Ends of Schreier graphs of self-similar groups

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Abstract

Every self-similar group acts on the space X^ω of infinite words over some alphabet X. We study the Schreier graphs Γ_w for $w \in X^\omega$ of the action of self-similar groups generated by bounded automata on the space X^ω. Using sofic subshifts we determine the number of ends for every Schreier graph Γ_w. Almost all Schreier graphs Γ_w with respect to the uniform measure on X^ω have one or two ends, and we characterize bounded automata whose Schreier graphs have two ends almost surely. The connection with (local) cut-points of limit spaces of self-similar groups is established.

Keywords: self-similar group, Schreier graph, end of graph, bounded automaton, tile, cut-point
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1 Introduction

One of important asymptotic invariants of an infinite graph is its number of ends. Considering ends is a standard way to compactify a topological space. Roughly speaking, each end represents a topologically distinct way to move to infinity inside the space. The most convenient way to define an end in an infinite graph Γ is as an equivalence class of infinite rays, where two rays in Γ are equivalent if they can be connected by infinitely many disjoint paths. In other words, two rays belonging to the same end will have the tails in the same connected component of $\Gamma \setminus F$ for any finite subgraph F of Γ. The number of ends is a quasi-isometric invariant, and the celebrated theorem of Stallings characterizes infinite finitely generated groups according to the number of ends it may have: one, two or infinitely many.

In this paper we will be interested in Schreier graphs of certain groups of automorphisms of rooted trees. Given a group G generated by a finite set S, and a set M with a transitive action of G on M, one can associate to it the Schreier graph $\Gamma(G, M, S)$ that describes the

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action on M of the generators S: the vertex set of the graph is M, and two vertices m_2 are connected by and edge if and only if there exists a generator that maps m_1 to m_2. It can be realized as the Schreier graph of the stabilizer of any vertex $m \in M$, that is, the vertex set is the set of cosets $G/Stab_G(m)$. Conversely, the Schreier graph $\Gamma(G, G/H, S)$ can be considered for any subgroup $H < G$. A Schreier graph $\Gamma(G, G/H, S)$ is a Cayley graph of G/H if and only if the subgroup H is normal in G.

Let $G < AutT$ be a finitely generated group of automorphisms of a rooted tree, and assume that it acts transitively on each set of vertices at a given distance from the root (such sets are clearly preserved by automorphisms, we will call them "levels"). We then have a family of natural actions of G on the levels of the tree, and thus a family of corresponding Schreier graphs $\{\Gamma_n\}_{n\geq 1}$. Moreover, the action can be extended to the boundary ∂T, i.e., the space of ends of the rooted tree, and we can also consider the family of infinite Schreier graphs $\{\Gamma_w | w \in \partial T\}$ whose vertex sets are the orbits of the action of G on ∂T.

A very interesting class of groups of automorphisms of rooted trees consists of groups acting on regular trees in a self-similar way. A subgroup $G < AutT$ is called self-similar if the restriction of the action to the full subtree T_v rooted at an arbitrary vertex v of T gives again an element of G after identifying the isomorphic rooted trees T_v and T. These groups can alternatively be described as groups generated by the states of an invertible Mealy automaton (see Section 2.1 for all necessary definitions). Best understood are contracting self-similar groups, where "contracting" stands for contracting the length of a group element under restricting its action to subtrees T_v (see Section 2.1 for precise definitions). Contracting self-similar groups appear naturally in the study of expanding (partial) self-coverings of topological spaces and orbispaces, as their iterated monodromy groups [19]. Nekrashevych introduced limit spaces of contracting self-similar groups, and in the case when the group is an iterated monodromy group of an expanding partial self-covering f, the limit space is homeomorphic to the Julia set $J(f)$. Finite Schreier graphs of the group form a sequence of combinatorial approximations to the limit space. Therefore the limit space and the family orbital Schreier graphs represent two limiting constructions associated to the action and to the sequence of finite Schreier graphs $\{\Gamma_n\}_{n\geq 1}$.

Finite Schreier graphs and their infinite limits are interesting objects associated to a self-similar action of a finitely generated group on a regular rooted tree. Structure of these graphs as well as some of their properties such as their spectra, expansion, growth, their random weak limits, probabilistic models on them, have been studied in various works over the last ten years, see [2, 16, 15, 6, 10, 11, 18].

The aim of this paper is to investigate the ends of the orbital Schreier graphs of groups generated by bounded automata. This class of self-similar contracting groups is most amenable to investigation, in particular, the corresponding Schreier graphs Γ_n can be iteratively constructed using inflation of graphs, see Section 2.4 below. In the dual language of limit spaces, being generated by a bounded automaton guarantees that the limit space is post-critically finite or, equivalently, finitely ramified (see Section 2.4 for details).

Our main results are as follows.

Given a group generated by the states of a bounded automaton, we determine, for every $w \in \partial T$, the number of ends in the orbital Schreier graph Γ_w. The answer comes from a

finite state automaton constructed on the basis of the automaton that generates the group (Section 3).

The action of the group on the boundary of the tree is ergodic with respect to the uniform measure on the boundary, and therefore almost all orbital Schreier graphs $\Gamma_w, w \in \partial T$ have the same number of ends. For a group generated by a bounded automaton this "typical" number of ends is 1 or 2 and we show that in most cases it is 1, by characterizing completely the bounded automata generating groups whose infinite Schreier graphs have almost surely 2 ends (Section 4). Our algorithm also allows to maximize the number of ends in a Schreier graphs of a given bounded automaton group, and provides a description of graphs with more than 2 ends.

Establishing a connection between the global structure of infinite Schreier graphs and the local structure of the limit space, we show that the number of ends in a typical Schreier graph coincides with the number of connected components in a typical punctured neighborhood in the limit space (Section 5).

Section 1 is a collection of necessary definitions and results that will be used in the proofs. In Section 5 we illustrate our results by performing explicit computations for some concrete examples.

2 Preliminaries

In this section we review the basic definitions and facts concerning self-similar groups, bounded automata, their Schreier graphs and limit spaces. For a more detailed information and for the references, see [19].

2.1 Self-similar groups and automata

Let X be a finite set with q elements, $q \geq 2$. Denote by $X^* = \{x_1x_2 \dots x_n | x_i \in X, n \geq 0\}$ the set of all finite words over X (including the empty word denoted $\emptyset$). The length of a word $v = x_1x_2 \dots x_n \in X^n$ is denoted by $|v| = n$, and the set of elements of length n is denoted by X^n. Elements of X^* can be identified with the vertex set of the q-regular tree rooted at the empty word. The boundary of the tree is then identified with the space of all right-infinite words in the alphabet X that we denote X^ω. We shall also consider the set $X^{-\omega}$ of all left-infinite sequences (words) $\dots x_2x_1$, $x_i \in X$, with the product topology of discrete sets X. The *uniform Bernoulli measure* ν on each space X^ω and $X^{-\omega}$ is the product measure of uniform distributions on X. The *shift* σ on the space X^ω (respectively on $X^{-\omega}$) is the map which deletes the first (respectively the last) letter of a right-infinite (respectively left-infinite) word.

A faithful action of a group G on the set $X^* \cup X^\omega$ is called *self-similar* if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for all $w \in X^* \cup X^\omega$. The element h is called the *restriction* of g on x and is denoted by $h = g|_x$. Inductively one defines the restriction $g|_{x_1x_2\ldots x_n} = g|_{x_1}|_{x_2}\ldots|_{x_n}$ for every word $x_1x_2\ldots x_n \in X^*$. Restrictions have the following properties

$$g(vu) = g(v)g|_v(u), \qquad g|_{vu} = g|_v|_u, \qquad (g \cdot h)|_v = g|_{h(v)} \cdot h|_v$$

for all $g, h \in G$ and $v, u \in X^*$ (we are using left actions so that $(g \cdot h)(v) = h(g(v))$).

It follows from the definition that every self-similar group G preserves the length of the words under its action on the space X^*, so that we have an action of the group G on the set X^n for every n. In particular, every self-similar group acts by homeomorphisms on the space X^ω.

Another way to introduce self-similar groups is through input-output automata over the alphabet X and automata groups. An (input-output) *automaton* is a quadruple $\mathsf{A} = (S, X, \mu, \nu)$, where S is the set of states; X is an alphabet; $\mu : S \times X \to S$ is the transition map; and $\nu : S \times X \to X$ is the output map. The automaton A is *finite* if S is finite and it is *invertible* if, for all $s \in S$, the transformation $\nu(s, \cdot) : X \to X$ is a permutation of X. An automaton A can be represented by its *Moore diagram*, a directed labeled graph whose vertices are identified with the states of A. For every state $s \in S$ and every letter $x \in X$, the diagram has an arrow from s to $\mu(s, x)$ labeled by $x|\nu(s, x)$. This graph contains complete information about the automaton, and we will identify the automaton with its Moore diagram. A natural action on the words over X is induced, so that the maps μ and ν can be extended to $S \times X^*$ by the rules

$$\begin{aligned} \mu(s, xw) &= \mu(\mu(s, x), w), \\ \nu(s, xw) &= \nu(s, x)\nu(\mu(s, x), w), \end{aligned} \tag{1}$$

where we set $\mu(s, \emptyset) = s$ and $\nu(s, \emptyset) = \emptyset$, for all $s \in S$, $x \in X$ and $w \in X^*$. Moreover, (1) uniquely defines a map $\nu : S \times X^\omega \to X^\omega$.

If we fix an initial state $s \in S$ in an automaton A, then the transformation $\nu(s, \cdot)$ on the set $X^* \cup X^\omega$ is defined by (1); it is denoted by A_s. The image of a word $x_1x_2\ldots$ under A_s can be easily found using the Moore diagram. Consider the directed path starting at the state s with consecutive labels $x_1|y_1$, $x_2|y_2, \ldots$; then the image of the word $x_1x_2\ldots$ under the transformation A_s is the word $y_1y_2\ldots$. More generally, given an invertible automaton $\mathsf{A} = (S, X, \mu, \nu)$, one can consider the group generated by the transformations A_s for all $s \in S$, which is called the *automaton group* generated by A and is denoted by $G(\mathsf{A})$. Every automaton group is self-similar and vise versa, every self-similar group G can be given (generated) by its complete automaton $\mathsf{A}(G)$ of the action. The states of $\mathsf{A}(G)$ are the elements of the group G, and there is an edge $g \to g|_x$ labeled by $x|g(x)$ for every $g \in G$ and $x \in X$.

A self-similar group G is called *contracting* if there exists a finite set $\mathcal{N} \subset G$ with the property that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words v of length greater or equal to n. The smallest set $\mathcal{N}$ with this property is called the *nucleus* of the group. It follows from the definition that every contracting group is generated by a finite automaton, that is, by an automaton with finitely many states. It is also clear from

the definition that $h|_x \in \mathcal{N}$ for every $h \in \mathcal{N}$ and $x \in X$. Therefore the nucleus $\mathcal{N}$ is a subautomaton of the complete automaton of the group. Moreover, every state of $\mathcal{N}$ has an incoming arrow, because otherwise minimality of the nucleus would be violated. Also, the nucleus is symmetric, i.e., $h^{-1} \in \mathcal{N}$ for every $h \in \mathcal{N}$.

A self-similar group G is called *self-replicating* (or recurrent) if it acts transitively on X, and the map $g \mapsto g|_x$ from the stabilizer $Stab_G(x)$ to the group G is surjective for some (every) letter $x \in X$. It can be shown that a self-replicating group acts transitively on X^n for every $n \geq 1$. It is also easy to see ([19, Proposition 2.11.3]) that if a finitely generated contracting group is self-replicating then its nucleus $\mathcal{N}$ is a generating set.

2.2 Schreier graphs and tile graphs of self-similar groups

Let G be a group generated by a finite set S and let H be a subgroup of G. The *(simplicial) Schreier graph* $\Gamma(G, S, H)$ of the group G is the graph whose vertices are the right cosets $G/H = \{Hg : g \in G\}$, and two vertices Hg_1 and Hg_2 are adjacent if there exists $s \in S$ such that $g_2 = g_1 s$ or $g_1 = g_2 s$. If the group G acts on a set M, then the corresponding *(simplicial) Schreier graph* $\Gamma(G, S, M)$ is the graph with the set of vertices M, and two vertices v and u are adjacent if and only if there exists $s \in S$ such that $s(v) = u$ or $s(u) = v$. If the action (G, M) is transitive, then the Schreier graph $\Gamma(G, S, M)$ is isomorphic to the Schreier graph $\Gamma(G, S, Stab_G(m))$ of the group with respect to the stabilizer $Stab_G(m)$ for every $m \in M$.

Let G be a self-similar group generated by a finite set S. The sets X^n are invariant under the action of G, and we denote the associated Schreier graphs by $\Gamma_n = \Gamma_n(G, S)$. For a point $w \in X^\omega$ we consider the action of the group G on the G-orbit of w, and the associated Schreier graph is called *orbital Schreier graph* denoted $\Gamma_w = \Gamma_w(G, S)$. For every $w \in X^\omega$ we have $Stab_G(w) = \bigcap_{n \geq 1} Stab_G(w_n)$, where w_n denotes the prefix of length n of the infinite word w. The connected component of the rooted graph (Γ_n, w_n) around the root w_n is exactly the Schreier graph of G with respect to the stabilizer of w_n. It follows immediately that the graphs (Γ_n, w_n) converge to the graph (Γ_w, w) in the pointed Gromov-Hausdorff topology.

Besides the Schreier graphs, we will also work with their subgraphs called the *tile graphs*. Define the tile graph $T_n = T_n(G, S)$ to have the set of vertices X^n, where two vertices v and u are connected by an edge if and only if there exists $g \in S$ such that $g(v) = u$ (as in Schreier graphs) and $g|_v = 1$. The tile graph T_n is thus a subgraph of the Schreier graph Γ_n. To define a tile graph for the action on the boundary, consider the same set of vertices as in Γ_w and connect vertices v and u by an edge if there exists $s \in S$ such that $s(v) = u$ and $s|_{v'} = 1$ for some finite beginning $v' \in X^*$ of the sequence v. The connected component of this graph containing the vertex w is called the *orbital tile graph* T_w. It is clear from the construction that we also have the convergence $(T_n, w_n) \to (T_w, w)$ in the pointed Gromov-Hausdorff topology.

