THE PATTERSON-SULLIVAN EMBEDDING AND MINIMAL VOLUME ENTROPY FOR OUTER SPACE

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Abstract. Motivated by Bonahon's result for hyperbolic surfaces, we construct an analogue of the Patterson-Sullivan-Bowen-Margulis map from the Culler-Vogtmann outer space $CV(F_k)$ into the space of projectivized geodesic currents on a free group. We prove that this map is a topological embedding. We also prove that for every $k \geq 2$ the minimum of the volume entropy of the universal covers of finite connected volume-one metric graphs with fundamental group of rank k and without degree-one vertices is equal to $(3k-3)\log 2$ and that this minimum is realized by trivalent graphs with all edges of equal lengths, and only by such graphs.

1. Introduction

A *geodesic current* on a word-hyperbolic group G is a positive G-invariant Borel measure on the space $\partial^2 G := \{(x,y) : x, y \in \partial G, x \neq y\}$, where ∂G is the hyperbolic boundary of G endowed with the canonical boundary topology. The study of geodesic currents on free groups is motivated by investigating geometry and dynamics of individual automorphisms, as well as of groups of automorphisms of a free group. A similar program proved to be successful in the case of fundamental groups of hyperbolic surfaces. Bonahon's foundational work [4, 5] showed the relevance of geodesic currents to the study of the geometry of the Teichmüller space and of the dynamical properties of surface homeomorphisms. Results about geodesic currents in the hyperbolic surface case can be also found in [7, 8, 21, 41, 38] and other sources. Interesting applications of geodesic currents to the study of free group automorphisms were recently obtained in [25, 26, 27, 24].

Patterson-Sullivan measures were introduced by Patterson [36] and Sullivan [44] in the context of a Kleinian group acting on the boundary of a hyperbolic space. The notion was extended by Coornaert [11] to the case of a group G acting geometrically (that is isometrically, properly discontinuously and cocompactly) on a Gromov-hyperbolic geodesic metric space. The case of groups acting on $CAT(-1)$ spaces was treated by Bourdon [6], Burger and Mozes [10], Coornaert and Papadopoulos [12, 13] and other authors (see, for example [22, 37]). Further results in the general Gromov-hyperbolic context were obtained by Furman [17]. The Patterson-Sullivan measures on the universal covers of finite simplicial graphs were considered by Lyons [31] and by Coornaert and Papadopoulos [12].

2000 *Mathematics Subject Classification.* Primary 20F65, Secondary 05C, 37A, 37E, 57M.

Key words and phrases. free groups, metric graphs, Patterson-Sullivan measures, geodesic currents, volume entropy.

The first author was supported by the NSF grant DMS#0404991. Both authors acknowledge the support of the Centre de Recerca Matemàtica at Barcelona and of the Swiss National Foundation for Scientific Research.

Let us remind here the definition in the case of a non-elementary group G acting geometrically on a $CAT(-1)$ space X. For $s > 0$ an *s-conformal density* is a G-equivariant family of regular Borel measures $(\mu_x)_{x\in X}$ on ∂X that are pairwise absolutely continuous and with the property that their mutual Radon-Nykodim derivatives satisfy

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-sB_\xi(x,y)}, \qquad \text{for every } x, y \in X,$$

where for a point $\xi \in \partial X$ and for $x, y \in X$, $B_\xi(x,y)$ is a Busemann function defined by

$$B_\xi(x,y) := \lim_{z\to\xi, z\in X}[d(x,z) - d(y,z)].$$

It turns out that there is a unique $s > 0$ such that a nonzero s-conformal density exists, namely the *critical exponent* $h(X)$ of X (see Section 3 below for definition). Moreover, when G acts on X geometrically, there is a unique nonzero $h(X)$-conformal density up to scalar multiplication. Any of these proportional families $(\mu_x)_{x\in X}$ is said to be a family of *Patterson-Sullivan measures* on ∂X. In this setting $h(X)$, which is defined in terms of the Poincaré series, is equal to the Hausdorff dimension of ∂X.

Furman [17] proved that in this case (as well as in the more general situation of a geometric action on a Gromov-hyperbolic space) there is a unique up to scalar multiple nonzero G-invariant measure ν on $\partial^2 X := \{(\zeta_1,\zeta_2)|\zeta_1,\zeta_2 \in \partial X, \zeta_1 \neq \zeta_2\}$ in the same measure class as μ_x^2, where $(\mu_x)_{x\in X}$ is a family of Patterson-Sullivan measures on ∂X. Any of the nonzero scalar multiples of ν is called an *X-Patterson-Sullivan current.* Via the identification between $\partial^2 G$ and $\partial^2 X$, this measure ν pulls back to a canonical, up to a scalar multiple, geodesic current on G, any nonzero scalar multiple of which is called a *G-Patterson-Sullivan current.*

In the case of closed hyperbolic surfaces Patterson-Sullivan currents admit several other equivalent characterizations. Let S be a closed surface with a fixed hyperbolic metric ρ, so that $\widetilde{(S,\rho)} = \mathbb{H}^2$. Thus $G = \pi_1(S)$ acts on $\mathbb{H}^2$ geometrically and $\mathbb{H}^2/G = S$. In this situation there is a natural identification between the space of G-invariant measures on $\partial^2\mathbb{H}^2$ and the space of shift-invariant measures on the unit tangent bundle $\mathbb{U}S$, where the $\mathbb{R}$-shift action is given by the geodesic flow on (S,ρ). As shown by Kaimanovich [23], under this identification Patterson-Sullivan currents correspond precisely to *Bowen-Margulis measures* (or *maximal entropy measures*) on $\mathbb{U}S$, that is the only shift-invariant measures on $\mathbb{U}S$ whose entropy is equal to the topological entropy of the geodesic flow on (S,ρ). For this reason Furman suggests, even in the general setting of a group acting geometrically on a Gromov-hyperbolic space, to call Patterson-Sullivan currents *Bowen-Margulis* or *Patterson-Sullivan-Bowen-Margulis* currents.

For closed hyperbolic surfaces $\mathbb{H}^2$-Patterson-Sullivan currents coincide with *Liouville currents* corresponding to the hyperbolic structure ρ. Bonahon [4, 5] proved that the map sending a marked hyperbolic structure to the corresponding projective class of Liouville currents provides a topological embedding $L : \mathcal{T}(S) \to \mathbb{P}Curr(G)$ of the Teichmüller space $\mathcal{T}(S)$ to the projectivized space of geodesic currents $\mathbb{P}Curr(G)$. The space $\mathbb{P}Curr(G)$ is compact and, as was also proved by Bonahon, the closure of $L(\mathcal{T}(S))$ turns out to be homeomorphic to the Thurston compactification $\widehat{\mathcal{T}(S)}$ of $\mathcal{T}(S)$.

The Culler-Vogtmann outer space [14] is a free group analogue of the Teichmüller space. For a free group F of finite rank $k \geq 2$ the *outer space* $CV(F)$ consists of equivalence classes of free, discrete and minimal isometric actions of F on $\mathbb{R}$-trees for which the quotient metric graph has volume one. Two such actions are equivalent if there is an F-equivariant isometry between the two trees in question.

Let F be a free group of finite rank $k \geq 2$, let Γ be a finite connected graph with no degree-one and degree-two vertices, and let $\alpha : F \to \pi_1(\Gamma, p)$ be an isomorphism. Thus α defines a free and discrete action of F on $\tilde{\Gamma}$ with the quotient Γ. Every choice $\mathcal{L}$ of a volume-one metric graph structure on Γ (that is, assignment of positive lengths to non-oriented edges of Γ, so that the sum of the lengths of all edges is equal to 1) turns $\widetilde{\Gamma}$ into an $\mathbb{R}$-tree so that the above action of F on $\widetilde{\Gamma}$ becomes an action by isometries. Hence $\mathcal{L}$ defines a point in $CV(F)$. Varying the lengths of edges of Γ gives an open simplex W_α in $CV(F)$ of dimension $N-1$, where N is the number of non-oriented edges of Γ. Thus the outer space $CV(F)$ is a union of open simplices of bounded dimension.

There is a natural map $\tau : CV(F) \to \mathbb{P}Curr(F)$ that takes a point of $CV(F)$ represented by the action of F on a tree X, to the projective class of F-Patterson-Sullivan currents corresponding to this action. We call τ the *Patterson-Sullivan map* or the *Patterson-Sullivan-Bowen-Margulis map*.

Our main result is the following statement, which parallels the above mentioned theorem of Bonahon for hyperbolic surfaces:

Theorem A. *The Patterson-Sullivan map $\tau : CV(F) \to \mathbb{P}Curr(F)$ is a topological embedding. The Hausdorff dimension map $h : CV(F) \to \mathbb{R}$ is continuous and, moreover, the restriction of h to any open simplex in $CV(F)$ is real-analytic.*

Injectivity of τ follows from a general result of Furman [17] proved in the context of geometric actions on Gromov-hyperbolic spaces. The main work in the present paper is in proving the continuity of τ. Any isomorphism $\alpha : F \to \pi_1(\Gamma, p)$ provides a sort of "coordinate system" for $\mathbb{P}Curr(F)$. We give an essentially explicit computation of $\tau|_{W_\alpha}$ in this coordinate system. The continuity of h can also be derived by a direct argument using the volume entropy interpretation of h given in Lemma 3.9 below. The key point is that if $\mathcal{L}$ and $\mathcal{L}'$ are two "nearby" metric structures on a finite graph Γ defining the metrics d and d' on $X = \widetilde{\Gamma}$ then the identity map $Id : (X, d) \to (X, d')$ is bi-Lipschitz with a bi-Lipshitz constant close to 1.

As we mentioned earlier, in the case of a closed hyperbolic surface S with $G = \pi_1(S)$ Bonahon proved that the Liouville map $L : \mathcal{T}(S) \to \mathbb{P}Curr(G)$ extends to a homeomorphism from Thurston's compactification $\widehat{\mathcal{T}(S)}$ of $\mathcal{T}(S)$ to the closure of the image of L. It is well-known that $\widehat{\mathcal{T}(S)}$ coincides with the length-function compactification of $\mathcal{T}(S)$. We expect that, unlike the map L in the surface case, the map $\tau : CV(F) \to \mathbb{P}Curr(F)$ does not extend to a homeomorphism from the length function compactification of $CV(F)$ to the closure of $\tau(CV(F))$ in $\mathbb{P}Curr(F)$. We intend to pursue this nontrivial question in future work. Reiner Martin [32] constructed a different family of continuous $Out(F)$-equivariant embeddings from $CV(F)$ to $\mathbb{P}Curr(F)$. Unlike the Patterson-Sullivan embedding τ, Martin's embeddings are not based on a natural geometric construction and use an ad-hoc procedure, where a point of $CV(F)$ is sent to an explicitly defined infinite linear combination of "counting" currents determined by conjugacy classes of elements of F.

It is well-understood that in a fairly general negatively curved setting the Hausdorff dimension of the boundary coincides with the volume entropy. If (M, g) is a closed connected Riemannian manifold, then the *volume entropy of* g is defined as

$$h(g) := \liminf_{R\to\infty} \frac{\log Vol_{\tilde{g}}(B(x,R))}{R},$$

where $x \in \widetilde{M}$ is a base-point and $B(x, R)$ is the ball of radius R and center x in $\widetilde{M}$, equipped with the pullback $\tilde{g}$ of the Riemannian metric g. This definition does not depend on the choice of $x \in \widetilde{M}$, and $h(g) > 0$ if and only if the group $\pi_1(M)$ has exponential growth. If g has strictly negative sectional curvature, then $(\widetilde{M}, \tilde{g})$ is a $CAT(-1)$ space and the Hausdorff dimension of its boundary (which is also equal to the critical exponent of $(\widetilde{M}, \tilde{g})$) is equal to the volume entropy $h(g)$. A similar statement is true for the universal cover of a compact locally $CAT(-1)$-space K, as stated in Lemma 3.9 below. In that case volume has to be interpreted as counting the number of $\pi_1(K)$-orbit points in the ball of radius n around the basepoint in $\widetilde{K}$.

For a compact connected Riemannian manifold M it is natural to ask what the infimum of $h(g)$ is when g varies over metrics with $Vol_g(M) = 1$ and whether this infimum is achieved. This is known as the *minimal entropy problem* (see discussion in [1]). A famous theorem of Besson, Courtois and Gallot [1] shows that if M admits a locally symmetric volume-one metric g_0 of negative curvature, then g_0 minimizes volume entropy among all volume-one metrics (see an earlier paper of Katok [28] for the case of surfaces.)

A particular case of their theorem (see [2], Section 5) says that if (M, g_0) and (M', g) are homotopically equivalent negatively curved compact connected Riemannian manifolds of the same dimension $n \geq 3$, and if (M, g_0) is locally symmetric, then $h^n(g)Vol(M', g) \geq h^n(g_0)Vol(M, g_0)$. Besson, Courtois and Gallot also show that $h(g) = h(g_0)$ and $Vol(M', g) = Vol(M, g_0)$ if and only if (M', g) is isometric to (M, g_0).

