Abelian Sandpile Model on Randomly Rooted Graphs and Self-Similar Groups

M. Matter ∗† and T. Nagnibeda ∗‡

May 20, 2011

Abstract

The main result of this paper is a rigorous proof of criticality and an explicit computation of critical exponents for the decay of avalanches in the Abelian sandpile model (ASM) on finite approximations of a large family of infinite graphs. We begin by introducing the notion of criticality of the ASM for limits of finite graphs in local convergence, which naturally leads to the question about criticality of the ASM in the random weak limit. Our main technical ingredient is a sufficient condition for almost sure criticality of the ASM on sequences of finite cacti (i.e., separable graphs whose blocks are cycles or single edges) under the assumption that the random weak limit is almost surely 1-ended. Examples that allow explicit computations of the critical exponents come from actions of finitely generated groups on regular rooted trees, by automorphisms. Restricting the action to the consecutive levels of the tree defines a sequence of finite graphs whose limits in the local convergence are orbital Schreier graphs for the action of the group on the boundary of the tree. In the case of iterated monodromy groups of complex polynomials, these graphs are cacti, and we show that for 1-ended ones, the critical exponent for the decay of the mass of avalanches depends on the growth of the graph. The well-known Basilica group related to $z^2 - 1$ gives rise to uncountably many 4-regular one-ended graphs of quadratic growth with the critical exponent for the mass of avalanches equal to 1; as well as uncountably many new non-critical examples of quadratic growth (thus not quasi-isometric to $\mathbb{Z}$). Another iterated monodromy group that we consider provides uncountably many graphs with the critical exponent equal to $2\log 2 / \log 3 > 1$. Finally, we also exhibit graphs of polynomial growth with arbitrarily small critical exponent.

keywords: Abelian sandpile model, critical exponent, avalanche, random weak limits of graphs, self-similar group, Schreier graph, polynomial growth.

2010 Mathematics Subject Classification: Primary: 60K35; Secondary: 82C22, 20E08.

1 Introduction

The Sandpile Model was introduced in the late eighties by physicists Bak, Tang and Wiesenfeld [3] in the aim of constructing an analytically tractable model of a phenomenon often observed in nature and called self-organized criticality. Its mathematical study was initiated by Dhar in [14]; in particular, he proved that the model is abelian. This result was also recovered independently in the work of Björner, Lovasz and Schor [7] where the same model was studied under the name of chip-firing game on graphs. A detailed treatment of the ASM can be found in [28], [15], [34].

A configuration is a distribution of an amount of chips (or of grains of sand) on the vertices of a

∗The authors acknowledge the support of the Swiss National Science Foundation Grant PP0022-118946.
†Section de mathématiques, 2-4 Rue du Lièvre CP 64, 1211 Genève 4, Michel.Matter@unige.ch
‡Section de mathématiques, 2-4 Rue du Lièvre CP 64, 1211 Genève 4, Tatiana.Smirnova-Nagnibeda@unige.ch
The term **abelian** stands for the following convenient feature of the model: the order in which we stabilize unstable vertices of a configuration does not affect the result [14]. In order to be sure that any unstable configuration will eventually stabilize (at least in the case of a finite graph), we choose a number of vertices to be **dissipative**: a chip reaching such a vertex leaves the game.

Once a stable configuration is reached, the game can be reactivated by adding an extra chip on a randomly chosen vertex. In the case of a finite graph, this defines an irreducible Markov chain whose stationary distribution is the uniform distribution supported by the unique recurrent class (more details in Section 2.1 below). Recurrent configurations can also be characterized in a deterministic way by the so-called **Burning Algorithm** [14] (Theorem 2.1.1 below), establishing a bijection between recurrent configurations and spanning trees in the graph.

The dynamics of the model is described by **avalanches**, that is, sequences of consecutive firings triggered by adding an extra chip on some vertex to a recurrent configuration. Given a growing sequence of subgraphs $\{\Gamma_n\}_{n \geq 1}$ of an infinite graph $\Gamma$, **criticality** of the ASM on $\Gamma$ is manifested in that various spatial statistics associated with avalanches (such as their mass, length, amplitude, etc.) decay asymptotically according to a power law (with a cut-off), as $\Gamma_n \nearrow \Gamma$. More precisely, if $X$ is a random variable encoding some feature of an avalanche and $\mu_n$ is the uniform probability distribution on the recurrent configurations on $\Gamma_n$, then it is expected that in criticality, for large values of $X$, $\lim_{n \to \infty} P_{\mu_n}(X) \sim X^{-\delta_X}$ for some critical exponent $\delta_X$. Although many numerical simulations have been done in order to exhibit criticality of ASM on lattices, as well as to determine various critical exponents, there are only very few rigorously proven cases so far.

In the case of the Bethe lattice, Dhar and Majumdar proved in [16] that the critical exponent corresponding to the mass $M$ of an avalanche is $\delta_M = 3/2$ in large volume limit. On the one-dimensional lattice $\mathbb{Z}$, the probability of observing an avalanche of mass $M > 0$ (respectively length $L > 0$) on a segment of length $n$ is independent of $M$ (respectively $L$) and behaves as $n^{-1}$ (respectively $n^{-2}\log\log n$) as $n$ tends to infinity [36], so no critical behaviour is observed in this case.

Numerical experiments as well as non-rigorous scaling arguments yield the conjecture that on the two-dimensional lattice $\mathbb{Z}^2$, the critical exponent for the mass of an avalanche is $\delta_M = 5/4$, [33], whereas for $d > 4$ this critical exponent is expected to be $3/2$, by universality [32]. Let us also mention a result of Majumdar and Dhar confirming criticality of ASM on $\mathbb{Z}^d$: it is proven in [17] that the correlation between the indicator functions of having one chip on vertex 0 and one chip on a vertex $x \in \mathbb{Z}^d$ behaves as $C|x|^{-2d}$ in large volume limit.

Another family of graphs on which extensive simulations of avalanches have been performed is the Sierpiński gasket where it is shown that $\delta_M \approx 1.46$ [12]. See also [23].

Typically, in all these examples, the passage from finite to infinite volume is realized by exhaustion: the infinite graph is approximated by a growing family of finite subgraphs $\{\Gamma_n\}_{n \geq 1}$. For every finite subgraph $\Gamma_n$, vertices situated on the boundary are taken to be dissipative. In this paper we propose to study the asymptotic properties of the ASM in a more general context of limits of finite graphs. Namely, we will be working in the space $X$ of (rooted isomorphism classes of) connected **rooted graphs** with uniformly bounded degree. The convergence in this space can be formulated in terms of isomorphisms of balls of growing radius centered in the roots, which is an easy case of the pointed Gromov-Hausdorff convergence ([22], Chapter 3), see Subsection 2.2 for a precise definition. In Subsection 2.2 below, we introduce the ASM on sequences of graphs converging in this sense and discuss such issues as the choice of dissipative vertices, the choice of root, criticality, etc. Note that the usual treatment of the ASM on increasing families of finite graphs exhausting an infinite graph is a particular case of the setup discussed here.

Our viewpoint allows us to exhibit a host of examples of infinite graphs for which we can explicitly compute the critical exponent for the decay of avalanches (on the approximating sequence of finite graphs). Our first examples share many properties with $\mathbb{Z}^2$, namely they are 4-regular graphs of
quadratic growth and with one end. We prove that the critical exponent is 1. Another family of examples consists of 4-regular one-ended graphs of polynomial growth of degree $\log 3/\log 2$, with the critical exponent equal to $2 \log 2/\log 3$.

The proposed approach might present a drawback if one wants to study the ASM on a sequence of finite graphs that do not have one distinguished choice of root (it is for example the case for Schreier graphs of self-similar groups studied in this paper). A natural way to overcome this is to choose the root uniformly at random, thus considering each unrooted graph $\Gamma_n$ in the sequence as a probability distribution $\rho_n$ on $\mathcal{X}$. Random choice of root combined with the convergence in the space of (rooted isomorphism classes of) connected rooted graphs of bounded degree is exactly what lies behind the notion of random weak limit (or distributional limit) introduced by Benjamini and Schramm in [5].

The random weak limit of a sequence $\{\Gamma_n\}_{n \geq 1}$ of connected graphs of bounded degree is defined to be the weak limit of the measures $\rho_n$ in the space of probability measures on $\mathcal{X}$. The random weak limit is thus a probability distribution on the space $\mathcal{X}$ of rooted graphs, supported by the set of the limits in $\mathcal{X}$ of $\{(\Gamma_n, v_n)\}_{n \geq 1}$ for all possible choices of the roots $v_n$.

In this setting, it then becomes natural to ask the question about criticality of the ASM on the sequence of graphs $\{\Gamma_n\}_{n \geq 1}$ in the random weak limit, more on that in Subsection 2.2.

A natural source of families of finite graphs with interesting infinite limits of self-similar nature is the theory of self-similar groups developed in the past ten years by Grigorchuk, Nekrashevych and others (see [19], [31] and references therein). More precisely, a finitely generated group $G < \text{Aut}(T)$ acting by automorphisms on a regular rooted tree $T$ defines a covering sequence $\{\Gamma_n\}_{n \geq 1}$ of finite Schreier graphs describing the action of $G$ on each level of the tree. These graphs converge, in the space $\mathcal{X}$ of rooted graphs, to infinite graphs $\{\Gamma_\xi\}_{\xi \in \partial T}$, that form a graphing of the orbital equivalence relation for the limit action of $G$ on the boundary $\partial T$ of the tree. The random weak limit of $\{\Gamma_n\}_{n \geq 1}$ is the image of the uniform distribution on the boundary of the tree under the identification of rooted-isomorphic copies of $(\Gamma_\xi, \xi), \xi \in \partial T$.

One eminent example in the class of self-similar groups is the so-called Basilica group introduced by Grigorchuk and Żuk. They proved in [21] that this group is not subexponentially amenable, that is cannot be constructed from groups of subexponential growth by taking extensions and direct limits, whereas Bartholdi and Virág [4] later showed that the group is amenable. The Basilica group can also be realized as the iterated monodromy group of the complex polynomial $z^2 − 1$, which means in particular that its Schreier graphs form an approximating sequence of the Julia set of $z^2 − 1$, the so-called Basilica fractal [31]. It is a 2-generated group which acts by automorphisms on the binary tree.

Limits of Schreier graphs $\Gamma_n$ of the Basilica group in the space of (rooted isomorphism classes of) rooted graphs are explicitly described and classified in [13]. There exist uncountably many non-isomorphic limits $\Gamma_\xi$, so that the random weak limit of $\{\Gamma_n\}_{n \geq 1}$ is a continuous distribution. The limit graph $(\Gamma_\xi, \xi)$ has 1, 2 or 4 ends depending on the infinite word $\xi$ (see Theorem 4.0.7), the case of 1 end being generic.

In this paper, we consider the ASM on Schreier graphs of the Basilica group and show the following:

**Theorem 1.0.1.** Let $\xi \in \{0, 1\}^\omega$ and let $\xi_n$ denote the prefix of $\xi$ of length $n$ so that the sequence of finite Schreier graphs $(\Gamma_n, \xi_n)$ converges to the limit $(\Gamma_\xi, \xi)$ in the space of rooted graphs. Then, almost surely with respect to the random weak limit of $\Gamma_n$’s, the ASM is critical, in the sense that the probability distribution of the mass of an avalanche on $(\Gamma_n, \xi_n)$ obeys a power-law decay as $n \to \infty$ (see Definition 2.2.3). More precisely,

- if the limit $(\Gamma_\xi, \xi)$ has one end, then the probability distribution of the mass of an avalanche on $(\Gamma_n, \xi_n)$ almost surely has a power-law decay as $n \to \infty$, with critical exponent $\delta = 1$ (see Theorem 5.3.1);
- if the limit $(\Gamma_\xi, \xi)$ has either two or four ends, then the probability distribution of the mass of an avalanche on $(\Gamma_n, \xi_n)$ does not have a power-law decay as $n \to \infty$ (see Theorems 5.1.2 and 5.2.1).
We thus exhibit dependence of the asymptotic of avalanches on the number of ends of the limit graph. We also obtain an uncountable family of 4-regular one-ended graphs of quadratic growth [8] where the critical exponent is explicitly computed (= 1). On the other hand, we also obtain an uncountable family of 2-ended graphs of quadratic growth (thus not quasi-isometric to $\mathbb{Z}$) with non-critical ASM.

Technically, our approach relies on the fact that the Schreier graphs of the Basilica group are cacti graphs: these are separable graphs whose blocks are either cycles or single edges (see Section 2.4). A general treatment of the ASM on sequences of cacti graphs is addressed in [27].

The groups of automorphisms of rooted trees whose Schreier graphs are cacti graphs, and to which our method therefore applies, form a large class of groups characterized by Nekrashevych as iterated monodromy groups of post-critically finite backward iterations of topological polynomials [29]. Another example from this class of groups is the so-called “interlaced adding machines”, or the $IMG(-z^3/2 + 3z/2)$ described by Nekrashevych in [30]. This group shares many properties with the Basilica group, and the same goes for their Schreier graphs. This group provides examples of graphs with the critical exponent $2 \log 2 / \log 3 > 1$ (see Section 6).

**Theorem 1.0.2.** Let $\xi \in \{0, 1\}^\omega$ and let $\xi_n$ denote the prefix of $\xi$ of length $n$ so that the sequence of finite Schreier graphs $(\Gamma_n, \xi_n)$ of the group $IMG(-z^3/2 + 3z/2)$ converges to the orbital Schreier graph $(\Gamma_\xi, \xi)$ in the space of rooted graphs. Then, almost surely with respect to the random weak limit of $\Gamma_n$’s, the ASM is critical, in the sense that the probability distribution of the mass of an avalanche on $(\Gamma_n, \xi_n)$ almost surely has a power-law decay as $n \to \infty$, with critical exponent $\delta = 2 \log 2 / \log 3$ (see Theorem 6.2.1).

Both examples of the Basilica group and the interlaced adding machines group satisfy the conditions of our Theorem 7.1.1 that establishes a connection between the critical exponent for the mass of avalanches in the ASM on one-ended Schreier graphs and the degree of their polynomial growth. Quadratic polynomials $z^2 + c$ with the values of $c$ taken in smaller and smaller hyperbolic components attached to the main cardioid of the Mandelbrot set provide examples of iterated monodromy groups whose Schreier graphs have polynomial growth of arbitrarily high degree. Consequently, probability distributions of the mass of avalanches on these Schreier graphs decay as power laws with arbitrarily small critical exponent. These examples are discussed in Subsection 7.2.

Let us mention that all graphs considered in this paper are recurrent, as orbital Schreier graphs of groups generated by bounded automata (see Theorem V.24 in [8]). Note also that random weak limits of planar graphs (and cacti are planar), are almost surely recurrent, by a result of Benjamini and Schramm [5]. Also, all graphs considered here are non-transitive.

The paper is structured as follows: in Section 2, we collect some facts and notations about the ASM and then consider general properties of the model on separable graphs and, in particular, cacti graphs. In Subsection 2.2, we introduce and discuss the ASM on sequences of graphs converging in the space $X$ of rooted graphs, as well as criticality of the ASM in the random weak limit. Section 3 recalls basic notions about groups of automorphisms of rooted trees, self-similar groups and their Schreier graphs. We show that any covering sequence of finite regular graphs of even degree can be realized as Schreier graphs for an action of a finitely generated group on a spherically homogeneous rooted tree, by automorphisms. In Subsection 3.2, we go back to the study of avalanches in the random weak limit and show that for covering sequences of regular cacti critical in the random weak limit the critical exponent is constant in the random weak limit. Section 4 recalls results from [13] about the structure of finite and infinite Schreier graphs of the Basilica group. In Section 5, we prove Theorem 1.0.1, in particular we show that almost all orbital Schreier graphs of the Basilica group are critical with the critical exponent equal to 1. In Section 6, we consider the group generated by two interlaced adding machines and exhibit examples with the critical exponent equal to $2 \log 2 / \log 3 > 1$. In Section 7, a relation is established between the critical exponent for the mass of avalanches and the degree of polynomial growth, for 1-ended limits of sequences of cacti; and graphs with arbitrarily small critical exponents are discussed.
2 Abelian Sandpile Model

2.1 Chip-firing Game on a graph

Let $\Gamma = (V, E)$ be a finite connected graph, possibly with multiple edges and loops, with a vertex set $V \equiv V(\Gamma)$ and an edge set $E \equiv E(\Gamma)$. Let $P \subset V$ be a non-empty set of vertices that will be called dissipative vertices. We will write $V_0 := V \setminus P$. A configuration on $\Gamma$ is a function $\eta : V_0 \rightarrow \mathbb{N}$. We say that $\eta$ is stable if $\eta(v) < \deg(v)$ for all $v \in V_0$ where $\deg(v)$ denotes the degree of $v$, that is, the number of edges incident to $v$ (each loop contributes two to the degree.) An unstable configuration evolves by firing its unstable vertices as long as there are some. Firing an unstable vertex $v$ corresponds to sending one chip along each edge incident to $v$ to the corresponding neighbour. We will adopt the convention that all chips reaching a dissipative vertex $p \in P$ leave the graph. The basic theorem about the game asserts that every configuration reaches through a finite number of firings a stable configuration. Moreover, the resulting stable configuration, the set of vertices fired in the stabilization and the number of times each of these vertices were fired are all independent of the order in which the unstable vertices are fired [14]. Given a configuration $\eta$, a consecutive sequence of firings resulting in the stabilization of $\eta$ is called an avalanche. The number of vertices (respectively distinct vertices) fired during the avalanche is called its length (respectively mass). By the result cited above, both the mass and the length are the same for all avalanches leading to the stabilization of a given configuration.

Let $\Omega$ denote the set of all stable configurations, and let us consider the following Markov chain on $\Omega$ [28]. Starting from some initial stable configuration $\eta_0$, we add an extra chip to $\eta_0$ on a vertex $v \in V_0$ chosen accordingly to some initially fixed probability distribution $\pi : V_0 \rightarrow [0, 1]$ satisfying the condition $\pi(v) > 0$ for all $v \in V_0$. Then, we let the configuration $\eta_0 + \delta_v$ stabilize and denote by $\eta_1 \in \Omega$ the resulting stable configuration. We then repeat the previous operation with $\eta_1$, and so on. This Markov chain is irreducible and its stationary measure $\mu$ is supported by a unique class of recurrent (or critical) configurations, which is independent of the initial distribution $\pi$ and is denoted by $\mathcal{R}_\Gamma$.

It turns out that recurrent configurations can be characterized deterministically by the so-called Burning Algorithm [14]. Given a configuration $\eta$ and a subgraph $H \subset \Gamma$ not containing dissipative vertices, we say that the restriction $\eta_H$ is a forbidden sub-configuration of $\eta$ if $\eta(v) < \deg_H(v)$ for every $v \in H$ (where $\deg_H(v)$ denotes the degree of $v$ in $H$). Dhar has shown that a stable configuration on $\Gamma$ is recurrent if and only if it does not contain any forbidden sub-configuration. The Burning Algorithm decides, given a configuration $\eta$, whether it contains a forbidden sub-configuration or not, as follows. For $t \geq 1$, we define inductively the sets $B_t$ and $U_t$, where $B_t$ stands for the set of vertices “burnt” at time $t$, and $U_t$ stands for the set of vertices “un-burnt” up to time $t$. We also denote by $\Gamma_t$ the subgraph of $\Gamma$ spanned by the vertices $U_t$, whereas $\Gamma_0$ denotes the subgraph of $\Gamma$ spanned by $V_0$.

$$B_1 := \{ v \in V_0 | \eta(v) \geq \deg_{\Gamma_0}(v) \};$$
$$U_1 := \Gamma_0 \setminus B_1;$$
$$U_t := \Gamma_0 \setminus \bigcup_{s=1}^{t-1} B_s;$$
$$B_{t+1} := \{ v \in U_t | \eta(v) \geq \deg_{\Gamma_t}(v) \}.$$ 

If there exists $t_0$ such that $B_{t_0}$ is empty, then $\eta_{U_{t_0}}$ is forbidden. Otherwise, every vertex of $\Gamma$ is eventually burnt, which implies that $\eta$ does not contain any forbidden configuration.