2.3 Limit spaces and tiles of self-similar groups

Let G be a contracting self-similar group with nucleus $\mathcal{N}$. Consider the space $X^{-\omega} \times G$ with the product topology of discrete sets X and G. The *limit G-space* $\mathcal{X}_G$ of the group G is defined as the quotient of the space $X^{-\omega} \times G$ by the equivalence relation, where two sequences $\dots x_2x_1 \cdot g$ and $\dots y_2y_1 \cdot h$ of $X^{-\omega} \times G$ are equivalent if there exists a left-infinite path $\dots e_2e_1$ in the nucleus $\mathcal{N}$ that ends in the vertex hg^{-1} and every edge e_i is labeled by $x_i|y_i$. The group G naturally acts on the space $\mathcal{X}_G$ by multiplication from the right.

The image of $X^{-\omega} \times 1$ in $\mathcal{X}_G$ is called the *tile* $\mathcal{T}$ of the group. It can be described as the quotient of $X^{-\omega}$ by the equivalence relation, where two sequences $\dots x_2x_1$ and $\dots y_2y_1$ are equivalent if and only if there exists a path $\dots e_2e_1$ in the nucleus $\mathcal{N}$ that ends in the trivial state and the edge e_i is labeled by $x_i|y_i$. The image of $X^{-\omega}v \times 1$ for $v \in X^n$ is called the *tile* $\mathcal{T}_v$ of n-th level. The tile $\mathcal{T}$ decomposes in the union $\cup_{v \in X^n} \mathcal{T}_v$ of the tiles of n-th level for every n. All tiles $\mathcal{T}_v$ are compact and homeomorphic to $\mathcal{T}$.

Two tiles $\mathcal{T}_v$ and $\mathcal{T}_u$ of the same level $v, u \in X^n$, have nonempty intersection if and only if there exists $h \in \mathcal{N}$ such that $h(v) = u$ and $h|_v = 1$. This is precisely how we connect vertices in the tile graph $T_n(G, \mathcal{N})$ with respect to the nucleus. Hence the graphs $T_n(G, \mathcal{N})$ can be used to approximate the tile $\mathcal{T}$, which justifies the term "tile graph". The tile $\mathcal{T}$ is connected if and only if all the tile graphs $T_n = T_n(G, \mathcal{N})$ are connected (see [19, Proposition 3.3.10]).

A contracting self-similar group G satisfies the *open set condition* if for any element g of the nucleus $\mathcal{N}$ there exists a word $v \in X^*$ such that $g|_v = 1$, i.e., in the nucleus $\mathcal{N}$ there is a path from any vertex to the trivial state. If a group satisfies the open set condition then the tile $\mathcal{T}$ is the closure of its interior, and any two different tiles of the same level have disjoint interiors; otherwise for large enough n there exists a tile $\mathcal{T}_v$ for $v \in X^n$ which is covered by other tiles of n-th level (see [19, Proposition 3.3.7]). Under the open set condition the boundary $\partial\mathcal{T}$ of the tile $\mathcal{T}$ consists equivalence classes of sequences $\dots x_2x_1$ for which there exists a path $\dots e_2e_1$ in the nucleus $\mathcal{N}$ that ends in a non-trivial state and the edge e_i is labeled by $x_i|y_i$ for some $y_i \in X$ (here the sequence $\dots y_2y_1$ also represents a point of $\partial\mathcal{T}$).

The *limit space* $\mathcal{J}_G$ is the quotient of the limit G-space $\mathcal{X}_G$ by the action of the group G. Similarly to the above, it can be defined as the quotient of the space $X^{-\omega}$ by the equivalence relation, where two sequences $\dots x_2x_1$ and $\dots y_2y_1$ are equivalent if there exists a left-infinite path $\dots e_2e_1$ in the nucleus $\mathcal{N}$ such that every edge e_i is labeled by $x_i|y_i$. The limit space $\mathcal{J}_G$ is compact, metrizable, finite-dimensional space. If the group G is finitely generated and self-replicating then the space $\mathcal{J}_G$ is connected and locally connected (see [19, Theorem 3.5.1]). Under the open set condition, the limit space $\mathcal{J}_G$ can be obtained from the tile $\mathcal{T}$ by gluing some of its boundary points. Namely, we need to glue two points represented by the sequences $\dots x_2x_1$ and $\dots y_2y_1$ for every path $\dots e_2e_1$ in the nucleus $\mathcal{N}$ that ends in a non-trivial state and every edge e_i is labeled by $x_i|y_i$.

We consider the uniform Bernoulli measure ν on the space $X^{-\omega}$ and the counting measure on the group G, and we put the product measure on the space $X^{-\omega} \times G$. The push-forward of this measure under the canonical projection $X^{-\omega} \times G \to \mathcal{X}_G$ defines the

measure μ on the limit G-space $\mathcal{X}_G$. The tile $\mathcal{T}$ has integer measure and $\mu(\mathcal{T}) = |X|^n \mu(\mathcal{T}_v)$ for every $v \in X^n$.

The push-forward of the uniform Bernoulli measure ν under the factor map $X^{-\omega} \to \mathcal{J}_G$ defines the *self-similar measure* m on the limit space $\mathcal{J}_G$. The shift σ on the space $X^{-\omega}$ induces a continuous surjective measure-preserving map $\mathsf{s} : \mathcal{J}_G \to \mathcal{J}_G$, and every point of $\mathcal{J}_G$ has at most $|X|$ preimages under s. The *limit dynamical system* $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ is conjugate to the one-sided Bernoulli shift (see [7, Theorem 21]). More information about properties of the measures m and μ can be found in [7].

Important examples of self-similar groups are the iterated monodromy groups $IMG(f)$ of post-critically finite rational functions f. The limit space of the group $IMG(f)$ is homeomorphic to the Julia set $J(f)$ of the function f (see [19, Section 6.4] for more details).

2.4 Self-similar groups generated by bounded automata

In this paper we study the Schreier graphs of self-similar groups generated by bounded automata. A finite invertible automaton A is called *bounded* if the number of paths of length n in the automaton avoiding the trivial state is bounded independently on n, or, equivalently, the number of left- (or right-) infinite paths in the automaton avoiding the trivial state is finite. Bounded automata can be characterized by their cyclic structure. A cycle in the automaton is called trivial if it is a loop at the trivial state. Then a finite invertible automaton is bounded if and only if any two non-trivial cycles in the automaton are disjoint and not connected by a directed path (see [21]). The states of a bounded automaton S can be classified as follows:

1) a state $s \in S$ is *finitary* if there exists n such that $s|_v = 1$ for all $v \in X^n$;
2) a state $s \in S$ is *circuit* if there exists a nonempty word $v \in X^n$ such that $s|_v = s$, in this case $s|_u$ is finitary for every $u \in X^n$, $u \neq v$;
3) for every state $s \in S$ there exists n such that for every $v \in X^n$ the state $s|_v$ is either finitary or circuit.

By passing to a power X^m of the alphabet X every bounded automaton can be brought to the *basic form* (see [19, Proposition 3.9.11]) in which the above items 1),2),3) hold with $n = 1$, in particular all cycles are loops, and $s|_x = 1$ for every finitary s and every $x \in X$; here for m we can take an integer number which is greater than the diameter of the automaton and is a multiple of the length of every cycle.

Every self-similar group G generated by a bounded automaton is contracting (see [19, Theorem 3.9.12]), and we can consider the limit spaces of these groups. Moreover, the nucleus is also a bounded automaton, and, in particular, every group generated by a bounded automaton satisfies the open set condition. The nucleus contains only finitary or circuit states, because every state should have an incoming edge.

The limit spaces of groups generated by bounded automata are connected with important classes of fractals, namely post-critically finite and finitely-ramified self-similar sets. A left-infinite sequence $\ldots x_2 x_1 \in X^{-\omega}$ is called *post-critical*, if there exists a left-infinite path $\ldots e_2 e_1$ in the nucleus $\mathcal{N}$, which begins in a nontrivial state and is labeled by $\ldots x_2 x_1$. The set $\mathcal{P}$ of all post-critical sequences is called *post-critical*. Then for a contracting self-similar

group G with nucleus $\mathcal{N}$ the following statements are equivalent: every two tiles of the same level have finite intersection (the limit space is finitely-ramified); the post-critical set $\mathcal{P}$ is finite (the limit space is post-critically finite); the nucleus $\mathcal{N}$ is a bounded automaton (or the generating automaton of the group is bounded). Under the open set condition, the post-critical set $\mathcal{P}$ is the preimage of the boundary of the tile $\mathcal{T}$ under the canonical projection $X^{-\omega} \to \mathcal{T}$, and the above statements are also equivalent to the finiteness of the boundary $\partial\mathcal{T}$ (see [6, Chapter IV]).

In what follows, except a few special cases directly indicated, we make the following assumptions about the studied self-similar groups G and their generating sets S:

1. **The group G is generated by a bounded automaton S.**
2. **The tile $\mathcal{T}$ of the group is connected.**
3. **Every state of the automaton S has an incoming edge, and $S^{-1} = S$.**

Instead of the assumption 2 it is enough to require that **the group acts transitively on X^n for every** $n \geq 1$, i.e., the Schreier graphs $\Gamma_n(G, S)$ are connected. Then, even if the tile graphs $T_n(G, S)$ are not connected, there is a uniform bound on the number of connected components in $T_n(G, S)$, and one can apply the developed methods to each component. The assumption 3 is technical, it guaranties that every right-infinite path in the automaton S can be continued to the left. If the generating set S contains a state s', which does not contain incoming edges, then $s|_x \in S \setminus \{s'\}$ for every $s \in S$ and $x \in X$, and hence the state s' does not interplay on the asymptotic properties of the tile or Schreier graphs. Moreover, if the group is self-replicating then the property 3 is always satisfied when we take the nucleus $\mathcal{N}$ as the generating set S.

Under these assumptions the tile graphs $T_n = T_n(G, S)$ and the Schreier graphs $\Gamma_n = \Gamma_n(G, S)$ are connected. Then we can describe the vertex sets of the orbital tile graphs $T_w = T_w(G, S)$. Two right- (or left-) infinite sequences are called *cofinal* if they differ only in finitely many letters. Cofinality is an equivalence relation on X^ω and $X^{-\omega}$. The respective equivalence classes are called the cofinality classes and they are denoted by $Cof(\cdot)$. Then for every $w \in X^\omega$ the set $Cof(w)$ is the set of vertices of the orbital tile graph T_w.

To describe the vertex sets of the orbital Schreier graphs Γ_w let us classify the sequences $w \in X^\omega$. A right-infinite sequence $w = x_1x_2\ldots \in X^\omega$ is called *critical* if there exists a right-infinite path $e_1e_2\ldots$ in the automaton $\mathcal{N} \setminus \{1\}$ labeled by $x_1x_2\ldots$. It follows that every shift $\sigma^n(w)$ of a critical sequence w is again critical, and for every n there exists $v \in X^n$ such that vw is critical (here we use that every right-infinite path in the nucleus can be continued to the left). A sequence $w \in X^\omega$ is called *regular* if the cofinality class of w does not contain critical sequences, or, equivalently, if the shifted sequence $\sigma^n(w)$ is not critical for every $n \geq 0$. Notice that the cofinality class of a critical sequence contains sequences with are neither regular nor critical.

Proposition 1. *Every post-critical sequence is pre-periodic, and can be read along a left-infinite path in the automaton $S \setminus \{1\}$.*

The set of critical sequences is finite. Every critical sequence is periodic, and can be read along a right-infinite path in the automaton $S \setminus \{1\}$. Every cofinality class contains not more than one critical sequence.

The cofinality class of a regular point contains only regular points. If w is regular, then there exists a finite beginning $v \in X^*$ of w such that $s|_v = 1$ for every $s \in S$.

Proof. The pre-periodicity follows from the structure of bounded automata. For the second statement about post-critical sequences see [6, Proposition IV.18].

The number of right-infinite sequences in every bounded automaton avoiding the trivial state is finite, and hence the number of critical sequences is finite. Every critical sequence is read along a non-trivial cycle in the nucleus $\mathcal{N}$, or, equivalently, in the automaton S (the proof is the same as for post-critical sequences). Since the cycles are disjoint and not connected by a directed path, these sequences are periodic. The statement about the cofinality class of a critical sequence follows immediately, because different periodic sequences can not differ only in finitely many letters.

If $w = x_1x_2\ldots \in X^\omega$ is regular, then starting from any state $s \in S$ and following the edges labeled by $x_1, x_2, \ldots$ we will end at the trivial state. Hence there exists n such that $s|_{x_1x_2\ldots x_n} = 1$. □

If the automaton S is in the basic form, then every post-critical sequence is of the form $y^{-\omega}x$ for some letters $x, y \in X$, and every critical sequence is of the form x^ω for some $x \in X$.

Proposition 2. *The Schreier graph Γ_w coincides with the tile graph T_w for every regular sequence $w \in X^\omega$.*

Let $w \in X^\omega$ be a critical sequence, and let $O(w)$ be the set of all critical sequences $v \in X^\omega$ such that $s(w) = v$ for some $s \in S$. To construct the Schreier graph Γ_w we need to take the disjoint union of the orbital tile graphs T_v for $v \in O(w)$, and connect critical sequences $v_1, v_2 \in O(w)$ if $s(v_1) = v_2$ for some $s \in S$.

Proof. If the point w is regular, then the set of vertices of Γ_w is the cofinality class $Cof(w)$, which is the set of vertices of T_w. Suppose there is an edge between v and u in the graph Γ_w. Then $s(v) = u$ for some $s \in S$. Since the sequence w is regular, all the sequences in $Cof(w)$ are regular, and hence there exists a beginning $v' \in X^*$ of v such that $s|_{v'} = 1$. Hence there is an edge between v and u in the tile graph T_w.

