \gamma_w$*.*

Corollary

The iterated monodromy group of any backward iteration of the given form is isomorphic to $\mathcal{D}_w$ *for some* w*.*

Proposition

For any $w \in \{0,1\}^\infty$ the set of $w' \in \{0,1\}^\infty$ such that $\mathcal{D}_w$ is conjugate with $\mathcal{D}_{w'}$ is at most countable.

Theorem

Groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are isomorphic if and only if they are conjugate in $\mathrm{Aut}(X^*)$.

The Family $\mathcal{R}_w$

Let $w \in \{0,1\}^\infty$ and $w = x\overline{w}$.
Define

$$\alpha_w = \sigma\left(1, \gamma_{\overline{w}}\right),$$
$$\beta_w = \begin{cases} (\alpha_{\overline{w}}, 1) & \text{if } x = 0, \\ (1, \alpha_{\overline{w}}) & \text{if } x = 1, \end{cases}$$
$$\gamma_w = \left(1, \beta_{\overline{w}}\right).$$

Let $\mathcal{R}_w = \langle \alpha_w, \beta_w, \gamma_w \rangle$.
$\mathcal{R}_{11\ldots} = \mathrm{IMG}\left(z^2 + (-0.1226\ldots + 0.7449\ldots i)\right)$ and
$\mathcal{R}_{00\ldots} = \mathrm{IMG}\left(z^2 - 1.7549\ldots\right)$.

Grigorchuk groups

Let $w \in \{0, 1, 2\}^\infty$ and $w = x\overline{w}$.
Define

$$a_w = \sigma,$$
$$b_w = \begin{cases} (1, b_{\overline{w}}) & \text{if } x = 0, \\ (a_{\overline{w}}, b_{\overline{w}}) & \text{otherwise}, \end{cases}$$
$$c_w = \begin{cases} (1, c_{\overline{w}}) & \text{if } x = 1, \\ (a_{\overline{w}}, c_{\overline{w}}) & \text{otherwise}, \end{cases}$$
$$d_w = \begin{cases} (1, d_{\overline{w}}) & \text{if } x = 2, \\ (a_{\overline{w}}, d_{\overline{w}}) & \text{otherwise}, \end{cases}$$

Denote $G_w = \langle a_w, b_w, c_w, d_w \rangle$.

The Space of Finitely Generated Groups

Let $F_n = \langle a_1, a_2, \ldots, a_n \mid \emptyset \rangle$.
The set $\mathfrak{G}_n$ of quotients of F_n, i.e., the set of *marked* n-generated groups

$$\mathfrak{G}_n = \{(G, a_1, \ldots, a_n) \ : \ \langle a_1, \ldots, a_n \rangle = G\}$$

is identified with the set of normal subgroups of F_n, which has a natural topology of a subset of 2^{F_n}.
Two groups are close if their Cayley graphs coincide on a large ball.

Theorem

The map $\{0,1\}^\infty \to \mathfrak{G}_3$

$$w \mapsto (\mathcal{R}_w, \alpha_w, \beta_w, \gamma_w)$$

is a homeomorphic embedding.

Let $\Omega \subset \{0,1\}^\infty$ be the set of sequences which have infinitely many zeros. Denote by D_w the limit of $\mathcal{D}_{w_n}$ for $w_n \in \Omega$ and $w_n \to w$.

Theorem

We have $D_w = \mathcal{D}_w$ *for* $w \in \Omega$. *The map* $\{0,1\}^\infty \to \mathfrak{G}_3$

$$w \mapsto (D_w, \alpha_w, \beta_w, \gamma_w)$$

is a homeomorphic embedding.

Similar result holds also for the family of Grigorchuk groups. The exceptional groups in this case are G_w for eventually constant w.

Theorem

Two groups D_{w_1}, D_{w_2} are isomorphic, if and only if the sequences w_1 and w_2 are cofinal, i.e., if they are of the form $w_1 = v_1 u$ and $w_2 = v_2 u$ for $|v_1| = |v_2|$.
The isomorphism classes are dense and countable in the family $\{\mathcal{R}_w\}_{w\in\{0,1\}^\infty}$.