We will use the following equivalent reformulation of Dhar’s theorem [28].

**Theorem 2.1.1.** A configuration $c$ on $\Gamma$ is recurrent if and only if there exists a sequence of firings $p_1 \ldots p_k v_1 \ldots v_{|V_0|}$ which is an enumeration of $V$. Here firing a dissipative vertex $p_i$ means that we add on each neighbour $v$ of $p_i$ as many chips as there are edges between $v$ and $p_i$ in $\Gamma$. We call such a sequence a burning sequence for $c$ on $\Gamma$. 

5
2.2 Avalanches on converging sequences of rooted graphs

As explained in the introduction, in this paper we propose to study avalanches on converging sequences of rooted graphs. Let $X$ denote the space (of rooted isomorphism classes) of connected graphs with uniformly bounded degrees and having a distinguished vertex called the root; $X$ can be endowed with the following metric: given two rooted graphs $(\Gamma, v)$ and $(\Gamma', v')$,

$$\text{Dist}((\Gamma, v), (\Gamma', v')) := \inf \left\{ \frac{1}{r+1} : B_\Gamma(v, r) \text{ is isomorphic to } B_{\Gamma'}(v', r) \right\}$$

(1)

where $B_\Gamma(v, r)$ is the ball of radius $r$ in $\Gamma$ centered in $v$. We say that a sequence of rooted graphs $\{(\Gamma_n, v_n)\}_{n\geq 1}$ converges to a limit graph $(\Gamma, v)$ if $\lim_{n \to \infty} \text{Dist}((\Gamma_n, v_n), (\Gamma, v)) = 0$. The assumption of uniformly bounded degrees guarantees that $(X, \text{Dist})$ is a compact metric space.

We shall now proceed to define the Abelian sandpile model on an arbitrary sequence of graphs converging in the space $(X, \text{Dist})$. To begin with, we will explain how to choose dissipative vertices. If the sequence $\{\Gamma_n\}_{n\geq 1}$ is increasing, $\Gamma_n \subseteq \Gamma_{n+1}$, and exhausts $\Gamma$, $\Gamma = \bigcup_{n \geq 1} \Gamma_n$, then the dissipative vertices are usually defined as the boundary of $\Gamma_n$ in $\Gamma$. We will imitate this choice of dissipative vertices for an arbitrary sequence of finite graphs $(\Gamma_n, v_n)$ converging to an infinite graph $(\Gamma, v)$, by declaring the dissipative vertices in $\Gamma_n$ to be the boundary in $\Gamma_n$ of a sequence of subgraphs $H_n \subseteq \Gamma_n$ that form an “exhaustion in the limit”, as defined below. As we will see later, the choice of a particular subsequence can vary depending on the geometry of the graphs.

Definition 2.2.1. Let $\{(\Gamma_n, v_n)\}_{n\geq 1}$ be a sequence of finite graphs converging in $(X, \text{Dist})$ to an infinite graph $(\Gamma, v)$. Suppose that for any $n \geq 1$, there is a subgraph $H_n \subseteq \Gamma_n$ satisfying the following requirements:

1. $H_n$ is connected and $v_n \in H_n$.

2. For any $m \geq n$, there exists a subgraph of $\Gamma_m$ containing $v_m$ and isomorphic to $H_n$. (Consequently, there exists a subgraph $\tilde{H}_n$ of $\Gamma$ containing $v$ and isomorphic to $H_n$.)

3. $\{\tilde{H}_n\}_{n\geq 1}$ is an exhaustion of $\Gamma$.

Then, we say that $\{H_n\}_{n\geq 1}$ is an exhaustion in the limit.
Convention 2.2.2. (Choice of dissipative vertices)

Given a sequence of finite graphs \(\{(\Gamma_n, v_n)\}_{n \geq 1}\) converging in \((X, \text{Dist})\) to an infinite graph \((\Gamma, v)\), and given a sequence of subgraphs \(\{H_n\}_{n \geq 1}\), \(H_n \subset \Gamma_n\) for all \(n\), that is an exhaustion in the limit, set \(P_n\), the set of dissipative vertices in each \(\Gamma_n\), to be the internal boundary of \(H_n\) in \(\Gamma_n\), i.e. the vertices of \(H_n\) that have neighbours in the complement \(\Gamma_n \setminus H_n\).

A natural choice that satisfies these conditions is to take for \(H_n\) the ball \(B_{\Gamma_n}(v_n, r_n) \subset \Gamma_n\) where \(r_n := \max\{r \in \mathbb{N} | B_{\Gamma_n}(v_n, r) \supset B_{\Gamma_n}(v_m, r) \forall m \geq n\}\). But we can also choose other subgraphs than the balls.

We shall next discuss how avalanches are triggered in our model and how criticality will be understood. For all \(n \geq 1\), consider the probability space \((\mathcal{R}_{\Gamma_n}, \mu_n)\) (with the natural \(\sigma\)-algebra) where \(\mu_n\) denotes the uniform distribution on \(\mathcal{R}_{\Gamma_n}\). Define the random variable \(\text{Mass}_{\Gamma_n}(\cdot, v_n) : (\mathcal{R}_{\Gamma_n}, \mu_n) \rightarrow \mathbb{N}\) that maps a recurrent configuration to the mass (i.e. the number of distinct vertices fired) of the avalanche triggered by adding to this configuration an extra chip on the root \(v_n\). In general, the graphs \((\Gamma_n, v_n)\) are non transitive, so that the choice of the vertex on which we add an extra chip to some recurrent configuration, (which is irrelevant in the case of the lattice \(\mathbb{Z}^d\) or of the regular tree, as long as it is far from the boundary), becomes important. In the context of convergence in the space of rooted graphs, it is natural to choose the sequence of roots as the sequence of vertices on which we add an extra chip to a recurrent configuration, (which is irrelevant in the case of the lattice \(\mathbb{Z}^d\) or of the regular tree, as long as it is far from the boundary), becomes important. In the context of convergence in the space of rooted graphs, it is natural to choose the sequence of roots as the sequence of vertices on which we add this extra chip. Together with the choice of dissipative vertices specified above, we thus ensure that the vertex on which we add an extra chip to a recurrent configuration lies farther and farther away from the dissipative vertices, as \(n \rightarrow \infty\).

Definition 2.2.3. We say that the ASM on the sequence of finite rooted graphs \(\{(\Gamma_n, v_n)\}_{n \geq 1}\) converging in \((X, \text{Dist})\) to the infinite limit graph \((\Gamma, v)\) has critical behaviour (with respect to the mass of avalanches) if \(\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_n}(\text{Mass}_{\Gamma_n}(\cdot, v_n) = M) \sim M^{-\delta}\) for some exponent \(\delta > 0\) (called the critical exponent).

Note that, depending on the geometry of the underlying graphs, it may happen that not every integer \(M \geq 1\) can be realized as the mass of an avalanche. In such situations, we restrict our considerations to those integers which can be realized as the mass of an avalanche. In particular, we will write \(\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_n}(\text{Mass}_{\Gamma_n}(\cdot, v_n) = M) \sim M^{-\delta}\) if there exist \(C_1, C_2 > 0\) and \(M_0 \geq 1\) such that \(C_1 M^{-\delta} \leq \lim_{n \rightarrow \infty} \mathbb{P}_{\mu_n}(\text{Mass}_{\Gamma_n}(\cdot, v_n) = M) \leq C_2 M^{-\delta}\) for all \(M \geq M_0\) which occur as the mass of an avalanche.

Proposition 2.2.4. For any \(M > 0\), the limit \(\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_n}(\text{Mass}_{\Gamma_n}(\cdot, v_n) = M)\) exists and does not depend on the dissipative sets, provided they are chosen according to Convention 2.2.2.

Proof. Suppose that we want to study avalanches of mass not greater than some fixed positive integer \(M\). By definition, the set of vertices of \(\Gamma_n\) fired during an avalanche triggered by adding an extra chip on \(v_n\) induces a connected subgraph containing \(v_n\). For every \(n\) large enough, there exists \(r_M\) such that the ball \(B_{\Gamma_n}(v_n, r_M - 1)\) in \(\Gamma_n\) contains all vertices which may be involved in an avalanche of mass not greater than \(M\). Moreover, we can suppose that \(B_{\Gamma_n}(v_n, r_M) \supset B_{\Gamma_n}(v, r_M)\). Note that the distance between \(v_n\) and any set of dissipative vertices satisfying Convention 2.2.2 tends to infinity as \(n \rightarrow \infty\). Thus, we can suppose that the ball \(B_{\Gamma_n}(v_n, r_M)\) does not intersect dissipative vertices.

It follows now from the description of recurrent configurations as configurations without forbidden subconfigurations, that the restriction of any recurrent configuration to \(B_{\Gamma_n}(v_n, r_M - 1)\) is the same for any choice of dissipative vertices satisfying 2.2.2. As the behaviour of avalanches of mass not greater than \(M\) is entirely determined by the subconfigurations on \(B_{\Gamma_n}(v_n, r_M - 1)\), we conclude that the probability of observing an avalanche of mass not greater than \(M\) on \(\Gamma_n\) is independent of \(n\) for every \(n\) large enough, and is the same for any choice of dissipative vertices satisfying 2.2.2.

Remark 2.2.5. Note that the classical setup for studying the ASM, that is a sequence of finite graphs exhausting an infinite (usually transitive) graph, fits into our, more general setup, in the choice of dissipative vertices, as well as in the choice of a vertex that is used to trigger avalanches and thus to define criticality.
Finally, for a sequence of unrooted graphs, we can also formulate the question of criticality of the ASM for random rooting. Recall that, given a sequence \( \{\Gamma_n\}_{n \geq 1} \) of finite unrooted graphs, \( \rho \) is the random weak limit of \( \{\Gamma_n\}_{n \geq 1} \) if the sequence \( \{\rho_n\}_{n \geq 1} \) converges weakly to \( \rho \) where, for every \( n \), \( \rho_n \) is the probability distribution on \( \mathcal{X} \) induced by choosing a root in \( \Gamma_n \) uniformly at random [5].

**Definition 2.2.6.** Given a sequence \( \{\Gamma_n\}_{n \geq 1} \) of finite unrooted graphs, we will say that the ASM on the sequence \( \{\Gamma_n\}_{n \geq 1} \) is critical in the random weak limit (with critical exponent \( \delta \)) if it is critical (with critical exponent \( \delta \)) on almost every (with respect to the random weak limit \( \rho \) of \( \{\Gamma_n\}_{n \geq 1} \)) sequence \( \{(\Gamma_n, v_n)\}_{n \geq 1} \) of rooted graphs converging to an infinite limit graph \( (\Gamma, v) \).

**Example 2.2.7.** *Infinite regular tree.* In their paper [16], Dhar and Majumdar consider avalanches of the ASM on a sequence of graphs \( \{T_n\}_{n \geq 1} \) exhausting the infinite \( q \)-regular tree \( T \), where \( T_n \)'s are balls of radius \( n \) centered in some fixed vertex of \( T \) (so that \( T_n \subset T_{n+1} \)). For any \( n \geq 1 \), they set as dissipative all vertices in \( T_n \) of degree 1 (note that Convention 2.2.2 includes their choice.) As the limit graph \( T \) is transitive, the choice for the vertex \( v_n \) on which one adds an extra chip in order to destabilize a recurrent configuration on \( T_n \) is irrelevant, provided that this vertex is located far enough from the dissipative vertices, in order to avoid boundary perturbations. More precisely, by far enough, one means that, for any \( M > 0 \), one considers avalanches of mass \( M \) on \( T_n \) where \( n \) is sufficiently large to ensure that the ball in \( T_n \) centered in \( v_n \), and containing all vertices possibly touched by any avalanche of mass \( M \) triggered at \( v_n \), does not contain dissipative vertices. Under these assumptions, Dhar and Majumdar establish that

\[
\lim_{n \to \infty} \mathbb{P}_{\mu_n}(\text{Av}_{T_n}(\cdot, v_n) = M) \sim M^{-2}. \tag{2}
\]

They note also that the case of the regular tree is different from that of cubic lattices in the sense that the size of the boundary of \( T_n \) (i.e. the number of vertices of degree 1) is large in comparison with the cardinality of \( T_n \). Indeed, the tree is non-amenable. It can be deduced moreover, that in the random weak limit of the sequence \( \{T_n\}_{n \geq 1} \), the infinite \( q \)-regular tree \( T \) (with any root) occurs with \( \rho \)-probability 0. More precisely, \( \rho \) is supported by one-ended trees \( (T_v, v) \) that look as in Figure 1.

![Figure 1: A limit graph \((T_v, v)\).](image)

If \((T_v, v)\) is such that the distance between the root \( v \) and the set of vertices of degree one is \( d \), then \( \rho((T_v, v)) = 2^{-d-1} \). (The limit tree shown in Figure 1 occurs then with \( \rho \)-probability 1/8.)

Let us consider avalanches of the ASM on the sequence \( \{T_n\}_{n \geq 1} \) in the random weak limit, as in Definition 2.2.6. Given some rooting of every \( T_n \), we choose in each \((T_n, v_n)\) the subgraph \( H_n \) from Definition 2.2.1 such that its internal boundary is a single vertex \( p_n \). Accordingly to Convention 2.2.2, \( p_n \) is dissipative (the image of \( p_n \) in the limit graph \((T_v, v)\) is situated on the left-most infinite path on Figure 1.) It easily follows from Theorem 2.1.1 that for any \( n \geq 1 \), there is a single recurrent
Thus, any avalanche triggered by adding an extra chip on \( v_n \) has maximal mass in the sense that every vertex in \( H_n \setminus \{ p_n \} \) is fired at least once during the avalanche. It follows that the probability measure \( P_{\mu_n}(\text{Mav}_{T_n}(\cdot, v_n) = \cdot) \) is the Dirac measure on \( |H_n| - 1 \).

### 2.3 ASM on separable graphs

For \( k \in \mathbb{N}^* \), a graph \( \Gamma = (V, E) \) is \( k \)-connected if \( |V| > k \) and \( \Gamma \setminus X \) is connected for every subset \( X \subset V \) with \( |X| < k \). A connected graph \( \Gamma \) is separable if it can be disconnected by removing a single vertex. Such a vertex is called a cut vertex. Note that non-separability of a connected graph is the same as 2-connectedness. The largest 2-connected components of a separable graph are called blocks. Any cut vertex belongs to at least two different blocks.

Separable graphs belong to a wider class of tree-like graphs. Computations of certain critical values for percolation and Ising model for such graphs can be found in the Ph.D. thesis of Spakulova [37]. The study of the ASM on separable graphs is also simplified thanks to its tree-like structure, and in particular by the fact that the critical group of such a graph is a direct product of the critical groups of its blocks [2]. In this paper, we will need a more precise statement explained below (see Lemma 2.3.2).

#### Remark 2.3.1.

From now until the end of Section 2, we will assume that \( |P| = 1 \). Indeed, the results of the two forthcoming subsections will be applied in Sections 5, 6 and 7 to graphs for which we will be able to choose one-element dissipative sets satisfying our Convention 2.2.2. The choice of the unique dissipative vertex will be explained in Convention 2.4.5 below.

Consider a finite separable graph \( \Gamma \) with blocks \( C_1, \ldots, C_s \). Fixing one of the vertices (denote it \( p \) and think it to be the dissipative vertex), induces the following partial order on the vertices of \( \Gamma \). For \( u, u' \in V \), we put \( u' \geq u \) if and only if \( u \) lies on any path in \( \Gamma \) joining \( u' \) to \( p \). For any \( 1 \leq i \leq s \), let \( p_i \) be the smallest element of \( V(C_i) \) in this order. Then the following holds:

#### Lemma 2.3.2.

Given \( \Gamma \) a finite separable graph with blocks \( C_1, \ldots, C_s \) and a dissipative vertex \( p \), a configuration \( c \) on \( \Gamma \) is recurrent if and only if for all \( 1 \leq i \leq s \), the subconfiguration \( c^i : V_0(C_i) \rightarrow \mathbb{N} \) defined by \( c^i(v) := c(v) - \text{outdeg}_{C_i}(v) \) is recurrent on the subgraph \( C_i \) with \( p_i \) considered as the dissipative vertex. (Here, for a subgraph \( H \) of \( \Gamma \) and a vertex \( v \) of \( H \), \( \text{outdeg}_H(v) \) stands for the number of edges connecting \( v \) to the complement of \( H \) in \( \Gamma \).)

#### Proof.

Let \( c \) be a configuration on \( \Gamma \). Suppose that \( c \) is recurrent, take a block \( C_i \) and let \( v \in V(C_i) \). By Theorem 2.1.1, there exists a burning sequence for \( c \) on \( \Gamma' := \Gamma \setminus \{ v, \ldots, v_{|V|-1} \} \). Since \( c \) is stable, any vertex \( v \in V \) must have some of its neighbours fired before being fired itself. Since every path joining \( v \) to \( p_i \) contains the vertex \( p_i \), \( v \) cannot be fired before \( p_i \) does. On the other hand, once \( p_i \) is fired, then every vertex of \( C_i \) can be fired in the order provided by the sequence \( pv_1 \ldots v_{|V|-1} \). In particular, there is a subsequence of \( pv_1 \ldots v_{|V|-1} \) which is a burning sequence for \( c^i \) on \( C_i \) with \( p_i \) set as the unique dissipative vertex.

Conversely, if for each block \( C_i \) the subconfiguration \( c^i \) is recurrent, then one can fire vertices of \( \Gamma \) as follows: after firing the vertex \( p_i \), fire vertices belonging to the blocks containing \( p_i \) according to the burning sequences provided by the Burning Algorithm applied consecutively to each of these blocks. Then, repeat the previous operation with the blocks sharing a vertex with the already fired blocks. Since there is a burning sequence for each block of \( \Gamma \), all vertices of \( \Gamma \) are eventually fired.

The following definition and observation will be crucial in our study of avalanches further on.

#### Definition 2.3.3.

A block-path of length \( k \) in a separable graph \( \Gamma \) is a sequence of \( k \) distinct blocks of \( \Gamma \) such that two consecutive blocks intersect.
Given \( w, w' \in V \), there is a unique block-path \( C_1 \ldots C_r \) of minimal length such that \( w \in C_1 \) and \( w' \in C_r \). We say then that \( C_1 \ldots C_r \) joins \( w \) to \( w' \). Similarly, given \( w \in V \) and \( C \) a block of \( \Gamma \), there is a unique block-path \( C_1 \ldots C_r = C \) of minimal length such that \( w \in C_1 \). We say then that \( C_1 \ldots C_r \) joins \( w \) to \( C \).

**Proposition 2.3.4.** Given a finite separable graph \( \Gamma \) with a dissipative vertex \( p \), and given a vertex \( v \in V_0 \), let \( CP_v := C_1 \ldots C_r \) be the block-path joining \( v \) to \( p \). Then, the avalanche triggered by adding an extra chip on \( v \) to some recurrent configuration \( c \) depends only on the subconfigurations of \( c \) on the blocks constituting \( CP_v \).