If the point w is critical, then the set of vertices of Γ_w is the union of cofinality classes $Cof(v)$ for $v \in O(w)$. Consider an edge $s(u_1) = u_2$ in Γ_w. If this is not an edge of T_v for $v \in O(w)$, then the restriction of s on every beginning of u_1 is not trivial. Hence v is critical, and this edge was added under construction. □

Hence it is enough to study the number of ends and connected components in the (orbital) tile graphs.

The study of orbital tile graphs T_w is based on the approximation by finite tile graphs T_n. The graphs T_n can be iteratively constructed using the inflation of graphs developed in [6, Chapter V] and [8]. Let us remind this construction.

For every post-critical sequence $\ldots x_2x_1 \in \mathcal{P}$ the vertex $x_n \ldots x_2x_1$ is called *post-critical* in the Schreier graph Γ_n and in the tile graph T_n. For all large enough n, the post-critical

vertices are in 1-to-1 correspondence with the set $\mathcal{P}$. With a slight abuse in notations we consider the elements of the post-critical set $\mathcal{P}$ as the vertices of the graphs Γ_n and T_n.

Let E_S be the set of all pairs $\{(p,x),(q,y)\}$ for $p,q \in \mathcal{P}$ and $x,y \in X$ such that there exists a left-infinite path $\dots e_2 e_1$ in the automaton S, which ends in the trivial state and is labeled by the pair $px|qy$.

Theorem 3. *To construct the tile graph T_{n+1} take $|X|$ copies of the tile graph T_n, identify their sets of vertices with $X^n x$ for $x \in X$, and connect two vertices vx and uy by an edge if and only if $v,u \in \mathcal{P}$ and $\{(v,x),(u,y)\} \in E_S$.*

The procedure of inflation of graphs given in Theorem 3 can be described by the *model graph* M with the set of vertices $\mathcal{P} \times X$, where we put an edge between (p,x) and (q,y) if the pair $\{(p,x),(q,y)\}$ belongs to the set E_S. The vertex (p,x) of M is called *post-critical* if the sequence px is post-critical. In this way we can consider the elements of $\mathcal{P}$ as vertices of the graph M. Then to construct the graph T_{n+1} we can place the graph T_n in the model graph instead of the vertices $\mathcal{P} \times x$ for each $x \in X$ such that the post-critical vertices of T_n fit with the set $\mathcal{P} \times x$. The post-critical vertices of M will correspond to the post-critical vertices of T_{n+1}.

To construct the Schreier graph Γ_n take the tile graph T_n and add the edges between post-critical vertices p and q if $s(p) = q$ for $s \in S$. Indeed, if $s(v) = u$ and $s|_v \neq 1$ (the edge that does not appear in T_n) then v and u are post-critical vertices, and they should be connected in Γ_n. Notice that the added edges can be described directly through the generating set S, and do not depend on n.

3 Ends of Schreier graphs and tile graphs

For a graph Γ and its vertex v we denote by $\Gamma \setminus v$ the graph obtained from Γ by removing the vertex v.

Proposition 4. *Every tile graph T_w for $w \in X^\omega$ has finitely many ends, which is equal to*

$$\#Ends(T_w) = \lim_{n\to\infty} \#infcomp(T_{\sigma^n(w)} \setminus \sigma^n(w)).$$

Proof. Let us show that the number of infinite connected components of the graphs $T_{\sigma^n(w)} \setminus \sigma^n(w)$ and $T_w \setminus X^n \sigma^n(w)$ is the same for every n. Consider the natural partition of the set of vertices of T_w given by

$$Cof(w) = \bigsqcup_{w' \in Cof(\sigma^n(w))} X^n w'.$$

Using the graph T_w construct a new graph $\mathcal{G}$ with the set of vertices $Cof(\sigma^n(w))$, where two vertices v and u are connected by an edge if there exist $v', u' \in X^n$ such that $v'v$ and $u'u$ are connected in T_w. The graph $\mathcal{G}$ is isomorphic to the tile graph $T_{\sigma^n(w)}$ under the identity map on $Cof(\sigma^n(w))$. Indeed, let v and u are connected in $\mathcal{G}$ and $s(v'v) = u'u$ for some $s \in S$. Then $s|_{v'}(v) = u$ and $s|_{v'} \in S$, because S is self-similar. Conversely, suppose

$s(v) = u$ for some $s \in S$. Since each element of S has an incoming edge, there exists $s' \in S$ and $v', u' \in X^n$ such that $s'(v'v) = u'u$.

The subgraph of T_w spanned by every set of vertices $X^n w'$ for $w' \in Cof(\sigma^n(w))$ is connected, because the tile graphs T_n are connected. Hence, the number of infinite connected components in $T_w \setminus X^n\sigma^n(w)$ is equal to the number of infinite connected components in $T_{\sigma^n(w)} \setminus \sigma^n(w)$. In particular, it is bounded by the size of the generating set S.

Every infinite component of $T_w \setminus X^n\sigma^n(w)$ contains at least one end. Hence the estimate

$$\#Ends(T_w) \geq \#infcomp(T_w \setminus X^n\sigma^n(w)) = \#infcomp(T_{\sigma^n(w)} \setminus \sigma^n(w))$$

holds for all n. For the converse consider the ends $\gamma_1, \ldots, \gamma_k$ of the graph T_w. They can be made disconnected by removing finitely many vertices. Take n large enough so that the set $X^n\sigma^n(w)$ disconnects the ends γ_i. Since every end belongs to an infinite component, we get at least k infinite components of $T_{\sigma^n(w)} \setminus \sigma^n(w)$. In particular, the number of ends is finite and the statement follows. □

In particular, the number of ends of every tile graph T_w is not greater than the maximal degree of vertices, i.e., $\#Ends(T_w) \leq |S|$.

Connected components in tile graphs without a vertex. We denote by $c(\Gamma)$ the number of connected components in the graph Γ.

Proposition 5. *For $w = x_1x_2\ldots \in X^\omega$ we have*

$$c(T_w \setminus w) = \lim_{n\to\infty} c(T_n \setminus w_n).$$

Proof. Let us show that the sequence $c(T_n \setminus w_n)$ is non-increasing for large enough n, and eventually is equal to $c(T_w \setminus w)$. We can consider the graph T_n as the subgraph of T_{n+k} (and T_w) under the inclusion $v \mapsto vx_{n+1}x_{n+2}\ldots x_{n+k}$. Notice that every edge of T_w appears in T_n for all large enough n. Choose n such that the subgraph T_n contains all edges of T_w adjacent to the vertex w. Hence, every component of $T_w \setminus w$ and $T_{n+1} \setminus w_{n+1}$ contains a vertex of the subgraph $T_n \setminus w_n$, and thus is connected with some component of $T_n \setminus w_n$. It follows $c(T_w \setminus w) \leq c(T_{n+k} \setminus w_{n+k}) \leq c(T_n \setminus w_n)$ for all $k \geq 1$. The bounded non-increasing integer sequence $c(T_n \setminus w_n)$ stabilizes, and we can suppose n satisfies $c(T_n \setminus w_n) = c(T_{n+k} \setminus w_{n+k})$ for all $k \geq 1$. That means that the components of $T_n \setminus w_n$ cannot be connected in $T_{n+k} \setminus w_{n+k}$, and hence in $T_w \setminus w$. Thus $c(T_w \setminus w) = c(T_n \setminus w_n)$, and the limit is proved. □

Now we want to treat infinite components of $T_w \setminus w$. Notice that if C is a component of $T_n \setminus w_n$ without post-critical vertices, then all the edges of the graph $T_w \setminus w$ adjacent to the component C are contained in the graph $T_n \setminus w_n$. Then C is a finite component of $T_w \setminus w$. Hence the number of infinite components of $T_w \setminus w$ is not greater than the number of components of $T_n \setminus w_n$ with post-critical vertices.

Proposition 6. *Let $w = x_1x_2\ldots \in X^\omega$ be a regular or a critical sequence. Then*

$$\#infcomp(T_w \setminus w) = \lim_{n\to\infty} ic(T_n \setminus w_n),$$

where $ic(T_n \setminus v)$ is the number of connected components of $T_n \setminus v$ that contain post-critical vertices.

Proof. Suppose w is regular. Let C be a finite component of $T_w \setminus w$. Since C is finite and does not contain critical sequences, for all large enough n the vertex v_n of T_n is not post-critical for every vertex v in C. Hence $\#infcomp(T_w \setminus w) = ic(T_n \setminus w_n)$ for large enough n.

The same holds if w is critical, because every cofinality class contains not more than one critical sequence, and hence the graph $T_w \setminus w$ has no critical sequences. □

With a slight modification the last proposition also works for a sequences w, which is not critical but is cofinal to some critical sequence u. In this case, we can count the number of connected components of $T_n \setminus w_n$ that contain post-critical vertices other than u_n, and then pass to the limit to get the number of infinite components in $T_w \setminus w$. Indeed, it is enough to notice that if the graph $T_n \setminus w_n$ contains a connected component C with precisely one post-critical vertex u_n for large enough n, then C is a finite component in the graph $T_w \setminus w$. Under this modification the proposition may be applied to any sequence.

Also to find the number of ends it is enough to know that the limit in Proposition 6 is valid for regular and critical sequences. For any sequence w cofinal to a critical sequence u we just consider the graph $T_w = T_u$ centered at the vertex u and apply the proposition.

Let us describe how to compute the numbers $c(T_n \setminus v)$ and $ic(T_n \setminus v)$.

Finite automaton recognizing the number of components in tile graphs without a vertex. Using the iterative construction of tile graphs given by Theorem 3 we can provide a recursive procedure to find the numbers $c(T_n \setminus v)$ and $ic(T_n \setminus v)$.

Let n be large enough so that different post-critical sequences $p, q \in \mathcal{P}$ induces different post-critical vertices p_n, q_n of the graph T_n.

Consider the tile graph T_n and its vertex v. There are two cases that we need to treat a little bit differently. First, suppose that v is not post-critical. The components of $T_n \setminus v$ define a partition of the post-critical set $\mathcal{P} = \bigsqcup_{i=1}^{k} \mathcal{P}_i$, where $\mathcal{P}_i$ is the set of all post-critical vertices from the same component. It is also possible, that there are some components without post-critical vertices, denote their number by $fc(T_n \setminus v)$. Under our notations $ic(T_n \setminus v) = k$ and $c(T_n \setminus v) = k + fc(T_n \setminus v)$. Let us describe how these quantities change when we pass to the tile graph T_{n+1} and its vertex vx for $x \in X$. Consider the model graph M, add an edge between $p \times y$ and $q \times y$ for every $p, q \in \mathcal{P}$ and $y \in X, y \neq x$, and an edge between $p \times x$ and $q \times x$ for $p, q \in \mathcal{P}_i$ for every i. Let $\mathcal{P} = \bigsqcup_{i=1}^{s} P_i$ be a partition of the post-critical set in the model graph M, and let $fc(M)$ be a number of components without post-critical vertices. Then it is the partition of $\mathcal{P}$ given by $T_{n+1} \setminus vx$, and $fc(T_{n+1} \setminus vx) = fc(M) + fc(T_n \setminus v)$.

Suppose v is a post-critical vertex. The components of $T_n \setminus v$ define a partition of $\mathcal{P} \setminus v = \bigsqcup_{i=1}^{k} \mathcal{P}_i$, where $\mathcal{P}_i$ is the set of all post-critical vertices from the same component. Consider the previous model graph M and delete the vertex $v \times x$. Let $\mathcal{P} = \bigsqcup_{i=1}^{s} P_i$ be a partition of the post-critical set in the model graph M, and let $fc(M)$ be a number of

components without post-critical vertices. Then it is the partition of $\mathcal{P}$ given by $T_{n+1} \setminus vx$, and $fc(T_{n+1} \setminus vx) = fc(M) + fc(T_n \setminus v)$.

Notice that our construction does not depend on n, but only on a given partition of $\mathcal{P}$ or $\mathcal{P} \setminus p$ for $p \in \mathcal{P}$. We can construct a recognition automaton A_{ic}, whose vertices correspond to the partitions of $\mathcal{P}$ and $\mathcal{P} \setminus p$, and there is an edge labeled by x from one partition to the constructed above partition. To start the process we need to add initial vertex, and for every letter $x \in X$ put an edge from the initial vertex to the vertex corresponding to the partition given by $T_1 \setminus x$. Since we are interested only in the number of components in $T_n \setminus v$ containing post-critical vertices, we label every vertex of A_{ic} by the number of components in the corresponding partition. The automaton A_{ic} contains enough information to find the number of ends of every tile graph T_w.

To find the number of all components, we just trace the numbers $fc(T_n \setminus v)$ by the formula above. To do that we can construct the automaton A_c, whose vertices correspond to the partitions of $\mathcal{P}$ and $\mathcal{P} \setminus p$ together with the integer number $fc(T_n \setminus v)$, and we put an edge labeled by x to the constructed above partition and the number $fc(T_{n+1} \setminus vx)$. In the obvious way we introduce the initial vertex, and label every vertex by the number of corresponding components, which is the number of components in the corresponding partition plus the number $fc(T_n \setminus v)$.

We get the following statement.

Proposition 7. *The graph $T_n \setminus v$ has k components if and only if the path in the automaton A_c corresponding to the word v ends in the vertex labeled by number k. In particular, for every k the set $C(k)$ of all words $v \in X^*$ such that the graph $T_{|v|} \setminus v$ has k components is a regular language recognized by the automaton A_c.*

The same statement holds for the number of components containing post-critical vertices and the automaton A_{ic}.

Using automata A_c and A_{ic} one can construct similar automata for the number of components in the Schreier graphs Γ_w without a vertex. For every vertex of A_c or A_{ic} take the corresponding partition of the post-critical set, and glue components according to the edges described after Theorem 3. We get a new partition and we label the vertex by the number of components in this partition. Basically, we get the same automata, but vertices will be labeled in a different way.

The number of ends. To treat the number of (infinite) components in $T_w \setminus w$ and the number of ends of T_w for $w \in X^\omega$ let us describe some properties of the automata A_c and A_{ic}.

Lemma 1.

