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GREEN FUNCTIONS ON TREES WITH FINITELY MANY CONE TYPES

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ABSTRACT

We prove that the Green function of a nearest neighbour random walk on a tree with finitely many cone types is algebraic. Analysis of this function for a special class of trees associated to groups leads to new estimates of spectral radii of groups with finitely many cone types.

1. Introduction

Let Γ be an infinite, locally finite, connected, simple graph with the vertex set $V(\Gamma)$. A random walk on Γ is given by a stochastic transition matrix

$$P = \big(p(x,y)\big)_{x,y\in V(\Gamma)} .$$

In this paper we will only consider *nearest neighbour* random walks, i.e., such that $p(x,y) \neq 0$ implies $x \sim y$. We also suppose P irreducible, i.e., for every $x, y \in V(\Gamma)$ there exists $n \in \mathbb{N}$ such that the probability to go from x to y in n steps $p^{(n)}(x,y)$ is positive. The transition matrix P acts on real valued functions on $V(\Gamma)$ by

$$P\phi(x) = \sum_{y\in V(\Gamma)} p(x,y)\phi(y) .$$

A function ϕ is called *harmonic* if $P\phi = \phi$.

The *Green kernel* on Γ associated with the random walk P is defined by

$$G(x,y \mid z) = \sum_{n=0}^{\infty} p^{(n)}(x,y)z^n .$$

The radius of convergence of the series $G(x,y \mid z)$ is $R_G = 1/\rho \geq 1$ with

$$\rho = \limsup_{n\to\infty} \big(p^{(n)}(x,y)\big)^{\frac{1}{n}} .$$

It is independent of x and y because P is irreducible. The number ρ is the *spectral radius* of the random walk P. It can be viewed as the spectral radius of the transition operator P acting on the space $l^2_\sharp(V(\Gamma))$ with the inner product

$$\langle \phi, \psi\rangle_\sharp = \sum_{x\in V(\Gamma)} \phi(x)\psi(x)deg(x) .$$

Equivalently, the spectral radius is characterized by (see [Pr])

$$\rho = \min\{\lambda \mid \exists\phi : V(\Gamma) \to \mathbb{R}_+ \text{ such that } P\phi = \lambda\phi\} .$$

Denote by $q^{(n)}(x,y)$ the probability, starting from x, to reach y for the first time after n steps; and consider the kernel

$$F(x,y \mid z) = \sum_{n=0}^{\infty} q^{(n)}(x,y)z^n .$$

Then equality

$$G(x,x \mid z) = \big(1 - F(x,x \mid z)\big)^{-1} \tag{1}$$

holds in the disc of convergence of G. The function $F(x,x \mid z)$ is analytic, thus continuous in its disc of convergence; $F(x,x \mid 0) = 0$; and it is increasing in the intersection of $\mathbb{R}_+$ with its disc of convergence. Therefore, either $F(x,x \mid R_F) \leq 1$, or, by continuity, there exists a unique real positive z_0 in the disc of convergence of $F(x, x \mid z)$, such that $F(x, x \mid z_0) = 1$. Thus, the above formula implies

$$R_G = \begin{cases} R_F & \text{if } F(x, x \mid R_F) \leq 1, \\ z_0 & \text{otherwise.} \end{cases}$$

The power series $F(x, y \mid z)$ (respectively $G(x, y \mid z)$) has non-negative coefficients, hence, by Prinsgheim's Theorem (see [Rem, page 235]), R_F (respectively R_G) is a singular point of $F(x, y \mid z)$ (respectively $G(x, y \mid z)$).

Our main object of attention in this paper will be the Green function of a nearest neighbour random walk on a tree with finitely many cone types. Random walks on trees have been extensively studied in the past two decades. The properties of the Green functions were described in [St], [GW], [W], [L] for random walks on homogeneous trees. As for trees with finitely many cone types, the asymptotic behaviour of random walks on such trees was studied in [Ly] and [T].

For $x, y \in V(\Gamma)$ a *path* of length n from x to y is a sequence $x = x_1, x_2, ..., x_{n+1} = y$ of vertices, such that, for each $1 \leq i \leq n$, the vertices x_i and x_{i+1} are neighbours, $x_i \sim x_{i+1}$. The *distance* between two vertices $d(x, y)$ is the length of a shortest path from x to y. A shortest path is called *geodesic*.

