RANDOM WALKS ON TREES, SPECTRAL RADII OF GROUPS, AND RAMANUJAN GRAPHS

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Abstract. We investigate properties of random walks on trees with finitely many cone types and apply our results to get estimates on spectral radii of groups and to check whether a given finite graph is Ramanujan.

1. Introduction

The notion of a cone type in a graph was introduced by Cannon in [Can1] (see also [Can2]). Probabilities on infinite trees with finitely many cone types were first studied by Lyons in [Ly]. An investigation of random walks on such trees was undertaken in the author's PhD thesis [Nag2] and was pursued in a joint paper with Woess [SNW]. Two main motivations for this investigation were given by two important classes of trees with finitely many cone types: trees of geodesics of hyperbolic groups (more generally, groups with finitely many cone types, e.g., also all Coxeter groups), and universal covers of finite graphs. Computation of the spectral radius of a certain random walk on the tree allows, in the first case, to get a good estimate on the spectral radius of the group and, in the second case, to decide whether a given finite graph is Ramanujan.

We shall begin by reviewing relevant results on random walks on trees with finitely many cone types and then turn our attention to the two applications.

2. Random walks on trees with finitely many cone types

Let Γ be an infinite, locally finite, connected, simple graph with the vertex set $V(\Gamma)$. For $x, y \in V(\Gamma)$ a *path* of length n from x to y is a sequence $x = x_1, x_2, ..., x_{n+1} = y$ of vertices, such that, for each $1 \leq i \leq n$, the vertices x_i and x_{i+1} are neighbours, $x_i \sim x_{i+1}$. The *distance* between two vertices $d(x, y)$ is the length of a shortest path from x to y. A shortest path is called *geodesic*. With a base point $x_0 \in V(\Gamma)$ fixed, the *norm* of a vertex x is $|x| = d(x_0, x)$.

The author would like to acknowledge the hospitality of the Erwin Schrödinger Institute and thank Vadim Kaimanovich, Klaus Schmidt and Wolfgang Woess for organizing an excellent semester on Random Walks in Vienna in 2000.

A random walk on Γ is described by a stochastic transition matrix

$$P = \big(p(x,y)\big)_{x,y\in V(\Gamma)} .$$

The *Green kernel* on Γ associated with the random walk P is defined by

$$G(x,y \mid z) = \sum_{n=0}^{\infty} p^{(n)}(x,y) z^n .$$

The radius of convergence of the series $G(x,y \mid z)$ is $R_G = 1/\rho \geq 1$ with

$$\rho = \limsup_{n\to\infty} \left(p^{(n)}(x,y)\right)^{\frac{1}{n}} .$$

It is independent of x and y if P is irreducible. The number ρ is called the *spectral radius* of the random walk P. It is indeed the spectral radius of the transition operator P acting on the space $l^2_\sharp(V(\Gamma))$ with the inner product $\langle\phi,\psi\rangle_\sharp = \sum_{x\in V(\Gamma)} \phi(x)\psi(x)deg(x)$.

Denote by $q^{(n)}(x,y)$ the probability, starting from x, to reach y for the first time after n steps; and consider the kernel

$$F(x,y \mid z) = \sum_{n=0}^{\infty} q^{(n)}(x,y) z^n .$$

In the disc of convergence of G we have

$$\begin{aligned} G(x,y \mid z) &= I(x,y) + F(x,y \mid z)G(y,y \mid z) \\ G(x,x \mid z) &= \big(1 - F(x,x \mid z)\big)^{-1} . \end{aligned} \tag{1}$$

The function $F(x,x \mid z)$ is analytic, thus continuous in its disc of convergence; $F(x,x \mid 0) = 0$; and it is increasing in the intersection of $\mathbb{R}_+$ with its disc of convergence. Therefore, either $F(x,x \mid R_F) \leq 1$, or, by continuity, there exists a unique real positive z_0 in the disc of convergence of $F(x,x \mid z)$, such that $F(x,x \mid z_0) = 1$. Therefore R_G either coincides with R_F, if $F(x,x \mid R_F) \leq 1$, or is equal to z_0. The power series $F(x,y \mid z)$ (respectively $G(x,y \mid z)$) has non-negative coefficients, hence by Prinsgheim's Theorem R_F (respectively R_G) is a singular point of $F(x,y \mid z)$ (respectively $G(x,y \mid z)$).

Geometry of the graph is important for the study of random walks on it. *Trees* are particularly nice in this respect as any two vertices x,y are connected by a unique path $(x_1 = x, x_2, ..., x_k = y)$ in the tree, and consequently $F(x,y \mid z) = \prod_{i=1}^{k-1} F(x_i, x_{i+1} \mid z)$. The book [W] contains much information and extensive bibliography on random walks on trees.