In the last section of our paper we prove an analogue of these statements in the outer space setting. Theorem A implies that the volume entropy function h (which again coincides with the Hausdorff dimension of the boundary and with the critical exponent) factors to a continuous function on the moduli space $\mathcal{M} = CV(F)/Out(F)$

$$\overline{h} : \mathcal{M} \to \mathbb{R}_{>0}.$$

A point in $\mathcal{M}$ is a finite connected graph Γ without degree-one and degree-two vertices and with $\pi_1(\Gamma) \cong F$, endowed with the structure $\mathcal{L}$ of a volume-one metric graph. Then $\overline{h}(\mathcal{L})$ is the volume entropy of the tree $\partial\widetilde{\Gamma}$, where the metric on $\widetilde{\Gamma}$ is given by the lift of $\mathcal{L}$. The analogue of a locally symmetric manifold is a regular graph (i.e., such that all vertices are of the same degree) with all edges of equal length. The volume entropy of a regular tree with all edges of the same length is easy to compute explicitly. In particular, assigning the length $1/(3k-3)$ to each of $(3k - 3)$ non-oriented edges in a trivalent graph with the fundamental group free of rank k gives a volume-one metric graph with volume entropy of its universal cover equal to $(3k - 3) \log 2$. We prove that this is precisely the minimum of the volume entropy over all finite connected metric volume-one graphs without vertices of degree one or two and with fundamental group free of rank k.

Theorem B. *Let F be a free group of rank $k \geq 2$. Then:*

(1) *For the function $\overline{h} : CV(F)/Out(F) \to \mathbb{R}$ we have*

$$\min \overline{h} = (3k-3)\log 2.$$

This minimum is realized by any regular trivalent connected graph Γ with $\pi_1(\Gamma) \cong F$, (so that Γ has $3k-3$ non-oriented edges), where each edge of Γ is given length $1/(3k-3)$.

(2) *If a point of $CV(F)/Out(F)$ realizes the minimum of h then this point is a regular trivalent graph with all edges of equal lengths.*

(3) *We have*

$$\sup_{\mathcal{M}} \overline{h} = \infty.$$

As an intermediate step in proving Theorem B we establish that among all the volume-one metric structures on an m-regular graph Γ with $m \geq 3$, the volume entropy is minimized by assigning all the edges of Γ equal lengths. This fact is a particular case of solving the minimal volume entropy problem for an arbitrary finite connected graph which was carried in a recent work of Lim [30]. Related results have also been obtained by Rivin [39]. The results of Rivin [39] and of Lim [30] can be used to provide alternative proofs of parts (1) and (2) of Theorem B. Nonetheless, we choose to present our proof for completeness and because it uses rather different arguments from those of Rivin and of Lim.

The authors are grateful to Florent Balacheff, David Berg, Pierre de la Harpe, Vadim Kaimanovich, Jérôme Los, Martin Lustig, Paul Schupp and Dylan Thurston for helpful discussions. We are also grateful to Frederic Paulin and Seonhee Lim for informing us about Seonhee Lim's work and to Jean-Francçis Lafont for informing us about the entropy results of Rivin.

2. Geodesic currents

Convention 2.1. For the remainder of the paper let F be a finitely generated free group of rank $k \geq 2$. We will denote by ∂F the space of ends of F with the standard ends-space topology. Thus ∂F is a topological space homeomorphic to the Cantor set. We shall also think about ∂F as the hyperbolic boundary of F, endowed with the canonical boundary topology, in the sense of the theory of word-hyperbolic groups (see, for example [18]).

We set

$$\partial^2 F := \{(\zeta, \xi) : \zeta, \xi \in \partial F \text{ and } \zeta \neq \xi\}.$$

Definition 2.2 (Geodesic currents)**.** A *geodesic current* on F is a positive locally finite (that is, finite on compact subsets) F-invariant Borel measure on $\partial^2 F$. We denote the space of all geodesic currents on F by $Curr(F)$.

The space $Curr(F)$ comes equipped with a weak topology: for $\nu_n, \nu \in Curr(F)$ we have $\lim_{n\to\infty} \nu_n = \nu$ iff for every two disjoint closed-open sets $S, S' \subseteq \partial F$ we have $\lim_{n\to\infty} \nu_n(S \times S') = \nu(S \times S')$.

Definition 2.3 (Projectivized geodesic currents)**.** For two nonzero geodesic currents $\nu_1, \nu_2 \in Curr(F)$ we say that ν_1 is equivalent to ν_2, denoted $\nu_1 \sim \nu_2$, if there exists a positive scalar $r \in \mathbb{R}$ such that $\nu_2 = r\nu_1$. We denote

$$\mathbb{P}Curr(F) := \{\nu \in Curr(F) : \nu \neq 0\}/\sim$$

and call it the *space of projectivized geodesic currents on* F. The space $\mathbb{P}Curr(F)$ is endowed with the quotient topology. We will denote the $\sim$-equivalence class of a nonzero geodesic current ν by $[\nu]$.

Convention 2.4. For a (finite or infinite) graph Δ, denote by $V\Delta$ the set of all vertices of Δ, and denote by $E\Delta$ the set of all oriented edges of Δ (i.e., the set of all ordered pairs (u, v) where u and v are adjacent vertices in Δ.) A path γ in Δ is a sequence of oriented edges which connects a vertex $o(\gamma)$ (origin) with a vertex $t(\gamma)$ (terminus). A path is called *reduced* if it does not contain a back-tracking (a path of the form (e, e^{-1})). We denote by $\mathcal{P}(\Delta)$ the set of all finite reduced paths in Δ. For a vertex $x \in V\Delta$, we denote by $\mathcal{P}_x(\Delta)$ the collection of all $\gamma \in \mathcal{P}(\Delta)$ that begin with x. For $\gamma \in \mathcal{P}(\Delta)$, we denote by $a(\gamma)$ the set of all $e \in E\Delta$ such that $e\gamma \in \mathcal{P}(\Delta)$ and we denote by $b(\gamma)$ the set of all $e \in E\Delta$ such that $\gamma e \in \mathcal{P}(\Delta)$.

Definition 2.5 (Simplicial charts)**.** Let F be a free group of rank $k \geq 2$ and let Γ be a finite connected graph without degree-one vertices such that $\pi_1(\Gamma) \cong F$. Let $\alpha : F \to \pi_1(\Gamma, p)$ be an isomorphism, where p is a vertex of Γ. We call such α a *simplicial chart* for F.

Convention 2.6. Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart. We consider $X := \widetilde{\Gamma}$, a topological tree, and denote the covering map from X to Γ by $j : X \to \Gamma$. For $\gamma \in \mathcal{P}(X)$ we call the reduced path $j(\gamma)$ in Γ the *label* of γ. As there is only one reduced path connecting two arbitrary vertices in a tree, we will often write $[x, y]$ for a path in X with origin x and terminus y.

Let ∂X denote the space of ends of X with the natural ends-space topology. Then we obtain a canonical α-equivariant homeomorphism $\hat{\alpha} : \partial F \to \partial X$, as follows. Suppose we endow Γ with the structure of a metric graph, that is, we assign a positive length to each edge of Γ. This turns X into an $\mathbb{R}$-tree and the action of $\pi_1(\Gamma, p)$ on X becomes a discrete isometric action. Moreover, X is quasi-isometric to F and, if F is equipped with a word metric and x_0 is a lift of p to X then the orbit map $\tilde{\alpha} : F \to X$, $f \to \alpha(f)x_0$, is a quasi-isometry. This quasi-isometry extends to a homeomorphism $\hat{\alpha} : \partial F \to \partial X$. A crucial feature of this construction is that $\hat{\alpha}$ does not depend on the choice of a metric structure on Γ. If α is fixed, we will usually suppress explicit mention of $\hat{\alpha}$ and also of the map α itself when talking about the action of F on X and on ∂X arising from this situation. We also denote by $\partial^2 X$ the set of all pairs (ζ_1, ζ_2) such that $\zeta_1, \zeta_2 \in \partial X$ and $\zeta_1 \neq \zeta_2$. For $(\zeta_1, \zeta_2) \in \partial^2 X$ we denote by $[\zeta_1, \zeta_2]$ the simplicial (non-parameterized) geodesic from ζ_1 to ζ_2 in X. Thus $[\zeta_1, \zeta_2]$ is a subgraph of X isomorphic to the simplicial line, together with a choice of direction on that line. We also have the identification $\hat{\alpha} : \partial^2 F \to \partial^2 X$.

Definition 2.7 (Cylinder sets)**.** For every reduced path γ in X denote

$$\begin{gathered} Cyl_X(\gamma) := \{(\zeta_1, \zeta_2) \in \partial^2 X : \gamma \subseteq [\zeta_1, \zeta_2] \\ \text{and the orientations on } \gamma \text{ and on } [\zeta_1, \zeta_2] \text{ agree}\} \end{gathered}$$

Also, for $x = o(\gamma) \in X$ denote

$$Cyl_x(\gamma) := \{\zeta \in \partial X : \gamma \text{ is an initial segment of } [x, \zeta]\}$$

The collection of all sets $Cyl_X(\gamma)$, where γ varies over $\mathcal{P}(X)$, gives a basis of closed-open sets for $\partial^2 X$. Similarly, for any $x \in X$, the collection of all sets $Cyl_x(\gamma)$, where γ varies over $\mathcal{P}_x(X)$, gives a basis of closed-open sets for ∂X. Let us denote $Cyl_\alpha(\gamma) := \hat{\alpha}^{-1} Cyl_X(\gamma)$, so that $Cyl_\alpha(\gamma) \subseteq \partial^2 F$. It is easy to see that:

Lemma 2.8. *For $\nu_n, \nu \in Curr(F)$ $\lim_{n\to\infty} \nu_n = \nu$ iff $\lim_{n\to\infty} \nu_n(Cyl_\alpha(\gamma)) = \nu(Cyl_\alpha(\gamma))$ for every $\gamma \in \mathcal{P}(X)$. Moreover, for $\nu, \nu' \in Curr(F)$ we have $\nu = \nu'$ iff $\nu(Cyl_\alpha(\gamma)) = \nu'(Cyl_\alpha(\gamma))$ for every $\gamma \in \mathcal{P}(X)$.*

Remark 2.9. Note that for any $f \in F$ and $\gamma \in \mathcal{P}(X)$ we have $fCyl_\alpha(\gamma) = Cyl_\alpha(f\gamma)$. Since geodesic currents are, by definition, F-invariant, for a geodesic current ν and for $\gamma \in \mathcal{P}(X)$ the value $\nu(Cyl_\alpha(\gamma))$ only depends on the label $j(\gamma)$ of γ.

3. Patterson-Sullivan measures

We recall here the basics regarding Patterson-Sullivan measures in the context of $CAT(-1)$ spaces. Among numerous relevant articles on this subject cited in the Introduction, the primary references for us are Burger-Mozes [10], Coornaert-Papadopoulos [12], Coornaert [11] and Furman [17]. The work of Furman is particularly important in our context since he explains, among many other things, the connection between Patterson-Sullivan measures and the analogues of Bowen-Margulis currents. Moreover, the injectivity of the Patterson-Sullivan map in Theorem A is an immediate corollary of Furman's results [17].

Definition 3.1 (Metric and semi-metric graph structures)**.** A *quasi-metric structure $\mathcal{L}$ on a (finite or infinite) graph* Γ is an assignment of a *length* $L(e) \geq 0$ to each edge $e \in E\Gamma$ of Γ. We define the *volume* of $\mathcal{L}$ as $vol(\mathcal{L}) := \frac{1}{2}\sum_{e\in E\Gamma} L(e)$. A quasi-metric structure is called a *semi-metric structure* if $L(e) = L(e^{-1})$ for every $e \in E\Gamma$. A semi-metric structure is called a *metric structure* if $L(e) > 0$ for every $e \in E\Gamma$. We say that such a quasi-metric structure $\mathcal{L}$ is *non-singular* if there is a maximal tree T in Γ such that $L(e) > 0$ for every $e \in E(\Gamma - T)$. A quasi-metric structure is *positive* if $L(e) > 0$ for every $e \in E\Gamma$.

If $\mathcal{L}$ is a semi-metric structure on Γ, let Γ' be the graph obtained from Γ by contracting to points all edges of Γ of $\mathcal{L}$-length zero. Then Γ' comes equipped with a canonical metric graph structure $\mathcal{L}'$ coming from $\mathcal{L}$. We call $(\Gamma', \mathcal{L}')$ *the metric graph associated to* $(\Gamma, \mathcal{L})$.

Convention 3.2. Let $X = \widetilde{\Gamma}$ and let $j : X \to \Gamma$ be the covering map. Let $\mathcal{L}$ be a nonsingular semi-metric graph structure on a finite graph Γ. Let $(\Gamma', \mathcal{L}')$ be the metric graph associated to $(\Gamma, \mathcal{L})$ and let $q : \Gamma \to \Gamma'$ be the canonical projection map.