**Proof.** If \( c(v) < \deg(v) - 1 \), then the avalanche is trivial. If \( c(v) = \deg(v) - 1 \), then \( v \) becomes unstable after adding an extra chip, and a non-trivial avalanche is initiated. Consider the block-path \( CP_v \) joining \( v \) to \( p \), let \( w \) be a separating vertex belonging to some block of \( CP_v \), and consider the subgraph \( D(w) \) of \( \Gamma \) induced by the set \( \{ v \in V_0 | v \succeq w \} \) of all descendants of \( w \). Since \( c \) is recurrent, we can conclude by Theorem 2.1.1 and the proof of Lemma 2.3.2, that each time \( w \) is fired, every successor \( v \succ w \) is fired exactly once, and as a result the subconfiguration on \( D(w) \) remains unchanged. This happens independently of the recurrent subconfiguration on \( D(w) \). The statement follows.

### 2.4 ASM on cacti graphs

In this paper, we will be interested in a particular class of separable graphs called “cacti”.

**Definition 2.4.1.** A separable graph \( \Gamma \), possibly with loops, is a cactus if its blocks are either cycles (possibly of length 2), or single edges.

Note that if \( (\Gamma, v) \) is the limit graph in \( (\mathcal{X}, \text{Dist}) \) of a sequence \( \{ (\Gamma_n, v_n) \}_{n \geq 1} \) of cacti, then \( (\Gamma, v) \) is obviously a cactus as well.

The ASM on cacti is addressed in [27] where the identity of the critical group as well as inverses are explicitly realized in terms of configurations. Here we will be rather interested in finding the asymptotic of avalanches for a sequence of finite cacti (converging in the space of rooted graphs \( (\mathcal{X}, \text{Dist}) \)), with respect to either random or some particular rooting; see Theorem 2.4.7 below. Our results indicate that the answer depends on such invariant of the infinite limit graph as the number of ends. More results in this direction are to be found in the forthcoming paper [27].

#### 2.4.1 ASM on cycles

As the building blocks of a cactus graph are cycles, we will start by recalling and stating some easy facts about the ASM on cycles, [36], [27], which will be useful later.

Let \( C \) be the cycle of length \(|C|\) and let \( V(C) = \{ p, v_1, v_2, \ldots, v_{|C|-1} \} \), where \( p \) is the unique dissipative site and other vertices are numbered in the counterclockwise direction.

**Proposition 2.4.2.** 1) There are exactly \(|C|\) recurrent configurations \( c_0, \ldots, c_{|C|-1} \) on \( C \). They are given by

\[
e_j(v_i) = \begin{cases} 
0 & \text{if } i = j, \\
1 & \text{otherwise}, 
\end{cases}
\]

and \( c_0(v_i) = 1 \) for \( i, j = 1, \ldots, |C| - 1 \).

2) [27] Let \( \eta \) be a configuration on \( C \) and let \( c_j \) be a recurrent configuration. Then,
\[ [c_j + \eta] = c_{[j-\sum_{k=1}^{\lfloor \eta \rfloor} \eta(v_k) \mod |C|]}, \]

where \([\cdot + \cdot]\) denotes the result of adding configurations coordinatewise and then stabilizing.

**Corollary 2.4.3.** If \( \eta \cdot \delta_{v_k} \) for some \( 1 \leq k \leq |C| - 1 \) and \( t \geq 1 \) (i.e. \( \eta(v_k) = t \) and \( \eta(v_i) = 0 \) for \( i \neq k \)), then

\[ [c_j + t \cdot \delta_{v_k}] = c_{[j+tk] \mod |C|}, \]

(3)

We now turn to avalanches on cacti graphs. Let \( v \in V(C) \) be a recurrent configuration. If \( \eta = \delta_{v_k} \) \( i \neq k \), then the mass of the avalanche is given by \( \eta \cdot \delta_{v_k} \) \( i \neq k \), whereas if \( i = k \), it is given by \( \eta \cdot \delta_{v_k} \) \( i = k \). Thus, if we fix \( 0 < M < |C| - 1 \), there are at most two avalanches of mass \( M \), more precisely:

- if \( 0 < M < i_0 \), then there is no recurrent configuration providing an avalanche of mass \( M \);
- if \( i_0 \leq M < |C| - 1 - i_0 \), then there is one configuration providing an avalanche of mass \( M \), which is \( c_{M+1} \);
- if \( |C| - i_0 \leq M < |C| - 1 \), then there are two configurations providing an avalanche of mass \( M \), which are \( c_{M+1} \) and \( c_{|C|-1-M} \).

**Proposition 2.4.4.** [27] In the notations above,

\[ \mathbb{P}_\mu(M_{av}(\cdot, v_{i_0}) = M) = \begin{cases} 0 & \text{if } 0 < M < i_0, \\ \frac{1}{|C|} & \text{if } i_0 \leq M \leq |C| - 1 - i_0, \\ |C| & \text{if } |C| - i_0 \leq M < |C| - 1. \end{cases} \]

Moreover, \( \mathbb{P}_\mu(M_{av}(\cdot, v_{i_0}) = 0) = \mathbb{P}_\mu(M_{av}(\cdot, v_{i_0}) = |C| - 1) = \frac{1}{|C|}. \)

**Proof.** Since there are \( |C| \) different recurrent configurations on \( C \), there are at most \( |C| \) distinct avalanches. The mass of an avalanche is zero (respectively \( |C| - 1 \)) if and only if the configuration on which we add the extra chip is \( c_{i_0} \) (respectively \( c_0 \)). We thus have \( \mathbb{P}_\mu(M_{av}(\cdot, v_{i_0}) = 0) = \mathbb{P}_\mu(M_{av}(\cdot, v_{i_0}) = |C| - 1) = \frac{1}{|C|}. \)

Let \( c_j \) be a recurrent configuration. If \( i_0 > j \), then the mass of the avalanche is given by \( M_{av}(c_j, v_{i_0}) = |C| - 1 - j \), whereas if \( i_0 < j \), it is given by \( M_{av}(c_j, v_{i_0}) = j - 1 \). Thus, if we fix \( 0 < M < |C| - 1 \), there are at most two avalanches of mass \( M \), more precisely:

- if \( 0 < M < i_0 \), then there is no recurrent configuration providing an avalanche of mass \( M \);
- if \( i_0 \leq M < |C| - 1 - i_0 \), then there is one configuration providing an avalanche of mass \( M \), which is \( c_{M+1} \);
- if \( |C| - i_0 \leq M < |C| - 1 \), then there are two configurations providing an avalanche of mass \( M \), which are \( c_{M+1} \) and \( c_{|C|-1-M} \).

\[ \square \]

### 2.4.2 Avalanches on cacti graphs

The first part of Theorem 1.0.1 as well as Theorem 1.0.2 stated in the introduction are based on Theorem 2.4.7 below, a general result about avalanches on cacti graphs converging in \((X, \text{Dist})\) to an infinite one-ended limit (which, as we have noted before, is a cactus as well).

Let \( \{ (\Gamma_n, v_n) \}_{n \geq 1} \) be a sequence of finite cacti converging to a one-ended cactus \( (\Gamma, v) \). Note that there exists a unique block-path \( CP_v = C_1C_2 \ldots \) of infinite length in \( \Gamma \) starting at \( v \) (i.e., \( v \in C_1 \) but \( v \notin C_2 \)). We can choose the subgraphs \( H_n \) from Convention 2.2.2 so that, for any \( n \geq 1 \), the internal boundary of \( H_n \) consists of a unique vertex \( p(n) \); this vertex is a cut vertex in \( \Gamma_n \) and its image under the isomorphism mapping \( H_n \subset \Gamma_n \) to \( H_n \subset \Gamma \) (see Definition 2.2.1) is a cut vertex between two consecutive blocks of \( CP_v \).
Convention 2.4.5. (Choice of dissipative vertex for cacti converging to a one-ended limit)
Given a sequence of finite cacti \( \{ (\Gamma_n, v_n) \}_{n \geq 1} \) converging in \( (\mathcal{X}, \text{Dist}) \) to a one-ended cactus \( (\Gamma, v) \), and given a sequence of subgraphs \( \{ H_n \}_{n \geq 1}, H_n \subset \Gamma_n \) for all \( n \), that is an exhaustion in the limit and such that, for each \( n \), the internal boundary of \( H_n \) consists of a unique vertex \( p^{(n)} \), set \( p^{(n)} \) to be the unique dissipative vertex in \( \Gamma_n \).

Lemma 2.4.6. Let \( \{ (\Gamma_n, v_n) \}_{n \geq 1} \) be a sequence of finite cacti converging to a one-ended cactus \( (\Gamma, v) \). For any \( n \geq 1 \), let \( p^{(n)} \in V(\Gamma_n) \) be the dissipative vertex chosen accordingly to Convention 2.4.5 and let \( CP_{v_n} = C_1 \ldots C_n \) be the unique block-path in \( \Gamma_n \) joining vertex \( v_n \) to \( p^{(n)} \). Then, \( (CP_{v_n}, v) = \lim_{n \to \infty} (CP_{v_n}, v_n) \).

Proof. Since the limit \( (\Gamma, v) \) of the sequence \( \{ (\Gamma_n, v_n) \}_{n \geq 1} \) is a one-ended cactus, the number of blocks in \( CP_{v_n} \) diverges as \( n \to \infty \). By definition of the convergence in \( \mathcal{X} \), for any \( i \geq 1 \) there exists \( n_0 \geq 1 \) such that the sub-block-paths consisting of the \( i \) first blocks in \( CP_{v_n} \) are all isomorphic for every \( n \geq n_0 \). \( \square \)

We will use the notations from Subsection 2.3. Namely, given a finite cactus \( \Gamma \) with a unique dissipative vertex \( p \), a fixed vertex \( v \neq p \) and the unique block-path \( CP_v = C_1 \ldots C_r \) joining \( v \) to \( p \) in \( \Gamma \), for each \( 1 \leq i < r \), \( p_i \) is the separation vertex between \( C_i \) and \( C_{i+1} \), and \( D(p_i) \) is the subgraph induced by \( \{ v \in V(\Gamma) \mid v \geq p_i \} \). Denote by \( d_i \) the number of vertices in \( D(p_i) \). We thus have an increasing sequence of positive integers \( \{ d_i \}_{i=1} \) where \( d_i := |V(D(p))| = |V(\Gamma)| \). If \( (\Gamma, v) \) is an infinite one-ended cactus rooted at \( v \), and \( CP_v = C_1 C_2 \ldots \) is the unique block-path in \( \Gamma \) of infinite length starting at \( v \), then for each \( i \geq 1 \), \( p_i \) is the separation vertex between \( C_i \) and \( C_{i+1} \). Removing \( p_i \) disconnects \( \Gamma \) into several connected components; let \( D(p_i) \) denotes the subgraph consisting of the component together with \( p_i \) containing \( C_1 \ldots C_i \). In this case, the sequence \( \{ d_i \}_{i \geq 1} \) where \( d_i \) is the number of vertices in \( D(p_i) \) is infinite.

Theorem 2.4.7. Let \( \{ (\Gamma_n, v_n) \}_{n \geq 1} \) be a sequence of finite rooted cacti converging to a one-ended cactus \( (\Gamma, v) \). For any \( n \geq 1 \), let \( p^{(n)} \in V(\Gamma_n) \) be the dissipative vertex chosen accordingly to Convention 2.4.5. Denote by \( CP_{v_n} = C_1 \ldots C_r \) the unique block-path in \( \Gamma_n \) joining vertex \( v_n \) to \( p^{(n)} \). Suppose that \( \sum_{i=1, |C_i| > 0} \frac{1}{|C_i|} \) converges as \( r_n \to \infty \). Then, for an integer \( M \) that occurs as the mass of an avalanche, we have

\[
\lim_{n \to \infty} P_{\mu_n}(\text{Max}_{\Gamma_n}(\cdot, v_n) = M) \sim \frac{1}{|C_{i_M}| \cdot |C_{i_M+1}|}
\]

where the blocks \( C_{i_M} \) and \( C_{i_M+1} \) belong to the unique block-path \( CP_v \) of infinite length in \( \Gamma \), and where the index \( i_M \) is uniquely determined by the condition \( d_{i_M-1} \leq M < d_{i_M} \).

Proof. Consider the graph \( \Gamma_n \) for some fixed \( n \geq 1 \) and let \( c \) be a recurrent configuration on \( \Gamma_n \). If \( s \) is the number of blocks constituting \( \Gamma_n \), \( c \) can be decomposed into \( s \) subconfigurations \( c^1, \ldots, c^s \) where \( c^i \) is a recurrent configuration on the block \( C_i \) (see Lemma 2.3.2). If \( c(v_n) = \text{deg}(v_n) - 1 \), then upon adding an extra chip on \( v_n \), an avalanche starts on \( C_1 \) which possibly extends to further blocks of \( CP_{v_n} \). Since the order of firings does not matter, we can suppose that one starts stabilizing the subconfiguration on \( C_{i+1} \) only when the subconfiguration on \( C_j \) is already stable. Recall that, by Proposition 2.3.4, it is enough to keep track of the subconfigurations of \( c \) on the blocks of \( CP_{v_n} \). For any \( 1 \leq j \leq r_n \), we say that the avalanche reaches the block \( C_j \) if \( p_{j-1} \) is fired during the avalanche. Note that if \( C_j \) is a single edge, then \( c^j(p_{j-1}) = \text{deg}(p_{j-1}) - 1 \). Once an avalanche has reached \( C_j \) and if \( C_j \) is not a single edge, then the subavalanche on \( C_j \) has two "branches", each of them propagating in direction of \( p_j \) along a path joining \( p_{j-1} \) to \( p_j \). Since the subconfiguration \( c^j \) on \( C_j \) is recurrent, there is at most one vertex \( w \in V(C_j) \setminus \{ p_j \} \) such that \( c^j(w) = \text{deg}(w) - 2 \) (see Proposition 2.4.2). Hence, at least one of the branches of the subavalanche extends to \( p_j \) so that at least one chip reaches \( p_j \). Then, if \( p_j \) is not fired, we say that the avalanche stops on \( C_j \).
With every recurrent configuration \( c \), one associates a sequence of positive integers \( \{ t_j(c)\}_{j=0}^{r_n-1} \), where \( t_j(c) \) is the number of chips that have reached \( p_j \) during the avalanche triggered by adding an extra chip to \( c \). By convention, fix \( t_0(c) \equiv 1 \). Recall that, on a cycle \( C_j \), there are \( |C_j| \) recurrent configurations which are \( c_0^j, \ldots, c_{|C_j|-1}^j \). For \( 1 \leq j \leq r_n - 1 \), three following situations may occur:

(S1) \( t_j(c) - t_{j-1}(c) = 1 \): this occurs if and only if \( c^j = c_0^j \) and \( \{ c_0^j + t_{j-1}(c) \cdot d_{p_{j-1}} \} \neq c_0^j \);

(S2) \( t_j(c) - t_{j-1}(c) = 0 \): this occurs if and only if either \( c^j = c_k^j \) and \( \{ c_k^j + t_{j-1}(c) \cdot d_{p_{j-1}} \} = c_0^j \), or \( c^j = c_k^j \) for some \( 0 < k < |C_j| \) and \( \{ c_k^j + t_{j-1}(c) \cdot d_{p_{j-1}} \} \neq c_0^j \);

(S3) \( t_j(c) - t_{j-1}(c) = -1 \): this occurs if and only if \( c^j = c_k^j \) for some \( 0 < k < |C_j| \) and \( \{ c_k^j + t_{j-1}(c) \cdot d_{p_{j-1}} \} = c_0^j \).

The difference \( |t_j(c) - t_{j-1}(c)| \) cannot be greater than one, since the total amount of chips in a recurrent configuration on a cycle \( C_j \) is either \(|C_j| - 1\) or \(|C_j| - 2\) (see Proposition 2.4.2 and Lemma 2.3.2).

Finally, note that if the block \( C_j \) is a single edge, then \( t_j(c) = t_j^1(c) \).

We consider now avalanches of some fixed mass \( M \). Since we are interested in the asymptotic behaviour of avalanches as \( n \) tends to infinity and since we have supposed that \( r_n \) tends to infinity (as \( n \to \infty \)), we can suppose without loss of generality that \( M < d_{r_{n-1}} \); all these avalanches reach some block \( C_{i_M} \), \( 1 \leq i_M < r_n \) and stop on it (i.e. vertex \( p_{i_M} = 1 \) is fired but not vertex \( p_{i_M+1} \)). Note that an avalanche cannot stop on a cycle of length two.

Let us now find bounds on the number of recurrent configurations on \( \Gamma_n \) producing avalanches of mass \( M \). Let \( c \) be such that the avalanche triggered by adding an extra chip to \( c \) on \( v_n \) is of mass \( M \).

Then, its corresponding sequence \( \{ t_j(c)\}_{j=0}^{r_n-1} \) satisfies

- \( t_j(c) \geq 1 \) for all \( 0 \leq j \leq i_M - 1 \);
- \( t_{i_M}(c) = 1 \);
- \( t_j(c) = 0 \) for all \( i_M < j \leq r_n - 1 \).

We have to distinguish two cases. Suppose that there exists \( 1 \leq j_0 < i_M \) such that \( C_{j_0} \) is a cycle of length two, and suppose that \( j_0 \) is the smallest such index. Consider the sequence \( \{ t_j\}_{j=0}^{r_n-1} \) defined by \( t_j = 1 \) if \( j < j_0 \), \( t_j = 2 \) if \( j_0 \leq j < i_M \), \( t_{i_M} = 1 \) and \( t_j = 0 \) if \( j > i_M \). If there is no index \( j_0 \) such that \( C_{j_0} \) is a cycle of length two, then define \( \{ t_j\}_{j=0}^{r_n-1} \) by \( t_j = 1 \) if \( j \leq i_M \) and \( t_j = 0 \) if \( j > i_M \). We count the number of recurrent configurations \( c \) whose associated sequence \( \{ t_j(c)\}_{j=0}^{r_n-1} \) coincides with \( \{ t_j\}_{j=0}^{r_n-1} \). In the former case, it follows from Corollary 2.4.3 that, for each \( j < i_M \) such that \( C_j \) is not a single edge nor a cycle of length two, there are at least \( |C_j| - 2 \) recurrent subconfigurations on \( C_j \) satisfying the right-hand side of (S2). If \( C_j \) is a cycle of length two, and \( j_0 < j < i_M \), then both recurrent subconfigurations on \( C_j \) satisfy the right-hand side of (S2). The subconfiguration on \( C_{i_M} \) must be \( c^{j_0} = c_0^{j_0} \) (see (S1)) whereas the subconfigurations on \( C_{i_M}, C_{i_M+1} \) are uniquely determined by (S3). In the latter case, for each \( j < i_M \) such that \( C_j \) is not a single edge, there are at least \( |C_j| - 2 \) recurrent subconfigurations on \( C_j \) satisfying the right-hand side of (S2). Consider now the subavalanche on the cycle \( C_{i_M} \), denoting its mass by \( m \) (so that \( d(p_{i_M-1}, p_{i_M}) \leq m < |C_{i_M}| \)). By Proposition 2.4.4 and its proof, at least one but at most two subconfigurations on \( C_{i_M} \) provoke subavalanches of such mass. The subconfiguration on \( C_{i_M+1} \) is uniquely determined by (S3). Finally, in both cases, the configurations on the remaining blocks of \( \Gamma_n \) can be chosen freely since they do not influence the avalanche (see Proposition 2.3.4). Thus, the number \( N \) of recurrent configurations on \( \Gamma_n \) producing an avalanche of mass \( M \) is at least

\[
N \geq 2^R \cdot \prod_{j=1}^{i_M-1} (|C_j| - 2) \cdot \prod_{C_j \subset \Gamma_n} \prod_{C_j \neq C_1 \ldots C_{i_M+1}} |C_j|,
\]
where \( R = |\{ C_j; C_j \text{ is a cycle}, |C_j| = 2, j_0 < j < i_M \}| \) and the latter product runs over blocks of \( \Gamma_n \) which are not single edges. Since the total number of recurrent configurations on \( \Gamma_n \) (which is the number of spanning trees of \( \Gamma_n \)) is equal to the product of the lengths of the cycles in \( \Gamma_n \), the probability of observing an avalanche of mass \( M \) on \( \Gamma_n \) upon adding an extra chip on \( v_n \) is bounded from below by

\[
P_{\mu_n}(\text{Max}_{\Gamma_n}(\cdot, v_n) = M) \geq N \prod_{C_j \subset \Gamma_n} |C_j|^{-1} \geq \frac{1}{2 \cdot |C_{i_M}| \cdot |C_{i_M+1}|} \prod_{|C_j| > 2} (1 - \frac{2}{|C_j|}),
\]

where the former product runs over blocks of \( \Gamma_n \) which are not single edges. The upper-bound

\[
P_{\mu_n}(\text{Max}_{\Gamma_n}(\cdot, v_n) = M) \leq \frac{2}{|C_{i_M}| \cdot |C_{i_M+1}|}
\]

follows from the fact that in the former case, the subconfigurations on \( C_{i_M} \) and \( C_{i_M+1} \) are uniquely determined by (S3) whereas in the latter case, there are at most two subconfigurations on \( C_{i_M} \) producing a subavalanche on \( C_{i_M} \) of mass \( m \) and the subconfiguration on \( C_{i_M+1} \) is uniquely determined.