In this paper we shall turn our attention to *trees with finitely many cone types* which are defined as follows. Let T be an infinite, locally finite tree. We say that a vertex x is a *predecessor* of a vertex y and y is a *successor* of x, if x and y are neighbours (i.e. connected by an edge), and $|x| < |y|$. We define the *cone* $\mathcal{C}(x)$ of a vertex x of T as the induced subgraph of T rooted at x with $V\big(\mathcal{C}(x)\big) = \{y \in V(T) \mid x$ belongs to a geodesic from x_0 to $y\}$. Two vertices x and y are said to be of a same *cone type* if their cones are isomorphic as rooted graphs. Consider a function $t : V(T) \to \mathbb{Z}_+$, such that $t(x) = 0$ if and only if $x = x_0$ and $t(x_1) = t(x_2)$ if and only if $x_1, x_2 \in V(T) \setminus \{x_0\}$ and are of a same cone type. We will say that a vertex x is *of type* $t(x)$. Thus, we assume that the type of the base point is different from the type of any other vertex, even if the cone of x_0 is isomorphic to the cone of some other vertex. Since T is a tree, each vertex x of T except x_0 has exactly 1 predecessor $pr(x)$. The number $s_{i,j}$ of successors of type j of a vertex of type i is well defined for every $i \in \mathbb{Z}_+$ and $j \in \mathbb{N}$.

Suppose that there exists a type function t on T which takes values in a finite set $\{0, 1, ..., K\}$. We say then that T has finitely many cone types. There is a way to view every infinite tree with finitely many cone types as a cover of a finite digraph. Suppose G is a finite directed graph with a base vertex x_0. The *directed cover of G with respect to x_0* is a rooted infinite tree T^{G,x_0} such that the set of its vertices is in one-to-one correspondence with the set of directed paths in G starting in x_0. Two vertices are connected by an edge (a priori undirected) in T^{G,x_0} if one of the corresponding directed paths in G is the expansion of the other one by a directed edge.

Proposition 1. *For every finite digraph G and every $x_0 \in V(G)$, the tree T^{G,x_0} has finitely many cone types. Conversely, for every tree T with finitely many cone types there exists a finite digraph G and $x_0 \in V(G)$, such that $T = T^{G,x_0}$.*

Proof. Suppose that two directed paths starting at x_0 in G end at a same vertex. Then the cones of the corresponding vertices in T^{G,x_0} are isomorphic. Thus the number of the cone types in T^{G,x_0} is less or equal to $1 + |V(G)|$.

Let now T be a tree with $K + 1$ cone types. Let $s_{i,j}$ denote the number of successors of type j of a vertex of type i in T. Construct a finite directed graph G as follows. There are $K + 1$ vertices $\{x_0, ..., x_K\}$ in G. For each $i = 0, 1, ..., K$, $j = 1, ..., K$, there are $s_{i,j}$ directed edges connecting the vertex x_i with the vertex x_j. Obviously, $T = T^{G,x_0}$. □

Consider an infinite, locally finite tree T with $K+1$ cone types. On T consider a nearest neighbour random walk $P = \big(p^{(n)}(x, y)\big)_{x,y\in V(T)}$ such that $p(x, pr(x))$ depends only on the cone type of x for any $x \in V(T) \setminus \{x_0\}$, and $p(x, y)$ depends only on the cone types of x and y for any $x \in V(T)$ and for any successor y of x. Consequently, the probability $p(x, pr(x))$ for a vertex of type $i, i \geq 1$ will be denoted by p_{-i}, and the probability $p(x, y)$ will be denoted by $p_{i,j}$ for x of type i and y a successor of x of type j. We have

$$\begin{aligned} p_{-i} \;+ \sum_{j=1,...,K} s_{i,j} p_{i,j} \;&=\; 1, \quad j = 1, ..., K\ ; \\ \sum_{j=1,...,K} s_{0,j} p_{0,j} \;&=\; 1\ . \end{aligned}$$

Theorem 1. *For every $x, y \in V(T)$, the Green function $G(x, y \mid z)$ is an algebraic element over the field $\mathbb{Q}(z, \{p_{-i}z\}, \{p_{i,j}z\})$. More precisely, there exists a unique (up to a sign) non-constant irreducible polynomial $\mathcal{P}_{x,y}(w, z_1, z_2, ..., z_{1+K(K+2)})$ in $2+K(K+2)$ variables* *with integer coefficients, such that*

$$\mathcal{P}_{x,y}\big(G(x,y \mid z), z, p_{-1}z, ..., p_{-K}z, \{p_{i,j}z\}_{\substack{i=0,1,...,K \\ j=1,...,K}}\big) \;\equiv\; 0 \;.$$