From now on, let us fix a base point $x_0 \in V(\Gamma)$. The *norm* of a vertex x is $|x| = d(x_0, x)$. Let us say that a vertex x is a *predecessor* of a vertex y and y is a *successor* of x, if x and y are neighbours (i.e. connected by an edge), and $|x| < |y|$.

We define the *cone* $\mathcal{C}(x)$ of a vertex x of Γ as the induced subgraph of Γ rooted at x, such that $V\big(\mathcal{C}(x)\big) = \{y \in V(\Gamma) \mid x \text{ belongs to a geodesic from } x_0 \text{ to } y\}$. Two vertices x and y are said to be of a same *cone type* if their cones are isomorphic as rooted graphs. Suppose that there is only a finite number $K + 1$ of different cone types in Γ and consider a function $t : V(\Gamma) \to \{0, ..., K\}$, such that $t(x) = 0$ if and only if $x = x_0$ and $t(x_1) = t(x_2)$ if and only if $x_1, x_2 \in V(\Gamma) \setminus \{x_0\}$ and are of a same cone type. We will say that a vertex x is *of type* $t(x)$.

Trees with finitely many cone types are in many ways similar to covering trees.

Example. Consider a finite graph $G = (V(G), E(G))$. Denote by $\tilde{G}$ the universal covering tree of G. Then the number of cone types in $\tilde{G}$ with respect to any base point is at most $1 + \sum_{v \in V(G)} deg(v)$, where $deg(v)$ denotes the number of neighbours of a vertex v of G.

Indeed, let π denote the covering map $\tilde{G} \to G$. Fix a base point x_0 in G and some $\tilde{x}_0 \in \pi^{-1}(x_0)$. Consider the set $\mathbb{E}(G)$ of oriented edges of G, of cardinality $2|E(G)|$. A set $E_e(\tilde{G})$ of edges in $\tilde{G}$ is associated naturally to each edge e in $\mathbb{E}(G)$, so that $\pi(\gamma) = e$ for each $\gamma \in E_e(\tilde{G})$. Each edge $\gamma \in E(\tilde{G})$ can be represented as (γ_+, γ_-) with $\gamma_+, \gamma_- \in V(\tilde{G})$ and $d(\gamma_+, \tilde{x}_0) = d(\gamma_-, \tilde{x}_0) + 1$. With each such $\gamma \in E(\tilde{G})$, we can associate the cone of its extremity γ_+. The cones which correspond to the edges from some $E_e(\tilde{G})$ are isomorphic. As we assumed that the cone type of the base vertex is different from all others, it follows that the number of cone types is bounded from above by the number of oriented edges in G plus 1. The number of oriented edges in G is equal to $\sum_{v \in V(G)} deg(v)$.

Notice however that not every tree with finitely many cone types is the universal cover of a finite graph. For example, of the *trees of geodesics* associated to Cayley graphs of finitely generated groups which we study in the last section of this paper.

There is a way nevertheless, to view every tree with finitely many cone types as a cover of a finite digraph. More precisely, let now G be a finite directed graph with a base vertex x_0. The *directed cover of* G *with respect to* x_0 is a rooted infinite tree T^{G,x_0} such that the set of its vertices is in one-to-one correspondance with the set of directed paths in G starting in x_0.

Two vertices are connected by an edge (a priori undirected) in T^{G,x_0} if one of the corresponding directed paths in G is the expansion of the other one by a directed edge.

Proposition 1. *For every finite digraph G and every $x_0 \in V(G)$, the tree T^{G,x_0} has finitely many cone types. Conversely, for every tree T with finitely many cone types there exist a finite digraph G and $x_0 \in V(G)$, such that $T = T^{G,x_0}$.*

Proof. Suppose that two directed paths starting at x_0 in G end at a same vertex. Then the cones of the corresponding vertices in T^{G,x_0} are isomorphic. Thus the number of the cone types in T^{G,x_0} is less or equal to $1 + |V(G)|$.