By Lemma 2.46, the limits as \( n \to \infty \) of the right-hand sides in (4) and (5) are well-defined. The product \( \prod_{j=1}^{i_M-1} \sum_{|C_j| > 2} \left(1 - \frac{2}{|C_j|}\right) \) converges as \( i_M \to \infty \) to a limit \( L > 0 \) if and only if, the series \( \sum_{i \geq 1, |C_i| > 2} \frac{2}{|C_i|} \) converges (see for instance [38]). In such a case, it is bounded by \( L \) from below for every \( i_M \geq 2 \). This completes the proof.

\[\square\]

## 3 Actions on Rooted Trees, Schreier Graphs and Ergodicity of the Random Weak Limit

### 3.1 Groups of automorphisms of rooted trees and their Schreier graphs

Let \( \{ q_n \}_{n \geq 0} \) be a sequence of positive integers and let \( T \) be a rooted tree such that all vertices of the \( n \)-th level of \( T \) (i.e. vertices situated at distance \( n \) from the root) have \( q_n \) children; \( T \) is called spherical homogenous and \( \{ q_n \}_{n \geq 0} \) is the spherical index of \( T \). For any \( n \geq 1 \), let \( X_n \) be a \( q_n \)-letters alphabet. Then, any vertex of the \( n \)-th level of \( T \) can be regarded as an element of \( \prod_{i=1}^{n-1} X_i =: L_n \) (the root is viewed as the empty word.) Also, write \( X^\omega := \prod_{n \geq 1} X_n \), which is the set of infinite words \( \xi \) such that, for any \( n \geq 1 \), the \( n \)-th letter of \( \xi \) belongs to \( X_n \). The set \( X^\omega \) can be identified with the boundary \( \partial T \) of the tree, which is defined as the set of infinite geodesic rays starting at the root of \( T \). The cylindrical sets \( \bigcup_{n \geq 1} \{ v \prod_{i \geq n} X_{|v|} \mid v \in L_n \} \) generate a \( \sigma \)-algebra of Borel subsets of the space \( X^\omega \). We shall denote by \( \lambda \) the uniform measure on \( X^\omega \).

Consider the group \( Aut(T) \) of all automorphisms of \( T \), i.e., the group of all bijections of the set of vertices of \( T \) preserving the incidence relation. The root, and hence the levels of the tree are preserved by any automorphism of \( T \). A group \( G \leq Aut(T) \) is said to be spherical transitive if it acts transitively on each level of the tree.

For \( G < Aut(T) \) we define the following subgroups of \( G \): the stabilizer of a vertex \( v \in T \) in \( G \) by \( Stab_G(v) = \{ g \in G \mid g(v) = v \} \); the stabilizer of the \( n \)-th level of the tree in \( G \) by \( Stab_G(L_n) = \bigcap_{v \in L_n} Stab_G(v) \); finally, the stabilizer of a boundary point \( \xi \in X^\omega \) in \( G \) by \( Stab_G(\xi) = \{ g \in G \mid g(\xi) = \xi \} \). The following properties hold:

- The subgroups \( Stab_G(v) \), for \( |v| = n \), are all of index \( \prod_{i=1}^{n} q_i \). Moreover, if the action of \( G \) on \( T \) is spherically transitive, they are all conjugate.
- \( \bigcap_{\xi \in \partial T} Stab_G(\xi) \) is trivial.
• Denote by $\xi_n$ the prefix of $\xi$ of length $n$. Then $\text{Stab}_G(\xi) = \bigcap_{n \in \mathbb{N}} \text{Stab}_G(\xi_n)$.

• $\text{Stab}_G(\xi)$ has infinite index in $G$.

Consider a finitely generated group $G$ with a set $S$ of generators such that $id \notin S$ and $S = S^{-1}$, and suppose that $G$ acts on a set $M$. Then, one can consider a graph $\Gamma(G, S, M)$ with the set of vertices $M$, and two vertices $m, m'$ joined by an edge labeled by $s$ if there exists $s \in S$ such that $s(m) = m'$. If the action of $G$ on $M$ is transitive, then $\Gamma(G, S, M)$ is the Schreier graph $\Gamma(G, S, \text{Stab}_G(m))$ of the group $G$ with respect to the subgroup $\text{Stab}_G(m)$ for some (any) $m \in M$. If the action of $G$ on $M$ is not transitive, and $m \in M$, then we denote by $\Gamma(G, S, m)$ the Schreier graph of the action on the $G$-orbit of $m$, and we call such a graph an orbital Schreier graph.

Suppose now that $G$ acts spherically transitively on a spherically homogeneous rooted tree $T$ with spherical index $\{q_n\}_{n \geq 0}$. Then, the $n$-th Schreier graph of $G$ is by definition $\Gamma_n := \Gamma(G, S, \Gamma_n) = \Gamma(G, S, P_n)$ where $P_n$ denotes the subgroup stabilizing some word $w \in L_n$. For each $n \geq 1$, let $\pi_{n+1} : \Gamma(G, S, L_{n+1}) \to \Gamma(G, S, L_n)$ be the map defined on the vertex set of $\Gamma(G, S, L_{n+1})$ by $\pi_{n+1}(x_1 \ldots x_n x_{n+1}) = x_1 \ldots x_n$. Since $P_{n+1} \leq P_n$, $\pi_{n+1}$ induces a surjective morphism between $\Gamma(G, S, L_{n+1})$ and $\Gamma(G, S, L_n)$. This morphism is a graph covering of degree $q_n$.

We also consider the action of $G$ on $\partial T = X^\omega$ and the orbital Schreier graphs $\Gamma_{t} := \Gamma(G, S, G \cdot t) = \Gamma(G, S, P_t)$ where $P_t$ denotes the stabilizer of $\xi$ for the action of $G$ on $X^\omega$. Recall that, given a ray $\xi$, we denote by $\xi_n$ the prefix of $\xi$ of length $n$, and that $P_t = \bigcap_{n} P_{n}$. It follows that the infinite Schreier graph $(\Gamma_{t}, \xi)$ rooted at $\xi$ is the limit of finite Schreier graphs $(\Gamma_n, \xi_n)$ rooted at $\xi_n$, as $n \to \infty$, in the compact metric space $(X, \text{Dist})$ (of rooted isomorphism classes) of rooted connected graphs with uniformly bounded degrees (see Subsection 2.2). Orbital Schreier graphs are interesting infinite graphs that contain information about the group and its action on the tree. The random weak limit of the sequence $\{\Gamma_n\}_{n \geq 1}$, concentrated on the classes of rooted-isomorphism of the orbital Schreier graphs $\{(\Gamma_{t}, \xi), \xi \in \partial T\}$, is often a continuous measure, see e.g. [13].

**Proposition 3.1.1.** Let $\{\Gamma_n\}_{n \geq 1}$ be a covering sequence of finite $2k$-regular graphs ($k \in \mathbb{N}^*$). Then, there exists a rooted tree $T$, and a group $G$ of automorphisms of $T$ such that the $\Gamma_n$’s are realized as Schreier graphs (with respect to an appropriate set of generators) of the action of $G$ on $T$.

**Proof.** For any $n \geq 1$, let $q_n$ be the degree of the covering $\pi_{n+1} : \Gamma_{n+1} \to \Gamma_n$. One associates a tree of preimages $T$ with the covering sequence $\{\Gamma_n\}_{n \geq 1}$ as follows: $T$ is an infinite rooted tree with vertex set $\bigcup_{n \geq 1} V(\Gamma_n) \cup \{\ast\}$ such that the $n$-th level of $T$ is $V(\Gamma_n)$ and every vertex $v$ of the $n$-th level has $q_n$ children corresponding to the fibre of $v$ in $\Gamma_{n+1}$ (by convention, the root $\ast$ of $T$ has $|V(\Gamma_1)|$ children.) For any $n \geq 1$, we denote by $T_{[n]}$ the rooted subtree of $T$ of height $n$.

We proceed by induction on $n$. Consider the graph $\Gamma_1$. By a theorem of Petersen (see for instance [18]), every 2k-regular graph has a 2-factor that is, a 2-regular spanning subgraph. Denote by $F_1^1, \ldots, F_1^k$ the decomposition of $\Gamma_1$ into 2-factors. Any $F_i^1$ is a collection $C_1^1, \ldots, C_k^1$ of disjoint cycles. Assign an arbitrary orientation to each of them, so that each 2-factor $F_i^1$ determines a unique permutation $\sigma_i^1$ of the vertex set of $\Gamma_1$. For $1 \leq i \leq k$, label the edges of the cycles $C_i^1, \ldots, C_i^k$ in $\Gamma_1$ by $\sigma_i^1$. Consider the subgroup $G_1$ of automorphisms of $T_{[1]}$ generated by the set of permutations $\{\sigma_i^1, \ldots, \sigma_i^k\}$. The Schreier graph $\Gamma(G_1, \{\sigma_i^1, \ldots, \sigma_i^k\}, V(\Gamma_1))$ coincides with $\Gamma_1$ labeled as above.

Suppose that there is a subgroup $G_n \equiv \langle \sigma_i^1, \ldots, \sigma_i^n \rangle$ of automorphisms of $T_{[n]}$ such that for every $1 \leq m \leq n$, $\Gamma(G_n, \langle \sigma_i^1, \ldots, \sigma_i^m \rangle, V(\Gamma_m))$ coincides with $\Gamma_m$. Every automorphism $\sigma_i^n$ corresponds to a 2-factor $F_i^n$ of $\Gamma_n$, i.e. $F_i^n$ with its edges labeled by $\sigma_i^n$ coincides with $\Gamma(\langle \sigma_i^n \rangle, \langle \sigma_i^n \rangle)$ of the automorphism $\sigma_i^n$ of $T_{[n]}$. Consider the 2-factor $F_i^n = \bigcup_{i=1}^{q_n} C_i^n$. For any $v \in V(\Gamma_n)$, number its children in $T$ by $v_1, \ldots, v_{q_n}$; consider the (unique) cycle $C_i^n$ containing $v$ together with its fibre $C_{sl}^i, \ldots, C_{sr}^i$ ($1 \leq rs \leq q_n$) in $\Gamma_{n+1}$. The orientation of $C_i^n$ induces an orientation on each cycle of the fibre. For $1 \leq l \leq rs$, consider $C_{sl}^i$ and a child $v_j \in V(C_{sl}^i)$ of $v$. If the neighbour of $v_j$ in $C_{sl}^i$ is a child $w_{lj}$ of $w \in V(\Gamma_n)$, then permute vertices $v_j$ and $v_{lj}$ in $T_{[n+1]}$. Then, consider $w_{lj}$ together with its next neighbour $w_{lj'}$ in $C_{sl}^i$ and permute vertices $w_{lj'}$ and $w_{lj''}$. Continue like this along $C_{sl}^i$ until $v_j$ is reached again. We
thus obtain a set \( \{ \sigma_1^{n+1}, \ldots, \sigma_k^{n+1} \} \) of automorphisms of \( T_{n+1} \) such that the restriction of every \( \sigma_i^{n+1} \) to \( T_n \) is \( \sigma_i^n \). For every \( 1 \leq i \leq k \), label the edges of \( \Gamma_{n+1} \) belonging to the fibre of \( F_i^n \) by \( \sigma_i^{n+1} \).

By construction, the subgroup \( G_{n+1} \) of \( Aut(T_{n+1}) \) generated by these automorphisms is such that \( \Gamma(G_{n+1}, \{ \sigma_1^{n+1}, \ldots, \sigma_k^{n+1} \}, V(\Gamma_{n+1})) \) coincides with \( \Gamma_{n+1} \) labeled as above.

For \( i = 1, \ldots, k \), consider the automorphisms of \( T \) defined by \( \sigma_i := \lim_{n \to \infty} \sigma_i^n \), and let \( G = \langle \sigma_1, \ldots, \sigma_k \rangle \) be the subgroup of \( Aut(T) \) generated by these elements. As, for any \( n \geq 1 \), \( \sigma_i^n \) is the restriction of \( \sigma_i \) to \( T_n \), replace in each \( \Gamma_n \) the labels \( \sigma_i^n \) by \( \sigma_i \) for \( 1 \leq i \leq k \). Then, for any \( n \geq 1 \), the Schreier graph \( \Pi(G, \{ \sigma_1, \ldots, \sigma_k \}, V(\Gamma_n)) \) of the action of \( G \) on the \( n \)-th level of \( T \) coincides with \( \Gamma_n \) newly labeled.

In his paper [29], Nekrashevych shows that if \( \{ \Gamma_n \}_{n \geq 1} \) is a covering sequence of finite \( 2k \)-regular cacti, and only then, the corresponding group of automorphisms of the tree of preimages (see Proposition 3.1.1) is an iterated monodromy group of a post-critically finite backward iteration of topological polynomials. A post-critically finite backward iteration is a sequence \( f_1, f_2, \ldots \) of complex polynomials (or orientation preserving branched coverings of planes) such that there exists a finite set \( P \) such that all critical values of \( f_1 \circ f_2 \circ \cdots \circ f_n \) belong to \( P \) for every \( n \). The iterated monodromy group of such a sequence is the automorphism group of the tree of preimages \( T_t = \bigcup_{n \geq 0} (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t) \) induced by the monodromy actions of the fundamental group \( \pi_1(\mathbb{C}\setminus P, t) \) (\( t \) is an arbitrary basepoint.)

**Theorem 3.1.2 (Nekrashevych).** An automorphism group \( G \) of a rooted tree \( T \) is an iterated monodromy group of a post-critically finite backward iteration of polynomials if and only if there exists a generating set of \( G \) with respect to which the Schreier graphs of the action of \( G \) on \( T \) are cacti.

Suppose now that the rooted tree \( T \) is \( q \)-regular (i.e. \( q_n = q \) for any \( n \geq 0 \)). Then, given a finite alphabet \( X = \{0, 1, \ldots, q - 1\} \), any vertex of the \( n \)-th level of \( T \) can be regarded as an element of \( X^n \), the set of words of length \( n \) in the alphabet \( X \) (\( X^0 \) consists of the empty word), whereas the boundary \( \partial T \) of \( T \) is identified with \( X^\omega \), the set of infinite words in \( X \); write \( X^* = \bigcup_{n \geq 0} X^n \).

Given \( g \in Aut(T) \) and \( v \in X^* \), define \( g|_v = g(v)g|_v \) for all \( w \in X^* \). For any vertex \( v \) of the tree, the subtree of \( T \) rooted at \( v \) is isomorphic to \( T \). Therefore, every automorphism \( g \in Aut(T) \) induces a permutation of the vertices of the first level of the tree and \( q \) restrictions, \( g|_0, \ldots, g|_{q-1} \), to the subtrees rooted at the vertices of the first level. It can be written as \( g = \tau(g|_0, \ldots, g|_{q-1}) \), where \( \tau \) describes the action of \( g \) on the first level of the tree. In fact, \( Aut(T) \) is isomorphic to the wreath product \( S_q \wr Aut(T) \) where \( S_q \) denotes the symmetric group on \( q \) letters, and thus \( Aut(T) \cong \bigodot_{\sigma \in S_\pi}(S_{q^n}) \).

For a subgroup \( G < Aut(T) \), the natural question whether restricting the action to a subtree isomorphic to \( T \) preserves \( G \), motivates the following definition. It was forged around 2000, see e.g. [20], though self-similar groups were known before – this class of groups contains many exotic examples of groups, including groups of intermediate growth, non-elementary amenable groups, amenable but not subexponentially amenable groups.

**Definition 3.1.3.** The action of a group \( G \) by automorphisms on a \( q \)-regular rooted tree \( T \) is self-similar if \( g|_v \in G \), \( \forall v \in X^* \), \( \forall g \in G \).

A self-similar group \( G \) can be embedded into the wreath product \( S_q \wr G \). Consequently, an automorphism \( g \in G \) can be represented as \( g = \tau(g|_0, \ldots, g|_{q-1}) \), where \( \tau \) describes the action of \( g \) on the first level of the tree, and \( g|_v \in G \) is the restriction of the action of \( g \) to the subtree \( T_v \) rooted at the \( i \)-th vertex of the first level. So, if \( x \in X \) and \( w \) is a finite word in \( X \), we have \( g(xw) = \tau(x)g(xw) \).

Self-similar groups can be also characterized as **automata groups**, i.e., groups generated by states of an invertible automaton (see e.g. [20]). An automaton over the alphabet \( X \) with the set of states \( S \) is defined by the transition map \( \mu : S \times X \to S \) and the output map \( \nu : S \times X \to X \). It is invertible if, for all \( s \in S \), the transformation \( \nu(s, \cdot) : X \to X \) is a permutation of \( X \). It can be represented by its Moore diagram where vertices correspond to states and for every state \( s \in S \) and every letter \( x \in X \), an oriented edge connects \( s \) with \( \mu(s, x) \) labeled by \( x|\nu(s, x) \). A natural action on the words
over $X$ is induced, so that the maps $\mu$ and $\nu$ can be extended to $S \times X^*$: $\mu(s, xw) = \mu(s, x)w)$, $\nu(s, xw) = \nu(s, x)\nu(s, x, w)$, where we set $\mu(s, \emptyset) = s$ and $\nu(s, \emptyset) = \emptyset$. If we fix an initial state $s$ in an automaton $A$, then the transformation $\nu(s, \cdot)$ on the set $X^*$ is thus defined; it is denoted by $A_s$. The image of a word $x_1x_2 \ldots$ under $A_s$ can be easily found using the Moore diagram: consider the directed path starting at the state $s$ with consecutive labels $x_1y_1$, $x_2y_2$, $\ldots$; the image of the word $x_1x_2 \ldots$ under the transformation $A_s$ is then $y_1y_2 \ldots$. More generally, given an invertible automaton $A = (S, X, \mu, \nu)$, one can consider the group generated by the transformations $A_s$, for $s \in S$; this group is called the automaton group generated by $A$ and is denoted by $G(A)$.

To a group with a self-similar action that is contracting, (which means the existence of a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $k \in \mathbb{N}$ such that $g|_k \in \mathcal{N}$, for all words $v$ of length greater or equal to $k$), Nekrashevych associates its limit space $\mathcal{J}(G)$, often a fractal. Rescaled finite Schreier graphs form a sequence of finite approximations to the compact $\mathcal{J}(G)$. Orbital Schreier graphs $\Gamma_\xi$ on the other hand describe the local structure of the limit space.