Corollary

For any $w_1, w_2 \in \{0,1\}^\infty$ and any finite set of relations and inequalities between the generators of D_{w_1} there are generators of D_{w_2} such that the same relations and inequalities hold.

Theorem (K.-U.Bux and R.Perez)

The group $\mathcal{D}_{000\ldots} = \mathrm{IMG}\left(z^2 + i\right)$ *has intermediate growth.*

Theorem (A.Erschler)

The group $\mathcal{D}_{111\ldots} = G_{0101\ldots}$ *has intermediate growth satisfying the estimates*

$$\exp\left(\frac{n}{\log^{2+\epsilon}(n)}\right) \preceq \gamma(n) \preceq \exp\left(\frac{n}{\log^{1-\epsilon}(n)}\right).$$

Theorem

The group $D_{111\ldots}$ *is of exponential growth, hence it has non-uniform exponential growth.*

The kernel of the epimorphism $D_{111\ldots} \to \mathcal{D}_{111\ldots}$ is C_4^∞.

Universal Groups of the Families

Let $\mathcal{D}$ be the subgroup of $\prod_{w\in\{0,1\}^\infty} \mathcal{D}_w$ generated by the "diagonal" elements

$$(\alpha_w)_{w\in\{0,1\}^\infty}, (\beta_w)_{w\in\{0,1\}^\infty}, (\gamma_w)_{w\in\{0,1\}^\infty}.$$

This group can be also defined as

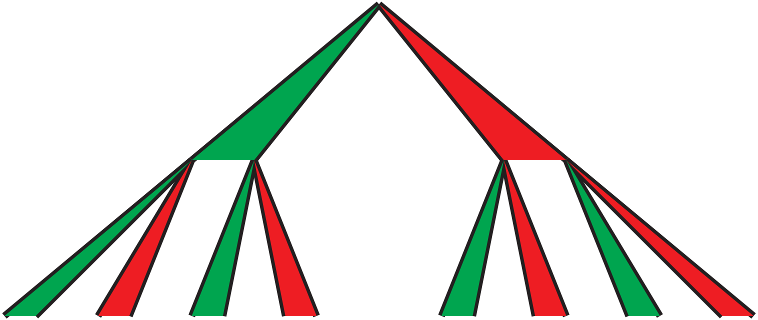
$$\langle \alpha, \beta, \gamma \mid \emptyset \rangle \Big/ \bigcap_{w\in\{0,1\}^\infty} N_w ,$$

where N_w is the kernel of the epimorphism $\alpha \mapsto \alpha_w, \beta \mapsto \beta_w, \gamma \mapsto \gamma_w$. Let us call $\mathcal{D}$ the *universal group* of the family $\{\mathcal{D}_w\}$.

The universal group $\mathcal{D}$ is also self-similar. It is generated by

$$\alpha = (1,2)(3,4)$$
$$\beta = (\alpha, \gamma, \alpha, \gamma)$$
$$\gamma = (\beta, 1, 1, \beta)$$

Identify $1 \leftrightarrows (0,0)$, $2 \leftrightarrows (1,0)$, $3 \leftrightarrows (0,1)$ and $4 \leftrightarrows (1,1)$.
Then $\mathcal{D}$ acts only on the first coordinates of letters.
Let $T_{y_1 y_2 \ldots}$ be the subtree consisting of the words $(x_1, y_1)(x_2, y_2)\ldots(x_n, y_n)$.
The subtrees T_w are $\mathcal{D}$-invariant.
Restriction of $\mathcal{D}$ onto T_w is $\mathcal{D}_w$.