Then $\mathcal{L}$ canonically lifts to a semi-metric graph structure $\widetilde{\mathcal{L}}$ on X defined as $\widetilde{L}(e) := L(j(e))$ for every $e \in EX$. Similarly let $X' = \widetilde{\Gamma}'$ and let $j' : X' \to \Gamma'$ be the associated covering map. Again, $\mathcal{L}'$ lifts to a metric graph structure $\widetilde{\mathcal{L}}'$ on X'.

It is easy to see that both j and j' preserve edge-lengths and that X' is obtained from X by contracting all edges of length zero in X to points. Thus $(X', \widetilde{\mathcal{L}}')$ is the metric graph associated to $(X, \widetilde{\mathcal{L}})$. We denote by $\widetilde{q} : X \to X'$ the canonical projection map.

The semi-metric structure $\widetilde{\mathcal{L}}$ defines a semi-metric $d = d_{\mathcal{L}}$ on X and $\widetilde{\mathcal{L}}'$ defines a metric $d' = d'_{\mathcal{L}}$ on X'. Moreover, $\widetilde{q} : (X, d) \to (X', d')$ is distance-preserving. Note that for both (X, d) and (X', d') there are obvious notions of *geodesic edge-paths*. In both cases we can metrize ∂X and $\partial X'$ by setting

$$d_x(\xi, \zeta) := e^{-d(x,[\xi,\zeta])} \quad \text{where } \xi, \zeta \in \partial X$$

$$d'_{x'}(\xi', \zeta') := e^{-d'(x',[\xi',\zeta'])} \quad \text{where } \xi', \zeta' \in \partial X',$$

where $x \in X, x' \in X'$. Note that d_x is a metric on ∂X, although $\mathcal{L}$ was just a semi-metric structure. Moreover, if $x' = \widetilde{q}(x)$ then the map $\widetilde{q} : (\partial X, d_x) \to (\partial X', d'_{x'})$ is a homeomorphism and an isometry.

Convention 3.3. For the remainder of this section we will fix G, X and the notations below to be one of the following:

(1) We denote by G a finitely generated group acting properly discontinuously and cocompactly by isometries on a $CAT(-1)$ space X.

If $x \in X$ is a base-point, the boundary ∂X is metrized as follows: for two points $\xi, \zeta \in \partial X$ put

$$d_x(\xi, \zeta) = \begin{cases} 0, & \text{if } \xi = \zeta, \\ exp(-d(x, [\xi, \zeta])), & \text{if } \xi \neq \zeta. \end{cases}$$

(2) We consider $G = F$ a free group of finite rank $k \geq 2$ and $\alpha : F \to \pi_1(\Gamma, p)$ a simplicial chart for F, as well as a non-singular semi-metric structure $\mathcal{L}$ defining a semi-metric d on $X = \widetilde{\Gamma}$. Thus F acts on X via α by d-preserving transformations. In this case let $q : \Gamma \to \Gamma', \widetilde{q} : X \to X', \mathcal{L}'$, d' and the metrics on ∂X and $\partial X'$ be as in Convention 3.2. Thus $q_\# \circ \alpha : F \to \pi_1(\Gamma', p')$ is another simplicial chart for F, where $p' = q(p)$ and the map $\widetilde{q} : X \to X'$ is F-equivariant.

Definition 3.4 (Busemann Functions)**.** Let X be a $CAT(-1)$ space as in part (1) of Convention 3.3. For a point $\xi \in \partial X$ and for $x, y \in X$ put

$$B_\xi(x, y) := \lim_{z \to \xi} \big(d(x, z) - d(y, z)\big).$$

where $z \in X$.

We will need the following simple but useful fact regarding the explicit form of the Busemann functions in this context which is an immediate corollary of the definitions.

Lemma 3.5. *Let $x, y \in X$ and $\xi \in \partial X$ be such that $y \in [x, \xi]$. Then $B_\xi(x, y) = d(x, y)$.*

Convention 3.6. We will denote by $M(\partial X)$ the space of all positive regular Borel measures on ∂X. The space $M(\partial X)$ is endowed with the weak topology.

If $\mu \in M(\partial X)$ and $g \in G$ then the measure $g_*\mu \in M(\partial X)$ on ∂X is defined as $(g_*\mu)(A) = \mu(g^{-1}A)$ for a Borel subset $A \subseteq \partial X$.

Proposition-Definition 3.7 (Critical Exponent)**.** Let X and G be as in Convention 3.3 and let $x \in X$ be a base-point. The *Poincaré series* of X with respect to x is

$$\Pi_x(s) := \sum_{g \in G} e^{-sd(x,gx)}$$

Then for every $x \in X$ there exists a unique number $h \geq 0$ such that $\Pi_x(s)$ converges for all $s > h$ and diverges for all $s < h$. This number h does not depend on $x \in X$ and is called the *critical exponent*. We denote it by $h = h(X) = h(G, X)$.

Remark 3.8. Coornaert discusses this definition in [11], and shows in particular that under assumptions of Convention 3.3 $\Pi_x(h)$ diverges for every $x \in X$.

Coornaert's results [11] also imply that the critical exponent coincides with the *volume entropy* of X defined by the right-hand side of the equality:

Lemma 3.9. *Let G, X be as in Convention 3.3 and let $x \in X$. Then*

$$h(X) = \lim_{R \to \infty} \frac{1}{R} \log \#\{g \in G : d(x, gx) \leq R\}$$

Definition 3.10 (Conformal density)**.** Let G, X be as in Convention 3.3 and let $s \geq 0$. A continuous map $X \to M(\partial X)$, $x \mapsto \mu_x$ is called an *s-dimensional conformal density on ∂X for G* if:

(1) The family $(\mu_x)_x$ is G-equivariant, that is $\mu_{gx} = (g^{-1})_* \mu_x$ for every $x \in X, g \in G$.
(2) We have
$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-sB_\xi(x,y)}$$
for every $x, y \in X$.
(3) We have $\mu_x = \mu_y$ if $d(x, y) = 0$.

In particular, we see that for each $x, y \in X$ the measures μ_x, μ_y are absolutely continuous with respect to each other with bounded Radon-Nikodym derivatives.

Remark 3.11. Let $G = F, X$ be as in part (2) of Convention 3.3.

Then $h(F, X) = h(F, X')$ since for every $g \in F$ and for every $x \in X$ with $x' = \widetilde{q}(x) \in X'$ we have $d(x, gx) = d'(x', gx')$. We shall denote this critical exponent by $h_{\mathcal{L}}(X)$.

Moreover, suppose $(\mu_x)_x$ is a conformal s-density on ∂X. Then for any $x, y \in X$ with $d(x, y) = 0$ we have $\mu_x = \mu_y$ and hence $(\mu_x)_x$ canonically factors to a conformal s-density $(\mu'_{x'})_{x'}$ on $\partial X'$. Similarly, if $(\mu'_{x'})_{x'}$ is a conformal s-density on $\partial X'$, then it canonically pulls back to a conformal s-density $(\mu_x)_x = ((\tilde{q}^{-1})_* \mu'_{q(x)})_x$ on ∂X.

The following two statements follow from the basic results established in [11, 12, 10].

Proposition-Definition 3.12 (Patterson-Sullivan measures)**.** Let G, X be as in Convention 3.3 and let $h = h(X)$ be the critical exponent of X.

Then $s = h(X)$ is the only value of $s \geq 0$ such that there exists a nonzero s-dimensional conformal density on ∂X. Moreover, up to scalar multiplication, the nonzero h-dimensional conformal density $(\mu_x)_x$ is unique. The measures $(\mu_x)_x$ are called *Patterson-Sullivan measures* on ∂X.

Proposition 3.13. *Let $(\mu_x)_x$ be a family of Patterson-Sullivan measures on ∂X. Then:*

(1) *The measures μ_x belong to the same measure class for all $x \in X$. Each μ_x has no atoms, has full support on ∂X, and $\mu_x(A) > 0$ for every nonempty open subset $A \subseteq \partial X$.*
(2) *For every $x \in X$ the critical exponent h is equal to the Hausdorff dimension of $(\partial X, d_x)$. In particular, $0 < h(X) < \infty$.*
(3) *Let $x, y \in X$ and let m_y be the h-dimensional Hausdorff measure on $(\partial X, d_y)$. Then m_y and μ_x are absolutely continuous with respect to each other and their mutual Radon-Nikodym derivatives are bounded.*

Here is another useful characterization of Patterson-Sullivan measures (see, for example [17]):

Proposition 3.14. *Let G, X be as in Convention 3.3 and let $h = h(X)$ be the critical exponent. Let $(\mu_x)_x$ be a family of Patterson-Sullivan measures on ∂X. Then for every $x \in X$ the measure μ_x is, up to a scalar multiple, the weak limit as $s \to h+$, of the probability measures*

$$\frac{1}{\Pi_x(s)} \sum_{g \in G} e^{-sd(x,gx)} \mathrm{Dirac}(gx) \ .$$

4. Patterson-Sullivan measures and metric graphs

In this section we will concentrate our attention on the case when the acting group is a nonabelian free group F of finite rank $k \geq 2$. Therefore, for the remainder of this section, we assume that F, Γ, Γ', X, X', $\mathcal{L}$ are as in part (2) of Convention 3.3.

Remark 4.1. Let $\gamma \in \mathcal{P}(X)$ be an edge-path from a vertex x to a vertex y in X. Let $s \geq 0$. Lemma 3.5 shows that the function $e^{-sB_\xi(x,y)}$ is constant and equal to $e^{-sd(x,y)}$ on the cylinder $Cyl_x(\gamma) \subseteq \partial X$. Hence, when restricted to $Cyl_x(\gamma)$, condition (2) of Definition 3.10 simplifies to

$$\mu_x = e^{-sd(x,y)} \mu_y \quad \text{ on } Cyl_x(\gamma)$$

and, in particular,

$$\mu_x(Cyl_x(\gamma)) = e^{-sd(x,y)} \mu_y(Cyl_x(\gamma)).$$

This shows that an s-conformal density $(\mu_x)_x$ is uniquely determined by the values $\mu_x(Cyl_x(f))$, where x varies over the vertices of X and f varies over all edges of X with origin x. Moreover, in view of F-equivariance of $(\mu_x)_x$, it suffices to take x from a bijective lift of the vertex set of Γ to X.

Convention 4.2. Let e be an oriented edge of Γ and let $(\mu_x)_x$ be an s-conformal density for X with $s > 0$. Let f be a lift of e to X and let x be the origin of f. We denote

$$w_e = w_{e,\mathcal{L}} := \mu_x(Cyl_x(f)) \ .$$

Note that because of F-equivariance of $(\mu_x)_x$ the value w_e does not depend on the choice of the lift f of e.

Proposition 4.3. *Let $s > 0$. Let $(\mu_x)_x$ be an s-conformal density for X. Then:*

(1) *We have $w_e > 0$ for every $e \in E\Gamma$.*

(2) *For every $e \in E\Gamma$ we have*

$$w_e = \exp(-sL(e)) \sum_{e' \in b(e)} w_{e'}. \tag{*}$$

Moreover, if $(w_e)_{e\in E\Gamma}$ satisfy conditions (1), (2) above, then there exists a unique s-conformal density $(\mu_x)_x$ such that for every $e \in E\Gamma$ and for every lift f of e to X with origin x we have $w_e = \mu_x(Cyl_x(f))$.

Proof. Suppose $(\mu_x)_x$ is an s-conformal density for X. Let e be an edge of Γ and let f be a lift of e to X with origin x. Since $Cyl_x(f) \subseteq \partial X$ is a nonempty open set, Proposition 3.13 implies that $w_e = \mu_x(Cyl_x(f)) > 0$ so that condition (1) holds. Let y be the terminal vertex of f. For every edge $e' \in b(e)$ there is a unique lift f' of e' to X with origin y. Then

$$Cyl_x(f) = \bigsqcup_{f'} Cyl_x(ff')$$

and hence

$$\mu_x(Cyl_x(f)) = \sum_{f'} \mu_x(Cyl_x(ff')).$$

Remark 4.1, applied to $Cyl_x(ff') = Cyl_y(f') \subseteq \partial X$, implies that

$$\mu_x(Cyl_x(ff')) = e^{-sd(x,y)}\mu_y(Cyl_y(f')) = e^{-sL(e)}\mu_y(Cyl_y(f')).$$

Therefore

$$w_e = e^{-sL(e)} \sum_{e' \in b(e)} w_{e'},$$

and condition (2) is verified.

If $(w_e)_{e\in E\Gamma}$ satisfy conditions (1) and (2), then it is not hard to check that the formulae from Remark 4.1 can be used to define an s-dimensional conformal density $(\mu_x)_{x\in X}$, as required. We leave the details of this verification to the reader. □

We conclude this section with a short note on Hausdorff measures. It follows from the definitions that for any $x, y \in X$ the metrics d_x, d_y on ∂X are Lipschitz-equivalent to each other and hence have the same Hausdorff dimension. Let $s > 0$ and let $\mathcal{H}^s_x$ be the s-dimensional Hausdorff measure on $(\partial X, d_x)$.