An important class of self-similar groups is formed by iterated monodromy groups of partial self-coverings of path connected and locally path connected topological spaces (e.g. of complex rational functions.) If the covering is expanding, its Julia set is homeomorphic to the limit space of its iterated monodromy group. Details about this very interesting subject can be found in [31].

### 3.2 Invariance property of avalanches of the ASM on cacti

We go back in this subsection to the study of avalanches on a sequence of finite cacti. Using the fact that any covering sequence of 2k-regular graphs can be realized as a sequence of Schreier graphs, and hence its random weak limit $\rho$ is ergodic, we show that, for a covering sequence of cacti whose random weak limit is supported by one-ended graphs, the probability distribution for the mass of avalanches is, asymptotically, almost surely the same for all limit graphs.

**Proposition 3.2.1.** Let $\{\Gamma_n\}_{n \geq 1}$ be a covering sequence of finite 2k-regular cacti $(k \in \mathbb{N}^*)$ such that the conditions of Theorem 2.4.7 are satisfied in the random weak limit $\rho$. Then, asymptotically in $M$, the probability distribution $\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_n}(\text{Max}_\Gamma_n(\cdot, v_n) = M)$ is $\rho$-almost surely the same for any choice of the root $v_n$ in $\Gamma_n$.

**Corollary 3.2.2.** In particular, if $\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_n}(\text{Max}_\Gamma_n(\cdot, v_n) = M) \sim M^{-\delta}$ for some $\delta > 0$, then the ASM on $\{\Gamma_n\}_{n \geq 1}$ is critical in the random weak limit with critical exponent $\delta$.

**Proof.** It follows from Proposition 3.1.1, that the sequence $\{\Gamma_n\}_{n \geq 1}$ is realized as Schreier graphs (with respect to an appropriate set of generators) of an action of a group $G$ of automorphisms of a rooted tree $T$. Since the graphs we consider are connected, the action of $G$ on $T$ is spherically transitive; it follows that the action of $G$ on the boundary $\partial T$ of $T$ is ergodic with respect to the uniform measure $\lambda$ (see for example Proposition 6.5. in [20].)

Consider now the application $\phi : \partial T \rightarrow X, \phi(\xi) := (\Gamma_\xi, \xi)$, mapping a point $\xi$ in the boundary of $T$ to the (rooted isomorphism class of the) orbital Schreier graph $\Gamma_\xi$ rooted at $\xi$. Then, the random weak limit $\rho$ of the sequence $\{\Gamma_n\}_{n \geq 1}$ is the image under $\phi$ of $\lambda$. It can be shown [5] that $\rho$ is invariant under the equivalence relation change of root that identifies different rootings of a given graph. The application $\phi$ induces an equivalence relation on $Im\phi \subset X$: $(\Gamma_\xi, \xi)$ and $(\Gamma_\eta, \eta)$ are equivalent if and only if $\xi$ and $\eta$ belong to the same orbit of the action of $G$ on $\partial T$. This relation coincides on $Im\phi$ with the change of root. It follows then that the random weak limit $\rho$ is ergodic with respect to the change of root.

We claim that the asymptotical behaviour (in $M$) of the distribution $\lim_{n \rightarrow \infty} \mathbb{P}_{\mu_n}(\text{Max}_\Gamma_n(\cdot, v_n) = M)$ is almost surely invariant with respect to the change of root. Let $\{(\Gamma_n, v_n)\}_{n \geq 1}$ and $\{(\Gamma_n, v'_n)\}_{n \geq 1}$ be sequences with different rootings of the $\Gamma_n$’s, such that the corresponding limit graphs $(\Gamma, v)$ and $(\Gamma', v')$ are both one-ended and equivalent for the change of root. For any $n \geq 1$, let $p(v)$ (respectively $p(v')$) be the dissipative vertex in $(\Gamma_n, v_n)$ (respectively in $(\Gamma_n, v'_n)$) chosen accordingly to Convention
2.4.5. Let \( CP_v = C_1C_2 \ldots \) be the unique block-path of infinite length in \((\Gamma, v)\) starting at \(v\) (respectively \( CP_{v'} = C'_1C'_2 \ldots \) in \((\Gamma', v')\) starting at \(v'\).) Since \((\Gamma, v)\) and \((\Gamma', v')\) are one-ended and isomorphic as unrooted graphs then, up to some initial segment, \( CP_v \) and \( CP_{v'} \) are isomorphic (i.e. there exist \(k, l \geq 1\) such that \(C_kC_{k+1} \ldots \) and \(C'_lC'_{l+1} \ldots \) are isomorphic.) Thus, by Lemma 2.4.6, for any \(n\) sufficiently large, the block-paths \( CP_{v_n} = C_1 \ldots C_r n \) and \( CP_{v'_n} = C'_1 \ldots C'_r n \) in \( \Gamma_n \) coincide up to some initial and final segments (i.e. there exist \(k, l \geq 1\) and \(s(n), t(n)\) with \(s(n), t(n) \to \infty\) as \(n \to \infty\), such that \(C_kC_{k+1} \ldots C_{s(n)}\) and \(C'_lC'_{l+1} \ldots C'_{t(n)}\) coincide in \( \Gamma_n\)). It follows from the proof of Theorem 2.4.7, that for any \(n\) and \(M\) sufficiently large, the block \(C_M\) on which any avalanche of mass \(M\) triggered on \( \Gamma_n\) by adding an extra chip either on \(v_n\) or on \(v'_n\) stops, is the same. Hence, by Theorem 2.4.7, \(\lim_{n \to \infty} P_{\mu_n}(Maw_{\Gamma_n}(\cdot, v_n) = M) \sim \lim_{n \to \infty} P_{\mu_n}(Maw_{\Gamma_n}(\cdot, v'_n) = M)\), and this proves the claim.

By ergodicity of \(\rho\), the asymptotical behaviour (in \(M\)) of the distribution \(\lim_{n \to \infty} P_{\mu_n}(Maw_{\Gamma_n}(\cdot, v_n) = M)\) is almost surely constant.

4 The Basilica Group and its Schreier Graphs

The Basilica group \(B\) is an automorphism group of the rooted binary tree which is generated by two automorphisms \(a\) and \(b\) having the following self-similar structure:

\[
a = (b, id) e \quad b = (a, id)(0\,1),
\]

where \(id\) denotes the trivial automorphism of the tree, whereas \(e\) is the identity permutation in \(S_2\). In other words, \(a\) fixes the first level, then acts as \(b\) on the subtree rooted at \(0\) and as the identity on the subtree rooted at \(1\), whereas \(b\) permutes the vertices of the first level, then acts as \(a\) on the subtree rooted at \(0\) and as the identity on the subtree rooted at \(1\). It can be easily checked that the action of \(B\) on the binary tree is spherically transitive.

The group \(B\) was introduced by Grigorchuk and Žuk [21] as the group generated by the three-state automaton represented in Figure 2. It can also be described as the iterated monodromy group \(IMG(z^2 - 1)\) of the complex polynomial \(z^2 - 1\) [31] (see Figure 3).

![Figure 2: The automaton generating the Basilica group.](image-url)
For each $n \geq 1$, we denote by $\Gamma_n \equiv \Gamma(\mathcal{B}, \{a, b\}, \{0, 1\}^n)$ the Schreier graph of the action of the Basilica group $\mathcal{B}$ on the $n$-th level of the binary tree. These graphs, appropriately rescaled, form an approximating sequence of the Basilica Julia set $\mathcal{J}(z^2 - 1)$ (this is used for example by Rogers and Teplyaev in [35] for defining laplacians on the Julia set). The graphs $\{\Gamma_n\}_{n \geq 1}$ can be constructed recursively as follows:

**Proposition 4.0.3.** [13] The Schreier graph $\Gamma_{n+1}$ is obtained from $\Gamma_n$ by applying to all subgraphs of $\Gamma_n$ given by single edges the rules represented in Figure 4.

![Rewriting rules for construction of the Basilica Schreier graphs and the Schreier graph $\Gamma_1$.](image-url)

Figure 4: Rewriting rules for construction of the Basilica Schreier graphs and the Schreier graph $\Gamma_1$. 

---

**Figure 3:** The Julia set $\mathcal{J}(z^2 - 1)$.
It follows that, for each \( n \geq 1 \), \( \Gamma_n \) is a 4-regular cactus such that removing any cut vertex disconnects \( \Gamma_n \) into exactly two components. Let us call the unique cycle of \( \Gamma_n \) containing vertices \( 0^n \) and \( 0^{n-1}1 \) the central cycle of \( \Gamma_n \). Given any vertex \( v \in V(\Gamma_n) \), there is a unique block-path (see Subsection 2.3) \( CP_v = C_1 \ldots C_r \) joining \( v \) to the central cycle of \( \Gamma_n \).

**Definition 4.0.4. "Decoration of a vertex".** 1) Let \( v \in V(\Gamma_n) \backslash \{0^n\} \) be a cut vertex. Denote by \( U_1 \) and \( U_2 \) the two connected components obtained by removing \( v \), so that moreover \( 0^n \in U_1 \). The decoration \( \mathcal{D}(v) \) of \( v \) is the subgraph induced by the vertex set \( V(U_2) \cup \{v\} \).

2) Let \( v \) be a vertex with a loop. Then \( \mathcal{D}(v) \) is the subgraph induced by \( \{v\} \).

3) If \( v = 0^n \), then \( \mathcal{D}(0^n) \) is the subgraph induced by \( V(U_1) \cup \{0^n\} \) where \( 0^{n-1}1 \notin U_i \).

A decoration of a given vertex \( v \in V(\Gamma_n) \) is called a \( k \)-decoration (or a decoration of height \( k \)) if it is isomorphic to the decoration of the vertex \( 0^k \) for some \( 1 \leq k \leq n \).

The following proposition collects some of the properties of the graph \( \Gamma_n \).

**Proposition 4.0.5.** For any \( n \geq 1 \), consider the Schreier graph \( \Gamma_n \). Then, the following hold:

1. [13] Every decoration in \( \Gamma_n \) is a \( k \)-decoration for some \( 1 \leq k \leq n \).
2. \[ |D(0^n)| = \begin{cases} \frac{1}{3}(2^n + 2) & \text{if } n \text{ is even,} \\ \frac{1}{3}(2^n + 1) & \text{if } n \text{ is odd.} \end{cases} \]

3. The lengths of the cycles constituting \( \Gamma_n \) are all powers of two; the number \( \nu_k \) of cycles of length \( 2^k \) \((k \geq 1)\) is

\[ \nu_k = \begin{cases} 3 \cdot 2^{n-2k-1} & \text{for } 1 \leq k \leq \frac{n}{2} - 1, \\ 3 & \text{for } k = \frac{n}{2}, \end{cases} \]

if \( n \) is even, and

\[ \nu_k = \begin{cases} 4 \cdot 2^{n-2k-1} & \text{for } 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1, \\ 4 & \text{for } k = \lfloor \frac{n}{2} \rfloor, \\ 1 & \text{for } k = \lceil \frac{n}{2} \rceil, \end{cases} \]

if \( n \) is odd.

The proofs of statements 2. and 3. are straightforward when using the substitutional rules described in Proposition 4.0.3 and induction on \( n \).

The structure of the critical group \( K(\Gamma_n) \) follows now immediately (see Subsection 2.3).

Proposition 4.0.6. If \( n \) is even, then \( K(\Gamma_n) \) is isomorphic to

\[ \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} (\mathbb{Z}/2^k\mathbb{Z})^{3 \cdot 2^{n-2k-1}} \times (\mathbb{Z}/2^{\frac{n}{2}}\mathbb{Z})^3, \]

and if \( n \) is odd, then \( K(\Gamma_n) \) is isomorphic to

\[ \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} (\mathbb{Z}/2^k\mathbb{Z})^{3 \cdot 2^{n-2k-1}} \times (\mathbb{Z}/2^{\lfloor \frac{n}{2} \rfloor}\mathbb{Z})^4 \times (\mathbb{Z}/2^{\lceil \frac{n}{2} \rceil}\mathbb{Z}). \]

Note that since the lengths of the cycles in \( \Gamma_n \) are all powers of two, the latter decomposition corresponds to the decomposition of \( K(\Gamma_n) \) into invariant factors.

Given a ray \( \xi \in \{0, 1\}^\omega \), the sequence \( \{\Gamma_n, \xi_n\}_{n \geq 1} \) of finite Schreier graphs, rooted at the \( n \)-th prefix \( \xi_n \) of \( \xi \), converges in \((\mathcal{X}, \text{Dist})\) to the infinite orbital Schreier graph \( \Gamma(\xi, \xi) = (\Gamma(\mathcal{B}, \{a, b\}, \mathcal{B} \cdot \xi), \xi) \). The following results classify all rays \( \xi \in \{0, 1\}^\omega \) with respect to the number of ends of the corresponding limit graph (can be equal to 4, 2 or, almost surely, to 1), as well as gives information about different types of isomorphisms of infinite orbital Schreier graphs.

Theorem 4.0.7. [13] Set \( E_i = \{ \xi \in \{0, 1\}^\omega \mid \text{the infinite Schreier graph } \Gamma_\xi \text{ has } i \text{ ends} \} \). Then,

1. \( E_4 = \{w0^\omega, w(01)^\omega \mid w \in \{0, 1\}^*\} \);
2. \( E_1 = \{\alpha_1\beta_1\alpha_2\beta_2 \ldots, \alpha_i, \beta_j \in \{0, 1\} \mid \{\alpha_i\}_{i \geq 1} \text{ and } \{\beta_j\}_{j \geq 1} \text{ both contain infinitely many } 1's\}; \)
3. \( E_2 = \{0, 1\}^\omega \setminus (E_1 \sqcup E_4) \).

Corollary 4.0.8. [13]

1. There exists only one class of isomorphism of 4-ended (unrooted) infinite Schreier graphs. It contains a single orbit.
2. There exist uncountably many classes of isomorphism of 2-ended (unrooted) infinite Schreier graphs. Each of these classes contains exactly two orbits.

3. There exist uncountably many classes of isomorphism of 1-ended (unrooted) infinite Schreier graphs. The isomorphism class of $\Gamma_{1}\omega$ is a single orbit, and every other class contains uncountably many orbits.

Recall that $\phi: \{0,1\}^\omega \rightarrow \mathcal{X}$, $\phi(\xi) := (\Gamma_{\xi}, \xi)$, is the application mapping an infinite binary sequence $\xi$ to the (rooted isomorphism class of the) orbital Schreier graph $\Gamma_{\xi}$ rooted at $\xi$, and that the random weak limit of the sequence of finite Schreier graphs $\{\Gamma_n\}_{n \geq 1}$ is the image under $\phi$ of $\lambda$, the uniform measure on $\{0,1\}^\omega$.

Proposition 4.0.9. [13] The random weak limit of the sequence of finite Schreier graphs $\{\Gamma_n\}_{n \geq 1}$ is concentrated on 1-ended graphs.

We first describe the limit graphs with four and two ends (proofs can be found in [13]). Given $\xi \in E_4$, any orbital Schreier graph $\Gamma_{\xi}$ is isomorphic to the four-ended graph $\Gamma_{(4)}$ constructed as follows (see Figure 8): take two copies $R_1$ and $R_2$ of the double ray whose vertices are naturally identified with the integers. Let these two double rays intersect at vertex 0. For every $k \geq 0$, define the subset of $Z$

$$A_k := \{ n \in Z | n \equiv 2^k \mod 2^{k+1} \}.$$ 

Attach to each vertex of $A_k$ in $R_1$ (respectively in $R_2$) a $(2k+1)$-decoration (respectively a $(2k+2)$-decoration) by its unique vertex of degree 2.

For any $\xi \in E_2$, $\xi$ can be written as $\xi = \alpha_1 \beta_1 \alpha_2 \beta_2 \ldots$ where exactly one of the sequences $\{\alpha_i\}_{i \geq 1}$ or $\{\beta_i\}_{i \geq 1}$ has finitely many 1’s. If $\{\alpha_i\}_{i \geq 1}$ has finitely many 1’s, the graph $\Gamma_{\xi}$ is isomorphic to the following graph $\Gamma(\xi)$: consider the subsets of $Z$

$$A'_0 := 2Z \quad \text{and} \quad A'_k := \{ n \in Z | n \equiv 2^k - 1 - \sum_{i=1}^{k} 2^i \beta_{i+1} \mod 2^{k+1} \} \text{ for each } k \geq 1.$$ 

Construct $\Gamma(\xi)$ as a double ray with integer vertices with, for each $k \geq 0$, a $(2k+2)$-decoration attached by its unique vertex of degree 2 to every vertex corresponding to an integer in $A'_k$.

In the case where $\{\beta_i\}_{i \geq 1}$ has finitely many 1’s, the graph $\Gamma(\xi)$ is defined similarly, replacing $\beta$ by $\alpha$ in the definition of $A'_k$ and by attaching $(2k+1)$-decorations instead of $(2k+2)$-decorations (see Figure 6).

Corollary 4.0.10. The two-ended orbital Schreier graphs $\Gamma_{\xi}$, $\xi \in E_2$, form an uncountable family of non-isomorphic graphs which are not quasi-isometric to the one-dimensional lattice.
Lemma 4.0.13. [13] It is enough to keep track of the sizes of blocks $C$ if there exists a unique triple $(\xi, \gamma, \mu)$ such that

$$\text{Lemma 4.0.14. [13]}$$

it is well-defined and the graph $CP_{\xi}$ is isomorphic to the unique block-path of infinite length in $(\Gamma, \xi)$ starting at $\xi$.

Remark 4.0.12. It follows from Theorem 2.4.7 that for understanding the asymptotic of avalanches it is enough to keep track of the sizes of blocks $C_1, C_2, \ldots$ constituting the block-path $CP_{\xi}$.

We will need the following technical lemmas:

Lemma 4.0.13. [13] An element $\xi \in \{0,1\}^\omega$, $\xi \neq w1^\omega$ for any $w \in \{0,1\}^*$, belongs to $E_1$ if and only if there exists a unique triple $(l, \{m_k\}_{k \geq 0}, \{t_k\}_{k \geq 0})$ where $l \geq 1$ and $m_0 \geq 0$ are integers and $m_0$ is even; $t_0 = 0$; and $\{m_k\}_{k \geq 1}, \{t_k\}_{k \geq 1}$ are sequences of strictly positive integers and the $m_k$'s are even, such that $\xi$ can be written as

$$\xi = 0^{l-1}(0x_1^00x_2^0\ldots0x_{m_0}^0)1^{t_1}(0x_1^10x_2^1\ldots0x_{m_0}^1)1^{t_2} \ldots$$

with $x_i^j \in \{0,1\}$ for all $i, j$.

If $\xi = w1^\omega$ for some $w \in \{0,1\}^*$, then there exists a unique triple $(l, \{m_k\}_{k = 0}^{k_0}, \{t_k\}_{k = 0}^{k_0})$ where $l \geq 1$ and $m_0 \geq 0$ are integers and $m_0$ is even; $t_0 = 0$; and $\{m_k\}_{k = 1}^{k_0}, \{t_k\}_{k = 1}^{k_0}$ are finite sequences of strictly positive integers and the $m_k$'s are even, such that $\xi$ can be written as

$$\xi = 0^{l-1}(0x_1^00x_2^0\ldots0x_{m_0}^0)1^{t_1}(0x_1^10x_2^1\ldots0x_{m_0}^1)1^{t_2} \ldots 1^{t_{k_0}}(0x_1^{k_0}0x_2^{k_0}\ldots0x_{m_0}^{k_0})1^w.$$ 

Lemma 4.0.14. [13] Let $\xi \in E_1$ and define a sequence of integers $a_i = a_i^\xi, i \geq 1$, as follows: if $\xi = 1^\omega$, then $a_i := i$ for all $i \geq 1$. If $\xi \neq 1^\omega$, then Lemma 4.0.13 provides a triple $(l, \{m_k\}, \{t_k\})$ associated with $\xi$. For all $j \geq 1, 0 \leq s < t_j$, let $a_{T_j-1+s+1} := l + M_{j-1} + T_{j-1} + s$, where $M_j := \sum_{k=0}^j m_k$ and $T_j := \sum_{k=0}^j t_k$. Then,
• The sequence \( \{a_i\}_{i \geq 1} \) is increasing. More precisely,
\[
a_{i+1} - a_i = \begin{cases} 
m_j + 1 & \text{if there exists } j > 0 \text{ such that } i = T_j, \\
1 & \text{otherwise.} \end{cases}
\]  
(9)

• For all \( i \geq 1 \), the size of \( C_i \) in \( CP_\xi \subset \Gamma_\xi \) is equal to \( 2^{[a_i/2]} \).