1. *In every strongly connected component of the automata A_c and A_{ic} all vertices are labeled by the same number.*
2. *All strongly connected components of the automaton A_{ic} without outgoing edges are labeled by the same number.*

Proof. 1. Suppose that there is a strongly connected component with two vertices labeled by different numbers. It would imply that there exists an infinite word such that the

corresponding path in the automaton passes through each of these vertices infinite number of times. We get a contradiction with the proof of Proposition 5, because the sequences $c(T_n \setminus w_n)$ and $ic(T_n \setminus w_n)$ are eventually monotonic (for the last one the proof is the same).

2. Suppose there are two strongly connected components in the automaton A_{ic} without outgoing edges. Let v and u be finite words such that starting at the initial vertex of A_{ic} we end at the first and the second components respectively. Then for the infinite sequence $vuvu\ldots$ the limit in Proposition 4 does not exist, and we get a contradiction. □

By Proposition 5 the graph $T_w \setminus w$ has k components if and only if the path in the automaton A_c corresponding to the sequence w will be eventually in some strongly connected component labeled by the number k. Similarly by Proposition 6 we get the description of sequences with infinite components using the automaton A_{ic} (but only for regular and critical sequences).

It is useful to notice that we can construct a finite word $v \in X^*$ such that starting at any state of the automaton A_{ic} and following the word v we end in some strongly connected component without outgoing edges. If these strongly connected components correspond to the partition of the post-critical set on k parts, then it follows that $ic(T_n \setminus u_1 v u_2) = k$ for all words $u_1, u_2 \in X^*$ with $|u_1 v u_2| = n$.

To describe the number of ends we first treat critical sequences. Since every critical sequence w is periodic we can find the number of ends of the graph T_w using Proposition 4 and the automaton A_{ic}. Fix $k \geq 1$ and let $EC_{=k}$ be the union of cofinality classes of critical sequences w whose tile graph T_w has k ends. Let $\mathsf{A}_{ic}(k)$ be the subgraph of A_{ic} spanned by the strongly connected components labeled by number $\geq k$. Consider the right sofic subshift $\mathcal{R}$ given by the graph $\mathsf{A}_{ic}(k)$, which is the set of all sequences that can be read along right-infinite paths in $\mathsf{A}_{ic}(k)$ starting at any vertex. Define the set $E_{\geq k}$ of all sequences which are cofinal to some sequence from $\mathcal{R}$. Since the set $\mathcal{R}$ is shift-invariant, the set $E_{\geq k}$ coincides with $X^*\mathcal{R} = \{vw | v \in X^*, w \in \mathcal{R}\}$.

Proposition 8. *The tile graph T_w has $\geq k$ ends if and only if $w \in E_{\geq k} \setminus EC_{<k}$. Hence, the tile graph T_w has k ends if and only if $w \in E_{\geq k} \setminus (E_{\geq k+1} \cup EC_{\neq k})$.*

Proof. Take a regular sequence $w \in E_{\geq k}$. Then $w = uw'$ for some $w' \in \mathcal{R}$, and w' is regular. There exists v such that for the sequence vw' the corresponding path in A_{ic} starting at the initial vertex eventually lies in the subgraph $\mathsf{A}_{ic}(k)$. Then the graph $T_{vw'} \setminus vw'$ has $\geq k$ infinite components by Proposition 6. Hence the graph $T_{w'} \setminus w'$ has $\geq k$ infinite components and the graph $T_{uw'}$ has $\geq k$ ends by Proposition 4.

For the converse, suppose the graph T_w has $\geq k$ ends and the sequence w is regular. Then $ic(T_{\sigma^n(w)} \setminus \sigma^n(w)) \geq k$ for some n by Propositions 4 and 6. Hence some shift of the sequence w is in $\mathcal{R}$, and $w \in E_{\geq k}$. □

Now we can find the number of ends of Schreier graphs Γ_w by Proposition 2. For regular sequences w the graphs Γ_w and T_w have the same number of ends, and for critical sequences w we get

$$\#Ends(\Gamma_w) = \sum_{w' \in O(w)} \#Ends(T_{w'}),$$

where the set $O(w)$ is defined in Proposition 2.

Example with $IMG(z^2+i)$ in Section 4 shows that we cannot expect to get a description using subshifts of finite type, and indeed the description using sofic subshifts is the best possible.

The number of ends almost surely. For a finitely generated self-similar group, which acts transitively on levels X^n for all n, the action on the boundary X^ω is ergodic (see [13]). Hence the Schreier graphs Γ_w have the same number of ends for almost all sequences $w \in X^\omega$. In our settings of bounded automata this also follows from Lemma 1 item 2. Indeed, only strongly connected components without outgoing edges in the automaton A_{ic} may contribute a positive measure. Since all these components are labeled by the same number, for almost all sequences w the graphs $T_w \setminus w$ have the same number of infinite components. Hence almost all tile graphs T_w and Schreier graphs Γ_w have the same number of ends.

Proposition 9. *There are only finitely many Schreier graphs Γ_w and tile graphs T_w with more than two ends.*

Proof. Let us prove that the graph $T_w \setminus w$ can have more than 2 infinite components only for finitely many sequences w. Suppose not, and choose any such sequences $w^{(1)}, \dots, w^{(m)}$ with m larger than the number of partitions of the post-critical set $\mathcal{P}$. Choose level n large enough so that all words $w_n^{(1)}, \dots, w_n^{(m)}$ are different and $ic(T_n \setminus w_n^{(i)}) \geq 3$ for all i (it is possible by Proposition 6). Notice that since the graph T_n is connected the deletion of different vertices gives different partitions of $\mathcal{P}$. Indeed, if $\mathcal{P} = \sqcup_{i=1}^k \mathcal{P}_i$ with $k \geq 3$ is the partition we got after deletion of some vertex v, then some $k-1$ sets $\mathcal{P}_i$ will be in the same component of the graph $T_n \setminus u$ for any other vertex u. We get a contradiction with the choice of number m.

It follows that there are only finitely many tile graphs with more than 2 ends. This also holds for Schreier graphs by Proposition 2. □

Corollary 10. *The Schreier graphs Γ_w and tile graphs T_w can have more than two ends only for pre-periodic sequences w.*

Proof. Since the graph $T_w \setminus w$ can have > 2 infinite components only for finitely many sequences w, we get that in the limit in Proposition 4 the sequence $\sigma^n(w)$ attains a finite number of values. Hence w is pre-periodic. □

The last corollary is related to cut points of Julia sets [5].

Example with $IMG(z^2+i)$ in Section 4 shows that the Schreier graph Γ_w and the tile graph T_w may have more than two ends even for regular sequences w.

Corollary 11. *The tile graphs T_w and Schreier graphs Γ_w have the same number of ends for almost all sequences $w \in X^\omega$, which is equal to 1 or 2.*

The last statement is true in a much more general situation, see e.g. Proposition 6.10 in [1].

Two ends almost surely. In this section we classify groups for which almost all Schreier (tile) graphs have two ends. Notice that in this case the post-critical set $\mathcal{P}$ cannot consist of one element (actually every finitely generated self-similar group with $|\mathcal{P}| = 1$ is finite, and cannot act transitively on X^n for all n).

Lemma 2. *If almost all Schreier (tile) graphs have two ends then* $|\mathcal{P}| = 2$.

Proof. We pass to a power of the alphabet so that every post-critical sequence is of the form $y^{-\omega}$ or $y^{-\omega}x$ for some letters $x, y \in X$ and different post-critical sequences end on different letters. In particular, in the model graph M every component $\mathcal{P} \times x$ for $x \in X$ contains not more than one post-critical sequence.

Suppose almost all tile graphs have two ends. Then the strongly connected components without outgoing edges in the automaton A_{ic} correspond to the partitions of the post-critical set $\mathcal{P}$ on two parts. Take any partition $\mathcal{P} = P_1 \sqcup P_2$ corresponding to some vertex in one of these components. We will use the fact that all paths in A_{ic} starting at the partition $P_1 \sqcup P_2$ end in partitions of $\mathcal{P}$ with two parts.

To be under settings of the construction of the automaton A_{ic} we add edges to the model graph M between all vertices inside the component $\mathcal{P} \times x$ for every $x \in X$. Let us show that in the graph M there are only two components $\mathcal{P} \times x$ for $x \in X$ such that if we remove such a component the graph remains connected. Suppose that there are three such components $\mathcal{P} \times x$, $\mathcal{P} \times y$, $\mathcal{P} \times z$. Consider the arrow in the automaton A_{ic} starting at $P_1 \sqcup P_2$ and labeled by x. It ends in the partition with two parts $\mathcal{P} = P_1' \sqcup P_2'$. By construction and since the graph $M \setminus \mathcal{P} \times x$ is connected, one of the sets P_i' is a subset of $\mathcal{P} \times x$, and hence it contains precisely one element (post-critical sequence), which we denote by a. The partitions $\mathcal{P} \times y$ and $\mathcal{P} \times z$ also contain some post-critical sequences b and c. Notice that the last letters of a, b, c are x, y, z respectively. We can suppose that az and bz are different from c (over three post-critical sequences there are always two with this property). Consider the edge in the automaton A_{ic} starting at the partition $\{a\} \sqcup \mathcal{P} \setminus \{a\}$ and labeled by z. To get partition with two parts, the only edges of the model graph M going outside the component $\mathcal{P} \times z$ should be at the vertex (a, z). By the same arguments for the partition $\{b\} \sqcup \mathcal{P} \setminus \{b\}$, this unique vertex should be (b, z). Hence $a = b$ and we get a contradiction.

By the same arguments the component $\mathcal{P} \times x$ has a unique vertex which is connected to the rest of the model graph M, and this vertex is of the form (a, x) or (b, x). The same holds for the component $\mathcal{P} \times y$. Every other component $\mathcal{P} \times z$ contains precisely two vertices (a, z) and (b, z), which have edges going outside the component $\mathcal{P} \times z$. However every post-critical sequence appears in one of such edges. Hence the post-critical set contains precisely 2 elements. □

Corollary 12. *If the post-critical set* $\mathcal{P}$ *contains at least* 3 *points, then almost all Schreier graphs and tile graphs have one end.*

The following example shows that almost all Schreier graphs may have two ends for contracting group generated by a non-bounded automaton, i.e., with infinite post-critical set.

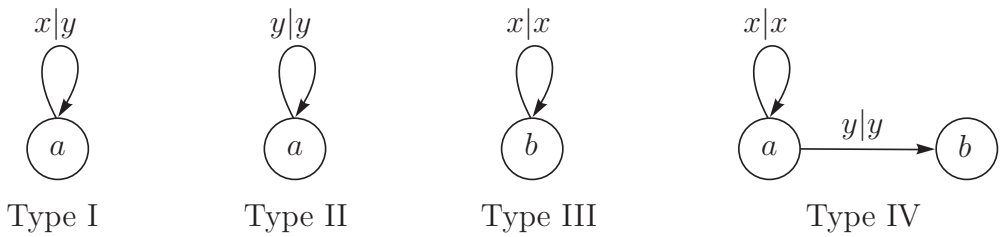

Figure 1: Bounded automata with $|\mathcal{P}| = 2$

Example 1. Consider the self-similar group G over $X = \{0, 1, 2\}$ generated by the element $a = (a^2, 1, a^{-1})(0, 1, 2)$ (see Example 7.6 in [4]). The group G is self-replicating and contracting with nucleus $\mathcal{N} = \{1, a^{\pm 1}, a^{\pm 2}\}$, but the generating automaton is not bounded. Every Schreier graph Γ_w is a line and has two ends. The tile graphs T_n are not connected and the tile $\mathcal{T}$ is a totally disconnected set. The limit space $\mathcal{J}_G$ is homeomorphic to a unit circle.

Notice that we have proved also in the lemma that if we connect by an edge vertices $a \times x$ and $b \times x$ of the model graph for every $x \in X$ than the obtained graph is an "interval", i.e., there are two vertices of degree 1, and the other vertices have degree 2. We use this in the next classification.

Theorem 13. *Almost all Schreier graphs have two ends if and only if the nucleus $\mathcal{N}$ of the group brought to the basic form (see Section 2.4) is one of the following.*

1. *The nucleus consists of the adding machine, its inverse, and the trivial element, where the adding machine is an element of type I (see Figure 1, where all edges not shown in the figure goes to the identity state) with transitive action on X.*

2. *There exists an order on the alphabet $X = \{x = x_1, x_2, \ldots, x_m = y\}$ such that one of the following cases holds.*

   (a) *The nucleus consists of elements of type II and III (see Figure 1); every pair $\{x_{2i-1}, x_{2i}\}$ is an orbit of the action of some element of type II and all nontrivial orbits of such elements on X are of this form; also every pair $\{x_{2i}, x_{2i+1}\}$ is an orbit of the action of some element of type III and all nontrivial orbits of such elements on X are of this form (in particular $|X|$ is an odd number).*

   (b) *The nucleus consists of elements of type III and IV; every pair $\{x_{2i}, x_{2i+1}\}$ is an orbit of the action of some element a of type III or IV and all nontrivial orbits of such elements on X are of this form; also every pair $\{x_{2i-1}, x_{2i}\}$ is an orbit of the action of some finitary element b of type IV and all nontrivial orbits of such elements on X are of this form (in particular $|X|$ is an even number).*

Moreover, in this case, all Schreier graphs are quasi-isometric to a line, except two Schreier graphs Γ_{x^ω} and Γ_{y^ω} in case 2 (a), and one Schreier graph Γ_{x^ω} in case 2 (b), which are quasi-isometric to a ray.