Let $\gamma = [x, y]$ be a geodesic segment in X. The definitions of d_x and d_y imply that

$$d_x = e^{-d(x,y)} d_y \quad \text{on } Cyl_x(\gamma) \subseteq \partial X.$$

Therefore, by definition of Hausdorff measures,

$$\mathcal{H}^s_x = e^{-sd(x,y)}\mathcal{H}^s_y \text{ on } Cyl_x(\gamma).$$

In view of Remark 4.1, this can be used to show that for s equal to the Hausdorff dimension of ∂X, the family $(\mathcal{H}^s_x)_x$ is a nonzero s-dimensional conformal density and thus provides a family of Patterson-Sullivan measures on ∂X. However, we will not use this fact in our arguments.

Remark 4.4. Note also, that if for every $e \in E\Gamma$ and $s \geq 0$ we take a lift f of e to X with origin x and denote $\theta_{e,s} := \mathcal{H}^s_x(Cyl(f))$, then the numbers θ_e satisfy the system of equations (*) from Proposition 4.3:

$$\theta_{e,s} = \sum_{e' \in b(e)} \exp(-sL(e))\theta_{e',s} \quad e \in E\Gamma.$$

5. Perron-Frobenius theory for metric trees

Systems of equations of the type (*) appearing in Proposition 4.3 arise in various contexts and can be studied by the theory of Perron-Frobenius-Ruelle. The matrix $A_{\mathcal{L}}(s)$ of the system of equations (*) in part (2) of Proposition 4.3 is a transfer operator, and the statements of Lemma 5.3 and Corollary 5.4 below are standard facts about transfer operators (see for example the article of Guillopé [19] where dynamics on metric trees is studied in detail.) In the probabilistic setting, Perron-Frobenius theory can be applied to study random walks on trees with finitely many cone types (among them trees with finite quotients). In particular it allows the computation of the rate of escape of a random walk and of the spectral radius of its transition operator, see [34, 35].

Below we shall give a self-contained exposition of the basic facts from the Perron-Frobenius theory that we need. We shall adapt to our situation the approach of Edgar [16] to the study of self-similar fractals through so-called Mauldin-Williams graphs [33]. In particular, the proof of Lemma 5.3 below follows closely the proof of Theorem 6.6.6 in [16].

First we need to recall some basic facts of the classical Perron-Frobenius theory (see [29, 43] for a detailed exposition). If A is a matrix with real coefficients, we will denote by $r(A)$ the *spectral radius* of A. Recall that a nonnegative matrix A is called *irreducible* if for every position ij there exists an integer $n > 0$ such that $(A^n)_{ij} > 0$. If A is a matrix, the notation $A \geq 0$ means that all entries of A are nonnegative and the notation $A > 0$ means that all entries of A are positive. If A and B are matrices of the same size, we write $A \leq B$ if $B - A \geq 0$ and $A < B$ if $B - A > 0$.

Proposition-Definition 5.1 (Perron-Frobenius Theorem)**.** Let $A \geq 0$ be an irreducible nonnegative $n \times n$-matrix, $n \geq 1$. Then:

(1) The number $r(A) > 0$ is an eigenvalue of A of multiplicity 1.
(2) There exists a (unique up to a scalar multiple) column vector $Y > 0$ such that $AY = r(A)Y$.
(3) If $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \geq 0$, a nonzero column vector, and $\lambda \in \mathbb{R}$ are such that $AY = \lambda Y$, then $\lambda = r(A)$.
(4) Suppose that $Y \geq 0$, a nonzero column vector and $\lambda \in \mathbb{R}$ are such that $AY \leq \lambda Y$ and such that for some coordinate i we have $(AY)_i < \lambda y_i$. Then $r(A) < \lambda$.
(5) Suppose that $Y \geq 0$, a nonzero column vector, and $\lambda \in \mathbb{R}$ are such that $AY \geq \lambda Y$. Then $r(A) \geq \lambda$.

The number $r(A)$ is called the *Perron-Frobenius eigenvalue* of A. A column eigenvector $Y > 0$ such that $AY = r(A)Y$ is called a *(right) Perron-Frobenius eigenvector* of A.

In this situation the transposed matrix A^T is also irreducible and $r(A) = r(A^T)$, so that A and A^T have the same Perron-Frobenius eigenvalue. If U is a right Perron-Frobenius eigenvector of A^T, the row-vector U^T is called a *left Perron-Frobenius eigenvector* of A.

Convention 5.2. For the remainder of this section, unless specified otherwise, let F, Γ, X be as in part (2) of Convention 3.3. Let $n = \#E\Gamma$ be the number of oriented edges of Γ and let us fix an ordering $e_1, \dots, e_n$ on $E\Gamma$. Also, let $\mathcal{L}$ be a positive quasi-metric or a non-singular semi-metric structure on Γ.

Let $H(\Gamma)$ denote the *reduced line graph* of Γ, which is defined as follows. The vertex set of $H(\Gamma)$ is $E\Gamma$. The set of oriented edges of $H(\Gamma)$ consists of those $\gamma \in \mathcal{P}(\Gamma)$ which contain exactly two edges of Γ. If $\gamma = ee'$ is an edge of $H(\Gamma)$, then the origin of γ in $H(\Gamma)$ is e and the terminus of γ in $H(\Gamma)$ is e'. The inverse edge of γ in $H(\Gamma)$ is the path $(e')^{-1}e^{-1}$ in Γ. Let M be the adjacency matrix of $H(\Gamma)$. Thus M is an $n \times n$-matrix where the entry in the position ij is defined as:

$$m_{ij} := \begin{cases} 1, & \text{if } e_ie_j \in \mathcal{P}(\Gamma) \\ 0, & \text{otherwise.} \end{cases}$$

Denote $A_{\mathcal{L}}(s) := Diag(e^{-sL(e_1)}, \dots, e^{-sL(e_n)})M$. The system (*) from part (2) of Proposition 4.3 rewrites as the matrix equation:

$$A_{\mathcal{L}}(s) \begin{bmatrix} w_{e_1} \\ \vdots \\ w_{e_n} \end{bmatrix} = \begin{bmatrix} w_{e_1} \\ \vdots \\ w_{e_n} \end{bmatrix}.$$

Let $\Phi_{\mathcal{L}}(s)$ denote the spectral radius of $A_{\mathcal{L}}(s)$.

Lemma 5.3. *The following hold:*

(1) *The matrices $A_{\mathcal{L}}(s)$ and $A_{\mathcal{L}}(s)^T$ are nonnegative and irreducible for every $s \in \mathbb{R}$.*
(2) *The function $\Phi_{\mathcal{L}}(s)$ is continuous and strictly monotone decreasing on the interval $0 \le s < \infty$.*
(3) *We have $\Phi_{\mathcal{L}}(0) > 1$.*
(4) *If $\mathcal{L}$ is a positive quasi-metric structure then $\lim_{s\to\infty} \Phi_{\mathcal{L}}(s) = 0$.*

Proof. Recall that Γ is finite, connected, has no degree-one vertices and $\pi_1(\Gamma)$ is a free group of rank $k \ge 2$. Therefore the graph $H(\Gamma)$ is strongly connected and hence its adjacency matrix M is nonnegative irreducible and the same is true for its transpose M^T. The matrix $A_{\mathcal{L}}(s)$ is obtained from M by multiplying the i-th row of M by a positive number $e^{-sL(e_i)}$ for each $i = 1, \dots, n$. Hence $A_{\mathcal{L}}(s)$ and $A_{\mathcal{L}}(s)^T$ are nonnegative and irreducible.

The continuity of $\Phi_{\mathcal{L}}(s)$ follows from its definition.

Suppose now that $0 \le s < s'$. Let $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be a positive Perron-Frobenius eigenvector of $A_{\mathcal{L}}(s)$, so that $A_{\mathcal{L}}(s)Y = \Phi_{\mathcal{L}}(s)Y$. Since $L(e_i) \ge 0$, the functions $e^{-sL(e_i)}$ are monotone non-increasing for each i. Hence component-wise $a_{ij}(s) \le a_{ij}(s')$ and therefore $A_{\mathcal{L}}(s')Y \le A_{\mathcal{L}}(s)Y = \Phi_{\mathcal{L}}(s)Y$. Moreover, there is some edge e_i with $L(e_i) > 0$ and hence $[A_{\mathcal{L}}(s')Y]_i < [A_{\mathcal{L}}(s)Y]_i = \Phi_{\mathcal{L}}(s)y_i$. Therefore $\Phi_{\mathcal{L}}(s') < \Phi_{\mathcal{L}}(s)$, as claimed.

Note that $A_{\mathcal{L}}(0) = M$ and $\Phi_{\mathcal{L}}(0)$ is the Perron-Frobenius eigenvalue of M. The fundamental group of $H(\Gamma)$ is free of rank at least two. Hence the universal cover of $H(\Gamma)$ has exponential growth, that is, the spectral radius of M is bigger than 1.

To see that (4) holds, note that if $L(e) > 0$ for every $e \in E\Gamma$ then we have $\lim_{s\to\infty} \sum_{ij} a_{ij}(s) = 0$. Also,

$$A(s)\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \le \sum_{ij} a_{ij}(s)\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}.$$

Hence $\Phi(s) \le \sum_{ij} a_{ij}(s)$ and so $\lim_{s\to\infty}\Phi(s) = 0$. $\square$

Corollary 5.4. *Let $\mathcal{L}$ be a positive quasi-metric structure or a non-singular semi-metric structure on Γ. Then there exists a unique $s > 0$ such that $\Phi_{\mathcal{L}}(s) = 1$. If $\mathcal{L}$ is a non-singular semi-metric structure then $s = h_{\mathcal{L}}(X)$.*

Proof. Lemma 5.3 implies that there is at most one $s > 0$ such that $\Phi_{\mathcal{L}}(s) = 1$. If $\mathcal{L}$ is a non-singular semi-metric structure, the existence of Patterson-Sullivan measures (Proposition-Definition 3.12) and Proposition 4.3 guarantee that when $s = h_{\mathcal{L}}(X)$, the Perron-Frobenius eigenvalue of $A_{\mathcal{L}}(s)$ is equal to 1, that is, that $\Phi_{\mathcal{L}}(h_{\mathcal{L}}(X))) = 1$. If $\mathcal{L}$ is a positive quasi-metric structure then parts (2), (3) and (4) of Lemma 5.3 guarantee the existence of $s > 0$ such that $\Phi_{\mathcal{L}}(s) = 1$. $\square$

Notation 5.5. Let $\mathcal{L}$ be a positive quasi-metric structure or a non-singular semi-metric structure on Γ. The unique value $s > 0$ such that $\Phi_{\mathcal{L}}(s) = 1$ will be denoted $h(\mathcal{L})$. We will also refer to $h(\mathcal{L})$ as the *volume entropy* of $\mathcal{L}$.

We will now rewrite the system (*) $A_{\mathcal{L}}(s)Y = Y$ in the form allowing to apply the Implicit Function Theorem. This system is equivalent to the following n equations:

$$e^{-sL(e_i)}(m_{i1}y_1 + ... + m_{in}y_n) - y_i = 0, \quad i = 1, \dots, n.$$

To express $y_1, \dots, y_n, s$ as implicit functions of $L(e_1), \dots, L(e_n)$ we need an extra normalizing equation: $y_1^2 + \cdots + y_n^2 = 1$.

Proposition 5.6. *Let $L_1 = L(e_1), \dots, L_n = L(e_n)$ be a non-singular semi-metric structure or a positive quasi-metric structure $\mathcal{L}$ on Γ. We set*

$$F_i(L_1, \dots, L_n, y_1, \dots, y_n, s) := e^{-sL_i}(m_{i1}y_1 + ... + m_{in}y_n) - y_i$$

for $i = 1, \dots, n$, and

$$F_{n+1}(L_1, \dots, L_n, y_1, \dots, y_n, s) := y_1^2 + \cdots + y_n^2 - 1.$$

Consider the following system of $n + 1$ equations in $2n + 1$ variables:

$$(!) \qquad F_i(L_1, \dots, L_n, y_1, \dots, y_n, s) = 0, \quad i = 1, \dots, n + 1$$

Let J be the Jacobian of this system, that is the $(n + 1) \times (n + 1)$-matrix consisting of the partial derivatives of $F_1, \dots, F_{n+1}$ with respect to $y_1, \dots, y_n, s$:

$$J_{ij} = \begin{cases} \frac{\partial F_i}{\partial y_j} & 1 \le i \le n+1, 1 \le j \le n \\ \frac{\partial F_i}{\partial s} & 1 \le i \le n+1, j = n+1. \end{cases}$$

Suppose $s > 0, y_i > 0$, for $i = 1, \dots, n$, are such that the point $z = (L_1, \dots, L_n, y_1, \dots, y_n, s)$ satisfies the system (!). Then $\det J|_z \ne 0$.

Proof. Let us compute the matrix J at z, using the information that z satisfies (!). We will denote $a_{ij} = (A_{\mathcal{L}}(s))_{ij} = e^{-sL_i} m_{ij}$.