The description from Lemma 4.0.13 allows to classify the words \( \xi \in E_1 \) giving rise to isomorphic orbital Schreier graphs \( \Gamma_\xi \) (see Theorem 5.4 in [13]).

**Proposition 4.0.15.** The orbital one-ended Schreier graphs \( \Gamma_\xi, \xi \in E_1 \), form an uncountable family of 4-regular graphs of quadratic growth (for a proof of this fact, see [8]).

![Figure 7: A finite part of \( \Gamma_{1^\omega} \).](image)

**Remark 4.0.16.** It follows also from Theorem 5.4 in [13], that any two non-isomorphic Schreier graphs \( \Gamma_\xi \) and \( \Gamma_\eta \) for \( \xi, \eta \in E_1 \) are not quasi-isometric. Also, none of them is quasi-isometric to \( \mathbb{Z}^2 \).

Indeed, let \( \Gamma_\xi \not\cong \Gamma_\eta \) and suppose that the sequences \( \{a_\xi^i\} \) and \( \{a_\eta^i\} \) do not coincide eventually (i.e. there do not exist \( i_0, j_0 \) such that \( a_{i_0+k}^\xi = a_{j_0+k}^\eta \) for all \( k \geq 0 \).) Since, under a quasi-isometry, \( CP_\xi \) must be mapped to \( CP_\eta \) and since the length of the \( i \)-th cycle of \( CP_\xi \) (respectively \( CP_\eta \)) is \( 2^{[a_i/2]} \) (respectively \( 2^{[a_i/2]} \)), we get a contradiction. On the other hand, if we suppose that the sequences \( \{a_\xi^i\} \) and \( \{a_\eta^i\} \) do eventually coincide, then condition (c) in Theorem 5.4 of [13] is not satisfied which means that the difference of the distances between successive cut vertices of \( CP_\xi \), respectively \( CP_\eta \), diverges.

To see that \( \mathbb{Z}^2 \) is not quasi-isometric to any orbital Schreier graph \( \Gamma_\xi \) for \( \xi \in E_1 \), note that any
quasi-isometry between infinite graphs maps a bi-infinite self-avoiding path to a bi-infinite self-avoiding path. However, there is no bi-infinite self-avoiding path in $\Gamma_\xi$, for any $\xi \in E_1$.

5 Avalanches on Basilica Schreier Graphs

In this section, we study avalanches of the ASM on sequences of finite Schreier graphs $\{(\Gamma_n, \xi_n)\}_{n \geq 1}$ of the Basilica group converging in the space $(X, \text{Dist})$ to the infinite orbital Schreier graph $\left(\Gamma_\xi, \xi\right)$, and prove Theorem 1.0.1.

Given $\xi \in \{0, 1\}^\omega$, we look at the probability distribution, as $n \to \infty$, of the random variable $M_{\text{av}}(\cdot, \xi_n)$ giving the mass of an avalanche triggered by adding a chip on the root $\xi_n$ to a recurrent configuration on $\Gamma_n$ chosen uniformly at random. Recall from Section 4 that, for almost every infinite binary sequence $\xi$, the orbital Schreier graph $\Gamma_\xi$ has 1 end (these boundary points are partitioned into uncountably many uncountable classes of isomorphic $\Gamma_\xi$’s); that there also exist an uncountable infinity of $\xi$’s that give rise to orbital Schreier graphs with 2 ends (partitioned into countable isomorphism classes); and a countable number of $\xi$’s with a 4-ended $\Gamma_\xi$ (all isomorphic as unrooted graphs). We examine separately the asymptotic distribution of the mass of avalanches depending on the number of ends in the orbital infinite graph $\Gamma_\xi$. The four-ended and two-ended graphs are shown to be non-critical (Theorems 5.1.2 and 5.2.1). However, almost every one-ended graph, and therefore also almost every orbital Schreier graph of the Basilica group is critical with the critical exponent equal to 1 (Theorem 5.3.1).

5.1 Limit graph with four ends

Recall that all orbital Schreier graphs $\Gamma_\xi$’s that have 4 ends are isomorphic to the graph $\Gamma_{(4)}$ described in Section 4. Therefore it is enough to examine one such $\Gamma_\xi$, and we will consider $\xi = 0^\omega$.

For any $n \geq 1$, we fix in $\Gamma_n$ four dissipative vertices as follows: consider the vertices $0_n$ and $0_n$; for each of them, its neighbours which are situated on a path from it to $0_n$ are dissipative. As $n$ tends to infinity, both cycles in $\Gamma_n$ containing $0_n$ grow and split in the limit, sending vertices $0_n$ and $0_n$ to infinity and giving in the limit the four infinite paths in $\Gamma_0^\omega$ intersecting at $0^\omega$ (see Figure 8 and [13]). Consequently, our choice of dissipative vertices corresponds to our Convention 2.2.2.

It is further convenient to merge in $\Gamma_n$ all four dissipative vertices into a single dissipative vertex $p$. The graph $\Gamma_n$ obtained in this way is still separable but is not a cactus anymore. More precisely, all blocks of $\Gamma_n$ but one are cycles denoted by $C_1, \ldots , C_s$. Denote the exceptional block by $B$; it consists of vertices $p$ and $p$, and of four disjoint paths, $\mathcal{P}_1$ to $\mathcal{P}_4$, where $|\mathcal{P}_1| = |\mathcal{P}_2| = 2 \lceil \frac{n}{2} \rceil - 1$ whereas $|\mathcal{P}_3| = |\mathcal{P}_4| = 2 \lceil \frac{n-1}{2} \rceil - 1$ (see Figure 8). Note that considering the ASM on the graph $\Gamma_n$ is equivalent to consider it on $\Gamma_n$. Indeed, merging all dissipative vertices into a single dissipative vertex does not affect neither the structure of chip configurations (as they are defined on non-dissipative vertices only) nor the firing rules (as dissipative vertices are never fired during the stabilization process) and hence avalanches. Also, $R_{\Gamma_n} = R_{\Gamma_n}$ since performing the Burning Algorithm on $\Gamma_n$ is equivalent to perform it on the graph $\Gamma_n$ (as the graphs spanned by the sets of vertices $V_0(\Gamma_n)$ and $V_0(\Gamma_n)$ are isomorphic.)
Figure 8: A part of $\Gamma_n$ isomorphic to a subgraph of $\Gamma_{0\omega}$.

The description of recurrent configurations on $\bar{\Gamma}_n$ (and hence on $\Gamma_n$) follows now directly from Lemma 2.3.2, Proposition 2.4.2 and from Theorem 2.1.1. Given a block $C_i$ of $\bar{\Gamma}_n$, denote its vertices by $p_i, v_{i1}, \ldots, v_{i|C_i|-1}$ (recall from Subsection 2.3 that $p_i$ denotes the smallest element of $V(C_i)$ in the order $\succeq$).

**Proposition 5.1.1.** A chip configuration $c : V_0(\bar{\Gamma}_n) \to \mathbb{N}$ on $\bar{\Gamma}_n$ is recurrent if and only if it has the form

$$c = c^1_j + c^2_j + \cdots + c^s_j + c^B$$

where for every $1 \leq i \leq s$, $j_i \in \{0, 1, \ldots, |C_i| - 1\}$. If $j_i \neq 0$, then $c^i_{j_i} : V_0(\bar{\Gamma}_n) \to \mathbb{N}$ is given by

$$c^i_{j_i}(w) = \begin{cases} 
2 & \text{if } w = v^i_{j_i}, \\
3 & \text{if } w = v^i_k \text{ for } k = 1, \ldots, |C_i| - 1, k \neq j_i, \\
0 & \text{otherwise},
\end{cases}$$

whereas if $j_i = 0$,

$$c^i_0(w) = \begin{cases} 
3 & \text{if } w = v^i_k \text{ for } k = 1, \ldots, |C_i| - 1, \\
0 & \text{otherwise},
\end{cases}$$

The subconfiguration $c^B : V_0(\bar{\Gamma}_n) \to \mathbb{N}$ satisfies
1. $2 \leq c^B(v) \leq 3$ for every $v \in V_0(B) \backslash \{0^n\}$;

2. for $1 \leq i \leq 4$, $c^B(v) = 2$ for at most one vertex $v \in V(P_i) \backslash \{0^n\}$ with the additional condition that at least one path $P_i$ is such that $c(v) = 3$ for every $v \in V(P_i) \backslash \{0^n\}$;

3. $|\{1 \leq i \leq 4 | v \in V(P_i) \backslash \{0^n\} \mid c^B(v) = 2\}| \leq c^B(0^n) \leq 3$;

4. $c^B(w) = 0$ for all $w \notin V_0(B)$.

The following result shows that the ASM on the sequence of Schreier graphs $\{(\Gamma_n, 0^n)\}_{n \geq 1}$ is non-critical in the sense of Definition 2.2.3:

**Theorem 5.1.2.** Let $\xi = 0^\omega$ so that $\lim_{n \to \infty}(\Gamma_n, 0^n) = (\Gamma_\infty, 0^\omega)$. Then, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \cdot 2^{−\frac{\kappa(B)}{3n}} \leq P_{\mu_n}(M_{\text{av}}(\Gamma_n, , 0^n) = M) \leq C_2 \cdot 2^{−n}. \quad (10)$$

**Proof.** Consider the graph $\Gamma_n$ for $n > 5$. Given any recurrent configuration $c$, it follows from Proposition 2.3.4 that the avalanche triggered by adding an extra chip on $0^n$ to $c$ does only depend on the subconfiguration $c^B$ of $c$ on the block $B$ of $\Gamma_n$.

Given an integer $M > 0$, we count the number of recurrent configurations on $\Gamma_n$ producing an avalanche of mass $M$. We first compute the total number of recurrent configurations on the block $B$, which is the number $\kappa(B)$ of spanning trees of $B$. Recall that $|P_1| = |P_2| = 2(\frac{n}{2}) − 1 − 1$ and $|P_3| = |P_4| = 2(\frac{n}{2}) − 1 − 1$. As it does not influence the final result, we omit the additive constant for technical convenience, and we get

$$\kappa(B) = 2|P_1|^2|P_3| + 2|P_3|^2|P_1| = \begin{cases} 2^{3n/2−1} & \text{if } n \text{ is even}, \\ 3 \cdot 2^{(3n−5)/2} & \text{if } n \text{ is odd}. \end{cases}$$

Given a recurrent configuration $c^B$ on the block $B$, denote by $l_i(c^B)$ the distance between $0^n$ and the vertex situated on $P_i$ with only 2 chips on it (see 2. in Proposition 5.1.1). If there is no such vertex on some of the paths $P_i$, then set $l_i(c^B) = |P_i|$. We look now at how the mass of avalanches triggered by adding an extra chip on $0^n$ depends on the $l_i$’s: if at least one of the $l_i$’s grows (respectively decreases), then the mass grows (respectively decreases). Thus, in order to keep the mass $M$ of the avalanche unchanged while modifying the values of the $l_i$’s, we must let some of them grow as well as some of them decrease. Suppose without loss of generality that decorations of odd heights are attached to the paths $P_1$ and $P_2$, whereas decorations of even heights are attached to the paths $P_3$ and $P_4$. It follows (see proof of Proposition 2.3.4) that an increase of $l_1$ (respectively $l_3$) must be thus compensated by a decrease of $l_2$ (respectively $l_4$) whereas an increase of $l_2$ (respectively $l_4$) must be compensated by a decrease of $l_1$ (respectively $l_3$).

Observe now that if $c_1$ and $c_2$ are two recurrent configurations on $\Gamma_n$ such that $l_1(c^B_1) + l_2(c^B_1) \neq l_1(c^B_2) + l_2(c^B_2)$ (or similarly $l_3(c^B_1) + l_4(c^B_1) \neq l_3(c^B_2) + l_4(c^B_2)$), then the masses of the avalanches triggered respectively by $c_1$ and $c_2$ are different.

It follows from the previous observation, that avalanches which are less likely to occur are those of small mass. We derive the lower-bound in (10) by counting the number of recurrent configurations on $B$ leading to avalanches on $\Gamma_n$ of minimal mass. There are exactly two such recurrent configurations $c^B_{\text{min}}$ and $d^B_{\text{min}}$; $c^B_{\text{min}}$ satisfies $l_1(c^B_{\text{min}}) = l_2(c^B_{\text{min}}) = l_3(c^B_{\text{min}}) = 1$ whereas $d^B_{\text{min}}$ satisfies $l_1(d^B_{\text{min}}) = l_2(d^B_{\text{min}}) = l_4(d^B_{\text{min}}) = 1$. Normalizing by $\kappa(B)$ yields the lower bound in (10).

On the other hand, the most likely avalanches arise from recurrent configurations $c$ with $c^B$ on $B$ satisfying $l_1(c^B) + l_2(c^B) = |P_1| + 1$. There are not more than $2|P_2| + 2(|P_1| − 2) \leq 2(\frac{n}{2}) + 1$ such recurrent configurations on $B$. Normalizing by $\kappa(B)$ yields the upper-bound in (10).

**Remark 5.1.3.** A careful computation yields approximate values for the constants $C_1 \approx 3.77$ and $C_2 \approx 5.65$. 

27
5.2 Limit graphs with two ends

The Basilica group provides us with an uncountable family of two-ended graphs not quasi-isometric to $\mathbb{Z}$ (see Section 4). We prove in this subsection that the ASM on the sequences of finite graphs approximating these infinite graphs (in the sense of pointed Gromov-Hausdorff convergence) does not exhibit a critical behaviour with respect to the mass of avalanches.

Some particular cases of sequences of cacti graphs approximating a 2-ended graph were already studied by Ali and Dhar in [1] where they considered graphs obtained from $\mathbb{Z}$ by replacing even edges by cycles of fixed length $L$. (Note that if $L = 2$, the corresponding graphs are essentially the Schreier graphs associated with the self-similar action on the binary rooted tree of the so-called Grigorchuk group, the first example of a group of intermediate growth). Ali and Dhar have found that the ASM on these sequences of decorated chains is not critical; in particular, they have shown that $P_{\mu_n}(M_{\text{av}_C}(\cdot, v_n) = M) \approx f(\frac{1}{n})n^{-1}$ where $f$ denotes some scaling function. The behaviour of avalanches with respect to their mass is thus similar to what one obtains on a sequence of growing cycles $C_n$ of length $n$ approximating the lattice $\mathbb{Z}$ where $P_{\mu_n}(M_{\text{av}_C}(\cdot, v_n) = M) \sim 1/n$ (see Subsection 2.4.1).

Let $\xi \in E_2$, let $\Gamma _{\xi}$ be the corresponding two-ended orbital Schreier graph and let $\{(\Gamma _n, \xi_n)\}_{n \geq 1}$ be the sequence of finite graphs approximating $(\Gamma _{\xi}, \xi)$. Recall that $\mathcal{CP}_{\xi_n}$ denotes the block-path in $\Gamma _n$ joining the vertex $\xi_n$ to the central cycle of $\Gamma _n$.

For any $n \geq 1$, we fix in $\Gamma _n$ two dissipative vertices $p_1$ and $p_2$; these are the two neighbours of $0^n$ such that any path joining $\xi_n$ to $0^n$ contains one of them. As $n$ tends to infinity, the length of the central cycle in $\Gamma _n$ grows and splits in the limit, sending vertex $0^n$ to infinity and giving the bi-infinite path in $\Gamma _{\xi}$ (see Figure 6 and [13]). Consequently, our choice of dissipative vertices corresponds to our Convention 2.2.2.

The recurrent configurations on $\Gamma _n$ are given by Lemma 2.3.2 and Proposition 2.4.2 (as in Subsection 5.1), we may merge both dissipative vertices $p_1$ and $p_2$ into a single one, $p$; the resulting graph $\Gamma _n$ is still separable and $\mathcal{R}_{\Gamma _n} \equiv \mathcal{R}_{\Gamma _n}$.

As in the case of $\Gamma _{\bar{w}}$, the ASM on the sequence of Schreier graphs $\{(\Gamma _n, \xi_n)\}_{n \geq 1}$ has non-critical behaviour if $\xi \in E_2$.

**Theorem 5.2.1.** Let $\xi \in E_2$. Then the ASM on the sequence $\{(\Gamma _n, \xi_n)\}_{n \geq 1}$ converging to the two-ended orbital Schreier graph $(\Gamma _{\xi}, \xi)$ exhibits non-critical behaviour. More precisely, the probability distribution of the mass of an avalanche on $\Gamma _n$ satisfies

$$P_{\mu_n}(M_{\text{av}_{\Gamma_n}}(\cdot, \xi_n) = M) \sim 2^{-\frac{2}{n}}.$$

**Proof.** Let $\xi \in E_2$. Observe that there exists a subsequence $\{n_i\}_{i \geq 1}$ of $\mathbb{N}$ such that, for every $i \geq 1$, the vertex separating the penultimate cycle of the block-path $\mathcal{CP}_{\xi_{n_i}}$ from the last cycle of $\mathcal{CP}_{\xi_{n_i}}$ is different from $0^n$. Let $n \geq 1$ belong to such a subsequence and consider the graph $\Gamma _n$. The root $\xi_n$ belongs to some $k$-decoration attached to the central cycle of $\Gamma _n$ by some vertex $v \neq p_i$, $i = 1, 2$. Note that if we choose $n$ large enough, $k$ does not depend on $n$.

Let $c$ be a (randomly chosen) recurrent configuration on $\Gamma _n$. By Proposition 2.3.4, the mass of the avalanche triggered by adding to $c$ an extra chip on $\xi_n$ depends only on the subconfigurations of $c$ on $\mathcal{CP}_{\xi_n}$.

As the avalanche propagates along the $k$-decoration attached at $v$, a certain amount of chips migrates in the direction of the central cycle of $\Gamma _n$ and finally reaches $v$. If the amount of chips eventually reaching $v$ is greater than one, then necessarily, the avalanche will propagate in both directions on the whole central cycle and the mass of the avalanche will be maximal (denote this mass by $M_{\max}$). The same happens if only one chip reaches $v$ but every vertex on the central cycle has three chips on it. On the other hand, if only one chip reaches $v$ and if there is a vertex on the central cycle with only two chips on it, then the avalanche will propagate along the central cycle in such a way that in one direction it will reach one of the dissipative vertices but in the other direction, it will be stopped at
the vertex with only two chips. Denote by $P$ the probability that at least two chips reach $v$ during an avalanche. Similarly, denote by $\tilde{P}$ the probability that the mass $M$ of the avalanche is greater than the cardinality of the decoration attached to $v$ (which, by Proposition 4.0.5, is equal to $1/3(2^k + 1)$ or $1/3(2^k + 2)$ depending on the parity of $k$). Note that neither $P$ nor $\tilde{P}$ depend on $n$.