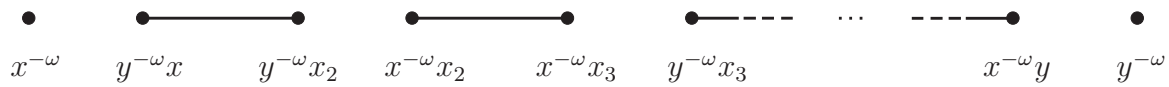

Figure 2: Case 2 (a)

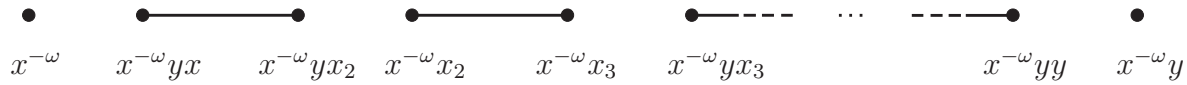

Figure 3: Case 2 (b)

Proof. After passing to a power of the alphabet, the nucleus of the group with $|\mathcal{P}| = 2$ consists of the elements shown in Figure 1 and the identity state. There are two cases that we need to treat a little bit differently depending on whether both post-critical sequences are periodic or not.

Consider the case when both post-critical sequences are periodic, here $\mathcal{P} = \{x^{-\omega}, y^{-\omega}\}$. In this case the nucleus may consist only of elements of type I, II, and III. Suppose the nucleus contains an element a of type I. It contributes the edges $\{(x^{-\omega}, z), (y^{-\omega}, a(z))\}$ to the model graph M for every $z \in X$. If there exists a nontrivial orbit of the action of a on X, which does not contain x, then it contributes a cycle to the model graph. If there exists a fixed point $a(z) = z$, then under construction of the automaton A_{ic} starting at the partition $\mathcal{P} = \{x^{-\omega}\} \sqcup \{y^{-\omega}\}$ and following the arrow labeled by z we get a partition with one part. Hence the element a should act transitively on X (it is the adding machine). Every other element of type I in the nucleus should have the same action on X, and hence coincide with a, otherwise we would got a vertex in the model graph of degree ≥ 3. If the nucleus additionally contains an element of type II or III, then there is an edge $\{(x^{-\omega}, z_1), (x^{-\omega}, z_2)\}$ or $\{(y^{-\omega}, z_1), (y^{-\omega}, z_2)\}$ in the model graph M for some different letters $z_1, z_2 \in X$ and we get a vertex of degree ≥ 3. Hence in this case the nucleus consists of the adding machine, its inverse, and the identity state.

Now suppose the nucleus contains elements of type II and III. These elements contribute edges $\{(x^{-\omega}, z), (x^{-\omega}, a(z))\}$ and $\{(y^{-\omega}, z), (y^{-\omega}, b(z))\}$ to the model graph. Since the model graph should be an interval, these edges should consequently connect all components $\mathcal{P} \times z$ for $z \in X$ (see Figure 2). It follows that there exists an order on the alphabet such that (a) holds.

Consider the case $\mathcal{P} = \{x^{-\omega}, x^{-\omega}y\}$. In this case the nucleus consists of elements of type III and IV. These elements contribute edges $\{(x^{-\omega}, z), (x^{-\omega}, a(z))\}$ and $\{(x^{-\omega}y, z), (x^{-\omega}y, b(z))\}$ to the model graph. These edges should consequently connect all components $\mathcal{P} \times z$ for $z \in X$ (see Figure 3). It follows that there exists an order on the alphabet such that (b) holds.

For the converse, one can directly check the following facts. In item 1, every Schreier graph Γ_w with respect to the nucleus is a line, and hence the Schreier graphs Γ_w with respect to any finite generating set are quasi-isometric to a line. In case 2 (a), the Schreier graphs Γ_{x^ω} and Γ_{y^ω} are quasi-isometric to a ray, while all the other Schreier graphs are

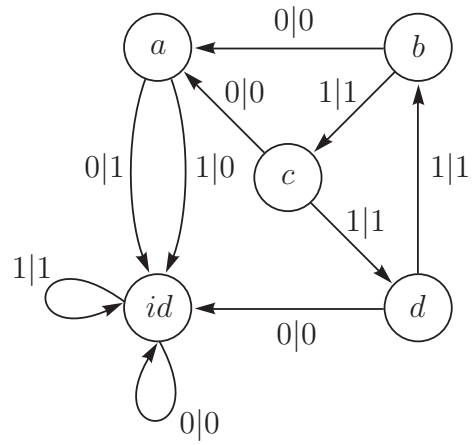

Figure 4: The generating automaton of the Grigorchuk group

quasi-isometric to a line. In case 2 (b), the Schreier graph Γ_{x^ω} is quasi-isometric to a ray, while all the other Schreier graphs are quasi-isometric to a line. $\square$

Example 2. The Grigorchuk group is a non-trivial example satisfying the conditions of the theorem. It is generated by the automaton shown in Figure 4, which also coincides with the nucleus of the group. After passing to the alphabet $\{0,1\}^3 \leftrightarrow X = \{0,1,2,3,4,5,6,7\}$, the nucleus $\mathcal{N}$ consists of the trivial element and

$$\begin{aligned} a &= (1,1,1,1,1,1,1,1)(0,4)(1,5)(2,6)(3,7) \\ b &= (1,1,1,1,1,1,1,b)(0,2)(1,3)(4,5) \\ c &= (1,1,1,1,1,1,a,c)(0,2)(1,3) \\ d &= (1,1,1,1,1,1,a,d)(4,5) \end{aligned}$$

We see that the nucleus satisfies case 2 (b) of the theorem when we choose the order $6,2,0,4,5,1,3,7$ on X. The Schreier graph Γ_{7^ω} is a ray, while the other orbital Schreier graphs are lines.

Connection with cut-points of tiles and limit spaces. In this section we show how the number of ends of Schreier and tile graphs is related to the number of connected components in tile and limit space without a point. Since the limit space $\mathcal{J}_G$ is obtained from the tile $\mathcal{T}$ by gluing finitely many specific boundary points, it is sufficient to consider the problem for the tile $\mathcal{T}$. Let $c(\mathcal{T}\backslash x)$ denotes the number of connected components in $\mathcal{T}\backslash x$.

We consider the tile $\mathcal{T}$ as its own topological space (with the induced topology from the space $\mathcal{X}_G$), and every tile $\mathcal{T}_v$ for $v \in X^*$ as a subset of $\mathcal{T}$ with induced topology. Hence the boundary of $\mathcal{T}$ is empty, but the points represented by post-critical sequences we still call *boundary points* of the tile. Every point in the intersection of tiles $\mathcal{T}_v \cap \mathcal{T}_u$ of the same level $|v| = |u|$ we call *critical*. These points are precisely the boundary points of the tiles $\mathcal{T}_v$ for $v \in X^*$ and they are represented by sequences of the form pv for $p \in \mathcal{P}$ and $v \in X^*$. In

particular the number of critical points is countable and hence they are of measure zero. All the other points we call *regular*.

If a sequence $\dots x_2x_1 \in X^{-\omega}$ represents a regular point $x \in \mathcal{T}$, then x is an interior point of the tile $\mathcal{T}_{x_n\dots x_2x_1}$ for all $n \geq 1$. Hence for a regular point x every connected component of $\mathcal{T} \setminus x$ intersects the tile $\mathcal{T}_{x_n\dots x_2x_1}$. Since the number of boundary points of all tiles $\mathcal{T}_v$ is not greater than $|\mathcal{P}|$, it follows that $c(\mathcal{T} \setminus x)$ coincides with the number of components in the partition of the boundary of the tile $\mathcal{T}_{x_n\dots x_2x_1}$ in $\mathcal{T} \setminus x$ for large enough n (in particular $c(\mathcal{T} \setminus x) \leq |\mathcal{P}|$). The last problem can be divided on two parts. First, we can find how the boundary of the tile $\mathcal{T}_{x_n\dots x_2x_1}$ decomposes in $\mathcal{T} \setminus \operatorname{int} \mathcal{T}_{x_n\dots x_2x_1}$ (here "int" stands for the interior of a set). Similar to the construction of the automaton A_{ic} one can construct a finite automaton for this problem, which given a word v returns the partition of $\partial\mathcal{T}_v$ (partition of a subset of $\mathcal{P}$) in $\mathcal{T} \setminus \operatorname{int} \mathcal{T}_v$. Second, we can find how the boundary $\partial\mathcal{T}_{x_n\dots x_2x_1}$ decomposes in $\mathcal{T}_{x_n\dots x_2x_1} \setminus x$. This can be done by the proposition below. Combining these two partitions of $\partial\mathcal{T}_{x_n\dots x_2x_1}$ we can compute the number of connected components in $\mathcal{T} \setminus x$.

Let T_n be the tile graph with respect to the nucleus of the group.

Proposition 14. *Let $x \in \mathcal{T}$ be a regular point represented by a sequence $\dots x_2x_1 \in X^{-\omega}$. Then*

$$bc(\mathcal{T} \setminus x) = \lim_{n\to\infty} ic(T_n \setminus x_n \dots x_2x_1),$$

where $bc(\mathcal{T} \setminus x)$ is the number of components of $\mathcal{T} \setminus x$ that contain the boundary points of $\mathcal{T}$.

Proof. Recall that two tiles $\mathcal{T}_v$ and $\mathcal{T}_u$ for $v, u \in X^n$ have non-empty intersection if and only if the vertices v and u are connected by an edge in the graph T_n. The interior $\operatorname{int} \mathcal{T}_v$ of the tile $\mathcal{T}_v$ consists of all points except finitely many those, which also belong to the other tiles of the same level. It follows that two vertices v and u of the graph T_n are connected by a path, which avoids the vertex $x_n \dots x_2x_1$, if and only if the tiles $\mathcal{T}_v$ and $\mathcal{T}_u$ lie in the same connected component of $\mathcal{T} \setminus \operatorname{int} \mathcal{T}_{x_n\dots x_2x_1}$.

Choose n large enough so that the tile $\mathcal{T}_{x_n\dots x_2x_1}$ does not contain the boundary points of $\mathcal{T}$ containing in $\mathcal{T} \setminus x$ and every tile $\mathcal{T}_v$ for $v \in X^n$ contains at most one boundary point of $\mathcal{T}$. Since the point x is regular, if two boundary points of the tile $\mathcal{T}$ lie in the same connected component of $\mathcal{T} \setminus \operatorname{int} \mathcal{T}_{x_n\dots x_2x_1}$, they lie in the same connected component of $\mathcal{T} \setminus x$. The inequality "$\leq$" follows.

Conversely, since the number of boundary points is finite, if two boundary points of $\mathcal{T}$ lie in the same components of $\mathcal{T} \setminus x$, then for sufficiently large n these two points lie in the same components of $\mathcal{T} \setminus \operatorname{int} \mathcal{T}_{x_n\dots x_2x_1}$. The statement follows. □

The critical points can be treated as follows. Notice that every point x of the tile $\mathcal{T}$ represented by a periodic post-critical sequences is regular. We can apply the above method to find the number $c(\mathcal{T} \setminus x)$ and the partition of the boundary of $\mathcal{T}$ in $\mathcal{T} \setminus x$. Now let the point x be represented by a pre-periodic post-critical sequence $p_1x_1 \in \mathcal{P}$ with periodic sequences $p_1 \in \mathcal{P}$ and $x_1 \in X$. Consider vertices $(p_i, x_i), i = 2, \dots, k$ of the model graph M connected with the vertex (p_1, x_1). Notice that since p_1 is periodic all p_i are periodic. The sequences p_ix_i are precisely all sequences that represent the point x. Hence the point x is an interior

point of the set $U = \cup_{i=1}^{k} \mathcal{T}_{x_i}$. We know how the boundary of the set U decomposes in $\mathcal{T} \setminus \operatorname{int} U$. Since every tile $\mathcal{T}_{x_i}$ is homeomorphic to the tile $\mathcal{T}$ via the homeomorphism s, we can find $c(\mathcal{T}_{x_i} \setminus x) = c(\mathcal{T} \setminus p_i)$ and deduce the decomposition of the boundary of $\mathcal{T}_{x_i}$ in $\mathcal{T}_{x_i} \setminus x$. Combining this information we find $c(\mathcal{T} \setminus x)$ and the corresponding decomposition of $\partial\mathcal{T}$ (in particular, $c(\mathcal{T} \setminus x) \leq k|\mathcal{P}|$). In this way we can find $c(\mathcal{T} \setminus x)$ for every point x represented by a post-critical sequence. Now let x be a critical point represented by the sequence pyu with $p \in \mathcal{P}$, $y \in X$, $u \in X^*$, and $py \notin \mathcal{P}$. As described above there is a finite automaton, which reading the word u returns the decomposition of the boundary of $\mathcal{T}_u$ in $\mathcal{T} \setminus \operatorname{int} \mathcal{T}_u$. Using the fact that $\mathcal{T}_u \setminus x$ and $\mathcal{T} \setminus py$ are homeomorphic we can find the number $c(\mathcal{T}_u \setminus x)$ as we did with pre-periodic sequences. Since the post-critical set is finite, one can construct a finite automaton, which given a critical point x by its representation pv for $p \in \mathcal{P}$ and $v \in X^*$ returns the number $c(\mathcal{T} \setminus x)$.

Propositions 6 and 14 establish the connection between cut-points of the tile $\mathcal{T}$ and the number of ends of tile graphs T_w. To describe the limit in Proposition 6 we constructed the automaton A_{ic}, which returns $ic(T_n \setminus x_n \dots x_2 x_1)$ by reading the word $x_n \dots x_2 x_1$ from left to right, so that we can apply it to right-infinite sequences. Similarly one can construct the finite automaton RA_{ic}, which returns $ic(T_n \setminus x_n \dots x_2 x_1)$ by reading the word $x_n \dots x_2 x_1$ from right to left (the reversion of a regular language is a regular language) so that we can apply it to left-infinite sequences and describe the limit in Proposition 14.

We can use the results from the previous subsections to get information about cut-points up to measure zero. If almost all tile graphs T_w have one end, then there exists a word $v \in X^*$ such that $ic(T_n \setminus u_1 v u_2) = 1$ for all $u_1, u_2 \in X^*$ with $n = |u_1 v u_2|$. The set of all sequences of the form $u_1 v u_2$ for $u_1 \in X^{-\omega}$ and $u_2 \in X^*$ is of full measure, and by Proposition 14 we get that $c(\mathcal{T} \setminus x) = 1$ for almost all points $x \in \mathcal{T}$. If almost all tile graphs have two ends, then we are in the settings of Theorem 13. In both cases of this theorem the tile $\mathcal{T}$ is an interval, and hence $c(\mathcal{T} \setminus x) = 2$ for almost all points. Hence the typical number of ends of Schreier graphs coincides with the typical number of connected components in the tile without a point. We have proved the following statement.