Moreover, if $(x_0, x_1, ..., x_k = x)$ *denotes the geodesic from* x_0 *to* x *and* $(x_0, y_1, ..., y_r = y)$ *denotes the geodesic from* x_0 *to* y*, then the coefficients of the polynomial* $\mathcal{P}_{x,y}$ *depend only on the cone types of the vertices* $x_1, ..., x_k, y_1, ..., y_r$*.*

Lemma 1. *The probability* $q^{(n)}(x, pr(x))$ *depends only on the cone type of* x *for any* $x \in V(T) \setminus \{x_0\}$ *and for any* $n \in \mathbb{N}$*.*

Proof. Obviously $q^{(n)}(x, pr(x)) \neq 0$ only if n is odd. For $n = 1$ we have

$$q^{(1)}(x, pr(x)) \;=\; p(x, pr(x)) \;=\; p_{-i} \;,$$

where $t(x) = i$. By induction, for $n \geq 3$, n odd,

$$q^{(n)}(x, pr(x)) \;=\; \sum_{y \text{ sucessor of } x} p(x,y) \sum_{k=1}^{\frac{n-1}{2}} q^{(2k-1)}(y, pr(y)) q^{(n-2k)}(x, pr(x))$$

depends only on the cone type of x. □

We can now denote by $q_{-i}^{(n)}$ the probability, starting from a point of type i, to approach x_0 for the first time after n steps. We also denote by $F_{-i}(z)$ the function $F\big(x, pr(x) \mid z\big)$ for any x of type i, and we have

$$F_{-i}(z) \;=\; \sum_{n=0}^{\infty} q_{-i}^{(n)} z^n \;.$$

We shall see that the finite collection of functions $\{F_{-i}(z)\}_{i=1,...,K}$ contains all the information about the Green kernel on T and its radius of convergence. (Note that as T is a tree, it therefore also contains all the information about the Martin boundary of (T, P).)

Proposition 2. *The power series* $F_{-i}(z)$*,* $i = 1, ..., K$*, satisfy the following system of polynomial equations:*

$$w_i \;=\; p_{-i} z \;+\; z \sum_{j=1}^{K} s_{i,j} p_{i,j} w_i w_j \;, \quad i = 1, ..., K \;. \tag{2}$$

Proof. As in Lemma 1, we have

$$q_{-i}^{(1)} \;=\; p_{-i} \;,$$

and, for $n \geq 3$

$$q_{-i}^{(n)} \;=\; \sum_{j=1}^{K} s_{i,j} p_{i,j} \sum_{k=1}^{\frac{n-1}{2}} q_{-j}^{(2k-1)} q_{-i}^{(n-2k)} \;.$$

Thus

$$F_{-i}(z) = \sum_{n=0}^{\infty} q_{-i}^{(n)} z^n = zp_{-i} + z\sum_{j=1}^{K} s_{i,j}p_{i,j} \sum_{n=3}^{\infty}\sum_{k=1}^{\frac{n-1}{2}} q_{-j}^{(2k-1)} z^{2k-1} q_{-i}^{(n-2k)} z^{n-2k}$$
$$= zp_{-i} + z\sum_{j=1}^{K} s_{i,j}p_{i,j} F_{-i}(z)F_{-j}(z) .$$

□

Let us now express functions $F(x, y \mid z)$, $x, y \in V(T), x \sim y$, in terms of the functions $\{F_{-i}(z)\}_{i=1,\dots,K}$. First, we have

$$F(x_0, x_0 \mid z) = \sum_{j=1}^{K} s_{0j}p_{0j}zF_{-j}(z) . \tag{3}$$

Now let x be a vertex of T and y a successor of x. Then, for $n \geq 3$, n odd,

$$q^{(n)}(x, y) = \sum_{y' \text{ a successor of } x} p(x, y') \sum_{r=1}^{\frac{n-1}{2}} q^{(2r-1)}(y', pr(y'))q^{(n-2r)}(x, y)$$
$$+ p(x, pr(x)) \sum_{r=1}^{\frac{n-1}{2}} q^{(2r-1)}(pr(x), x)q^{(n-2r)}(x, y) .$$

As follows from Lemma 1, the probabilities $p(x, y')$ and $q^{(r)}(y', pr(y'))$ depend only on the cone type of x and its successors (which are determined by the cone type of x). On the contrary, the cone type of $pr(x)$ is not determined by the cone type of x, thus the probabilities $p(x, pr(x))$ and $q^{(r)}(pr(x), x)$ depend also on the cone type of $pr(x)$. The probabilities $q^{(r)}(pr(x), x)$ depend in turn on the cone types of x, $pr(x)$, and $pr(pr(x))$. Each x in $V(T)$ is uniquely determined by the sequence $(x, pr(x), pr(pr(x)), \dots, pr^n(x) = x_0)$, which coincides with the unique geodesic from x to x_0 in T. Hence, the probabilities $q^{(n)}(x, y)$ depend on the cone types of y, x, and all the vertices which lie on the geodesic connecting x and x_0 in T.