Observe that, by Proposition 2.4.4 and its proof, there are at most two subconfigurations on the central cycle producing avalanches of the same mass. Collecting together all previous observations, we have, for $M$ sufficiently large,

$$
\mathbb{P}_{\mu_n}(\text{Mav} \cdot, \xi_n) = M = \begin{cases} 
\tilde{P} \cdot \frac{\alpha}{|C| + 2} & \text{if } M < M_{\text{max}}, \\
\frac{\alpha}{|C| + 2} & \text{if } M = M_{\text{max}}, 
\end{cases}
$$

where $\alpha \in \{1, 2\}$ and $|C|$ denotes the length of the central cycle. Since $|C| \sim 2^{2^k}$, the result follows. \hfill \Box

### 5.3 Limit graphs with one end

Recall from Section 4, that $E_1 \subset \{0, 1\}^\omega$ denotes the subset of full measure consisting of such rays $\xi$ that the infinite orbital Schreier graph $\Gamma_\xi$ has one end. For $\xi \in E_1$, consider the sequence $\{\xi_n\}_{n \geq 1}$ of vertices of the ray belonging to the consecutive levels of the tree, and the rooted finite Schreier graphs $\{(\Gamma_n, \xi_n)\}_{n \geq 1}$ converging to $\{(\Gamma_\xi, \xi)\}$. Let $\mathcal{CP}_{\xi_n} = C_1 \ldots C_{r_n}$ be the unique block-path in $\Gamma_n$ joining $\xi_n$ to the central cycle of $\Gamma_n$. By Lemma 4.0.11, $(\mathcal{CP}_{\xi}, \xi) = \lim_{n \to \infty} (\mathcal{CP}_{\xi_n}, \xi_n)$ is a well-defined block-path isomorphic to the unique block-path of infinite length in $(\Gamma_\xi, \xi)$ starting at $\xi$. Recall that there exists a subsequence $\{n_i\}_{i \geq 1}$ of $\mathbb{N}$ such that for every $i \geq 1$, the vertex separating the penultimate cycle of the block-path $\mathcal{CP}_{\xi_{n_i}}$ from the last cycle of $\mathcal{CP}_{\xi_{n_i}}$ is different from $0^n$. Let $n \geq 1$ belong to such a subsequence. According to Convention 2.4.5, we set $p^{(n)} := 0^n$ in $\Gamma_n$ to be dissipative. The following statement is the main result of this section:

**Theorem 5.3.1.** For almost every $\xi \in E_1$ (with respect to the uniform measure $\lambda$ on $\{0, 1\}^\omega$), we have

$$
\lim_{n \to \infty} \mathbb{P}_{\mu_n}(\text{Mav} \cdot, \xi_n) = M \sim M^{-1}.
$$

As an immediate consequence of Theorem 5.3.1 and Proposition 4.0.9, we have:

**Corollary 5.3.2.** The ASM on the sequence $\{\Gamma_n\}_{n \geq 1}$ of Schreier graphs of the Basilica group is critical in the random weak limit, with critical exponent equal to 1.

Given $\xi \in E_1$, let $(l, \{m_k\}, \{t_k\})$ be the triple provided by Lemma 4.0.13 and let $\{a_i\}_{i \geq 1}$ be the sequence associated with $\xi$ as defined in Lemma 4.0.14, so that the size of the $i$-th block of $\mathcal{CP}_{\xi}$ is $2^{2^i}$. In order to prove Theorem 5.3.1, we will need the following lemma:

**Lemma 5.3.3.** Choose $\xi \in E_1$ uniformly at random. Then, there is almost surely only a finite number of indices $j$ such that the corresponding terms of the sequence $\{m_k\}_{k \geq 0}$ associated with $\xi$ satisfy $m_j \geq 2j$.

**Proof.** With any $\xi \in E_1$ is associated a triple $(l, \{m_k\}, \{t_k\})$ given by Lemma 4.0.13. For any $j \geq 1$, define the event $A_j := \{\xi \in E_1 | m_j \geq 2j\}$. By definition of the sequence $\{m_k\}$ (see (8) in Lemma 4.0.13), for all $r > 0$, we have $\mathbb{P}(m_j \geq 2r) \leq 2^{-r}$. Thus, $\mathbb{P}(A_j) \leq 2^{-j}$ and, by Borel-Cantelli Lemma, $\mathbb{P}(\limsup_{j \to \infty} A_j) = 0$. \hfill \Box

We turn now to the proof of Theorem 5.3.1:
Proof. Choose $\xi \in E_1$ uniformly at random. For any $n \geq 1$, consider the finite Schreier graph $(\Gamma_n, \xi_n)$, the block-path $\mathcal{C}\mathcal{P}_{\xi_n}$ and the sequence $\{a_i\}_{i \geq 1}$ associated with $\xi$ (see Lemma 4.0.14). For further convenience, we interpolate the sequence $\{a_i\}_{i \geq 1}$ by an increasing continuous function $a : [0, +\infty) \to [0, +\infty)$ such that $a(0) = 0$.

Recalling that $|C_i| = 2^{\lceil a(i) / 2 \rceil}$ for every $i \geq 1$, the series

$$\sum_{i=1}^{\infty} \frac{1}{|C_i|} = \sum_{i=1}^{\infty} 2^{-\lceil a(i) \rceil}$$

converges, and it follows from the proof of Theorem 2.4.7 that

$$\frac{L}{2 \cdot |C_{i_1}| \cdot |C_{i_2} + 1|} \leq \mathbb{P}_n (M a v_{\Gamma_n} (\cdot, \xi_n) = M) \leq \frac{2}{|C_{i_1}| \cdot |C_{i_2} + 1|}, \quad (12)$$

where $C_{i_M}$ denotes the block on which each avalanche of mass $M$ stops, and $0 < L \leq 1$ is a constant depending on the sequence $\{a_i\}_{i \geq 1} \equiv \{a(i)\}_{i \geq 1}$. From (12), we get

$$\frac{L}{4} \cdot 2^{-a(i_M) + a(i_M + 1)} \leq \mathbb{P}_n (M a v_{\Gamma_n} (\cdot, \xi_n) = M) \leq 2 \cdot 2^{-a(i_M + 1) + a(i_M + 1)}, \quad (13)$$

On the other hand, the mass of an avalanche which stops on $C_{i_M}$ is bounded by

$$|\mathcal{D}(0^a(i_M - 1) + 1)| < M < |\mathcal{D}(0^a(i_M + 1)|, \quad (14)$$

where $|\mathcal{D}(0^a(i_M + 1)|$ is the number of vertices in the decoration of vertex $0^a(i_M + 1)$ in $\Gamma_{a(i_M + 1)}$. By Proposition 4.0.5, (14) implies

$$\frac{1}{3} (2^a(i_M - 1) + 1) < M < \frac{1}{3} (2^a(i_M + 1) + 2).$$

The previous inequalities can be rewritten as

$$\left\{ \begin{array}{l} a(i_M - 1) < \log(3M - 1) - 1, \\ a(i_M) > \log(3M - 2) - 1 \end{array} \right.$$

where $\log(\cdot) \equiv \log_2(\cdot)$. Since $a$ is increasing, one may write

$$\left\{ \begin{array}{l} i_M < a^{-1}(\log(3M - 1) - 1) + 1, \\ i_M > a^{-1}(\log(3M - 2) - 1). \end{array} \right.$$ 

The difference $a^{-1}(\log(3M - 1) - 1) + 1 - a^{-1}(\log(3M - 2) - 1)$ tends to 1 as $M \to \infty$. We can then assume that $i_M = \lfloor a^{-1}(\log(3M)) \rfloor$ for $M$ sufficiently large.

We show that, almost surely, $a(i_M + 1)/a(i_M)$ tends to 1 as $M \to \infty$. Recall that (see Lemma 4.0.14), for all $j \geq 1$, $0 \leq s < t_j$, $a(T_j - 1 + s + 1) = l + M_j + T_j - 1 + s$, where $M_j := \sum_{k=0}^{j-1} m_k$ and $T_j := \sum_{k=0}^{j-1} t_k$. Writing $i := T_j - 1 + s + 1$, $a(i) = l + M_j + i - 1$; we consider $j \equiv j(i)$ as a (non-decreasing) function of $i$ (corresponding to the number of terms in the sum $M_j$.) Note that $j(i) \leq i$. By Lemma 4.0.14,

$$a(i + 1) - a(i) = \left\{ \begin{array}{l} m_j(i) + 1 \quad \text{if } i \text{ is such that } i = T_j(i), \\ 1 \quad \text{otherwise.} \end{array} \right.$$

30
On the other hand, it follows from Lemma 5.3.3 that, almost surely, there exists \( j_0 \geq 1 \) such that \( m_j \leq 2^j \) for all \( j > j_0 \). We thus have

\[
1 \leq \frac{a(i + 1)}{a(i)} \leq \frac{a(i) + m_{j(i)} + 1}{a(i)} \leq 1 + \frac{2j(i)}{a(i)} + \frac{1}{a(i)}
\]

where the last inequality holds almost surely for any \( i \) sufficiently large. Clearly, \( 2j(i)/a(i) \leq 2j(i)/i \).

We check that \( j(i)/i \), which is non-increasing, tends to 0 as \( i \to \infty \). For the sake of contradiction, suppose that \( j(i)/i \) tends (from above) to \( C > 0 \) as \( i \to \infty \). It is easy to check that, given any finite word \( w \in \{0, 1\}^* \), \( w \) appears almost surely as a subword in \( \xi \in \{0, 1\}^\omega \) situated as far as we want in \( \xi \), i.e., given \( n_0 \geq 1 \), \( \mathbb{P}(\xi = \xi_n w \xi', n \geq n_0, \xi' \in \{0, 1\}^\omega) = 1 \). It follows that, almost surely, the sequence \( \{t_k\}_{k \geq 0} \) (see Lemma 4.0.13) is not bounded. Thus, we can find \( t_{k_0} \) large enough such that \( j(i_0)/i_0 < C \) where \( i_0 = T_{k_0 - 1} + t_{k_0} \) and \( j(i_0) = k_0 \), and we get a contradiction. As \( i_M \) tends to infinity as \( M \to \infty \), we conclude that, almost surely, \( a(i_M + 1)/a(i_M) \) tends to 1 as \( M \to \infty \).

Write \( x_M := a^{-1}(\log(3M)) \) so that \( i_M = \lfloor x_M \rfloor \). For any \( \epsilon > 0 \), there exists \( M_0 \) such that for all \( M > M_0 \), \( a(x_M) < a(i_M + 1) \leq (1 + \epsilon)a(i_M) \). It follows then from (13) that for \( n \) and \( M \) sufficiently large,

\[
\frac{L}{4} \cdot 2 \cdot \frac{n(x_M)(2^{z + 1})}{2} \leq \mathbb{P}_{\mu_n}(\text{Mav}_{\Gamma_n}(\cdot, \xi_n) = M) \leq 2 \cdot 2 \cdot \frac{n(x_M)}{1 + \epsilon},
\]

and hence,

\[
\frac{L}{4} \cdot (3M)^{-\frac{z + 4}{z + 1}} \leq \mathbb{P}_{\mu_n}(\text{Mav}_{\Gamma_n}(\cdot, \xi_n) = M) \leq 2 \cdot (3M)^{-\frac{1}{z + 1}}.
\]

We thus conclude that, almost surely, \( \lim_{n \to \infty} \mathbb{P}_{\mu_n}(\text{Mav}_{\Gamma_n}(\cdot, \xi_n) = M) \sim M^{-1} \).

6 Schreier Graphs of \( IMG(-z^3/2 + 3z/2) \) – Examples with the Critical Exponent > 1

In this section, we examine the ASM on Schreier graphs of still another (though similar to the Basilica) self-similar group, and compute the critical exponent for the mass of avalanches in the random weak limit to be \( 2\log 2 / \log 3 > 1 \).

6.1 Interlaced adding machines

The adding machine \( A \) is a group of automorphisms of the binary rooted tree generated by an automorphism \( a \) defined self-similarly by \( a = (\text{id}, a)(0, 1) \). Thus, the action of \( a \) on the \( n \)-th level of the tree corresponds to adding one to the binary representation of integers modulo \( 2^n \) (recall that vertices of the \( n \)-th level are identified with binary words of length \( n \)). It follows that, for any \( n \geq 1 \), the Schreier graph \( \Gamma(A, \{a\}, \{0, 1\}^\omega) \) is a cycle of length \( 2^n \). The action of the automorphism \( a \) on the boundary of the tree is free and the group generated by \( a \) is \( \mathbb{Z} \). It follows that the orbital Schreier graphs \( \Gamma(A, \{a\}, A \cdot \xi) \), for \( \xi \in \{0, 1\}^\omega \), are all isomorphic (as unlabeled graphs) to the bi-infinite path. In other words, the random weak limit of the sequence \( \{\Gamma(A, \{a\}, \{0, 1\}^n)\}_{n \geq 1} \) is atomic and supported by a single graph, which is \( \mathbb{Z} \). As mentioned in the Introduction, it is easy to see that the ASM is not critical in this case.

The interlaced adding machines group \( I \) is a spherically transitive group of automorphisms of the ternary rooted tree \( T \) generated by two automorphisms \( a \) and \( b \) with the following self-similar structure:
The group $\mathcal{I}$ is the iterated monodromy group of the complex polynomial $-z^3/2 + 3z/2$ (see [30]), whose Julia set is represented in Figure 9.

\[
a = (id, a, id)(0 \ 1), \quad b = (id, id, b)(0 \ 2).
\]

One notices that this Julia set looks very much like the Basilica Julia set (see Figure 3). The Basilica Schreier graphs and the Schreier graphs $\tilde{\Gamma}_n := \Gamma(\mathcal{I}, \{a, b\}, \{0, 1, 2\}^n)$ are also very similar.

It follows directly from the definition of the group $\mathcal{I}$, that for any $n \geq 1$, the Schreier graph $\tilde{\Gamma}_n$ is a 4-regular cactus and has all its edges labeled either by $a$ or by $b$. The number of vertices of $\tilde{\Gamma}_n$ is $3^n$, so that the covering map $\pi_{n+1} : \tilde{\Gamma}_{n+1} \rightarrow \tilde{\Gamma}_n$ is of degree 3.

By [9], the Schreier graphs $\tilde{\Gamma}_\xi := \Gamma(\mathcal{I}, \{a, b\}, \mathcal{I} \cdot \xi)$ have either 1, 2 or 4 ends, and the number of ends is one for almost all $\xi$ with respect to the uniform measure on the boundary $\partial T$ of the tree. More precisely, we have a classification in terms of ternary sequences of the orbital Schreier graphs with respect to their number of ends, in the spirit of the Basilica case treated in [13]. Given a word $w \in \{0, 1, 2\}^*$, we say that $w$ is of type $A$ (respectively $B$) if it does not contain the letter 2 (respectively 1). Any word (finite or infinite) in $\{0, 1, 2\}$ can be decomposed into an alternative succession of blocks of type $A$ and $B$.

**Theorem 6.1.1.**

1. The orbital Schreier graph $\tilde{\Gamma}_\xi$ has one end if and only if the number of blocks in the decomposition of $\xi$ into blocks of type $A$ and $B$ is infinite;

2. the orbital Schreier graph $\tilde{\Gamma}_\xi$ has four ends if and only if $\xi \in \{w0^\omega, w1^\omega, w2^\omega | w \in \{0, 1, 2\}^*\}$;

3. in all other cases, the orbital Schreier graph $\tilde{\Gamma}_\xi$ has two ends.

For $i = 1, 2, 4$, denote by $\tilde{E}_i := \{\xi \in \{0, 1, 2\}^* | \text{the orbital Schreier graph } \tilde{\Gamma}_\xi \text{ has } i \text{ ends} \}$. Moreover, we also have:

**Proposition 6.1.2.** There exist uncountably many non-isomorphic orbital Schreier graphs with one end.

Proposition 6.1.2 follows from Theorem 6.1.1 together with the following lemma, proved similarly to Proposition 5.6 in [13].

**Lemma 6.1.3.** Let $w \in \{0, 1, 2\}^n$. Then,

1. the total number of blocks in the decomposition of $w$ into blocks of type $A$ and $B$ equals the number of blocks in the block-path $CP_w$ in $\tilde{\Gamma}_n$ joining $w$ to $0^n$.
2. the size of the $i$-th block in the block-path $CP_w$ is equal to $2\nu_i$, where $\nu_i$ denotes the length of the prefix of $w$ containing the $i$ first blocks.

We will also need the following result obtained by following the method developed in [8].

**Proposition 6.1.4.** For almost every $\xi \in \{0, 1, 2\}^\omega$ (with respect to the uniform measure $\lambda$ on $\xi \in \{0, 1, 2\}^\omega$), the degree of polynomial growth of $\tilde{\Gamma}_\xi$ is $\log 3/\log 2$.

### 6.2 Criticality of the ASM on the Schreier graphs of $IMG(-z^3/2 + 3z/2)$

In this subsection, we consider avalanches of the ASM on sequences of finite rooted Schreier graphs $\{\tilde{\Gamma}_n, \xi_n\}_{n \geq 1}$ converging to the infinite orbital Schreier graph $(\tilde{\Gamma}_\xi, \xi)$, where $\xi \in \tilde{E}_1$. For any $n \geq 1$, we choose $0^n$ as the unique dissipative vertex in the graph $\tilde{\Gamma}_n$. It follows from Lemma 6.1.3 and Theorem 6.1.1, that the number of blocks in the block-path $CP_{\xi_n}$ joining $\xi_n$ to $0^n$ in $(\tilde{\Gamma}_n, \xi_n)$ tends to infinity as $n \to \infty$. Our choice for the dissipative vertex is thus in accordance with Convention 2.4.5.

We will prove the following:

**Theorem 6.2.1.** For almost every $\xi \in \tilde{E}_1$ (with respect to the uniform measure $\lambda$ on $\{0, 1, 2\}^\omega$), we have

\[
\lim_{n \to \infty} P_{\mu_n}(\text{Max}_{\tilde{\Gamma}_n}(\cdot, \xi_n) = M) \sim M^{-\frac{2\log 2}{\log 3}}.
\]

**Corollary 6.2.2.** We thus exhibit an uncountable family of non-isomorphic $4$-regular, one-ended graphs of superlinear but subquadratic growth, such that the ASM on the sequences of finite graphs approximating them is critical with critical exponent equal to $2\log 2/\log 3 > 1$.