Let now T be a tree with $K+1$ cone types. For each $i = 0, ..., K$; $j = 1, ..., K$, denote by $s_{i,j}$ the number of successors of type j of a vertex of type i in T. Construct a finite directed graph G as follows. There are $K+1$ vertices $\{x_0, ..., x_K\}$ in G. For each $i = 0, 1, ..., K$, $j = 1, ..., K$, there are $s_{i,j}$ directed edges connecting the vertex x_i with the vertex x_j. Obviously, $T = T^{G,x_0}$. $\square$

2. Green function of a nearest neighbour random walk on a tree with finitely many cone types

From now on we consider an infinite, locally finite tree T with $K+1$ cone types and a base point x_0, such that $t(x_0) = 0$ and $t(x) \in \{1, ..., K\}$ for each $x \in V(T) \setminus \{x_0\}$. Thus, we assume that the type of the base point is different from the type of any other vertex, even if the cone of x_0 belongs to one of the cone types $1, ..., K$. Since T is a tree, each vertex x of T, except x_0 has exactly 1 predecessor $pr(x)$. The number $s_{i,j}$ of successors of type j of a vertex of type i is well defined for every $i = 0, 1, ..., K$ and $j = 1, ..., K$.

On T, we consider a nearest neighbour random walk $P = \big(p^{(n)}(x,y)\big)_{x,y \in V(T)}$, such that $p(x, pr(x))$ depends only on the cone type of x for any $x \in V(T) \setminus \{x_0\}$, and $p(x,y)$ depends only on the cone types of x and y for any $x \in V(T)$ and for any successor y of x. Consequently, the probability $p(x, pr(x))$ for a vertex of type $i, i \geq 1$ will be denoted by p_{-i}, and the probability $p(x,y)$ will be denoted by $p_{i,j}$ for x of type i and y a successor of x of type j. We have

$$p_{-i} \;+ \sum_{j=1,\dots,K} s_{i,j} p_{i,j} \;=\; 1, \quad j = 1, ..., K \;;$$

$$\sum_{j=1,\dots,K} s_{0,j} p_{0,j} \;=\; 1 \;.$$

Our aim in this Section is to prove the following theorem.

Theorem 1. *For every $x, y \in V(T)$, the Green function $G(x,y \mid z)$ is an algebraic element over the field $\mathbb{Q}(z, \{p_{-i}z\}, \{p_{i,j}z\})$. More precisely, there exists a unique (up to a sign) non-constant irreducible polynomial $\mathcal{P}_{x,y}(w, z_1, z_2, ..., z_{1+K(K+2)})$ in $2 + K(K+2)$ variables with integer coefficients, such that*

$$\mathcal{P}_{x,y}\big(G(x,y \mid z), z, p_{-1}z, ..., p_{-K}z, \{p_{i,j}z\}_{\substack{i=0,1,\dots,K \\ j=1,\dots,K}}\big) \;\equiv\; 0$$

for any random walk P on T satisfying the above hypothesis.

Moreover, if $(x_0, x_1, ..., x_k = x)$ denotes the geodesic from x_0 to x and $(x_0, y_1, ..., y_r = y)$ denotes the geodesic from x_0 to y, then the polynomial $\mathcal{P}_{x,y}$ depends in fact only on the cone types of the vertices $x_1, ..., x_k, y_1, ..., y_r$.

The first step on the way of proving Theorem 1 will be to prove the algebraicity of $F(x_0, x_0 \mid z)$:

Theorem 1′. *There exists a unique (up to a sign) non-constant irreducible polynomial $\mathcal{Q}(w, z_1, ..., z_{1+K(K+2)})$ in $2 + K(K+2)$ variables with integer coefficients, such that*

$$\mathcal{Q}\big(F(x_0, x_0 \mid z), z, p_{-1}z, ..., p_{-K}z, \{p_{i,j}z\}_{\substack{i=0,1,...,K \\ j=1,...,K}}\big) \equiv 0$$

for any random walk P on T satisfying the above hypothesis.

Observe that Theorem 1′ and formula (1) imply the algebraicity of $G(x_0, x_0 \mid z)$.

The following fundamental fact about trees is proven, for example, in [Ca, Ch. 2.4].