Theorem 2. *Let x be a vertex of T, y be a successor of x of type j, and let $(x_0, x_1, \dots, x_k = x)$ with $t(x_m) = i_m$ for $m = 1, \dots, k$ be the unique geodesic from x_0 to x. Then*

$$F(x, y \mid z) = \cfrac{zp_{i_k,j}}{H_k(z) + \cfrac{z^2 p_{i_{k-1},i_k}}{H_{k-1}(z) + \cfrac{z^2 p_{i_{k-2},i_{k-1}}}{\ddots \, H_1(z) + \cfrac{z^2 p_{0,i_1}}{H_0(z)}}}} ,$$

where

$$\begin{aligned}
H_k(z) \ &= 1 - z \sum_{\substack{l=1 \\ l\neq j}}^{K} p_{i_k,l} F_{-l}(z) - z p_{i_k,j}(s_{i_k,j}-1) F_{-j}(z) \ ; \\
&\qquad \textit{for } m = k-1, \ldots, 1 \ : \\
H_m(z) \ &= 1 - z \sum_{\substack{l=1 \\ l\neq i_{m+1}}}^{K} p_{i_m,l} s_{i_m,l} F_{-l}(z) - z p_{i_m,i_{m+1}}(s_{i_m,i_{m+1}}-1) F_{-i_{m+1}}(z) \ ; \\
H_0(z) \ &= 1 - z \sum_{\substack{l=1 \\ l\neq i_1}}^{K} p_{0,l} s_{0,l} F_{-l}(z) - z p_{0,i_1}(s_{0,i_1}-1) F_{-i_1}(z).
\end{aligned}$$

Proof. As shown above, the function $F(x,y \mid z)$ depends only on $j, i_1, ..., i_k$, and we denote $F(x,y \mid z) = F_{0,i_1,\ldots,i_k,j}(z)$. The easiest case is that of the function $F_{0,i_1}(z)$. We have the following formulas for the probability to go from x_0 to one of its neighbours x of type i:

$$\begin{aligned}
q_{0,i}^{(1)} \ &= \ p_{0,i} \ , \\
q_{0,i}^{(n)} \ &= \ \sum_{\substack{j=0 \\ j\neq i}}^{K} p_{0,j} s_{0,j} \sum_{k=1}^{\frac{n-1}{2}} q_{-j}^{(2k-1)} q_{0,i}^{(n-2k)} + p_{0,i}(s_{0,i}-1) \sum_{k=1}^{\frac{n-1}{2}} q_{-i}^{(2k-1)} q_{0,i}^{(n-2k)}, \quad n \geq 3.
\end{aligned}$$

Thus

$$F_{0,i}(z) \ = \ z p_{0,i} \ + \ z \sum_{\substack{j=1 \\ j\neq i}}^{K} p_{0,j} s_{0,j} F_{-j}(z) F_{0,i}(z) \ + \ z p_{0,i}(s_{0,i}-1) F_{-i}(z) F_{0,i}(z)$$

and

$$F_{0,i}(z) \ = \ \frac{z p_{0,i}}{1 - z \sum_{\substack{j=1 \\ j\neq i}}^{K} p_{0,j} s_{0,j} F_{-j}(z) - z p_{0,i}(s_{0,i}-1) F_{-i}(z)} \ .$$

The probability of going from x (to which we associate the geodesic $(x_0, x_1, ..., x_k = x)$ with $t(x_m) = i_m, m = 1, ..., k$) to one of its successors of type j in n steps can be computed similarly:

$$
\begin{aligned}
q^{(1)}_{0,i_1,\dots,i_k,j} &= p_{i_k,j} \ , \\
q^{(n)}_{0,i_1,\dots,i_k,j} &= \sum_{\substack{l=0\\ l\neq j}}^{K} p_{i_k,l} s_{i_k,l} \sum_{r=1}^{\frac{n-1}{2}} q^{(2r-1)}_{-l} q^{(n-2r)}_{0,i_1,\dots,i_k,j} \\
&+ \ p_{i_k,j}(s_{i_k,j}-1) \sum_{r=1}^{\frac{n-1}{2}} q^{(2r-1)}_{-j} q^{(n-2r)}_{0,i_1,\dots,i_k,j} \\
&+ \ p_{-i_k} \sum_{r=1}^{\frac{n-1}{2}} q^{(2r-1)}_{0,i_1,\dots,i_{k-1},i_k} q^{(n-2r)}_{0,i_1,\dots,i_k,j}, \qquad n \geq 3 \ ,
\end{aligned}
$$

which implies

$$
\begin{aligned}
F_{0,i_1,\dots,i_k,j}(z) \ = \ zp_{i_k,j} \ &+ \ z \sum_{\substack{l=1\\ l\neq j}}^{K} p_{i_k,l} s_{i_k,l} F_{-l}(z) F_{0,i_1,\dots,i_k,j}(z) \\
&+ \ zp_{i_k,j}(s_{i_k,j}-1) F_{-j}(z) F_{0,i_1,\dots,i_k,j}(z) \\
&+ \ zp_{-i_k} F_{0,i_1,\dots,i_{k-1},i_k}(z) F_{0,i_1,\dots,i_k,j}(z) \ .
\end{aligned}
$$

