Abstracts

On subgroup structure of a 3-generated 2-group of intermediate growth

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1. Introduction.

A group is *branch* if it acts faithfully on a spherically homogeneous rooted tree and has the lattice of subnormal subgroups similar to the structure of the tree, [6]. A group G is *self-similar* if it has a faithful action on a d-regular rooted tree, $d \geq 2$, such that the section of any element $g \in G$ is again an element of the group modulo the canonical identification of the subtree and the original tree. Equivalently, it is generated by states of non-initial invertible Mealy type automaton, [11]. Precise definitions, more details, and relevant references can be found in [2, 11, 13].

Branch groups constitute one of three classes of just infinite groups [7]. Self-similar groups appear naturally in holomorphic dynamics [13]. Although quite different, these two classes of groups have large intersection, and many self-similar groups are also branch. In the class of finitely generated branch self-similar groups there are torsion groups and torsion free groups; groups of intermediate growth and groups of exponential growth; amenable and nonamenable groups. Branch self-similar groups have very interesting subgroup structure.

Among popular examples of branch self-similar groups is the 3-generated 2-group G of intermediate growth [5]. See [12] for an introduction to this group and [8] for detailed information and a list of open problems about it. Much is known about such subgroups of G as the stabilizers of vertices of the rooted binary tree (on which G acts) and of points on the boundary of the tree; the rigid stabilizers; the centralizers ; certain normal subgroups [4]). The group G has the congruence subgroup property (that is, every subgroup of finite index contains a stabilizer $st_G(n)$ of some level n), which allows to investigate its profinite completion $\hat{G}$ [10].

2. On closed subgroups of G. The result.

The main goal of this research is to understand subgroups of G closed in profinite topology (the group G is residually finite). One class of such subgroups consists of finitely generated subgroups, as proven in [9]. It is shown there that every finitely generated subgroup of G is (abstractly) commensurable with G. This unusual property relies on the fundamental result of Pervova [14] that every maximal subgroup of G has finite (hence $= 2$) index. For just infinite groups the property to have all maximal subgroups of finite index is preserved when passing to commensurable groups, and thus weakly maximal subgroups in G are closed in profinite topology (a subgroup is *weakly maximal* if it has infinite index and is maximal with respect to this property). For a branch group G, the stabilizers of points in the boundary of the tree are examples of weakly maximal subgroups. It would be interesting to describe all weakly maximal subgroups in G .

Torsion p-groups are of special interest in connection with the Kaplansky conjecture on Jacobson radical, which states that, in the case of a field of characteristic p, the Jacobson radical $JK[G]$ coincides with the augmentation radical $AK[G]$ if and only if the group is locally finite p-group. It is known that if $JK[G] = AK[G]$, any maximal subgroup of G is normal of finite index p . Therefore counterexamples (if they exist) to Kaplansky conjecture should lie in the class of p-groups with all maximal subgroups of finite index. If the group has as a homomorphic image onto a group which has maximal subgroup of infinite index then the group itself has a maximal subgroup of infinite index. Therefore it is natural to investigate which just infinite groups have this unusual property. In view of the trichotomy for just infinite groups mentioned above, and as finitely generated simple groups obviously are primitive, one should concentrate on the following two questions. Is it true that a finitely generated branch group has maximal subgroups only of finite index? Is it true that every finitely generated hereditary just infinite group has a maximal subgroup of infinite index?

The property of a group to have finitely generated subgroups closed in profinite topology is quite rare. It holds for free groups by a celebrated result of Marshall Hall Jr., as well as for a few other classes of groups. In [15] it is proven that a subset of a free group which is a product of finitely many finitely generated subgroups is closed in profinite topology. This remarkable property is known (in finitely generated case) only for free groups and their trivial generalizations. We believe that every subset of G which is a product of finitely many finitely generated groups is closed in profinite topology. Our results may be considered as positive evidence towards this statement.

Let $V(T)$ be a set of vertices of the rooted binary tree. The group G is regularly branch over the subgroup $K =< [a, b] >^G$. This implies that for any vertex $u \in V(T)$, the copy K_u of K acting on the subtree T_u with the root u and acting trivially outside T_u is a subgroup of K. We shall say that two vertices u, v are *orthogonal* if the subtrees T_u, T_v do not intersect. A subset $U \subset V(T)$ is called *orthogonal* if it consist of pairwise orthogonal vertices. It is called *a section* if every infinite geodesic ray from the root of the tree intersects U in exactly one point. It is clear that a section is a finite set. Two subsets $U, V \subset V(T)$ are *orthogonal* if every vertex of one set is orthogonal to every vertex of the other set. We consider the lexicographic order on $V(T)$.

Let $U = (u_1, \ldots, u_k)$ be an ordered orthogonal set of cardinality ≥ 2. Let $\Phi = (\phi_2, \ldots, \phi_k)$ be a set of isomorphisms $\phi_i : K_{u_1} \to K_{u_i}, i = 2, \ldots, k$. Then the pair (U, Φ) determines a *diagonal subgroup* D (abstractly isomorphic to K), consisting of elements g acting as $g \in K$ on the subtree T_{u_1}; as $\phi_i(g)$ on the subtree $T_{u_i}, i = 2, \ldots, k$; and trivially on the rest of the tree T.

Next we define a *block subgroup*. Let $(U_0, \{U_i\}_{i\in I})$ be a finite family of orthogonal, pairwise orthogonal subsets of $V(T)$ with $|U_i| \geq 2$. Let $\{\Phi_i\}_{i\in I}$ be a collection of isomorphisms corresponding to $\{U_i\}_{i\in I}$ and $\{D_i\}_{i\in I}$ be the set of corresponding diagonal subgroups . The union of sets $(U_0, \{U_i\}_{i\in I})$ can be extended to a section

S of the tree. These data determine a *block subgroup*

$$B = \prod_{u \in U_0} K_u \times \prod_{i \in I} D_i \times \prod_{v \in S \setminus (U_0 \cup \bigcup_{i \in I} U_i)} \{1\}$$

Theorem 1. *Let $H \leq G$ be a finitely generated subgroup of G. Then there is block subgroup H_1 of H of finite index.*

This subgroup can be found algorithmically, given generators of H.

In addition to techniques developed in [9], the proof of Theorem 1 uses the following new results.

Theorem 2. *Let $H \leq G$ be a subgroup of finite index in G, and suppose that $H \simeq K^m$ for some $m \leq 1$. Then there is a section S, $|S| = m$ such that $H = \prod_{v \in S} K_v$. In particular if a subgroup of finite index in G is isomorphic to K then it is equal to K.*

Theorem 3. *The group G has no proper subgroups of finite index isomorphic to the group.*

Note that as G is self-similar, it has many proper subgroups isomorphic to it, but, by Theorem 3, all of them are of infinite index. This last result relates to investigation of a strong version of co-hopfianity called scale-invariance which asks for a group to have proper subgroups of finite index isomorphic to the group, with an additional condition that intersections of nested sequences of such subgroups should be finite [3].

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