For $i \neq j$, $1 \leq i,j \leq n$ we get $\frac{\partial F_i}{\partial y_j} = e^{-sL_i} m_{ij} = a_{ij}$. For $i = j$ we get $\frac{\partial F_i}{\partial y_i} = e^{-sL_i} m_{ii} - 1 = a_{ii} - 1$. Thus in the upper left corner of J we see the $n \times n$ matrix $A_{\mathcal{L}}(s) - I_n$.

Let us compute $\frac{\partial F_i}{\partial s}$. We have

$$\frac{\partial F_i}{\partial s} = -L_i e^{-sL_i}(m_{i1}y_1 + ... + m_{in}y_n) = -L_i y_i \quad \text{ for } i = 1, \dots, n,$$

where the last equality holds since $F_i(z) = 0$.

Finally, the last row of J obtained by differentiating $F_{n+1} = y_1^2 + \cdots + y_n^2 - 1$ along $y_1, .., y_n, s$ is $[2y_1 \ 2y_2 \ \dots \ 2y_n \ 0]$.

Thus

$$J = \begin{bmatrix} a_{11} - 1 & a_{12} & a_{13} & \dots & a_{1n} & -L_1 y_1 \\ a_{21} & a_{22} - 1 & a_{23} & \dots & a_{2n} & -L_2 y_2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} & -L_i y_i \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - 1 & -L_n y_n \\ 2y_1 & 2y_2 & 2y_3 & \dots & 2y_n & 0 \end{bmatrix}$$

We claim that the rows of the matrix J are linearly independent and hence $\det J \neq 0$. The column vector $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ satisfies $(A_{\mathcal{L}}(s) - I_n)Y = 0$. This implies that the last row of J is perpendicular to the first n rows.

Since $Y > 0$, it therefore suffices to show that the first n rows of J are linearly independent.

Note that $\det(A_{\mathcal{L}}(s) - I_n) = 0$. However the matrix $A_{\mathcal{L}}(s) - I_n$ has rank $n - 1$ since 1 is the Perron-Frobenius eigenvalue of $A_{\mathcal{L}}(s)$ and hence has multiplicity one. Thus, up to a scalar, there is only one nontrivial linear relation between the rows of $A_{\mathcal{L}}(s) - I_n$. This relation is given by the left Perron-Frobenius eigenvector $Z = [z_1, \dots, z_n]$ of $A_{\mathcal{L}}(s)$. Indeed $ZA_{\mathcal{L}}(s) = Z$ and $Z[A_{\mathcal{L}}(s) - I_n] = 0$. Note that $z_i > 0$ for all $i = 1, \dots, n$.

Suppose that the first n rows of J are linearly dependent and that we have a nonzero row vector Z' of length n such that $Z'J_n = 0$ where J_n is the $n \times (n+1)$ matrix consisting of the first n rows of J. Then Z' is also a relation between the first n rows of $A_{\mathcal{L}}(s) - I_n$ and hence Z' is a multiple of Z. Thus $ZJ_n = 0$.

However, when we multiply Z by the last column of J_n to compute the $(n+1)$-st entry in ZJ_n, we get $-L_1 y_1 z_1 - L_2 y_2 z_2 - \cdots - L_n y_n z_n$.

This number is strictly negative since $L_i \geq 0, y_i > 0, z_i > 0$ for all $i = 1, \dots, n$ and there is some i such that $L_i > 0$. This gives us a contradiction with the fact that $ZJ_n = 0$. □

For the remainder of this section we will denote an n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ by $\overline{p}$.

Corollary 5.7. *Let $L_1^{(0)} = L^{(0)}(e_1), \dots, L_n^{(0)} = L^{(0)}(e_n)$ be a non-singular semi-metric or a positive quasi-metric structure $\mathcal{L}^{(0)}$ on Γ. Suppose $s^{(0)} > 0, y_i^{(0)} > 0$,* *where $i = 1, \dots, n$, are such that the point $z^{(0)} = (\overline{L}^{(0)}, \overline{y}^{(0)}, s^{(0)}) \in \mathbb{R}^{2n+1}$ satisfies the system (!). Then there exist an open neighborhood U of $\overline{L}^{(0)}$ in $\mathbb{R}^n$ and real-analytic functions $s = s(\overline{L})$, $y_i = y_i(\overline{L})$ on U such that for every $\overline{L} \in U$ the point*

$$(\overline{L}, y_1(\overline{L}), \dots, y_n(\overline{L}), s(\overline{L})) \in \mathbb{R}^{2n+1}$$

satisfies (!) and such that $y_i(\overline{L}^{(0)}) = y_i^{(0)}$, $s(\overline{L}^{(0)}) = s^{(0)}$.

Moreover, whenever $\overline{L} \in U$ defines a non-singular semi-metric structure $\mathcal{L}$ on Γ, then $s(\overline{L})$ is equal to the critical exponent of $(\widetilde{\Gamma}, d_{\mathcal{L}})$ and $(y_1(\overline{L}), \dots, y_n(\overline{L}))$ is a scalar multiple of $(w_{e_1,\mathcal{L}}, \dots, w_{e_n,\mathcal{L}})$.

Proof. Proposition 5.6 implies that the Implicit Function Theorem is applicable at the point $z^{(0)}$. Thus there exists an open neighborhood U of $z^{(0)}$ and real-analytic functions $s = s(\overline{L})$, $y_i = y_i(\overline{L})$ on U such that for every $\overline{L} \in U$ the point

$$(\overline{L}, y_1(\overline{L}), \dots, y_n(\overline{L}), s(\overline{L})) \in \mathbb{R}^{2n+1}$$

satisfies (!) and such that $y_i(\overline{L}^{(0)}) = y_i^{(0)}$, $s(\overline{L}^{(0)}) = s^{(0)}$.

Moreover, since $y_i^{(0)} > 0$, we can choose U so that $y_i = y_i(\overline{L}) > 0$ on U. Let $\overline{L} \in U$ define a non-singular semi-metric structure $\mathcal{L}$ on Γ. By Proposition 4.3 the critical exponent $h = h_{\mathcal{L}}$ of $(\widetilde{\Gamma}, d_{\mathcal{L}})$ satisfies the property that $\Phi_{\mathcal{L}}(h) = 1$. Also, by construction, $\Phi_{\mathcal{L}}(s(\overline{L})) = 1$. Corollary 5.4 now implies that $h = s(\overline{L})$. Moreover, Proposition 4.3 and the definition of the functions $y_i(L_1, \dots, L_n)$ imply that both

$$(w_{e_1,\mathcal{L}}, \dots, w_{e_n,\mathcal{L}})$$

and

$$(y_1(\overline{L}), \dots, y_n(\overline{L}))$$

are Perron-Frobenius eigenvectors of the matrix $A_{\mathcal{L}}(h)$. Therefore they are scalar multiples of each other, as required. □

6. Patterson-Sullivan currents

The following is essentially a corollary of Proposition 1 of Furman [17].

Proposition-Definition 6.1 (Patterson-Sullivan current)**.** Let G, X be as in Convention 3.3. Let $(\mu_x)_{x\in X}$ be a family of Patterson-Sullivan measures on ∂X and let $\mu = \mu_y$ for some $y \in X$. Then there exists a unique, up to a scalar multiple, G-invariant and flip-invariant nonzero locally finite measure ν on $\partial^2 X$ in the measure class of $\mu \times \mu$.

Moreover, this measure ν is of the form

$$d\nu(\xi, \zeta) = e^{2hf_\mu(\xi,\zeta)} d\mu(\xi) d\mu(\zeta),$$

where $f_\mu : \partial^2 X \to \mathbb{R}_+$ is a symmetric Borel function which is within bounded distance from the function $d(x, [\xi, \zeta])$, and h is the critical exponent of X.

Such a measure ν is called an *X-Patterson-Sullivan current* for this action of G on X. Since ν is unique up to a scalar multiple, its projective class $[\nu]$ is called *the projective X-Patterson-Sullivan current.*

For the remaining part of this section let $F, \Gamma, \mathcal{L}, X, d, \alpha$ be as in Part 2) of Convention 3.3.

Definition 6.2. Recall that the choice of simplicial chart α defines a homeomorphism $\hat{\alpha} : \partial^2 F \to \partial^2 X$. Let ν be an X-Patterson-Sullivan current. Then its pull-back $\hat{\alpha}_*(\nu)$ is an F-invariant measure on $\partial^2 F$ which is called an *F-Patterson-Sullivan current* for the pair $(\alpha, \mathcal{L})$. Its projective class $[\nu]$ is called *the projective F-Patterson-Sullivan current* for the pair $(\alpha, \mathcal{L})$.

We now proceed to give an explicit formula for the X-Patterson-Sullivan current associated with the action of F on X.

Proposition 6.3. *Let $z \in X$, and let $h = h_{\mathcal{L}}(X)$ be the critical exponent of X. Let $(\mu_x)_x$ be a family of Patterson-Sullivan measures on ∂X and let w_e be defined as in Convention 4.2.*

Then the measure ν on $\partial^2 X$ given by the formula

$$d\nu(\xi, \zeta) = e^{2hd(z,[\xi,\zeta])} d\mu_z(\xi) d\mu_z(\zeta) \tag{$\clubsuit$}$$

is an X-Patterson-Sullivan current.

Moreover, for any path $\gamma = [x, y] \in \mathcal{P}(X)$ with label $j(\gamma) \in \mathcal{P}(\Gamma)$ we have

$$\nu(Cyl_X(\gamma)) = e^{-hL(\gamma)} \Big(\sum_{e \in b(e')} w_e \Big) \Big(\sum_{e \in b(e'')} w_e \Big) \tag{$\dagger$}$$

where $(e')^{-1} \in E\Gamma$ is the label of the first edge of γ and $e'' \in E\Gamma$ is the label of the last edge of γ.

Proof. We will first show that ($\dagger$) defines a geodesic current on ∂X. That is, we claim that there exists a unique geodesic current ν' such that for every γ as in the statement of the proposition

$$\nu'(Cyl_X(\gamma)) = e^{-hL(\gamma)} \Big(\sum_{e \in b(e')} w_e \Big) \Big(\sum_{e \in b(e'')} w_e \Big).$$

In view of the definition of w_e's the above formula is equivalent to

$$\nu'(Cyl_X(\gamma)) = e^{-hL(\gamma)} \mu_x(Cyl_y(\gamma^{-1})) \mu_y(Cyl_x(\gamma)). \tag{$\ddagger$}$$

The uniqueness of ν' is obvious. Also, by construction ν' is F-invariant, provided that ν' is a measure. Thus it remains to show that the above formula does define a measure on $\partial^2 X$.

To do this we need to check (see, for example, [26]) that for every γ as above

$$\nu'(Cyl_X(\gamma)) = \sum_{f \in b(\gamma)} \nu'(Cyl_X(\gamma f))$$

and

$$\nu'(Cyl_X(\gamma)) = \sum_{f \in a(\gamma)} \nu'(Cyl_X(f\gamma)).$$

We will verify the first formula, as the second one is completely analogous. By ($\ddagger$) applied to each of the paths γf, where $f \in b(\gamma)$, we have

$$\nu'(Cyl_X(\gamma f)) = e^{-hL(\gamma f)} \mu_x(Cyl_{t(f)}(f^{-1}\gamma^{-1})) \mu_{t(f)}(Cyl_x(\gamma f)) =$$
$$e^{-hL(\gamma f)} \mu_x(Cyl_y(\gamma^{-1})) \mu_{t(f)}(Cyl_x(\gamma f)) =$$
$$e^{-hL(\gamma f)} \mu_x(Cyl_y(\gamma^{-1})) \mu_{t(f)}(Cyl_y(f)).$$

Since

$$Cyl_x(\gamma) = \bigsqcup_{f \in b(\gamma)} Cyl_x(\gamma f),$$

it follows that

$$\mu_y(Cyl_x(\gamma)) = \sum_{f \in b(\gamma)} \mu_y(Cyl_x(\gamma f)).$$

Therefore

$$\begin{gathered}
\nu'(Cyl_X(\gamma)) = e^{-hL(\gamma)}\mu_x(Cyl_y(\gamma^{-1}))\mu_y(Cyl_x(\gamma)) = \\
e^{-hL(\gamma)}\mu_x(Cyl_y(\gamma^{-1}))\big(\sum_{f \in b(\gamma)} \mu_y(Cyl_x(\gamma f))\big) = \\
e^{-hL(\gamma)}\mu_x(Cyl_y(\gamma^{-1}))\big(\sum_{f \in b(\gamma)} \mu_y(Cyl_y(f))\big) = \\
\text{by Remark 4.1} \\
e^{-hL(\gamma)}\mu_x(Cyl_y(\gamma^{-1}))\big(\sum_{f \in b(\gamma)} e^{-hL(f)}\mu_{t(f)}(Cyl_y(f))\big) = \\
\sum_{f \in b(\gamma)} e^{-hL(\gamma f)}\mu_x(Cyl_y(\gamma^{-1}))\mu_{t(f)}(Cyl_y(f)) = \\
\sum_{f \in b(\gamma)} \nu'(Cyl_X(\gamma f)).
\end{gathered}$$

Thus ν' is indeed a geodesic current. We will now show that $\nu' = \nu$, where the measure ν on $\partial^2 X$ is defined by (♣). It suffices to show that $\nu(Cyl_X(\gamma)) = \nu'(Cyl_X(\gamma))$ for every $\gamma \in \mathcal{P}(X)$. Let $\gamma = [x, y] \in \mathcal{P}(X)$. We need to consider the following two cases.