Hence we have

$$
F_{0,i_1,\dots,i_k,j}(z) \ = \ \frac{zp_{i_k,j}}{\Delta} \ ,
$$

where

$$
\Delta \ = \ 1 - z \sum_{\substack{l=1\\ l\neq j}}^{K} p_{i_k,l} s_{i_k,l} F_{-l}(z) - zp_{i_k,j}(s_{i_k,j}-1) F_{-j}(z) - zp_{-i_k} F_{0,i_1,\dots,i_{k-1},i_k}(z) \ .
$$

In this formula for $F_{0,i_1,\dots,i_k,j}(z)$ we can replace $F_{0,i_1,\dots,i_{k-1},i_k}(z)$ by the similar expression, in which only the functions $\{F_{-i}(z)\}$ and the function $F_{0,i_1,\dots,i_{k-2},i_{k-1}}(z)$ appear. Repeating this procedure $k-1$ times, we get an expression for $F_{0,i_1,\dots,i_{k-1},i_k}$ in terms of the functions $\{F_{-i}(z)\}$ and the function $F_{0,i_1}(z)$, which in its turn can be expressed only in terms of the functions $\{F_{-i}(z)\}$. Putting all these expressions together, we get the desired formula for $F_{0,i_1,\dots,i_k,j}(z)$. □

Proof of Theorem 1. It follows from Proposition 2 that the functions $F_{-i}(z)$ are algebraic elements over the field $\mathbb{Q}(z, \{p_{-i}z\}, \{p_{i,j}z\})$. Theorem 2 shows that the functions $F(x, y \mid z)$ can be obtained from $\{F_{-i}(z)\}$ by a finite number of operations of addition, multiplication by a scalar from the field $\mathbb{Q}(z, \{p_{-i}z\}, \{p_{i,j}z\})$, and taking inverse. Thus each function $F(x, y \mid z)$ is an algebraic element over the same field. Consequently, $G(x, y \mid z)$ is also algebraic over the same field by (1). □

It can be deduced from Theorem 1 that the spectral radius of a tree with finitely many cone types is an algebraic number. Moreover, we shall explain presently how it can be computed from the system (2). Computations of this type had appeared in many different places, in a similar context e.g. in Lalley's study of finite range random walks on a free group [Lal], see also [W]. A similar method was used in [SNW] to compute the rate of escape of the random walk P on T.

Lemma 2. *Denote by R_i the radius of convergence of $F_{-i}(z)$, $i = 1, ..., K$. Suppose that for every vertex $x \in V(T)$ the cone $\mathcal{C}(x)$ contains vertices of all types $1, ..., K$. Then $R_F = R_1 = ... = R_K$.*

Proof. Let x be a vertex of type i. For $j \in \{1, ..., K\}$, denote by $n_{i,j}$ the minimal distance between x and a vertex of type j in the cone $\mathcal{C}(x)$. Let $y \in \mathcal{C}(x)$ be a vertex of type j which lies at distance $n_{i,j}$ from x. Denote by $(x_0 = x, x_1, ..., x_{n_{i,j}-1}, x_{n_{i,j}} = y)$ the geodesic between x and y, and by t_k the type of x_k, $k = 1, ..., n_{i,j} - 1$. Put $C_{i,j} := p_{it_1} p_{t_1 t_2} ... p_{t_{n_{i,j}-1} j} p_{-j} p_{-t_{n_{i,j}-1}} ... p_{-t_1} p_{-i}$. Then $F_{-i}(z) \geq C_{i,j} F_{-j}(z)$, but also $F_{-j}(z) \geq C_{j,i} F_{-i}(z)$. Thus $R_i = R_j$. We further have $R_F = R_1 = ... = R_K$ by (3). □

Denote by $J(z)$ the Jacobian matrix of the system (2)

$$J(z) \;=\; \Big(\frac{\partial \mathcal{P}_i(z, w_1, ..., w_K)}{\partial w_j}\Big)_{i,j=1,..,K} ,$$

where $\mathcal{P}_i(z, w_1, ...w_K) \;=\; p_{-i} z \;+\; z \sum_{j=1}^{K} s_{i,j} p_{i,j} w_i w_j$, $i = 1, ..., K$. Under the assumption of Lemma 2, Perron-Frobenius Theorem can be applied to the matrix $J(z)$ to conclude that $J(z)$ has a positive simple eigenvalue $\lambda_*(z)$ which maximizes the eigenvalues of $J(z)$ in absolute value.