Proposition 15. *The number of connected components in $\mathcal{T} \setminus x$ (or in $\mathcal{J}_G \setminus x$) is the same for almost all points x, and is equal to 1 or 2.*

Almost every point of the tile (or the limit space) is a bisection point, i.e., $c(\mathcal{T} \setminus x) = 2$, if and only if almost all orbital Schreier graphs Γ_w have two ends, and in this case the tile is homeomorphic to an interval.

Corollary 16. *The tile of a contracting self-similar group with open set condition is homeomorphic to an interval if and only if the nucleus of the group satisfies Theorem 13.*

The limit space of a contracting self-similar group with connected tiles and open set condition is homeomorphic to a unit circle if and only if the nucleus of the group consists of the adding machine, its inverse, and the identity state.

The limit space of a contracting self-similar group with connected tiles and open set condition is homeomorphic to an interval if and only if the nucleus of the group satisfies Case 2 of Theorem 13.

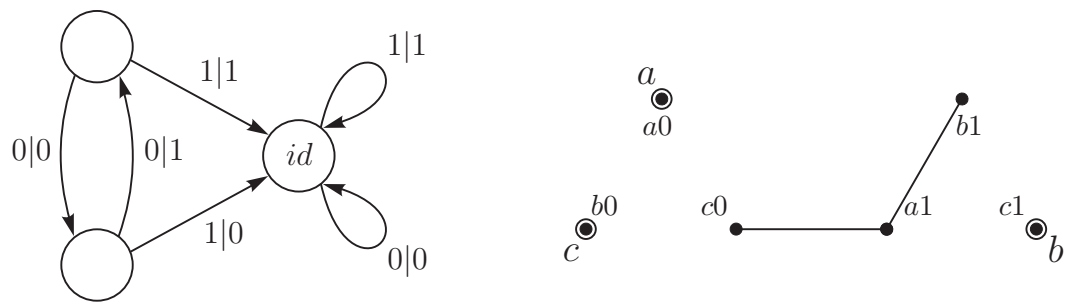

Figure 5: Basilica automaton and its model graph

Proof. Let G be a contracting self-similar groups with open set condition, and let the tile $\mathcal{T}$ of the group G is homeomorphic to an interval. Then the group G has connected tiles, all tiles $\mathcal{T}_v$ are homeomorphic to an interval. The boundary of tiles is finite, hence the group G is generated by a bounded automaton (here we use the open set condition) and we are under the settings of this section. Since $c(\mathcal{T} \setminus x) = 2$ for almost all points, almost all tile graphs have two ends, and hence the group satisfies Theorem 13.

It is left to prove the statements about limit spaces. In Case 1 of Theorem 13 the limit space is homeomorphic to a unit circle, and in Case 2 it is homeomorphic to an interval. Conversely, a small connected neighborhood of any point of a circle or of an interval is homeomorphic to an interval. Hence the tile should be homeomorphic to an interval, the group is generated by a bounded automaton, and we are in the settings of Theorem 13. □

The corollary agrees with the result of V. Nekrashevych and Z. Sunic about classification of self-similar groups whose limit dynamical system is conjugate to the tent map (see [20, Theorem 5.5]).

4 Examples

4.1 Basilica group

The Basilica group G is generated by the automaton shown in Figure 5. This group is the iterated monodromy group of $z^2 - 1$. It is torsion-free, has exponential growth, and is the first example of amenable but not subexponentially amenable group (see [17]). The orbital Schreier graphs Γ_w of this group have polynomial growth of degree 2 (see [6, Chapter VI]). The structure of Schreier graphs Γ_w was investigated in [10]. In particular it was shown that there are uncountably many pairwise non-isomorphic graphs Γ_w and the number of ends was described. Let us show how to get the result about ends using the developed method.

The alphabet is $X = \{0, 1\}$ and the post-critical set $\mathcal{P}$ consists of three elements $a = 0^{-\omega}$, $b = (01)^{-\omega}$, $c = (10)^{-\omega}$. The model graph is shown in Figure 5. The automata A_c and A_{ic} are shown in Figure 6. We get that every tile graph T_w has one or two ends, and we

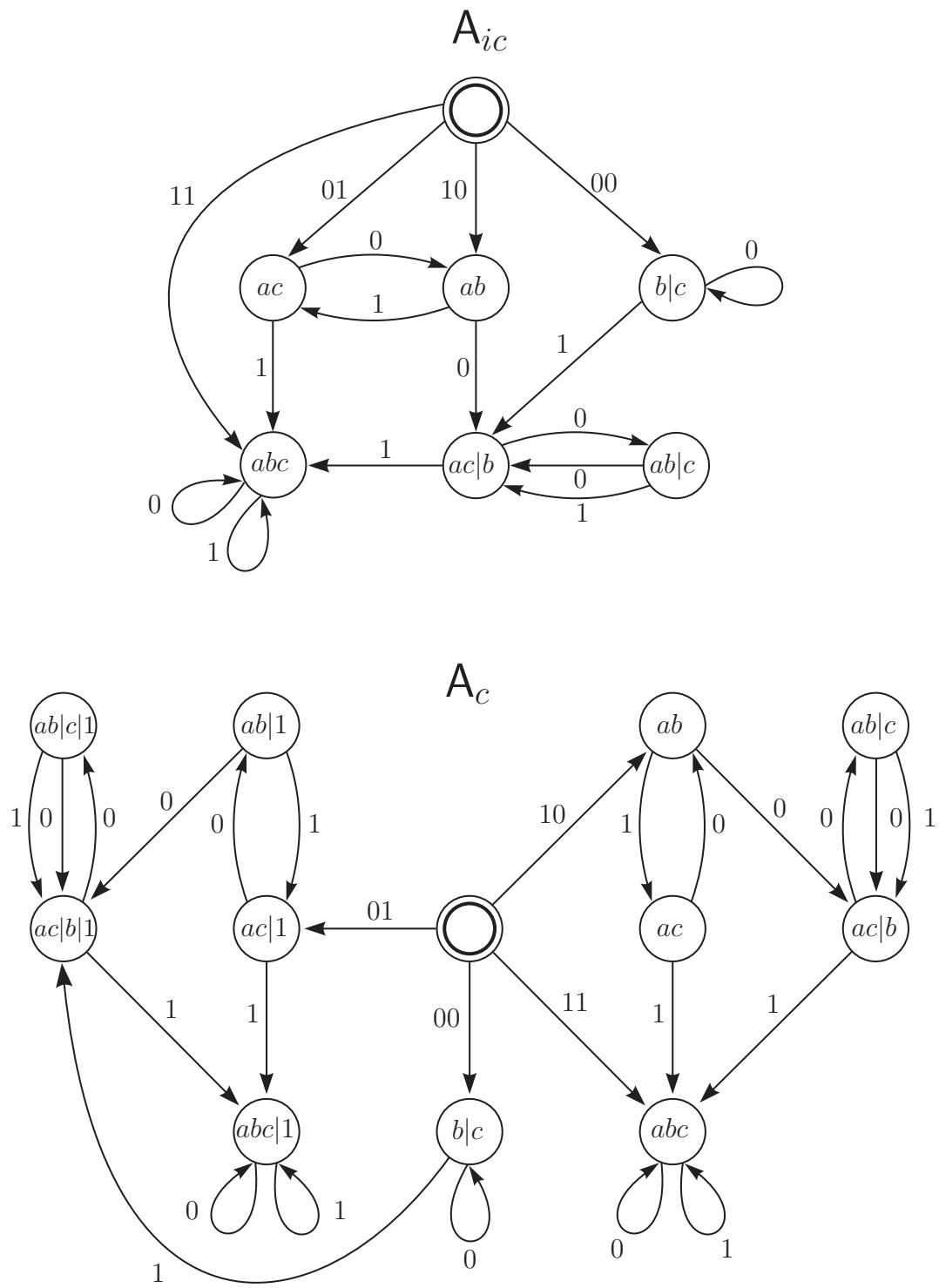

Figure 6: The automata A_{ic} and A_c for Basilica group

denote by E_1 and E_2 the corresponding sets of sequences. For the critical sequences $w = 0^\omega$ the tile graph T_w has two ends, while for the other critical sequences $(01)^\omega$ and $(10)^\omega$ the tile graph T_w has one end. Using automaton A_{ic} the sets E_1 and E_2 can be described by Proposition 8 as follows

$$\begin{aligned} E_2 &= \{x_1x_2\dots x_n0x_{n+1}0x_{n+2}\dots | n \geq 0 \text{ and } x_i \in X\} \setminus (Cof((01)^\omega \cup (10)^\omega)), \\ E_1 &= X^\omega \setminus E_2. \end{aligned}$$

Almost every tile graph T_w has one end, the set E_2 is uncountable but of measure zero.

Every graph $T_w \setminus w$ has 1, 2, or 3 connected components, and we denote by C_1, C_2, and C_3 the corresponding sets of sequences. Using automaton A_c these sets can be described precisely as follows

$$\begin{aligned} C_3 &= \left\{010(10)^k00x_10x_2\dots, 000^k10x_10x_2\dots | k \geq 0 \text{ and } x_i \in X\right\}, \\ C_2 &= \left(00X^\omega \cup 01X^\omega \cup \left\{(10)^k00x_10x_2\dots | k \geq 1 \text{ and } x_i \in X\right\}\right) \setminus C_3, \\ C_1 &= X^\omega \setminus (C_2 \cup C_3). \end{aligned}$$

The set C_3 is uncountable but of measure zero, while the sets C_1 and C_2 are of measure 1/2.

Every graph $T_w \setminus w$ has 1 or 2 infinite components. The corresponding sets IC_1 and IC_2 can be described using automaton A_{ic} as follows

$$\begin{aligned} IC_2 &= \left\{(10)^k00x_10x_2\dots, 0(10)^k00x_10x_2\dots, 00^k10x_10x_2\dots | k \geq 1 \text{ and } x_i \in X\right\} \setminus \\ &\quad \setminus (Cof((01)^\omega \cup (10)^\omega)), \\ IC_1 &= X^\omega \setminus IC_2. \end{aligned}$$

The set IC_2 is uncountable but of measure zero.

The finite Schreier graph Γ_n differs from the finite tile graph T_n by two edges $\{a_n, b_n\}$ and $\{a_n, c_n\}$. Assuming these edges one can relabel the states of the automaton A_c so that it returns the number of components in $\Gamma_n \setminus v$. In this way we get that $c(\Gamma_n \setminus v) = 1$ if the word v starts with 10 or 11, in the other cases $c(\Gamma_n \setminus v) = 2$. In particular the Schreier graph Γ_n has 2^{n-1} cut-vertices.

The orbital Schreier graph Γ_w coincides with the the tile graph T_w except when w is critical. The critical sequences 0^ω, $(01)^\omega$, and $(10)^\omega$ lie in the same orbit and the corresponding Schreier graph consists of three tile graphs T_{0^ω}, $T_{(01)^\omega}$, $T_{(10)^\omega}$ with two new edges $(0^\omega, (01)^\omega)$ and $(0^\omega, (10)^\omega)$. It follows that this graph has four ends.

The limit space $\mathcal{J}_G$ of the group G is homeomorphic to the Julia set of $z^2 - 1$ shown in Figure 7. The tile $\mathcal{T}$ can be obtained from the limit space by cutting the limit space in the way shown in the figure, or, vise versa, the limit space can be obtained from the tile by gluing points represented by post-critical sequences $0^{-\omega}$, $(01)^{-\omega}$, $(10)^{-\omega}$. Every point $x \in \mathcal{T}$ separates the tile $\mathcal{T}$ on 1, 2, or 3 connected components. Put $\mathcal{C} = \{0^{-\omega}1, (01)^{-\omega}1, (10)^{-\omega}0\}$. Then the sets $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ of sequences from $X^{-\omega}$, which represent the corresponding