A bigger group

Let $\widetilde{\mathcal{D}}$ be the group generated by

$$\begin{aligned} \alpha &= (12)(34), & a &= (13)(24), \\ \beta &= (\alpha, \gamma, \alpha, \gamma), & b &= (a\alpha, a\alpha, c, c), \\ \gamma &= (\beta, 1, 1, \beta), & c &= (b\beta, b\beta, b, b). \end{aligned}$$

Note that the group $\widetilde{\mathcal{D}}$ permutes the subtrees T_w.

Proposition

The following relations hold.

$$\begin{array}{lll}\alpha^a = \alpha, & \alpha^b = \alpha, & \alpha^c = \alpha,\\ \beta^a = \beta, & \beta^b = \beta, & \beta^c = \beta^\gamma,\\ \gamma^a = \gamma^\alpha, & \gamma^b = \gamma^\beta, & \gamma^c = \gamma.\end{array}$$

In particular, $\mathcal{D} \lhd \widetilde{\mathcal{D}}$.

The subgroup $\mathcal{D} \lhd \widetilde{\mathcal{D}}$ coincides with the set of elements acting trivially on the second coordinates of letters (i.e., leaving the subtrees T_w invariant). Hence, the quotient $H = \widetilde{\mathcal{D}}/\mathcal{D}$ acts naturally on the binary tree by the action

$$\begin{array}{ccc} a = \sigma, & b = (a, c), & c = (b, b). \\ a = (13)(24) & b = (a\alpha, a\alpha, c, c)\,, & c = (b\beta, b\beta, b, b)\,. \end{array}$$

The group $\widetilde{\mathcal{D}}$ permutes the subtrees T_w in the same way as H acts on $w \in \{0, 1\}^\infty$.

Consequently, if w_1 and w_2 belong to one H-orbit, then $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are isomorphic.

Questions

Are all groups $\mathcal{D}_w$ of intermediate growth?

Are all groups $\mathcal{R}_w$ amenable?

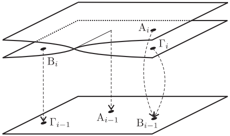
Are the universal groups of the three families amenable? They do not contain free subgroups.

Analytic interpretation

Let us identify C_0 with $\mathbb{C}$. Then there exist unique complex structures on C_i such that f_i are holomorphic (i.e., are polynomials).
We may assume that A_i and B_i coincide with 0 and 1, respectively.
Then position $p_i \in \mathbb{C}$ of Γ_i parametrizes the complex structure on C_i.

f_i is a quadratic polynomial such that

1. its critical value is 0,
2. $f_i(0) = 1$,
3. $f_i(1) = p_{i-1}$,
4. and $f_i(p_i) = 1$.



We get $f_i = (az+1)^2$ and $ap_i + 1 = -1$, hence $f_i(z) = \left(1 - \frac{2z}{p_i}\right)^2$, $p_{i-1} = \left(1 - \frac{2}{p_i}\right)^2$.

We get a map

$$F : \begin{pmatrix} z \\ p \end{pmatrix} \mapsto \begin{pmatrix} \left(1 - \frac{2z}{p}\right)^2 \\ \left(1 - \frac{2}{p}\right)^2 \end{pmatrix}.$$

$$F(z : p : u) = \left((p - 2z)^2 : (p - 2u)^2 : p^2\right)$$

$\mathrm{IMG}\,(F)$ coincides with $\langle ab, ac, \alpha, \beta, \gamma\rangle < \widetilde{\mathcal{D}}$ and $\mathrm{IMG}\,(F)\,/\mathcal{D} \cong \mathrm{IMG}\left(\left(1 - \frac{2}{p}\right)^2\right)$.

Family $\mathcal{R}_w$

Analogous computations for the family $\mathcal{R}_w$ give the rational function

$$F : \begin{pmatrix} z \\ p \end{pmatrix} \mapsto \begin{pmatrix} 1 - \frac{z^2}{p^2} \\ 1 - \frac{1}{p^2} \end{pmatrix}.$$

Theorem

The limit space of $\widehat{\mathcal{R}}$ is homeomorphic to the "Julia set" J_2 of F.