Theorem 3. *Under the assumption of Lemma 3*

$$R_F \;=\; \min\{z > 0 \mid \lambda_*(z) = 1\} .$$

Proof. The argument from [W, p.211] applies without any changes. □

Recall that the inverse of the spectral radius, R_G, is equal to R_F unless the random walk is ρ-positive recurrent (see [SNW, 2.4]). In this latter case R_G is a pole and the unique solution of $F(x_0, x_0 \mid z) = 1$.

3. Spectral radius of a group with finitely many cone types

Let G be a finitely generated group, S a finite symmetric system of generators in G, $|S| = m$. A random walk on (G, S) is understood as a random walk on the Cayley graph of G with respect to S. The spectral radius plays an important role in the study of random walks on the group. To compute it explicitely is often a difficult task, and efforts have been made to get good estimates on the spectral radius for different groups. A celebrated theorem of Kesten [Kes1, Kes2] states that $2\sqrt{m-1}/m \;\leq\; \rho_G \;\leq\; 1$ with equality on the left if and only if G is a free product of several copies of $\mathbb{Z}$ and $\mathbb{Z}_2$ and S are standard generators; and with equality on the right if and only if G is amenable.

The aim of this Section is to provide a method of estimating the spectral radius ρ_G of the simple random walk on (G, S) in the situation when the corresponding Cayley graph is bipartite and has finitely many cone types, as for example, when G is hyperbolic with all relations of even length, or when (G, S) is a Coxeter system. (Let us mention that no example is known of a group which has finitely many cone types with respect to some but not all generating sets.)

We are going to "simulate" the simple random walk on (G, S) by constructing a special random walk on its *tree of geodesics* $T_{G,S}$. The (infinite) vertex set $V(T_{G,S})$ of this tree is the set of geodesic segments of the form $\{[1_G, g]\}_{g\in G}$ in the Cayley graph $Cay(G, S)$. Two vertices γ_1, γ_2 in $T_{G,S}$ are connected by an edge if the corresponding geodesic segment γ_1 in $Cay(G, S)$ is a one-step extension of the segment γ_2. The tree $T_{G,S}$ has naturally a base vertex γ_0 corresponding to the null geodesic segment $[1_G, 1_G]$.

There is of course a natural projection of $T_{G,S}$ to $Cay(G, S)$:

$$\begin{array}{cccc} \theta: & V(T_{G,S}) & \to & G \\ & \gamma = [1_G, g] & \mapsto & g\ . \end{array}$$

This map is locally injective as the induced map on the set of edges is of the form $\big([1_G, g], [1_G, gs]\big) \mapsto (g, gs)$. However, θ is not a covering map because it is not a local isomorphism.

Observe that the notion of cone type defined in Section 2 for trees, can be repeated without any changes for any bipartite graph. The number of predecessors of a vertex is determined by its cone type and will be denoted r_i for a vertex of type i. The tree of geodesics $T_{G,S}$ has the same number of cone types as $Cay(G, S)$. Indeed, $\theta(\mathcal{C}(\gamma)) = \mathcal{C}(\theta(\gamma))$ for every $\gamma \in V(T_{G,S})$.

Suppose the number of cone types in $Cay(G, S)$ is $K + 1$. Consider a nearest neighbour random walk P_G on $T_{G,S}$ defined by the following probabilities:

$$\begin{array}{rcll} p_{i,j} & = & 1/m, & i = 0, ..., K;\ j = 1, ..., K. \\ p_{-i} & = & r_i/m & \qquad i = 1, ..., K. \end{array}$$

Theorem 4. *The spectral radius ρ_G of the simple random walk on G with respect to S is bounded from above by the spectral radius ρ_T of the random walk P_G on $T_{G,S}$.*

For $i = 0, ..., K$, consider the functions $f_i : \mathbb{R}_+^K \to \mathbb{R}_+$ defined by

$$f_i(c_1, ..., c_K) \ = \ r_i c_i + \sum_{j=1}^{K} \frac{s_{i,j}}{c_j}\ . \tag{4}$$

Lemma 3 ([Nag1, Section 2]). *The minimum $\min_{(c_1,...,c_K)\in\mathbb{R}_+^K} \max_{i=1,...,K} f_i(c_1, ..., c_K)$ exists and is at least $m\rho_G$.*

Proof of Theorem 4. Let us show that

$$\rho_T \ \geq \ \frac{1}{m} \min_{(c_1,...,c_K)\in\mathbb{R}_+^K} \max_{i=1,...,K} f_i(c_1, ..., c_K)\ . \tag{5}$$