Lemma 1. *For any $x, y \in V(T)$, let $x_1 = x, x_2, ..., x_k = y$ be the unique geodesic path in T from x to y. Then*

$$F(x, y \mid z) = \prod_{i=1}^{k-1} F(x_i, x_{i+1} \mid z) .$$

Hence, we will be only interested in computing the probabilities $q^{(n)}(x, pr(x))$ and $q^{(n)}(x, y)$, where y is a successor of x.

Lemma 2. *The probability $q^{(n)}(x, pr(x))$ depends only on the cone type of x for any $x \in V(T) \setminus \{x_0\}$ and for any $n \in \mathbb{N}$.*

Proof of Lemma 2. Remark first of all that $q^{(n)}(x, pr(x)) \neq 0$ only if n is odd. For $n = 1$ we have

$$q^{(1)}(x, pr(x)) = p(x, pr(x)) = p_{-i} ,$$

where $t(x) = i$.

By induction, for $n \geq 3$, n odd,

$$q^{(n)}(x, pr(x)) = \sum_{y \text{ sucessor of } x} p(x, y) \sum_{k=1}^{\frac{n-1}{2}} q^{(2k-1)}(y, pr(y)) q^{(n-2k)}(x, pr(x)) .$$

depends only on the cone type of x. $\square$

We can now denote by $q_{-i}^{(n)}$ the probability, starting from a point of type i, to approach x_0 for the first time after n steps. We also denote by $F_{-i}(z)$ the function $F\big(x, pr(x) \mid z\big)$ for any x of type i. Of course, $F_{-i}(z) = \sum_{n=0}^{\infty} q_{-i}^{(n)} z^n$.

Lemma 3. *The power series $F_{-i}(z)$, $i = 1, ..., K$, satisfy the following system of polynomial equations:*

$$w_i = p_{-i}z + z \sum_{j=1}^{K} s_{i,j} p_{i,j} w_i w_j , \quad i = 1, ..., K . \tag{2}$$

Proof of Lemma 3. As in Lemma 2, we have

$$q_{-i}^{(1)} = p_{-i} ,$$

and, for $n \geq 3$

$$q_{-i}^{(n)} = \sum_{j=1}^{K} s_{i,j} p_{i,j} \sum_{k=1}^{\frac{n-1}{2}} q_{-j}^{(2k-1)} q_{-i}^{(n-2k)} .$$

Thus

$$F_{-i}(z) \;=\; \sum_{n=0}^{\infty} q_{-i}^{(n)} z^n \;=\; zp_{-i} \;+\; z\sum_{j=1}^{K} s_{i,j}p_{i,j} \sum_{n=3}^{\infty} \sum_{k=1}^{\frac{n-1}{2}} q_{-j}^{(2k-1)} z^{2k-1} q_{-i}^{(n-2k)} z^{n-2k}$$

$$= \; zp_{-i} \;+\; z\sum_{j=1}^{K} s_{i,j}p_{i,j} F_{-i}(z) F_{-j}(z) \; .$$

□

Proof of Theorem 1′. The function $F\big(x_0, x_0 \mid z\big)$ may be expressed in terms of the functions $F_{-i}(z)$, $i = 1, ..., K$:

$$F(x_0, x_0 \mid z) \;=\; 1 \;+\; \sum_{j=1}^{K} s_{0,j}p_{0,j} z F_{-j}(z) \; . \tag{3}$$

The system (2) can be rewritten in the form

$$Y_i(w_1, ..., w_K; z_1, ..., z_M) \;=\; 0 \; , \quad i = 1, ..., K$$

with $M = 1 + K(K+2)$, $z_1 = z$; $z_i = p_{-(i-1)}z$ for $i = 2, ..., K+1$; $z_j = p_{k,l}z$ for $j = K+2, ..., K(K+2)+1$. It satisfies the conditions of Implicit Function Theorem (see [BM, page 39]). The solution guaranteed by the theorem is given by $\big(F_{-1}(z), ..., F_{-K}(z)\big)$.