Case 1: $d(z, [x, y]) > 0$.
Let $z' \in [x, y]$ be such that $d(z, z') = d(z, [x, y])$. Then

$$\begin{gathered}
\nu(Cyl_X(\gamma)) = e^{2hd(z,z')}\mu_z(Cyl_z([z, x]))\mu_z(Cyl_z([z, y])) = \\
e^{2hd(z,z')}e^{-hd(z,z')}\mu_{z'}(Cyl_{z'}([z', x]))e^{-hd(z,z')}\mu_{z'}(Cyl_{z'}([z', y])) = \\
\mu_{z'}(Cyl_{z'}([z', x]))\mu_{z'}(Cyl_{z'}([z', y])) = \\
e^{-hd(z',x)}\mu_x(Cyl_{z'}([z', x]))e^{-hd(z',y)}\mu_y(Cyl_{z'}([z', y])) = \\
e^{-hd(x,y)}\mu_x(Cyl_{z'}([z', x]))\mu_y(Cyl_{z'}([z', y])) = \\
e^{-hd(x,y)}\mu_x(Cyl_y([y, x]))\mu_y(Cyl_x([x, y])) = \\
e^{-hL(\gamma)}\mu_x(Cyl_y(\gamma^{-1}))\mu_y(Cyl_x(\gamma)) = \nu'(Cyl_X(\gamma)).
\end{gathered}$$

Case 2: $d(z, [x, y]) = 0$ (so that $z \in [x, y]$).
We will assume that $z \neq x, z \neq y$. The argument is easily adapted for the cases $z = x$ or $z = y$.

Then

$$\nu(Cyl_X(\gamma)) = e^{2hd(z,[x,y])}\mu_z(Cyl_z([z,x]))\mu_z(Cyl_z([z,y])) =$$
$$\mu_z(Cyl_z([z,x]))\mu_z(Cyl_z([z,y])) =$$
$$e^{-hd(z,x)}\mu_x(Cyl_z([z,x]))e^{-hd(z,y)}\mu_y(Cyl_z([z,y])) =$$
$$e^{-hd(x,y)}\mu_x(Cyl_z([z,x]))\mu_y(Cyl_z([z,y])) =$$
$$e^{-hd(x,y)}\mu_x(Cyl_y([y,x]))\mu_y(Cyl_x([x,y])) =$$
$$e^{-hL(\gamma)}\mu_x(Cyl_y(\gamma^{-1}))\mu_y(Cyl_x(\gamma)) = \nu'(Cyl_X(\gamma)).$$

Therefore $\nu = \nu'$, which completes the proof of Proposition 6.3 □

7. The Culler-Vogtmann outer space

The Culler-Vogtmann outer space, introduced by Culler and Vogtmann in a seminal paper [14], is a free group analogue of the Teichmüller space of a closed surface of negative Euler characteristic. We refer the reader to [14, 9, 3, 20] for a detailed discussion and the proofs of the basic facts that are listed in this section.

Definition 7.1 (Outer space)**.** Let F be a free group of finite rank $k \geq 2$. A *marked metric graph structure* on F is a pair $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$ is a simplicial chart for F and $\mathcal{L}$ is a metric graph structure on Γ. A marked metric graph structure is *minimal* if Γ has no degree-one and degree-two vertices.

Two marked metric graph structures $(\alpha_1 : F \to \pi_1(\Gamma_1, p_1), \mathcal{L}_1)$ and $(\alpha_2 : F \to \pi_1(\Gamma_2, p_2), \mathcal{L}_2)$ are *equivalent* if there exist an isometry $\iota : (\Gamma_1, \mathcal{L}_1) \to (\Gamma_2, \mathcal{L}_2)$ and a path v from $\iota(p_1)$ to p_2 in Γ_2 such that

$$(\iota_\# \circ \alpha_1)(f) = v\alpha_2(f)v^{-1}$$

for every $f \in F$. Clearly, minimality is preserved by equivalence of marked metric graph structures.

The *Culler-Vogtmann outer space* $CV(F)$ consists of equivalence classes of all volume-one minimal marked metric graph structures on F.

Definition 7.2 (Elementary charts)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F, where Γ has no degree-one and degree-two vertices.

For each non-singular semi-metric structure $\mathcal{L}$ on Γ let Γ', $\mathcal{L}'$ and q be as in Convention 3.2. Then $q_\# \circ \alpha : F \to \pi_1(\Gamma', q(p))$ is a simplicial chart for F and $(q_\# \circ \alpha, \mathcal{L}')$ is a minimal marked metric graph structure on F.

Denote by $S(\Gamma)$ the set of all volume-one non-singular semi-metric structures on Γ. Note that if Γ has N non-oriented edges, then $S(\Gamma)$ is embedded as a subset of $\mathbb{R}^n$. We topologize $S(\Gamma)$ accordingly.

It is not hard to see that for two non-singular semi-metric structures $\mathcal{L}_1, \mathcal{L}_2$ on Γ the pairs $(q_\# \circ \alpha, \mathcal{L}_1')$ and $(q_\# \circ \alpha, \mathcal{L}_2')$ are equivalent if and only if $\mathcal{L}_1 = \mathcal{L}_2$. Thus α defines an injective map $\lambda_\alpha : S(\Gamma) \to CV(F)$, $\lambda_\alpha : \mathcal{L} \mapsto (q_\# \circ \alpha, \mathcal{L}')$. This map λ_α is called the *elementary chart in* $CV(F)$ corresponding to α.

Let now $S_+(\Gamma)$ denote the set of all metric structures on Γ. If Γ has n oriented edges then $S_+(\Gamma)$ is an open simplex of dimension $n/2-1$ in $\mathbb{R}^n$ and $S_+(\Gamma)$ is dense in $S(\Gamma)$.

Definition 7.3 (Topology on the outer space)**.** The outer space $CV(F)$ is endowed with the weakest topology for which every elementary chart is a topological embedding.

As explained in [14], the outer space $CV(F)$ is a union of open simplices of the form $\lambda_\alpha(S_+(\Gamma))$, where λ_α is as in Definition 7.2. One can also view $CV(F)$ as the space of projectivized hyperbolic length functions on F corresponding to free and discrete isometric actions of F on $\mathbb{R}$-trees.

Definition 7.4 (Projectivized length functions)**.** Let $FLen(F)$ denote the space of all hyperbolic length functions $\ell : F \to \mathbb{R}$ on F corresponding to free and discrete isometric actions of F on $\mathbb{R}$-trees. The space $FLen(F)$ is endowed with the weak topology of pointwise convergence.

We will say that two length functions in $FLen(F)$ are equivalent if they are scalar multiples of each other, and will denote by $\mathbb{P}FLen(F)$ the space of equivalence classes of elements of $FLen(F)$, endowed with the quotient topology. The equivalence class of $\ell \in FLen(F)$ is denoted $[\ell]$. For each $\ell \in FLen(F)$ there exists a free discrete minimal isometric action of F on an $\mathbb{R}$-tree X_ℓ such that ℓ is the hyperbolic length function for this action. Moreover, the tree X_ℓ and the corresponding action of F are unique up to an equivariant isometry. Let Γ_ℓ denote the metric graph X_ℓ/F.

Let $FLen_1(F)$ denote the set of all $\ell \in FLen(F)$ such that Γ_ℓ has volume one. Note that every equivalence class $[\ell] \in \mathbb{P}FLen(F)$ has a unique representative in $FLen_1(F)$. For each $\ell \in FLen(F)$ the action of F on X_ℓ defines an isomorphism $\alpha_\ell : F \to \pi_1(\Gamma_\ell, p)$, where $p \in V\Gamma_\ell$. Let $\mathcal{L}_\ell$ denote the metric graph structure on Γ_ℓ inherited from X_ℓ. Note that the equivalence class of the marked metric graph structure $(\alpha_\ell, \mathcal{L}_\ell)$ on F does not depend on the choice of p.

The following result is well-known [14]:

Proposition 7.5. *The following hold:*

1. *The restriction of the quotient map* $[\,] : FLen(F) \to \mathbb{P}FLen(F)$ *to* $FLen_1(F)$ *is a homeomorphism whose image is* $\mathbb{P}FLen(F)$*. Thus* $FLen_1(F)$ *is canonically homeomorphic to* $\mathbb{P}FLen(F)$*.*
2. *Let* $\varrho : FLen_1(F) \to CV(F)$ *be the map that takes each* $\ell \in FLen_1(F)$ *to the equivalence class of the marked structure* $(\alpha_\ell, \mathcal{L}_\ell)$ *on* F*. Then* $\varrho : FLen_1(F) \to CV(F)$ *is a homeomorphism.*

Thus the outer space $CV(F)$ is homeomorphic to the spaces $FLen_1(F)$ and $\mathbb{P}FLen(F)$.

8. Proof of the main result

If $(\alpha_1 : F \to \pi_1(\Gamma_1, p_1), \mathcal{L}_1)$ and $(\alpha_2 : F \to \pi_1(\Gamma_2, p_2), \mathcal{L}_2)$ are two equivalent pairs representing the same point $\eta \in CV(F)$, then $\mathbb{R}$-trees $X_1 = \widetilde{\Gamma}_1$ and $X_2 = \widetilde{\Gamma}_2$ are F-equivariantly isometric and the corresponding hyperbolic length functions are equal. Hence it follows from Proposition 2 of Furman [17] (and it is also easy to see this directly) that the projective F-Patterson-Sullivan currents corresponding to $(\alpha_1, \mathcal{L}_1)$ and $(\alpha_2, \mathcal{L}_2)$ coincide (see Definition 6.2). Hence the following map is well-defined:

Definition 8.1 (Patterson-Sullivan map and Hausdorff dimension map)**.** Let F be a free group of finite rank $k \geq 2$ and let $CV(F)$ denote the outer space.

Let $\eta \in CV(F)$. Thus η is represented as an equivalence class of $(\alpha, \mathcal{L})$, where $\alpha : F \to \pi_1(\Gamma, p)$ is a simplicial chart on F such that Γ is a finite connected graph without degree-one and degree-two vertices and where $\mathcal{L}$ is a volume-one metric graph structure on Γ. Let $X = \widetilde{\Gamma}$ and let d be the metric on X induced by $\mathcal{L}$. Define $\tau(\eta)$ to be the projective F-Patterson-Sullivan current on F corresponding to $(\alpha, \mathcal{L})$. Also define $h(\eta)$ to be the Hausdorff dimension of ∂X (which, as we have seen, is equal to the critical exponent $h_{\mathcal{L}}(X)$.)

This defines a map $\tau : CV(F) \to \mathbb{P}Curr(F)$, which we will call the *Patterson-Sullivan map*, and a map $h : CV(F) \to \mathbb{R}$, which we will call the *Hausdorff dimension map*.

Theorem 8.2. *The Patterson-Sullivan map $\tau : CV(F) \to \mathbb{P}Curr(F)$ is a topological embedding. The Hausdorff dimension map $h : CV(F) \to \mathbb{R}$ is continuous and, moreover, the restriction of h to any open simplex in $CV(F)$ is real-analytic.*

Proof. Since $CV(F)$ is locally compact, in order to prove that τ is a topological embedding it suffices to show that τ is continuous and injective.

Injectivity follows from Theorem 2 of Furman [17]. Recall the identification of $CV(F)$ with $\mathbb{P}FLen(F)$ from Proposition 7.5. If $\tau([\ell_1]) = \tau([\ell_2])$ for $\ell_1, \ell_2 \in FLen(F)$ then Theorem 2 of [17] implies that there is $r > 0$ such that $r\ell_1 = \ell_2$ and hence $[\ell_1] = [\ell_2]$.

We now establish that τ and h are continuous. Since every point of the outer space is contained in finitely many elementary charts, it suffices to prove that τ and h are continuous on the image of every elementary chart in $CV(F)$.

Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F, where Γ has no degree-one and degree-two vertices. Let λ_α be the elementary chart in $CV(F)$ determined by α. Recall that the image $Im(\lambda_\alpha)$ of λ_α consists of all points of $CV(F)$ corresponding to volume-one semi-metric structures on Γ where all the edges with zero length are contained in a (possibly empty) subtree of Γ. Corollary 5.7 and formula (†) in Proposition 6.3 imply that $\tau|_{Im(\lambda_\alpha)}$ and $h|_{Im(\lambda_\alpha)}$ are continuous and, moreover, the restriction of h to the interior of $Im(\lambda_\alpha)$ is real-analytic. □

Remark 8.3 ($Out(F)$-Equivariance)**.** It is easy to see that the Patterson-Sullivan map τ is equivariant with respect to the left action of $Out(F)$ and, in fact, a similar statement holds in the general word-hyperbolic context considered by Furman [17]. It is even easier to see that h is constant on each $Out(F)$-orbit and thus factors to a continuous map on the moduli space $\overline{h} : CV(F)/Out(F) \to \mathbb{R}$.