Recall that by (1), $1/\rho_T \leq R_F$. Formula (3) implies in turn that R_F is equal to the minimum (over $i = 1, ..., K$) of the smallest real positive singularities of the functions $F_{-i}(z)$. These functions satisfy the system of equations (2). For a fixed z this system can be rewritten in the following form

$$\frac{m}{z} \;=\; \frac{r_i}{w_i} \;+\; \sum_{j=1}^{K} s_{i,j} w_j \;, \quad i = 1, ..., K \;.$$

For each $\epsilon > 0$, let $z_\epsilon = R_F - \epsilon \in \mathbb{R}_+$. Then there exists a solution $\big(F_{-1}(z_\epsilon), ..., F_{-K}(z_\epsilon)\big) \in \mathbb{R}_+^K$ of the system above. The functions $F_{-i}(z)$ are increasing in the intersection of their disc of convergence with $\mathbb{R}_+$, thus $F_{-i}(z_\epsilon) \neq 0$. We can put $c_i = 1/F_{-i}(z_\epsilon)$ in (4) and get

$$\begin{aligned} f_1\big(1/F_{-1}(z_\epsilon), ..., 1/F_{-K}(z_\epsilon)\big) \;&=\; ... \;=\; f_K\big(1/F_{-1}(z_\epsilon), ..., 1/F_{-K}(z_\epsilon)\big) \\ =\; \max_{i=1,\dots,K} f_i\big(1/F_{-1}(z_\epsilon), ..., 1/F_{-K}(z_\epsilon)\big) \;&=\; \frac{m}{z_\epsilon} \;=\; \frac{m}{R_F - \epsilon} \;. \end{aligned}$$

Consequently,

$$\min_{(c_1,\dots,c_K)\in\mathbb{R}_+^K} \max_{i=1,\dots,K} f_i(c_1, ..., c_K) \;\leq\; \frac{m}{R_F - \epsilon}$$

and

$$\rho_T \;\geq\; \frac{1}{m} \min_{(c_1,\dots,c_K)\in\mathbb{R}_+^K} \max_{i=1,\dots,K} f_i(c_1, ..., c_K) \;. \qquad \square$$

Remark. It can be shown that the inequality (5) is in fact an equality.

An advantage of the estimate provided by Theorem 4 is that it can be computed. Indeed, the tree of geodesics $T_{G,S}$ has finitely many cone types and the random walk P_G is of the type we have considered in Section 2. So the system of equations (2) can be written for it and the radius of convergence of its Green function can be found.

One particular instance in which this procedure can be carried out resulting in a good numerical estimate of the spectral radius ρ_G is that of surface groups. Consider the fundamental group of an orientable surface of genus $g \geq 2$

$$G_g \;=\; \langle a_1, ... a_g, b_1, ..., b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle \;.$$

The following estimates hold for the spectral radii ρ_g for the groups G_g of small genus:

Theorem 5.

$$\begin{aligned} 0.662420 \;\leq\; \rho_2 \;\leq\; 0.662816 \;, \\ 0.552773 \;\leq\; \rho_3 \;\leq\; 0.552792 \;. \end{aligned}$$

The upper bounds are deduced from Theorem 4. They first appeared in [Nag1] where a different method was used for computing (5). The lower bounds were obtained by L. Bartholdi [B].

4. How to recognize a Ramanujan graph?

Let now Γ be a finite connected graph on n vertices, and denote by $\lambda_1(\Gamma) > \lambda_2(\Gamma) \geq \lambda_3(\Gamma) ... \geq \lambda_n(\Gamma)$ the spectrum of its adjacency matrix. If Γ is k-regular, $\lambda_1(\Gamma) = k$. Recall that a k-regular connected finite graph Γ is called *Ramanujan* if

$$\lambda_2(\Gamma) \leq 2\sqrt{k-1} \ .$$

The number on the right-hand side of this inequality is nothing else than the spectral radius of the adjacency operator on the infinite k-regular tree, which is the universal covering tree of any k-regular graph.

In analogy with this classical definition, one can say that an arbitrary finite connected graph Γ is Ramanujan if

$$\lambda_2(\Gamma) \leq r(\tilde{\Gamma}) \ ,$$

where $\tilde{\Gamma}$ denotes the universal covering tree of Γ and r is the spectral radius of the adjacency operator on it. Note that if an infinite tree covers a finite graph then it covers infinitely many finite graphs (such trees are *uniform*), and so it makes sense to speak of infinite sequences of these, not necessarily regular Ramanujan graphs. Moreover, all finite graphs covered by the same tree have the same λ_2 ([Gr]).

Such graphs were studied by Greenberg ([Gr]), who proved a generalization of Alon-Boppana theorem, namely that $r(T)$ is the best upper bound on $\lambda_2(\Gamma)$ which holds for infinitely many graphs Γ covered by the same tree T. Later Lubotzky and the author showed that the famous classical problem of existence, for every regular tree, of an infinite family of Ramanujan quotients, gets negative solution if the assumption of regularity is dropped. More precisely, infinitely many uniform trees such that none of their quotient is Ramanujan were exhibited in [LN].