Let us show that there exists a unique (up to a sign) non-constant irreducible polynomial $\mathcal{Q}_i$ in $K+M$ variables with integer coefficients, such that $\mathcal{Q}_i\big(w_i(z_1, ..., z_M); z_1, ..., z_M\big) \equiv 0$ in the neighbourhood of the origin. For that, consider the set of common zeroes $X = \{Y_1 = ... = Y_K = 0\} \subset \mathbb{C}^{K+M}$. Implicit Function Theorem implies that X is a smooth curve in the neighbourhood of the origin. Consider the projection $\pi_0 : \mathbb{C}^{K+M} \to \mathbb{C}^M_z$ which projects $(w_1, ..., w_K; z_1, ..., z_M)$ to $z_1, ..., z_M$. Then Implicit Function Theorem implies moreover that the image of the tangent $T_{0,X}$ to X at the origin under the projection π_0 is of dimension M.

Consider now the projection $\pi_{0,i} : \mathbb{C}^{K+M} \to \mathbb{C}^{M+1}_{w_i,z}$. Of course, $dim\big(\overline{\pi_{0,i}(X)}\big) \leq M$. In fact, as $dim\big(\pi_0(T_{0,X})\big) = M$, we have exactly $dim\big(\overline{\pi_{0,i}(X)}\big) = M$. This means that there exists a non-constant polynomial $\mathcal{Q}_i \in \mathbb{C}[w_i; z_1, ..., z_M]$ such that $\overline{\pi_{0,i}(X)} = \{(w_i; z_1, ..., z_M) \mid \mathcal{Q}_i(w_i; z_1, ..., z_M) = 0\}$. As X is parametrized by $\big(w_1(z_1, ..., z_M), ..., w_K(z_1, ..., z_M); z_1, ..., z_M\big)$ in the neighbourhood of the origin, we have $\mathcal{Q}_i\big(w_i(z_1, ..., z_M); z_1, ..., z_M\big) = 0$.

Moreover, one can also conclude that the polynomials $\mathcal{Q}_i$ have integer coefficients because the polynomials Y_i have rational coefficients.

It follows that the functions $F_{-i}, i = 1, ..., K$ are algebraic elements over the field $\mathbb{Q}(z, \{p_{-i}\}_{i=1,...,K}, \{p_{i,j}\}_{i,j=1,...,K})$. A linear combination of algebraic elements is algebraic. Hence, there exists a polynomial $\tilde{\mathcal{Q}}(w)$ with coefficients in the field

$$\mathbb{Q}(z, \{p_{-i}\}_{i=1,...,K}, \{p_{i,j}\}_{\substack{i=0,1,...,K \\ j=1,...,K}})$$

such that $\tilde{\mathcal{Q}}(F(x_0, x_0 \mid z)) \equiv 0$. By writing the coefficients of the polynomial $\tilde{\mathcal{Q}}$ as rational functions of $z, \{p_{-i}z\}, \{p_{i,j}z\}$, multiplying them by the common denominator, and by a sufficiently large power of z, we get a polynomial $\hat{\mathcal{Q}}(w)$ with coefficients in the ring of polynomials of z, $\{p_{-i}z\}$, $\{p_{i,j}z\}$ with integer coefficients, such that $\hat{\mathcal{Q}}\big(F(x_0, x_0 \mid z)\big) \equiv 0$. In other words, we have constructed a polynomial $\mathcal{Q}(w_i, z_1, ..., z_{1+K(K+2)})$ with integer coefficients, such that

$$\mathcal{Q}\big(F(x_0, x_0 \mid z), z, \{p_{-i}z\}_{i=1,\dots,K}, \{p_{i,j}z\}_{\substack{i=0,1,\dots,K\\ j=1,\dots,K}}\big) \equiv 0 .$$

$\square$

Let us now proceed further and express functions $F(x, y \mid z)$, $x, y \in V(T), x \sim y$, in terms of the functions $\{F_{-i}(z)\}_{i=1,\dots,K}$.