**Proof.** Let $\xi \in \tilde{E}_1$ and let $\xi = A_1B_1A_2B_2 \ldots$ be its decomposition in blocks of type $A$ and $B$ ($A_1$ may be empty). For any $n \geq 1$, consider the Schreier graph $(\tilde{\Gamma}_n, \xi_n)$, the block-path $CP_{\xi_n}$ in $\tilde{\Gamma}_n$ and denote its blocks by $C_1C_2 \ldots C_r_n$, so that $r_n$ is the number of blocks in the above decomposition of the prefix $\xi_n$ of $\xi$. By Lemma 6.1.3, for any $i \geq 1$, the size of $C_i$ is given by

\[
\log_2(|C_i|) = \nu_i = \left\{ \begin{array}{ll}
\sum_{k=1}^{i/2} \frac{|A_k| + |B_k|}{|A_k| + |B_k| + |A_{i(k+1)/2}|} & \text{if } i \text{ is even}, \\
\sum_{k=1}^{(i-1)/2} \frac{|A_k| + |B_k|}{|A_k| + |B_k| + |A_{i(k+1)/2}|} & \text{if } i \text{ is odd,}
\end{array} \right.
\]  \hfill (15)

where $|A_k|$ (respectively $|B_k|$) denotes the length of the block $A_k$ (respectively $B_k$). For further convenience, we interpolate the sequence $\{\nu_i\}_{i \geq 1}$ by a continuous, increasing function $\nu: [0, +\infty) \to [0, +\infty)$ such that $\nu(0) = 0$. As the series $\sum_{i \geq 1} \frac{1}{\nu_i}$ converges, it follows from the proof of Theorem 2.4.7, that

\[
\frac{L}{2 \cdot |C_{i_M}| \cdot |C_{i_M+1}|} = P_{\mu_n}(\text{Max}_{\tilde{\Gamma}_n}(\cdot, \xi_n) = M) \leq \frac{2}{|C_{i_M}| \cdot |C_{i_M+1}|} \tag{16}
\]

where $C_{i_M}$ denotes the block on which each avalanche of mass $M$ stops, and $0 < L \leq 1$ is a constant depending on the sequence $\{\nu_i\}_{i \geq 1} \equiv \{\nu(i)\}_{i \geq 1}$. From (16), we get

\[
\frac{L}{2} \cdot 2^{- \nu(i_M) + \nu(i_M+1)} = P_{\mu_n}(\text{Max}_{\tilde{\Gamma}_n}(\cdot, \xi_n) = M) \leq 2 \cdot 2^{-\nu(i_M) + \nu(i_M+1)}. \tag{17}
\]

Observe that, for any $n \geq 1$ and $1 \leq k \leq n$, the cardinality of a $k$-decoration in $\tilde{\Gamma}_n$ (see Definition 4.0.4) is equal to $1/2(3^k + 1)$. It follows that the mass of an avalanche which stops on $C_{i_M}$ is bounded by
\[
1/2(3^{\nu(i_M)} + 1) < M < 1/2(3^{\nu(i_M)} + 1).
\]

Since \( \nu \) is increasing, this leads to
\[
\begin{align*}
\{ i_M &< \nu^{-1}(\log_3(2M - 1)) + 1, \\
i_M &> \nu^{-1}(\log_3(2M - 1)).
\end{align*}
\]

As \( i_M \) is an integer, we have \( i_M = \lfloor \nu^{-1}(\log_3(2M - 1)) \rfloor \).

**Lemma 6.2.3.** For almost every \( \xi \in \mathcal{E}_1 \), \( \lim_{i \to \infty} \frac{\nu(i+1)}{\nu(i)} = 1. \)

**Proof of the lemma:** By (15), \( \{ \nu(i) \}_{i \geq 1} \) satisfies
\[
\nu(i + 1) - \nu(i) = \begin{cases} \left| A_{i/2+1} \right| & \text{if } i \text{ is even,} \\
\left| B_{(i+1)/2} \right| & \text{if } i \text{ is odd.}
\end{cases}
\]

For every \( k \geq 1 \), define the event \( E_k := \{ \xi \in \{0,1,2\}^\omega \mid |A_k| \geq k \} \) (respectively \( \bar{E}_k := \{ \xi \in \{0,1,2\}^\omega \mid |B_k| \geq k \} \)). As \( \mathbb{P}(E_k) \leq \left(\frac{2}{3}\right)^k \) and \( \sum_{k \geq 1} \left(\frac{2}{3}\right)^k < \infty \), it follows from Borel-Cantelli Lemma, that \( \mathbb{P}(\lim \sup_{k \to \infty} E_k) = 0 \). Identically, \( \mathbb{P}(\lim \sup_{k \to \infty} \bar{E}_k) = 0 \). In other words, there almost surely exists \( i_0 \geq 1 \) such that, for all \( i > i_0 \), \( |A_{i/2+1}| < i/2 + 1 \) (respectively \( |B_{(i+1)/2}| < (i+1)/2 \)).

We have, for \( i \) even,
\[
1 \leq \frac{\nu(i + 1)}{\nu(i)} = \frac{\nu(i) + |A_{i/2+1}|}{\nu(i)} \leq 1 + \frac{i}{2\nu(i)} + \frac{1}{\nu(i)},
\]
where the last inequality holds almost surely for any \( i \) sufficiently large. The same bound holds for \( i \) odd. Using a similar argument than in the proof of Theorem 5.3.1, we check that \( i/\nu(i) \), which is non-increasing, tends to 0 as \( i \to \infty \).

Let \( x_M = \nu^{-1}(\log_3(2M - 1)) + 1 \), so that \( i_M = \lfloor x_M \rfloor \). By Lemma 6.2.3, for any \( \epsilon > 0 \), there exists \( M_0 \) such that for any \( M > M_0 \), \( \nu(i_M + 1) \leq (1 + \epsilon)\nu(i_M) \). From (17), we get
\[
\frac{L}{2} \cdot 2^{-(2+\epsilon)\nu(i_M)} \leq \mathbb{P}_{\mu_n}(MAv_{\Gamma_n}(\cdot, \xi_n) = M) \leq 2 \cdot 2^{-3\epsilon\nu(i_M) + 1}.
\]

As \( i_M \leq x_M \leq i_M + 2 \) and \( \nu \) is increasing, we have
\[
\frac{L}{2} \cdot 2^{-(2+\epsilon)\nu(x_M)} \leq \mathbb{P}_{\mu_n}(MAv_{\Gamma_n}(\cdot, \xi_n) = M) \leq 2 \cdot 2^{-3\epsilon\nu(x_M) - 1}.
\]

Using, for the lower bound, the fact that for any \( \epsilon' > 0 \), there is \( M'_0 \) such that for any \( M > M'_0 \), \( \nu(\nu^{-1}(\log_3(2M - 1)) + 1) \leq (1 + \epsilon') \log_3(2M - 1) \), we get
\[
\frac{L}{2} \cdot 2^{-(2+\epsilon)(1+\epsilon') \log_3(2M-1)} \leq \mathbb{P}_{\mu_n}(MAv_{\Gamma_n}(\cdot, \xi_n) = M) \leq 2 \cdot 2^{-2(2+\epsilon)(1+\epsilon') \log_3(2M-1)}
\]
which is equivalent to
\[
\frac{L}{2} \cdot (2M - 1)^{-(2+\epsilon)(1+\epsilon') \log_2(3)} \leq \mathbb{P}_{\mu_n}(MAv_{\Gamma_n}(\cdot, \xi_n) = M) \leq 2 \cdot (2M - 1)^{-(2+\epsilon)(1+\epsilon') \log_2(3)}.
\]

Finally, letting \( \epsilon, \epsilon' \to 0 \), we get the result. \( \square \)
7 Growth of Orbital Schreier Graphs and Critical Exponent of the ASM

In this section we show that, under some conditions, the critical exponent of the ASM on a sequence of finite rooted cacti converging in \((X, \text{Dist})\) to a one-ended limit graph is related to the growth of that limit graph. Then, we exhibit a family of iterated monodromy groups of quadratic polynomials such that the ASM on the corresponding sequences of Schreier graphs is critical in the random weak limit, with arbitrarily small critical exponent.

7.1 Degree of polynomial growth of orbital Schreier graphs and critical exponent

Given a locally finite graph \(\Gamma\) and \(v \in V(\Gamma)\), we say that \(\Gamma\) has polynomial growth of degree \(\alpha\) if the quantity

\[
\alpha := \limsup_{r \to \infty} \frac{\log(|B_r(v, r)|)}{\log r}
\]

is finite. Note that \(\alpha\) does not depend on the choice of \(v\).

Let \(\{(\Gamma_n, v_n)\}_{n \geq 1}\) be a sequence of finite rooted cacti converging in \((X, \text{Dist})\) to a one-ended cactus \((\Gamma, v)\). For any \(n \geq 1\), let \(p(n)\) be the unique dissipative vertex in \(\Gamma_n\) chosen accordingly to Convention 2.4.5. Recall from Subsection 2.4.2 that, for any \(n \geq 1\), \(C \cdot \mathcal{P}_{v_n} = C_1 \ldots C_r\) in \(\Gamma_n\) denotes the unique block-path joining vertex \(v_n\) to \(p(n)\); for each \(1 \leq i < r_n\), \(p_i\) is the separation vertex between \(C_i\) and \(C_{i+1}\), and \(D(p_i)\) is the subgraph induced by \(\{v \in V(\Gamma_n) | v \geq p_i\}\). Recall also that \(d_i\) denotes the number of vertices in \(D(p_i)\) (we omit indexation by \(n\) as, for any \(i \geq 1\), \(D(p_i)\) does not depend on \(n\), for any \(n\) sufficiently large.)

**Theorem 7.1.1.** Let \((\Gamma, v)\) be a one-ended infinite cactus obtained as the limit of a sequence of finite rooted cacti \(\{(\Gamma_n, v_n)\}_{n \geq 1}\). Suppose that \(\sum_{i=1}^{\infty} |C_i|^{-1}\) converges as \(r_n \to \infty\) and suppose that the subgraphs \(D(p_i)\), \(i \geq 1\), satisfy the following requirements:

1. there exists a constant \(c > 0\) such that, for any \(i\) sufficiently large, \(\text{Diam}(D(p_i)) \leq c|C_i|\);
2. \(\lim_{i \to \infty} \frac{\log d_{i-1}}{\log d_i} = 1\).

Then, for any \(\epsilon > 0\), there exists \(M_0\) such that, for any \(M > M_0\)

\[
C_1 \cdot M^{-\frac{\beta}{\beta'}} \leq \lim_{n \to \infty} \mathbb{P}_{\mu_n}(\text{Mat} \Gamma_n, (v_n) = M) \leq C_2 \cdot M^{-\frac{\beta}{\beta'}},
\]

where \(C_1, C_2 > 0\), \(\beta := \limsup_{i \to \infty} \frac{\log d_i}{\log \text{Diam}(D(p_i))}\) and \(\beta' := \liminf_{i \to \infty} \frac{\log d_i}{\log \text{Diam}(D(p_i))}\).

In particular, if \(\beta = \beta'\), then the ASM on the sequence \(\{(\Gamma_n, v_n)\}_{n \geq 1}\) is critical (in the sense of Definition 2.2.3) with critical exponent equal to \(\delta = 2/\beta\).

**Corollary 7.1.2.** Let \((\Gamma, v)\) be as in Theorem 7.1.1. Suppose that \(\Gamma\) has polynomial growth and that its degree of growth \(\alpha\) is given by the quantity \(\lim_{r \to \infty} \frac{\log \text{Diam}(D(p_i))}{\log d_i}\). Then, the critical exponent \(\delta\) is related to the growth degree \(\alpha\) of \(\Gamma\) by \(\delta = 2/\alpha\).

**Proof.** From Theorem 2.4.7 and its proof, for any integer \(M\) that occurs as the mass of an avalanche, we have...
\[
\frac{L}{2 \cdot |C_{i_M}| \cdot |C_{i_{M+1}}|} \leq \mathbb{P}_{\mu_n}(Mav_{\Gamma_n}(\cdot, v_n) = M) \leq \frac{2}{|C_{i_M}| \cdot |C_{i_{M+1}}|}
\]

where the index \(i_M\) is uniquely determined by the condition \(d_{i_M-1} \leq M < d_{i_M}\) and \(0 < L \leq 1\) is a constant. Applying logarithm to these inequalities and normalizing, we get

\[
\frac{\log d_{i_M-1}}{\log Diam(D(p_{i_M}))} \leq \frac{\log M}{\log Diam(D(p_{i_M}))} < \frac{\log d_{i_M}}{\log Diam(D(p_{i_M}))}.
\]

By condition 2., we have

\[
\liminf_{M \to \infty} \frac{\log d_{i_M-1}}{\log Diam(D(p_{i_M}))} = \liminf_{M \to \infty} \frac{\log d_{i_M}}{\log Diam(D(p_{i_M}))} = \beta'.
\]

On the other hand, condition 1. implies that for any \(M\) sufficiently large,

\[
\frac{\log M}{\log (c|C_{i_M}|)} \leq \frac{\log M}{\log Diam(D(p_{i_M}))} \leq \frac{\log M}{\log (\tilde{c}|C_{i_M}|)}
\]

(the upper bound follows from the fact that, by definition, \(C_i \subset D(p_i)\) for any \(i \geq 1\).) Hence, for any \(\epsilon > 0\), there exists \(M_0\) such that for any \(M > M_0\),

\[
\beta' - \epsilon < \frac{\log M}{\log (c|C_{i_M}|)} < \beta + \epsilon
\]

which is equivalent to

\[
\frac{1}{e} M^{\frac{\beta}{\beta' - \epsilon}} \leq |C_{i_M}| \leq \frac{1}{e} M^{\frac{\beta}{\beta' + \epsilon}}.
\]

If we normalize in (19) by \(\log Diam(D(p_{i_M+1}))\), we obtain similarly

\[
\frac{1}{e} M^{\frac{\beta}{\beta' + \epsilon}} < |C_{i_M+1}| \leq \frac{1}{e} M^{\frac{\beta}{\beta' - \epsilon}}.
\]

Thus, we have

\[
e^2 M^{\frac{\beta}{\beta' + \epsilon}} < \frac{1}{|C_{i_M}| \cdot |C_{i_{M+1}}|} < e^2 M^{\frac{\beta}{\beta' - \epsilon}}
\]

and replacing in (18), we get the result.

\[\Box\]

### 7.2 Examples with arbitrarily small critical exponent

We will now consider a particular family of self-similar groups of automorphisms of the binary rooted tree that gives rise to Schreier graphs of bigger and bigger degree and of bigger and bigger polynomial growth. These graphs satisfy the conditions of our Theorem 7.1.1, and thus provide examples of criticality with critical exponent arbitrarily close to 0.

The groups we are going to consider are realized as iterated monodromy groups of quadratic polynomials \(z^2 + c\), where the parameter \(c\) is chosen to be the center of one of the secondary \(p/q\)-components.
of the Mandelbrot set, so that the critical point 0 of the polynomial \( z^2 + c \) belongs to a super-attracting cycle of length \( q \geq 2 \). The case \( q = 2 \) corresponds to the Basilica group, see Figure 3, and the case \( q = 3 \) is the so-called Douady rabbit, see Figure 11.

If the orbit of 0 under iterations of the polynomial \( z^2 + c \) is a finite cycle, one can associate to the polynomial a \emph{kneading} automaton \( A_v \), where \( v \) is a finite binary word, and the self-similar group \( \mathcal{K}(v) \) generated by \( A_v \) is the iterated monodromy group of \( z^2 + c \) (see Chapters 6.6-6.11 in [31]). The length of the word \( v \) is equal to the size of the orbit of 0 under iterations of the polynomial. For a word \( v = x_1 x_2 \ldots x_{k-1} \in \{0,1\}^{k-1}, k > 1 \), the automaton \( A_v \) has \( k+1 \) states (including the identity state) and its Moore diagram is pictured in Figure 10 (for \( x \in \{0,1\} \), we write \( \bar{x} := 1 - x \)):

\[
\begin{array}{c}
\xymatrix{
  & a_1 \ar[r]^{0|1} & a_k \\
  \bar{x}_i \ar[r]_{x_{i-1}|x_{i-1}} & \bar{x}_i \\
  & \cdots & \\
  & id \ar[r]_{x_{k-1}|x_{k-1}} & \bar{x}_{k-1} \\
  a_1 \ar[r]_{x_1|x_1} & \cdots & a_{k-1} \\
  a_i \ar[r]_{x_i|x_i} & \cdots & a_{i+1} \\
  & \cdots & \\
  & id \ar[r]_{x_{k-1}|x_{k-1}} & \bar{x}_{k-1} \\
  & a_k \ar[r]^{1|0} & a_1
}\end{array}
\]

Figure 10: The automaton \( A_v \) corresponding to the word \( v = x_1 x_2 \ldots x_{k-1} \).

Consequently, the generators \( \{a_1, \ldots, a_k\} \) of the group \( \mathcal{K}(v) \) generated by \( A_v \) have the following self-similar structure:

\[
a_1 = (a_k, id)(0 1), \quad a_{i+1} = \begin{cases} (a_i, id)e & \text{if } x_i = 0, \\ (id, a_i)e & \text{if } x_i = 1, \end{cases} \quad \text{for } i = 1, \ldots, k - 1.
\]

We can, for example, consider the family of groups \( \mathcal{K}(0^{k-1}) \) for \( k > 1 \). The group \( \mathcal{K}(0) \) is the Basilica group that we have already studied in Sections 4 and 5, whereas \( \mathcal{K}(00) \) is the group \( \text{IMG}(z^2 + c) \) where \( c \approx -0.1225 + 0.7448i \). The Julia set of this group, called the \emph{Douady Rabbit}, is represented in Figure 11.
For any $k > 1$, the group $\mathcal{K}(0^{k-1})$ is the iterated monodromy group of a post-critically finite polynomial and, by Theorem 3.1.2, the Schreier graphs of the action of $\mathcal{K}(0^{k-1})$ on the levels of the binary rooted tree are cacti. By extending to the groups $\mathcal{K}(0^{k-1})$, for any $k > 1$, the analysis done for the Basilica group, we obtain the following description of the finite Schreier graphs:

**Proposition 7.2.1.** Let $k \geq 2$, and consider the Schreier graphs $\Gamma_n := \Gamma(\mathcal{K}(0^{k-1}), \{a_1, \ldots, a_k\}, \{0, 1\}^n)$ of the action of $\mathcal{K}(0^{k-1})$ on the levels of the binary rooted tree. Given $\xi_n \in \Gamma_n$, let $CP_{\xi_n} = C_1 \ldots C_r$ be the block-path joining $\xi_n$ to the vertex $0^n$ in $\Gamma_n$. Then, $r_n \leq n$ and the sizes of the blocks of $CP_{\xi_n}$ are given by $|C_j| = 2^{b_j}$, where the sequence $\{b_j\}_{j=1}^r$ is a non-decreasing sequence of positive integers with no constant segments of length greater than $k$.

By [9], almost all orbital Schreier graphs $\Gamma_\xi := \Gamma(\mathcal{K}(0^{k-1}), \{a_1, \ldots, a_k\}, \mathcal{K}(0^{k-1}) \cdot \xi)$ (with respect to the uniform distribution on the boundary $\partial T$ of the tree) have one end. Denote by $E_1 \subset \partial T$ the set of full measure constituted of infinite words $\xi$ such that the corresponding orbital Schreier graph $\Gamma_\xi$ has one end. For $\xi \in E_1$, consider the sequence of finite rooted Schreier graphs $\{(\Gamma_n, \xi_n)\}_{n \geq 1}$. According to Convention 2.4.5, we fix one dissipative vertex $p(\cdot) := 0^n$ in each $\Gamma_n$. It thus follows from Proposition 7.2.1, that the sequence $\{(\Gamma_n, \xi_n)\}_{n \geq 1}$ satisfies the assumptions of Theorem 2.4.7.

On the other hand, the orbital Schreier graphs $\Gamma_\xi$ have polynomial growth, and by applying an algorithm from [8], we show that the degree of polynomial growth grows with $k$ (essentially this is due to the fact that the graphs are $2k$-regular.)

**Proposition 7.2.2.** The degree of polynomial growth of the orbital Schreier graphs $\Gamma_\xi$ of the action of $\mathcal{K}(0^{k-1})$ on $\partial T$ is at least $k/2$.

One also verifies that, for almost every $\xi \in E_1$, the orbital Schreier graph $(\Gamma_\xi, \xi)$ satisfies the assumptions of Theorem 7.1.1 with $\beta = \beta' = k$, which then implies the following:

**Theorem 7.2.3.** For $k \geq 2$, the ASM on the sequence $\{\Gamma_n\}_{n \geq 1}$ of Schreier graphs of the action of $\mathcal{K}(0^{k-1})$ is critical in the random weak limit (in the sense of Definition 2.2.6) with critical exponent $\delta = 2/k$.

**Corollary 7.2.4.** $\{\mathcal{K}(0^{k-1})\}_{k \geq 2}$ is a family of self-similar groups such that the ASM on the associated sequences of Schreier graphs is critical in the random weak limit, and the critical exponent $\delta > 0$ can be arbitrarily small.
Acknowledgements

Figure 11 is reproduced under the terms of Creative Commons Attribution-ShareAlike 3.0 license. Figures 2, 4, 5 and 7 were published for the first time in [13].

References