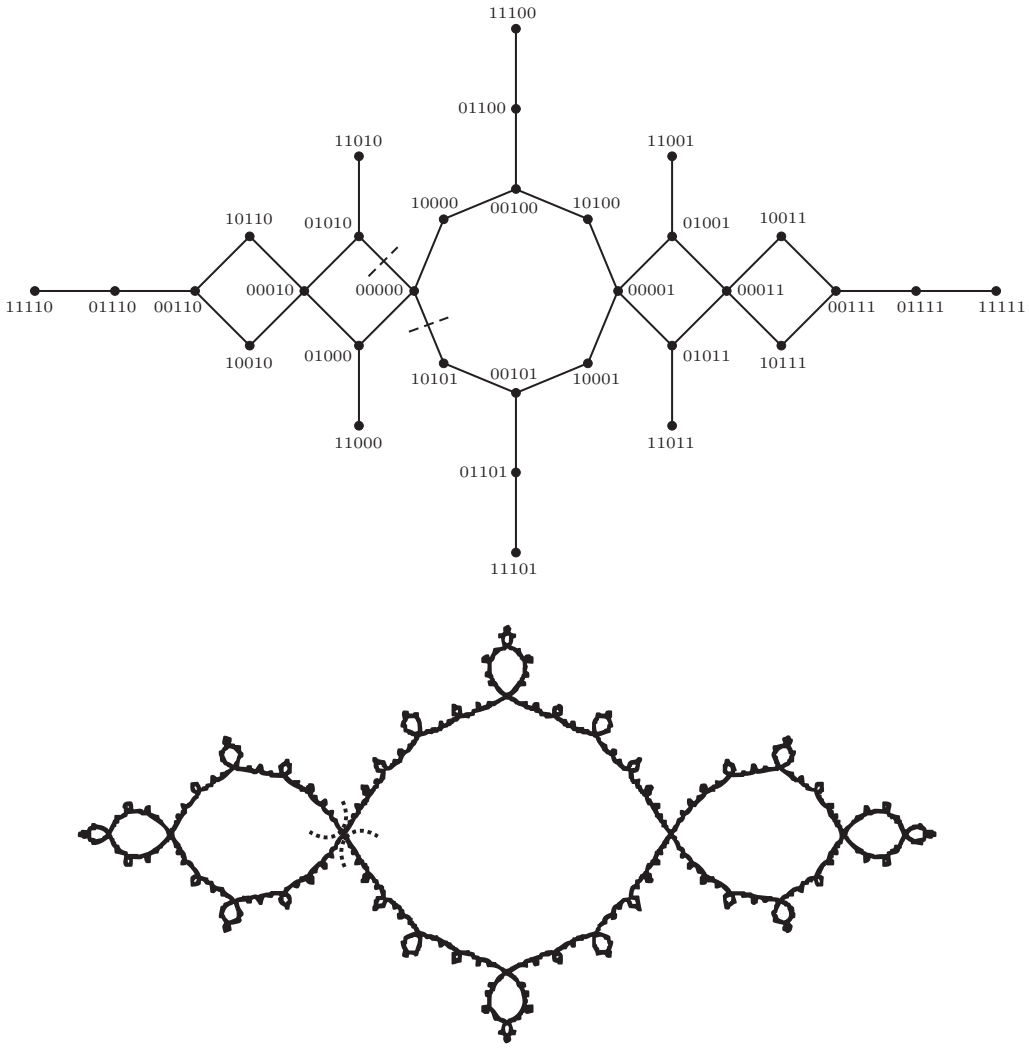

Figure 7: The Schreier graph Γ_5 of the Basilica group and its limit space

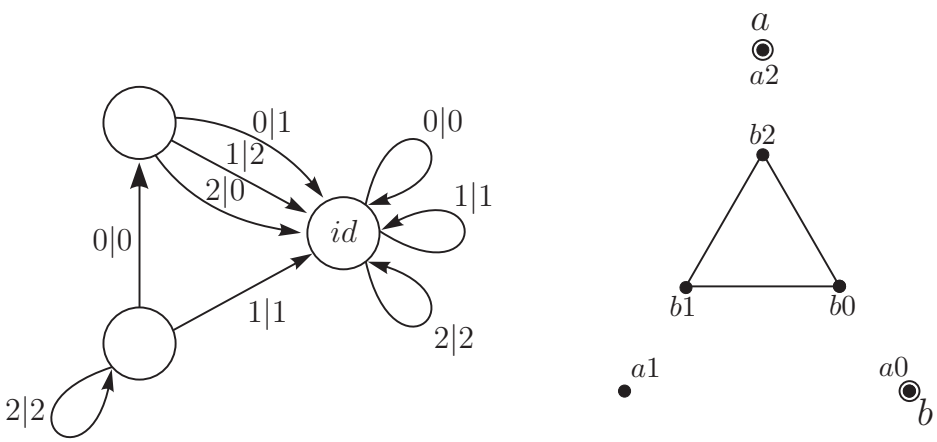

Figure 8: Gupta-Fabrykowski automaton and its model graph

points, can be described as follows

$$\begin{aligned}
\mathcal{C}_3 &= \bigcup_{n\geq 0} \mathcal{C}(0X)^n \cup \mathcal{C}(0X)^n 0,\\
\mathcal{C}_2 &= \bigcup_{n\geq 0} (\mathcal{C}(X0)^n \cup \mathcal{C}(X0)^n X) \bigcup \left((0X)^{-\omega} \cup (X0)^{-\omega}\right) \setminus \left(\mathcal{C}_3 \cup \left\{(10)^{-\omega}, (01)^{-\omega}\right\}\right),\\
\mathcal{C}_1 &= X^{-\omega} \setminus (\mathcal{C}_2 \cup \mathcal{C}_3).
\end{aligned}$$

The set $\mathcal{C}_3$ of three-section points is countable, the set $\mathcal{C}_2$ of bisection points is uncountable and of measure zero, and the tile $\mathcal{T} \setminus x$ is connected for almost all points x.

Every point $x \in \mathcal{J}_G$ separates the limit space $\mathcal{J}_G$ on 1 or 2 connected components. The corresponding sets $\mathcal{C}'_1$ and $\mathcal{C}'_2$ can be described as follows

$$\begin{aligned}
\mathcal{C}'_2 &= \bigcup_{n\geq 0} (\mathcal{C}(X0)^n \cup \mathcal{C}(0X)^n \cup \mathcal{C}(0X)^n 0 \cup \mathcal{C}(X0)^n X) \bigcup\\
&\quad \bigcup \left((0X)^{-\omega} \cup (X0)^{-\omega}\right) \setminus \left\{(10)^{-\omega}, (01)^{-\omega}\right\},\\
\mathcal{C}'_1 &= X^{-\omega} \setminus \mathcal{C}'_2.
\end{aligned}$$

The set $\mathcal{C}'_2$ of bisection points is uncountable and of measure zero, and the limit space $\mathcal{J}_G \setminus x$ is connected for almost all points x.

4.2 Gupta-Fabrykowski group

The Gupta-Fabrykowski group G is generated by the automaton shown on Figure 8. It was constructed in [12] as an example of a group of intermediate growth. Also this group is the iterated monodromy group of $z^3(-\frac{3}{2} + i\frac{\sqrt{3}}{2}) + 1$ (see [19, Example 6.12.4]). The Schreier graphs Γ_w of this group were studied in [2], where their spectrum and growth were computed (they have polynomial growth of degree $\frac{\log 3}{\log 2}$).

The alphabet is $X = \{0,1,2\}$ and the post-critical set $\mathcal{P}$ consists of two elements $a = 2^{-\omega}$ and $b = 2^{-\omega}0$. The model graph is shown in Figure 8. The automata A_c and A_{ic} are shown in Figure 9. Every Schreier graph Γ_w coincides with the tile graph T_w. We get that every tile graph T_w has one or two ends, and we denote by E_1 and E_2 the corresponding sets of sequences. For the only critical sequence 2^ω the tile graph T_w has one end. Using automaton A_{ic} the sets E_1 and E_2 can be described by Proposition 8 as follows

$$E_2 = X^*\{0,2\}^\omega \setminus Cof(2^\omega), \qquad E_1 = X^\omega \setminus E_2.$$

Almost every tile graph has one end, the set E_2 is uncountable but of measure zero.

Every graph $T_w \setminus w$ has 1 or 2 connected components, and we denote by C_1 and C_2 the corresponding sets of sequences. Using automaton A_c these sets can be described precisely as follows

$$C_2 = \bigcup_{k\geq 0} \left(2^k 01 X^\omega \cup 2^k 0\{0,2\}^\omega\right), \qquad C_1 = X^\omega \setminus C_2.$$

The sets C_1 and C_2 have measure $\frac{5}{6}$ and $\frac{1}{6}$ respectively.

Every graph $T_w \setminus w$ has 1 or 2 infinite components. The corresponding sets IC_1 and IC_2 can be described using automaton A_{ic} as follows

$$IC_2 = \bigcup_{k\geq 0} 2^k 0\{0,2\}^\omega \setminus Cof(2^\omega), \qquad IC_1 = X^\omega \setminus IC_2.$$

The set IC_2 is uncountable but of measure zero.

The limit space $\mathcal{J}_G$ and the tile $\mathcal{T}$ of the group G are homeomorphic to the Julia set of the map $z^3(-\frac{3}{2} + i\frac{\sqrt{3}}{2}) + 1$ shown in Figure 10. Every point $x \in \mathcal{J}_G$ separates the limit space on 1, 2, or 3 connected components. The sets $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ of sequences from $X^{-\omega}$, which represent the corresponding points, can be described as follows

$$\begin{aligned}
\mathcal{C}_3 &= 2^{-\omega}0X^* \setminus \{2^{-\omega}0\},\\
\mathcal{C}_2 &= \{0,2\}^{-\omega} \setminus \left(\mathcal{C}_3 \cup \{2^{-\omega}, 2^{-\omega}0\}\right),\\
\mathcal{C}_1 &= X^{-\omega} \setminus (\mathcal{C}_2 \cup \mathcal{C}_3).
\end{aligned}$$

The set $\mathcal{C}_3$ of three-section points is countable, the set $\mathcal{C}_2$ of bisection points is uncountable and of measure zero, and the limit space $\mathcal{J}_G \setminus x$ is connected for almost all points x.

4.3 Iterated monodromy group of $z^2 + i$

The iterated monodromy group of z^2+i is generated by the automaton shown in Figure 11. This group is one more example of a group of intermediate growth (see [9]). The algebraic properties of $IMG(z^2 + i)$ were studied in [14]. The Schreier graphs Γ_w of this group have polynomial growth of degree $\frac{\log 2}{\log \lambda}$, where λ is the real root of $x^3 - x - 2$ (see [6, Chapter VI]).

The alphabet is $X = \{0,1\}$ and the post-critical set $\mathcal{P}$ consists of three elements $a = (10)^{-\omega}0$, $b = (10)^{-\omega}$, and $c = (01)^{-\omega}$. The model graph is shown in Figure 11. The

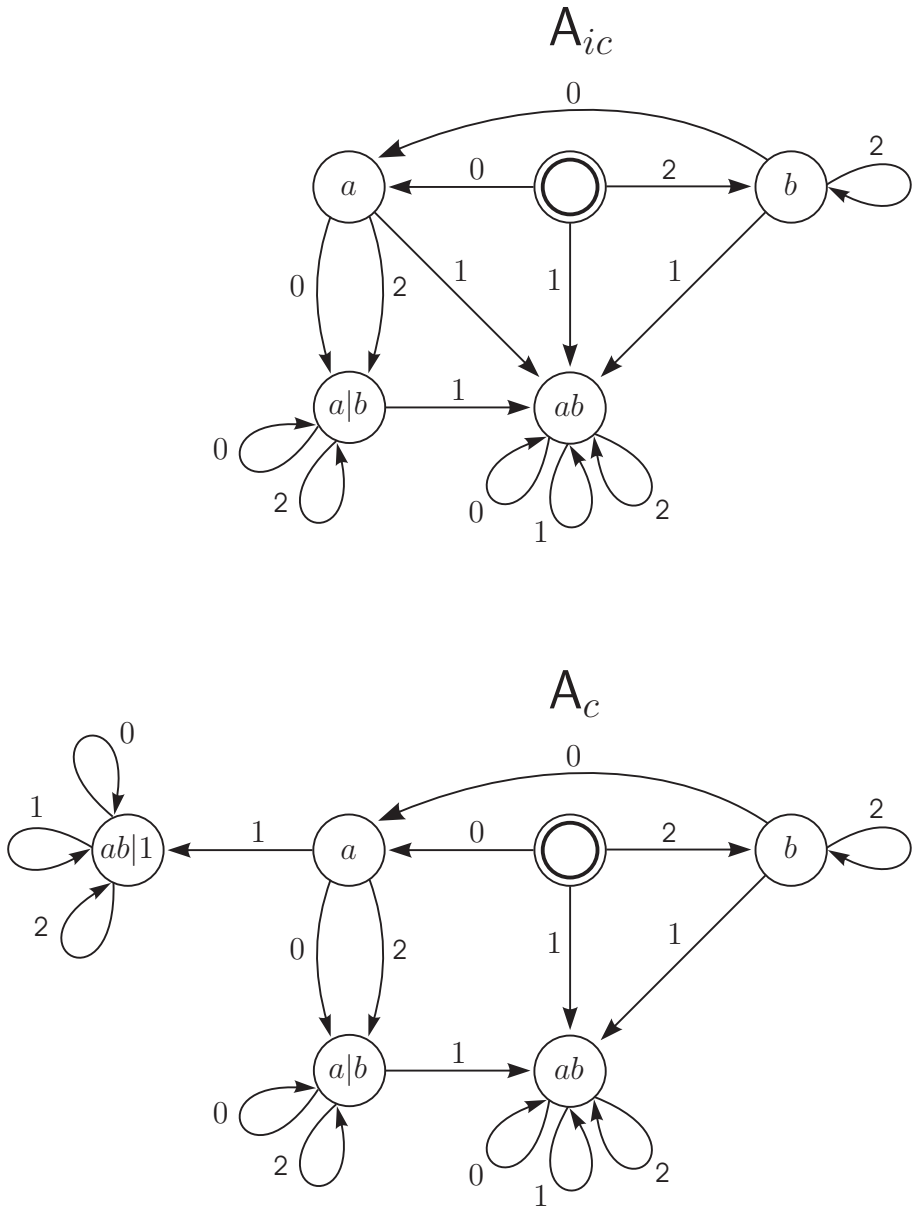

Figure 9: The automata A_{ic} and A_c for Gupta-Fabrykowski group

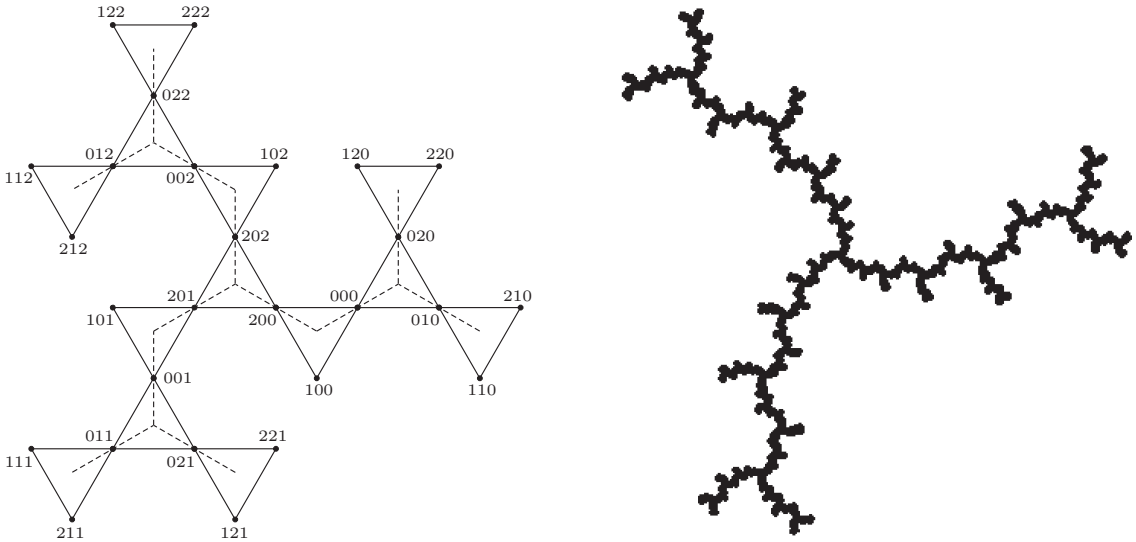

Figure 10: The Schreier graph Γ_3 of the Gupta-Fabrykowski group and its limit space