Let x be a vertex of T and y a successor of x. Then, for $n \geq 3$, n odd,

$$\begin{aligned} q^{(n)}(x, y) &= \sum_{y' \text{ a successor of } x} p(x, y') \sum_{r=1}^{\frac{n-1}{2}} q^{(2r-1)}(y', pr(y'))q^{(n-2r)}(x, y) \\ &\quad + p(x, pr(x)) \sum_{r=1}^{\frac{n-1}{2}} q^{(2r-1)}(pr(x), x)q^{(n-2r)}(x, y) . \end{aligned}$$

As follows from Lemma 2, the probabilities $p(x, y')$ and $q^{(r)}(y', pr(y'))$ depend only on the cone type of x and its successors (which are determined by the cone type of x). On the contrary, the cone type of $pr(x)$ is not determined by the cone type of x, thus the probabilities $p(x, pr(x))$ and $q^{(r)}(pr(x), x)$ depend also on the cone type of $pr(x)$. The probabilities $q^{(r)}(pr(x), x)$ depend in turn on the cone types of x, $pr(x)$, and $pr(pr(x))$. Each x in $V(T)$ is uniquely determined by the sequence $(x, pr(x), pr(pr(x)), ..., pr^n(x) = x_0)$, which coincides with the unique geodesic from x to x_0 in T. Hence, the probabilities $q^{(n)}(x, y)$ depend on the cone types of y, x, and all the vertices which lie on the geodesic connecting x and x_0 in T.

Theorem 2. *Let x be a vertex of T, y be a successor of x of type j, and let $(x_0, x_1, ..., x_k = x)$ with $t(x_m) = i_m$ for $m = 1, ..., k$ be the unique geodesic from x_0 to x. Then*

$$F(x, y \mid z) = \cfrac{zp_{i_k,j}}{H_k(z) + \cfrac{z^2 p_{i_{k-1},i_k}}{H_{k-1}(z) + \cfrac{z^2 p_{i_{k-2},i_{k-1}}}{\ddots \; H_1(z) + \cfrac{z^2 p_{0,i_1}}{H_0(z)}}}} ,$$

where

$$\begin{aligned} H_k(z) &= 1 - z \sum_{\substack{l=1\\ l\neq j}}^{K} p_{i_k,l}F_{-l}(z) - zp_{i_k,j}(s_{i_k,j} - 1)F_{-j}(z) \; ; \\ &\textit{for } m = k-1, \dots, 1 \; : \\ H_m(z) &= 1 - z \sum_{\substack{l=1\\ l\neq i_{m+1}}}^{K} p_{i_m,l}s_{i_m,l}F_{-l}(z) - zp_{i_m,i_{m+1}}(s_{i_m,i_{m+1}} - 1)F_{-i_{m+1}}(z) \; ; \\ H_0(z) &= 1 - z \sum_{\substack{l=1\\ l\neq i_1}}^{K} p_{0,l}s_{0,l}F_{-l}(z) - zp_{0,i_1}(s_{0,i_1} - 1)F_{-i_1}(z). \end{aligned}$$

Proof of Theorem 2. As shown above, the function $F(x,y \mid z)$ depends only on $j, i_1, ..., i_k$, and we denote $F(x,y \mid z) = F_{0,i_1,...,i_k,j}(z)$. The easiest case is that of the function $F_{0,i_1}(z)$. We have the following formulas for the probability to go from x_0 to one of its neighbours x of type i:

$$\begin{aligned} q_{0,i}^{(1)} &= p_{0,i} \ , \\ q_{0,i}^{(n)} &= \sum_{\substack{j=0 \\ j\neq i}}^{K} p_{0,j} s_{0,j} \sum_{k=1}^{\frac{n-1}{2}} q_{-j}^{(2k-1)} q_{0,i}^{(n-2k)} + p_{0,i}(s_{0,i}-1) \sum_{k=1}^{\frac{n-1}{2}} q_{-i}^{(2k-1)} q_{0,i}^{(n-2k)}, \quad n \geq 3. \end{aligned}$$

Thus

$$F_{0,i}(z) = zp_{0,i} + z \sum_{\substack{j=1 \\ j\neq i}}^{K} p_{0,j} s_{0,j} F_{-j}(z) F_{0,i}(z) + zp_{0,i}(s_{0,i}-1) F_{-i}(z) F_{0,i}(z)$$

and

$$F_{0,i}(z) = \frac{zp_{0,i}}{1 - z\sum_{\substack{j=1 \\ j\neq i}}^{K} p_{0,j} s_{0,j} F_{-j}(z) - zp_{0,i}(s_{0,i}-1) F_{-i}(z)} \ .$$