Indeed, suppose $(\alpha : F \to \pi_1(\Gamma, p), \mathcal{L})$ represents a point $\eta \in CV(F)$ and let $\phi \in Aut(F)$. Let $X = \widetilde{\Gamma}$, equipped with the metric d induced by $\mathcal{L}$.

By definition of the left action of $Aut(F)$ (and of $Out(F)$) on $CV(F)$, the point $\phi\eta \in CV(F)$ is the equivalence class of $(\phi^{-1} \circ \alpha, \mathcal{L})$. For both η and $\phi\eta$ the metric graph $(\Gamma, \mathcal{L})$ is the same. This already implies that $h(\eta) = h(\phi\eta)$.

The action of F on X corresponding to $\phi\eta$ is obtained from the F-action on X corresponding to η by a pre-composition with ϕ^{-1}. The definitions imply that if $(\mu_x)_x$ is a family of Patterson-Sullivan measures on ∂X corresponding to the action of F on X via α, then $(\mu_x)_x$ is also a family of Patterson-Sullivan measures on ∂X

corresponding to the action of F on X via $\phi^{-1} \circ \alpha$. Hence if ν is an X-Patterson-Sullivan current corresponding to the action of F on X via α, then ν is also an X-Patterson-Sullivan current corresponding to the action of F on X via $\phi^{-1} \circ \alpha$.

Denote $\nu_1 := \hat{\alpha}_*(\nu)$ and $\nu_2 := [\hat{\phi}^{-1} \circ \hat{\alpha}]_*(\nu)$, so that $\tau(\eta) = [\nu_1]$ and $\tau(\phi\eta) = [\nu_2]$. Definitions then imply that $\nu_2 = (\hat{\phi}^{-1})_*\nu_1$, that is, for any Borel subset $A \subseteq \partial^2 F$ we have $\nu_2(S) = \nu_1(\hat{\phi}^{-1}(A))$. By definition of the left action of $Aut(F)$ on $Curr(F)$ (see [26]) we have $(\phi\nu_1)(A) = \nu_1(\hat{\phi}^{-1}(A))$. Thus $\nu_2 = \phi\nu_1$ and hence $\tau(\phi\eta) = \phi(\tau\eta)$, as claimed.

9. The minimal volume entropy problem

Our goal in this section is to prove parts (1) and (3) of Theorem B from the Introduction.

Convention 9.1. For the remainder of this section let $k \geq 2$, and let Γ be a finite connected graph whose fundamental group $F = \pi_1(\Gamma, p)$ with respect to a base vertex $p \in V\Gamma$ is free of rank k. Let $X = \widetilde{(\Gamma, p)}$, and let $x_0 \in VX$ be a fixed lift of p.

Let $w \in \mathcal{P}(\Gamma)$ and $e \in E\Gamma$. We denote by $\langle e, w\rangle$ the number of occurrences of e in w. The following is an obvious but useful fact:

Lemma 9.2. *Let $\mathcal{L}$ be a positive quasi-metric structure on Γ. Let $w \in \mathcal{P}(\Gamma)$ be a reduced path. Then*

$$L_{\mathcal{L}}(w) = \sum_{e \in E\Gamma} \langle e, w\rangle L_{\mathcal{L}}(e).$$

The key step in the proof of Theorem B is the following statement, which provides a sharp bound for the volume entropy of (regular) m-valent metric graphs. Note that an m-valent graph with the fundamental group of rank k has $m(k-1)/(m-2)$ non-oriented edges.

The following lemma follows directly from the definition of the matrix $A_{\mathcal{L}}(s)$.

Lemma 9.3. *Let $\mathcal{L}$ be a positive quasi-metric structure on Γ. Then for any integer $t \geq 1$ and for any position ij we that*

$$[A_{\mathcal{L}}(s)^t]_{ij} = e^{sL(e_j)} \sum_v e^{-sL(v)}$$

where the summation is taken over all reduced paths v of edge-length t with the first edges e_i and the last edge e_j.

Proposition 9.4. *For $m \geq 3$ suppose Γ is a finite regular m-valent graph (i.e., a graph in which every vertex has degree m) with fundamental group free of rank $k \geq 2$. Let $\mathcal{L}$ be a volume-one positive quasi-metric structure on Γ. Then*

$$h(\mathcal{L}) \geq \frac{m(k-1)}{m-2} \log(m-1).$$

Proof. We consider the simple non-backtracking random walk on Γ. This walk can be thought of as a finite state Markov process with the state set $E\Gamma$ and with transition probabilities defined as

$$p(e, e') = \begin{cases} \frac{1}{m-1}, & \text{if } ee' \in \mathcal{P}(\Gamma), \\ 0, & \text{otherwise,} \end{cases}$$

where $e, e' \in E\Gamma$.

This Markov process is irreducible since for any $e, e' \in E\Gamma$ there exists a reduced path in Γ with initial edge e and terminal edge e'. The graph Γ has $(mk-m)/(m-2)$ nonoriented edges and $(2mk-2m)/(m-2)$ oriented edges. The uniform distribution μ_0 on $E\Gamma$, given by $\mu_0(e) = \frac{m-2}{2mk-2m}$ for every $e \in E\Gamma$, is obviously invariant with respect to our Markov process. Since the process is irreducible, μ_0 is the only invariant distribution on $E\Gamma$.

Let μ be the distribution on $E\Gamma$ which is uniformly distributed on the m oriented edges starting with the base-vertex p. In other words, $\mu(e) = 1/m$ if $o(e) = p$ and $\mu(e) = 0$ if $o(e) \neq p$.

Let $w_t = e_1, \ldots, e_t$ be a trajectory of our process of length t. Let $\epsilon > 0$ and $e \in E\Gamma$. By the Large Deviation principle (see Ch.IV-V in [15]) we have

$$\lim_{t\to\infty} \mathbb{P}_\mu(|\frac{\langle e, w_t\rangle}{t} - \frac{m-2}{2mk-2m}| > \epsilon) = 0.$$

Let $\epsilon > 0$ be arbitrary. Then there is $t_0 \geq 1$ such that for every $t \geq t_0$ and for every $e \in E\Gamma$ we have

$$\mathbb{P}_\mu(|\frac{\langle e, w_t\rangle}{t} - \frac{m-2}{2mk-2m}| \leq \epsilon) \geq \frac{1}{2}.$$

Denote

$$R(t,\epsilon) = \{w \in \mathcal{P}(\Gamma) : w \text{ consists of } t \text{ edges, and for every } e \in E\Gamma$$
$$|\frac{\langle e, w\rangle}{t} - \frac{m-2}{2mk-2m}| \leq \epsilon\}.$$

Thus for $t \geq t_0$

$$(\heartsuit) \qquad \#R(t,\epsilon) \geq (1/2)\cdot m \cdot (m-1)^{t-1} \geq (m-1)^{t-1}.$$

The volume of Γ is equal to one and hence $\sum_{e\in E\Gamma} L_{\mathcal{L}}(e) = 2$.

Then for every $w \in R(t,\epsilon)$ we have:

$$(\spadesuit) \qquad L_{\mathcal{L}}(w) = \sum_{e\in E\Gamma} \langle e, w\rangle L_{\mathcal{L}}(e) \leq$$
$$\sum_{e\in E\Gamma} (\frac{t(m-2)}{2mk-2m} + t\epsilon) L_{\mathcal{L}}(e) = (\frac{t(m-2)}{2mk-2m} + t\epsilon) \sum_{e\in E\Gamma} L_{\mathcal{L}}(e) =$$
$$(\frac{t(m-2)}{2mk-2m} + t\epsilon)\cdot 2 = \frac{t(m-2)}{mk-m} + 2t\epsilon.$$

Let $c = \min_{e\in E\Gamma} e^{sL(e)}$. Then Lemma 9.3 together with $(\heartsuit)$ and $(\spadesuit)$ imply that for every integer $t \geq 1$

$$\sum_{ij} (A^t)_{ij} \geq c(m-1)^{t-1} e^{-s\frac{t(m-2)}{mk-m} + 2t\epsilon} = \frac{c}{m-1}[(m-1)e^{-s\frac{(m-2)}{mk-m} + 2\epsilon}]^t$$

For $s = h(\mathcal{L})$ the matrix $A_{\mathcal{L}}(s)$ has spectral radius 1 and therefore

$$(m-1)e^{-s\frac{(m-2)}{mk-m} + 2\epsilon} \leq 1$$

and hence

$$s \geq \frac{mk-2}{m-2}(\log(m-1) + 2\epsilon).$$

Since this is true for all $\epsilon > 0$, this implies that

$$s \geq \frac{mk-m}{m-2}\log(m-1).$$

$\square$

The following easy computation shows that the bound in Proposition 9.4 is realized by the "uniform" volume-one metric structure, where all edges have equal lengths.

Lemma 9.5. *Let Γ be as in Proposition 9.4.*

Let $\mathcal{L}_0$ be the "uniform" volume-one metric structure on Γ, that is $L_{\mathcal{L}_0}(e) = \frac{m-2}{mk-m}$ for every $e \in E\Gamma$. Then

$$h_{\mathcal{L}_0}(X) = \frac{m(k-1)}{m-2}\log(m-1).$$

Proof. A direct check shows that for $s_0 = \frac{m(k-1)}{m-2}\log(m-1)$ we have

$$A_{\mathcal{L}_0}(s_0)\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}.$$

Therefore the Perron-Frobenius eigenvalue of $A_{\mathcal{L}_0}(s_0)$ is equal to 1 and hence $h(\mathcal{L}_0) = s_0$, as claimed. $\square$

Note that for the case $m = 3$, corresponding to regular trivalent graphs, we have $\frac{mk-m}{m-2}\log(m-1) = (3k-3)\log 2$. We are now ready to prove part (1) of Theorem B from the Introduction.

Theorem 9.6. *Let F_k be a free group of finite rank $k \geq 2$, and let $\mathcal{M}_k := CV(F_k)/Out(F_k)$ be the moduli space. For the function $\overline{h} : \mathcal{M}_k \to \mathbb{R}$ we have*

$$\min \overline{h} = (3k-3)\log 2.$$

This minimum is realized by any regular trivalent connected graph Γ with $\pi_1(\Gamma) \cong F_k$ (so that Γ has $3k-3$ non-oriented edges), where each edge of Γ is given length $1/(3k-3)$.

Proof. The moduli space $\mathcal{M}_k$ is a union of finitely many open simplices, corresponding to taking all volume-one metric structures on all the possible minimal graphs with fundamental group free of rank k.

Let $(\Gamma, \mathcal{L}) \in \mathcal{M}_k$. Then $(\Gamma, \mathcal{L})$ can be approximated in $\mathcal{M}_k$ by trivalent metric graphs. By Proposition 9.4 for all of these trivalent graphs the volume entropy is $\geq (3k-3)\log 2$. Since $\overline{h}$ is continuous on $\mathcal{M}_k$, it follows that $\overline{h}(\mathcal{L}) \geq (3k-3)\log 2$ as well. Together with Lemma 9.5 this implies the conclusion of Theorem 9.6.

$\square$

The following is part (3) of Theorem B from the Introduction.

Theorem 9.7. *Let F_k be a free group of finite rank $k \geq 2$, and let $\mathcal{M}_k := CV(F_k)/Out(F_k)$ be the moduli space. Then*

$$\sup_{\mathcal{M}_k} \overline{h} = \infty.$$

Proof. First note that it suffices to prove the statement of the theorem for $k = 2$. Indeed, suppose we know that $\sup_{\mathcal{M}_2} \overline{h} = \infty$ and let $k > 2$ be arbitrary.

Let $(\Gamma, \mathcal{L})$ be a finite volume-one connected metric graph with $\pi_1(\Gamma) \cong F_2$. Let $X = \widetilde{(\Gamma, p)}$, where p is a vertex of Γ, and let X be endowed with a metric $d_{\mathcal{L}}$ induced by $\mathcal{L}$. Denote by d the corresponding metric on ∂X.

Put Γ_1 to be the wedge at p of the graph Γ with $k - 2$ loop-edges. Let $\mathcal{L}_1$ be the metric structure on Γ_1 where each of the new loop-edges is given length $\frac{1}{2(k-2)}$ and where $\mathcal{L}_1$ restricted to Γ is $\mathcal{L}/2$. Then $\mathcal{L}_1$ has volume one and $\pi_1(\Gamma_1, p) \cong F_k$.

Let $X_1 = \widetilde{(\Gamma_1, p)}$, endowed with the induced metric $d_{\mathcal{L}_1}$. Denote by d_1 the corresponding metric on ∂X_1. Then X_1 contains an isometrically embedded copy of $(X, \frac{d_{\mathcal{L}}}{2})$ and hence $(\partial X_1, d_1)$ contains an isometrically embedded copy of $(\partial X, d^{1/2})$. Taking the square root of a metric doubles the Hausdorff dimension and therefore

$$h(X_1) \geq 2h(X).$$

In particular, $h(X) \to \infty$ implies $h(X_1) \to \infty$.