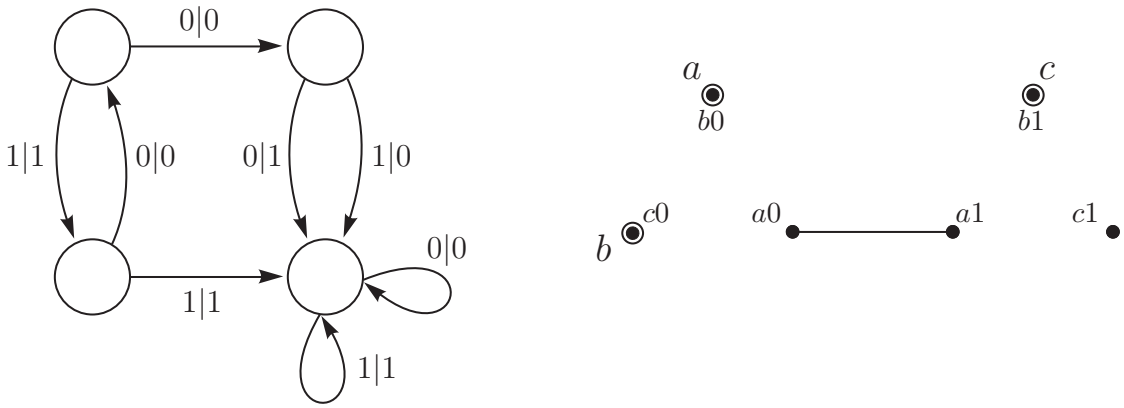

Figure 11: $IMG(z^2 + i)$ automaton and its model graph

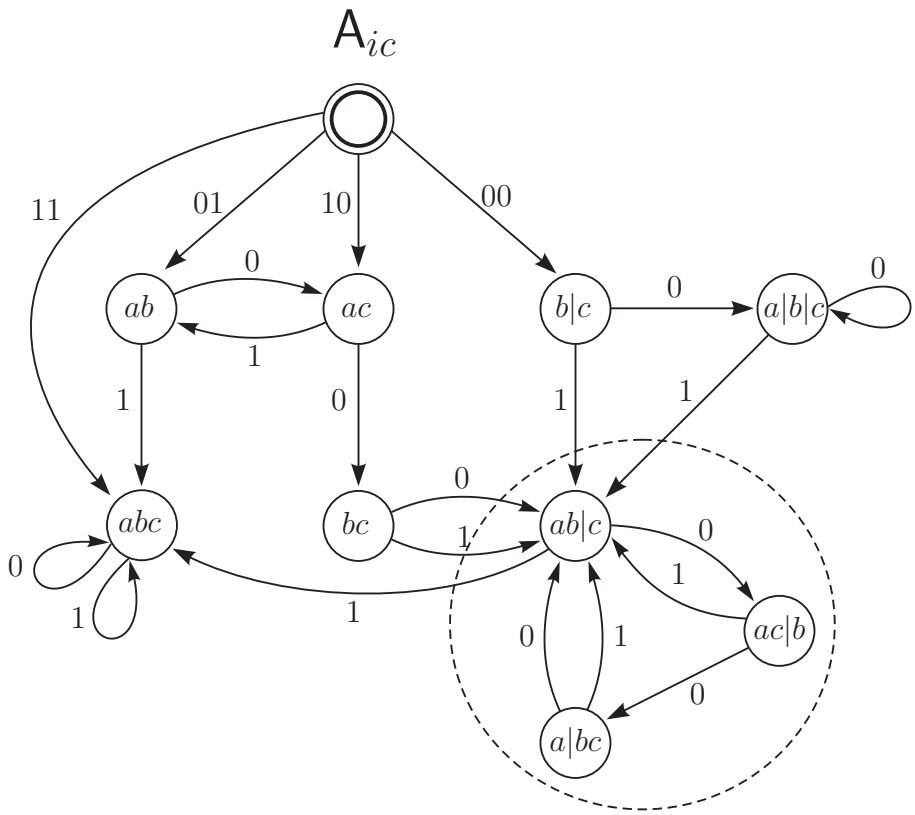

Figure 12: The automaton A_{ic} for $IMG(z^2+i)$

automata A_c and A_{ic} are shown in Figure 12. Every Schreier graph Γ_w coincides with the tile graph T_w and it is a tree. We get that every tile graph T_w has 1, 2, or 3 ends, and we denote by E_1, E_2, and E_3 the corresponding sets of sequences. Using automaton A_{ic} the sets E_1, E_2, E_3 can be described by Proposition 8 as follows. For the both critical sequences $(10)^\omega$ and $(01)^\omega$ the tile graph T_w has one end. Denote by $\mathcal{R}$ the right sofic subshift given by the subgraph emphasized in Figure 12 (we cannot describe it in the way we did with the previous examples). Then

$$E_3 = Cof(0^\omega), \quad E_2 = X^*\mathcal{R} \setminus Cof(0^\omega \cup (10)^\omega \cup (01)^\omega), \quad E_1 = X^\omega \setminus E_2.$$

Almost every tile graph has one end, the set E_2 is uncountable but of measure zero, and there is one graph T_{0^ω} with three ends. This example shows that Corollary 10 may hold for regular sequences, like here for the sequence 0^ω.

Every graph $T_w \setminus w$ has 1, 2, or 3 connected components, and we denote by C_1, C_2, and C_3 the corresponding sets of sequences. Using automaton A_c these sets can be described precisely as follows

$$C_3 = \bigcup_{k\geq 0} 0(10)^k 0X\mathcal{R} \bigcup_{k\geq 2} 0^k 1\mathcal{R} \bigcup 0^\omega, \quad C_2 = X^\omega \setminus (C_3 \cup C_1), \quad C_1 = \bigcup_{k\geq 0} 1(01)^k 1X^\omega.$$

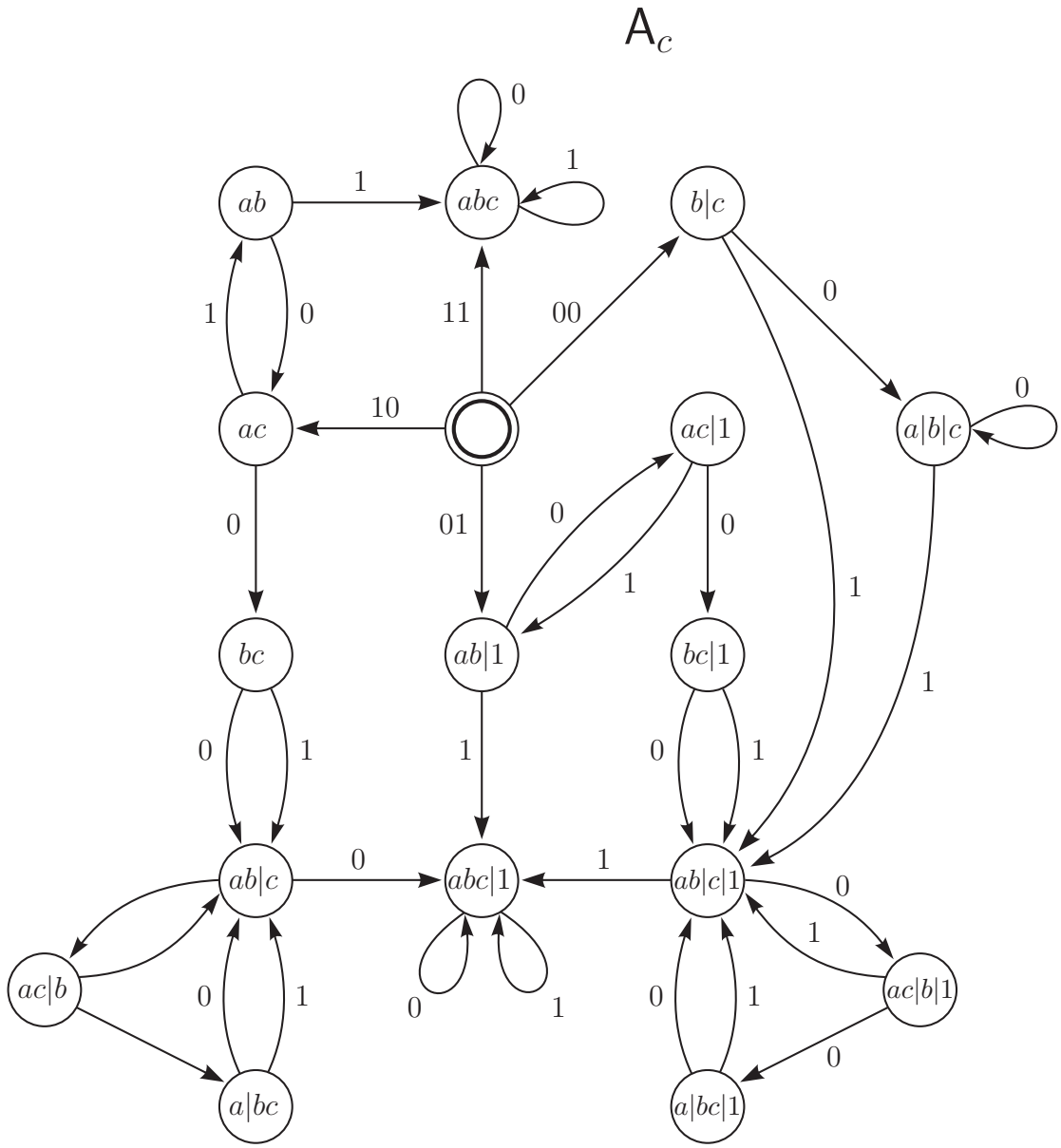

Figure 13: The automaton A_c for $IMG(z^2 + i)$

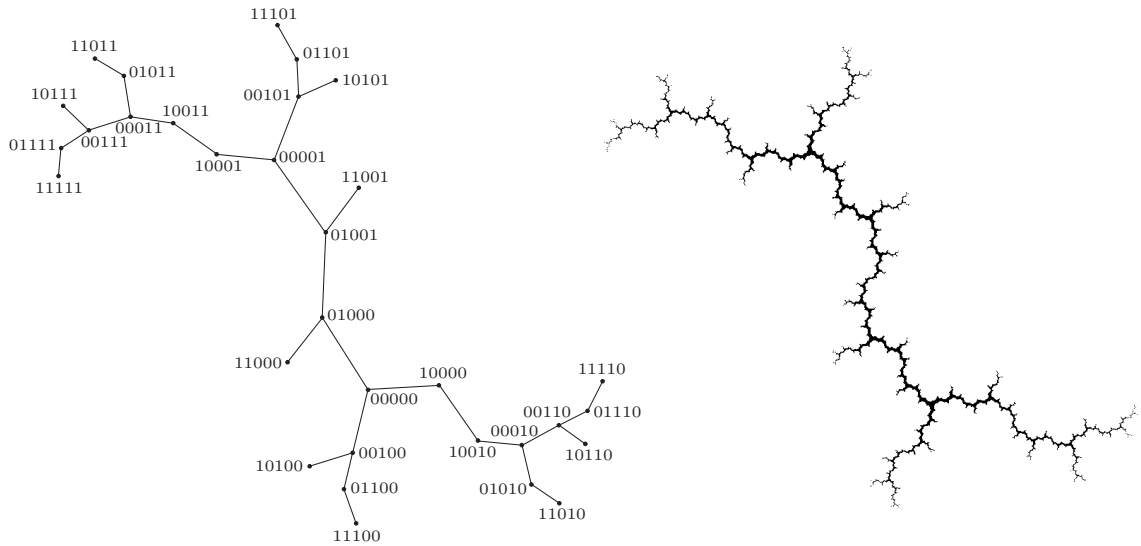

Figure 14: The Schreier graph Γ_5 of the IMG(z^2+i) and its limit space

The set C_3 is of measure zero, and the sets C_1 and C_2 have measure $\frac{1}{3}$ and $\frac{2}{3}$ respectively.

Every graph $T_w \setminus w$ has 1, 2, or 3 infinite components. The corresponding sets IC_1 and IC_2 can be described using automaton A_{ic} as follows

$$\begin{aligned} IC_2 &= \bigcup_{k\geq 1} \left(0^k 01\mathcal{R} \cup (10)^k 0X\mathcal{R} \cup 0(10)^k 0X\mathcal{R}\right), \\ IC_3 &= \{0^\omega\}, \quad IC_1 = X^\omega \setminus (IC_2 \cup IC_3)\,. \end{aligned}$$

The set IC_2 is uncountable but of measure zero.

The limit space $\mathcal{J}$ and the tile $\mathcal{T}$ of the group $IMG(z^2+i)$ are homeomorphic to the Julia set of the map z^2+i shown in Figure 14. Every point $x \in \mathcal{J}$ separates the limit space on 1, 2, or 3 connected components. The sets $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ of sequences from $X^{-\omega}$, which represent the corresponding points, can be described as follows

$$\mathcal{C}_3 = Cof(0^{-\omega}), \quad \mathcal{C}_2 = \mathcal{L}X^* \setminus \mathcal{C}_3, \quad \mathcal{C}_1 = X^{-\omega} \setminus (\mathcal{C}_2 \cup \mathcal{C}_3)\,,$$

where $\mathcal{L}$ is the left sofic subshift given by the subgraph emphasized in Figure 12. The set $\mathcal{C}_3$ of three-section points is countable, the set $\mathcal{C}_2$ of bisection points is uncountable and of measure zero, and the limit space $\mathcal{J} \setminus x$ is connected for almost all points x.

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