Alex Lubotzky has asked, while discussing different aspects of the notion of Ramanujan graph, whether an algorithm exists which, given a finite connected graph, determines in finite time whether this graph is Ramanujan. According to the definition above, the question is whether one can compute algorithmically the value of the spectral radius of a uniform tree. Results of Section 2 can be applied to solve this problem.

Proposition 3. *A uniform tree has finitely many cone types.*

Proof. Suppose T is a uniform tree, i.e., there exists a finite connected graph Γ such that $T = \tilde{\Gamma}$. Then the number of cone types in T (counted with respect to any base point) is at most $1 + \sum_{v \in V(\Gamma)} deg(v)$. Indeed, let π denote the covering map $\tilde{\Gamma} \to \Gamma$. Fix a base point x_0 in Γ and some $\tilde{x}_0 \in \pi^{-1}(x_0)$. Consider the set $\mathbb{E}(\Gamma)$ of oriented edges of Γ, of cardinality $2|E(\Gamma)|$. A set $E_e(\tilde{\Gamma})$ of edges in $\tilde{\Gamma}$ is associated naturally to each edge e in $\mathbb{E}(\Gamma)$, so that $\pi(\epsilon) = e$ for each $\epsilon \in E_e(\tilde{\Gamma})$. Each edge $\epsilon \in E(\tilde{\Gamma})$ can be represented as (ϵ_+, ϵ_-) with $\epsilon_+, \epsilon_- \in V(\tilde{\Gamma})$ and $d(\epsilon_+, \tilde{x}_0) = d(\epsilon_-, \tilde{x}_0) + 1$. With each such ϵ we can associate the cone of its extremity ϵ_+. The cones which correspond to the edges from some $E_e(\tilde{\Gamma})$ are isomorphic. As we assumed that the cone type of the base vertex is different from all others, it follows that the number of cone types minus 1 is bounded from above by the number of oriented edges in Γ, $\sum_{v \in V(\Gamma)} deg(v)$. $\square$

Note however that not every tree with finitely many cone types is the universal cover of a finite graph. For example, *trees of geodesics* of finitely generated groups studied in Section 3 are in general not uniform.

Proposition 3 above insures that results of Section 2 can be used for finding spectral radii of uniform trees. The proof shows that the assumption of irreducibility, as stated in Lemma 3 is satisfied. Thus Theorem 3 can be applied to $T = \tilde{\Gamma}$. (Theorem 3 concerns the spectral radius of the simple random walk, whereas it is the spectral radius of the adjacency operator which appears in the definition of a Ramanujan graph. Though there is no simple formula connecting these two numbers in the case when the tree is not regular, Theorem 3 can be repeated without any changes for the adjacency operator.)

The functions $\{F_{-i}(z)\}_{i=1,\dots,K}$ and $\lambda_*(z)$ satisfy the following system of polynomial equations:

$$\begin{cases} w_i \;=\; p_{-i}z \;+\; z\sum_{j=1}^{K} s_{i,j}p_{i,j}w_iw_j\ , \quad i=1,\dots,K \\ \qquad\qquad \det(J(z)-\lambda I) \;=\; 0\ . \end{cases}$$

The *elimination theory* of algebraic geometry ensures the existence of polynomials $\mathcal{Q}$, $\mathcal{Q}_i$, $i = 1,\dots K$, in two variables, with integer coefficients, such that $\mathcal{Q}(z, \lambda_*(z)) \equiv 0$ and $\mathcal{Q}_i(z, F_{-i}(z)) \equiv 0$. Moreover, there are algorithms running in polynomial time to find coefficients of these polynomials from the coefficients of the system of equations. The radius of convergence R_F can then be found as a solution of the equation $\mathcal{Q}(z, 1) = 0$.

Being an algebraic function, each $F_{-i}(z)$ has an expansion as a Puiseux series, and the constant term in it is exactly the value $F_{-i}(R_i)$. These values can be found for all $i = 1,\dots,K$ by plugging the Puiseux expansions in the system (2) (see [SNW, 2.7].) Therefore we also get the value $F(x_0, x_0 \mid R_F)$ via (3). If it is at most 1, the spectral radius is equal to $1/R_F$. Finally, if it appears that $F(x_0, x_0 \mid R_F) > 1$ and the radius of convergence of the Green function is a pole, polynomials $\mathcal{Q}_i$, $i = 1,\dots,K$ can be used to find the polynomial $\mathcal{P}$ such that $\mathcal{P}(z, F(x_0, x_0 \mid z)) \equiv 0$, and the spectral radius can be found as a solution of the equation $\mathcal{P}(z, 1) = 0$.

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