Thus we may assume that $k = 2$. Let Γ be the wedge of two loop-edges at a single vertex. Denote $E\Gamma = \{g, \bar{g}, f, \bar{f}\}$. Let $\mathcal{L}$ be a volume-one metric structure on Γ and denote $x = L(g), y = L(f)$, so that $x + y = 1$ and $x, y > 0$. Then $\overline{h}(\mathcal{L})$ is the unique number $s > 0$ such that $\Phi_{\mathcal{L}}(s) = 1$. The condition $\Phi_{\mathcal{L}}(s) = 1$ is equivalent to the existence of a positive vector $Y > 0$ such that $A_{\mathcal{L}}(s)Y = Y$.

The symmetry considerations imply that $Y_g = Y_{\bar{g}}$ and $Y_f = Y_{\bar{f}}$. Denote $a = Y_g = Y_{\bar{g}}$ and $b = Y_f = Y_{\bar{f}}$. Then the system $A_{\mathcal{L}}(s)Y = Y$ is:

$$\begin{cases} e^{-sx}(a + 2b) = a, \\ e^{-sy}(b + 2a) = b, \end{cases}$$

Up to re-scaling we may assume $b = 1$, so that the above system transforms into the equation

$$4 = (e^{sx} - 1)(e^{sy} - 1). \tag{\#}$$

Since the volume is equal to one we have $y = 1 - x$. For $0 < x < 1$ denote by $s(x)$ the unique value $s > 0$ such that the equation (#) holds.

We claim that $s(x) \to \infty$ as $x \to 0+$. Indeed, suppose not. Then there exists a sequence $x_n > 0$, with $\lim\limits_{n\to\infty} x_n = 0$ such that for the corresponding values $s_n = s(x_n)$ we have $s_n \leq M$, where $0 < M < \infty$. Also, denote $y_n = 1 - x_n$. Then $e^{s_n y_n} - 1 \leq e^M - 1 =: K$. Since $0 < s_n \leq M$ and $\lim_{n\to\infty} x_n = 0$, we have $\lim_{n\to\infty} e^{s_n x_n} - 1 = 0$. Therefore there exists $m > 1$ such that $0 < e^{s_m x_m} - 1 < 1/K$. Together with $0 < e^{s_m y_m} - 1 \leq K$ this implies

$$(e^{s_m x_m} - 1)(e^{s_m y_m} - 1) \leq K \cdot (1/K) = 1 < 4,$$

yielding a contradiction. □

10. Uniqueness of critical points for volume entropy

In this Section we will compute the derivative of the volume entropy function h (see also [39] for another proof) and, as a consequence, prove Part (2) of Theorem B.

Convention 10.1. For the remainder of this section, unless specified otherwise let Γ be a finite connected graph without degree-one vertices and with the fundamental group free of rank $k \geq 2$. Let n be the number of oriented edges of Γ and let

$N = n/2$ be the number of non-oriented edges of Γ. Let Q be the set of all volume-one positive quasi-metric structures on Γ. We identify Q with the open simplex of dimension $n-1$ in $\mathbb{R}^n$:

$$Q = \{(L_1, \dots, L_n) \in \mathbb{R}^n : \sum_{i=1}^{n} L_i = 2, \text{ and } L_i > 0 \text{ for } i = 1, \dots, n\}.$$

We will be using the notations from Section 5. The proof of Proposition 5.6 shows that h extends to a smooth function $s : V \to \mathbb{R}$ where V is an open neighborhood of Q in $\mathbb{R}^n$. Recall the notation $s = s(\overline{L})$ from Corollary 5.7.

We are now going to compute the partial derivatives $\frac{\partial s}{\partial L_i}$. For that we will use the proof of Proposition 5.6 and the following statement from Seneta's book [43]:

Proposition 10.2 (see Theorem 1.5 in [43])**.** *Let $A \geq 0$ be a nonzero irreducible $n \times n$ matrix. Then*

$$adj(r(A)I_n - r(A)) = \epsilon Y Z,$$

where Z and Y are left and right Perron-Frobenius eigenvectors of A accordingly and where $\epsilon \in \{-1, 1\}$.

Proposition 10.3. *Let $\mathcal{L} = (L_1, \dots, L_n) \in Q$. Then, at the point $\mathcal{L}$, for $i = 1, \dots, n$ we have*

$$\frac{\partial s}{\partial L_i} = s b_i e^{-sL_i} = 2\epsilon s z_i ||Y||^2 e^{-sL_i} \det(J)^{-1},$$

where $Z = [z_1, \dots, z_n]$ is a right Perron-Frobenius eigenvector of $A = A_{\mathcal{L}}(s)$; where J and Y are as in the proof of Proposition 5.6; where $[b_1 \dots b_n \; b_{n+1}]$ is the last row of J^{-1} and where $\epsilon \in \{-1, 1\}$.

Proof. Recall that in the proof of Proposition 5.6 the function $s : V \to \mathbb{R}$ is defined via the Implicit Function Theorem applied to the system of equations (!). By the Implicit Function Theorem via differentiating (!) we also have at $\mathcal{L} \in Q$:

$$\frac{\partial(y_1, \dots, y_n, s)}{\partial(L_1, \dots, L_n)} = -\left[\frac{\partial(F_1, \dots, F_n, F_{n+1})}{\partial(y_1, \dots, y_n, s)}\right]^{-1} \frac{\partial(F_1, \dots, F_n, F_{n+1})}{\partial(L_1, \dots, L_n)} =$$

$$= -J^{-1} \begin{bmatrix} -se^{-sL_1} & 0 & 0 & \dots & 0 \\ 0 & -se^{-sL_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -se^{-sL_n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

where

$$J = \begin{bmatrix} a_{11} - 1 & a_{12} & a_{13} & \dots & a_{1n} & -L_1 y_1 \\ a_{21} & a_{22} - 1 & a_{23} & \dots & a_{2n} & -L_2 y_2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} & -L_i y_i \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - 1 & -L_n y_n \\ 2y_1 & 2y_2 & 2y_3 & \dots & 2y_n & 0 \end{bmatrix}.$$

Thus to compute $\frac{\partial s}{\partial(L_1,\ldots,L_n)}$ we need to know the last row $[b_1 \ldots b_n\ b_{n+1}]$ of J^{-1}. Then

$$\frac{\partial s}{\partial L_i} = s b_i e^{-sL_i} \tag{+}$$

for $i = 1, \ldots, n$. Note that the value b_{n+1} is actually not needed since the last row of $\frac{\partial(F_1,\ldots,F_n,F_{n+1})}{\partial(L_1,\ldots,L_n)}$ consists entirely of zeros.

Since $J^{-1} = \det(J)^{-1} adj(J)$, we have:

$$b_i = -\det(J)^{-1} \det \begin{bmatrix} a_{11}-1 & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22}-1 & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 2y_1 & 2y_2 & 2y_3 & \ldots & 2y_n \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}-1 \end{bmatrix} \tag{!!}$$

where the ys occur in the i-th row. Recall that $Y = [y_1, y_2 \ldots, y_n]^T$ is a right Perron-Frobenius eigenvector of $A = A_{\mathcal{L}}(s)$. By Proposition 10.2 we have $adj(A - I_n) = -\epsilon YZ$ where $Z = [z_1, \ldots, z_n]$ is a left Perron-Frobenius eigenvector of A. Denote $R = adj(A - I_n)$. Then, by taking the i-th row decomposition of the determinant in (!!) we see that b_i is equal to the scalar product of $-\det(J)^{-1}[2y_1, \ldots, 2y_n]$ and the i-th column of R. Since $R = -\epsilon YZ$, the i-th column of R is $-\epsilon z_i Y$. Hence $b_i = 2\epsilon z_i ||Y||^2 det(J)^{-1}$.

Then by (+) we have $\frac{\partial s}{\partial L_i} = s b_i e^{-sL_i} = 2\epsilon s z_i ||Y||^2 e^{-sL_i} \det(J)^{-1}$, as required. □

Proposition 10.4. *Suppose that Γ is m-regular for some $m \geq 3$. Let $\mathcal{L} \in Q$ be a critical point of $s|_Q$. Then $L_1 = L_2 = \cdots = L_n$.*

Proof. By the Lagrange multipliers method at a critical point of $s|_Q$ we have that the gradient of s is parallel to the vector $(1, 1, \ldots, 1)$, that is, at such a point

$$\frac{\partial s}{\partial L_1} = \frac{\partial s}{\partial L_2} = \cdots = \frac{\partial s}{\partial L_n}. \tag{!!!}$$

We do know by the earlier computations that
$\frac{\partial s}{\partial L_i} = s b_i e^{-sL_i} = 2\epsilon s z_i ||Y||^2 e^{-sL_i} \det(J)^{-1}$. Then by (!!!) at a critical point $\mathcal{L}$ of $s|_Q$ we have

$$z_1 e^{-sL_1} = z_2 e^{-sL_2} = \cdots = z_n e^{-sL_n},$$

where $Z = [z_1, z_2, \ldots, z_n]$ is a left Perron-Frobenius eigenvector of A. Up to rescaling we can choose Z so that the first coordinate of Z is e^{sL_1}. Then by the above equation $Z = [e^{sL_1}, e^{sL_2}, \ldots, e^{sL_n}]$.

Now we use the fact that $ZA = Z\, diag(e^{-sL_i})M = Z$ which translates into $[1, 1, ..., 1]M = Z$. Since Γ is m-regular, in every column of M there are exactly $m - 1$ nonzero entries, each equal to 1. Then $[1, 1, ..., 1]M = Z$ translates into $z_1 = \cdots = z_n = m - 1$. Since we have chosen Z so that $z_i = e^{sL_i}$, this implies $L_1 = L_2 = \cdots = L_n$, as required. □

The following statement is a corollary of a result obtained by Robert [40] and Rivin [39].

Proposition 10.5. *Let Γ be a finite connected graph without degree-one vertices with $\pi_1(\Gamma)$ free of rank ≥ 2 and with N non-oriented edges.*

Let $U = U(\Gamma)$ be the set of all metric graph structures on Γ. Thus U is identified with

$$U = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i > 0, i = 1, \dots, N\}.$$

Then the volume entropy $h : U \to \mathbb{R}$ is a convex function on U.

The following is part (2) of Theorem B from the Introduction.

Theorem 10.6. *Let Γ be a finite connected graph without degree-one and degree-two vertices and with fundamental group free of rank $k \geq 1$. Let $\mathcal{L}$ be a volume-one metric structure such that $h(\mathcal{L}) = (3k-3)\log 2$. Then Γ is a regular trivalent graph and all edges of Γ have equal lengths in $\mathcal{L}$.*

Proof. **Case 1.** Suppose first that Γ is trivalent. Then Proposition 9.4 implies that $\mathcal{L}$ is a point of minimum of $s|_Q$. Therefore $\mathcal{L}$ is a critical point of $s|_Q$. Hence by Proposition 10.4 all edges have equal lengths in $\mathcal{L}$.

Case 2. Suppose now that Γ is not trivalent. Then there exists a trivalent graph Γ' and a sequence of volume-one metric structures $\mathcal{L}'_t$ on Γ' such that $(\Gamma', \mathcal{L}'_t)$ converges to $(\Gamma, \mathcal{L})$ in the moduli space $CV(F_k)/Out(F_k)$ as $t \to \infty$. Moreover, since Γ is not trivalent, there is an edge e of Γ' whose $\mathcal{L}_t$-length converges to 0 as $t \to \infty$, so that, after possibly passing to a subsequence, the points $\mathcal{L}'_t$ converge to a point in the boundary of the open simplex $U(\Gamma')$.

By continuity of h we have $\lim_{t\to\infty} h(\mathcal{L}'_t) = (3k-3)\log 2$. Let $\mathcal{L}_0$ be the volume-one metric structure on Γ' where all edges have equal length. Let S be the set of all points in $U = U(\Gamma)$ at distance $\epsilon > 0$ where ϵ is smaller than the distance from $\mathcal{L}_0$ to the boundary of U. For sufficiently large t the segments $[\mathcal{L}_0, \mathcal{L}'_t]$ intersect S in one point denoted $\mathcal{L}''_t$. By convexity of h we have $h(\mathcal{L}''_t) \leq \max\{h(\mathcal{L}_0), h(\mathcal{L}'_t)\}$. Since S is compact, after passing to a subsequence we have $\lim_{t\to\infty} \mathcal{L}''_t = \mathcal{L}'' \in S$. Since $\lim_{t\to\infty} h(\mathcal{L}'_t) = h(\mathcal{L}_0) = (3k-3)\log 2$, it follows that $h(\mathcal{L}'') \leq (3k-3)\log 2$ and therefore $h(\mathcal{L}'') = (3k-3)\log 2$ by Theorem 9.6. By Case 1 all edges of $\mathcal{L}''$ have equal length, which contradicts the fact that $\mathcal{L}'' \neq \mathcal{L}_0$. □

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