The probabilities of going from a vertex x (to which we associate the geodesic $(x_0, x_1, ..., x_k = x)$ with $t(x_m) = i_m, m = 1, ..., k)$ to one of its successors of type j in n steps can be computed similarly:

$$\begin{aligned} q_{0,i_1,\ldots,i_k,j}^{(1)} &= p_{i_k,j} \ , \\ q_{0,i_1,\ldots,i_k,j}^{(n)} &= \sum_{\substack{l=0 \\ l\neq j}}^{K} p_{i_k,l} s_{i_k,l} \sum_{r=1}^{\frac{n-1}{2}} q_{-l}^{(2r-1)} q_{0,i_1,\ldots,i_k,j}^{(n-2r)} \\ &\quad + p_{i_k,j}(s_{i_k,j}-1) \sum_{r=1}^{\frac{n-1}{2}} q_{-j}^{(2r-1)} q_{0,i_1,\ldots,i_k,j}^{(n-2r)} \\ &\quad + p_{-i_k} \sum_{r=1}^{\frac{n-1}{2}} q_{0,i_1,\ldots,i_{k-1},i_k}^{(2r-1)} q_{0,i_1,\ldots,i_k,j}^{(n-2r)}, \qquad n \geq 3 \ , \end{aligned}$$

which implies

$$\begin{aligned} F_{0,i_1,\ldots,i_k,j}(z) &= zp_{i_k,j} + z \sum_{\substack{l=1 \\ l\neq j}}^{K} p_{i_k,l} s_{i_k,l} F_{-l}(z) F_{0,i_1,\ldots,i_k,j}(z) \\ &\quad + zp_{i_k,j}(s_{i_k,j}-1) F_{-j}(z) F_{0,i_1,\ldots,i_k,j}(z) \\ &\quad + zp_{-i_k} F_{0,i_1,\ldots,i_{k-1},i_k}(z) F_{0,i_1,\ldots,i_k,j}(z) \ . \end{aligned}$$

Hence we have

$$F_{0,i_1,\ldots,i_k,j}(z) = \frac{zp_{i_k,j}}{\Delta} \ ,$$

where

$$\Delta \;=\; 1 - z\sum_{\substack{l=1\\ l\neq j}}^{K} p_{i_k,l}s_{i_k,l}F_{-l}(z) - zp_{i_k,j}(s_{i_k,j}-1)F_{-j}(z) - zp_{-i_k}F_{0,i_1,\dots,i_{k-1},i_k}(z)\;.$$

In this formula for $F_{0,i_1,\dots,i_k,j}(z)$ we can replace $F_{0,i_1,\dots,i_{k-1},i_k}(z)$ by the similar expression, in which only the functions $\{F_{-i}(z)\}$ and the function $F_{0,i_1,\dots,i_{k-2},i_{k-1}}(z)$ appear. Repeating this procedure $k-1$ times, we get an expression for $F_{0,i_1,\dots,i_{k-1},i_k}$ in terms of the functions $\{F_{-i}(z)\}$ and the function $F_{0,i_1}(z)$, which in its turn can be expressed only in terms of the functions $\{F_{-i}(z)\}$. Putting all these expressions together, we get the desired formula for $F_{0,i_1,\dots,i_k,j}(z)$. □

Proof of Theorem 1. In Theorem 1′ we proved that the functions $F_{-i}(z)$ are algebraic elements over the field $\mathbb{Q}(z,\{p_{-i}z\},\{p_{i,j}z\})$. Theorem 2 shows that the functions $F(x,y\mid z)$ can be obtained from $\{F_{-i}(z)\}$ by a finite number of operations of addition, multiplication by a scalar from the field $\mathbb{Q}(z,\{p_{-i}z\},\{p_{i,j}z\})$, and taking inverse. Thus each function $F(x,y\mid z)$ is an algebraic element over the same field. Consequently, $G(x,y\mid z)$ is also algebraic over the same field. □

3. Spectral radius of a group with finitely many cone types

Let G be a finitely generated group, S a finite symmetric system of generators in G. The aim of this Section is to provide a method of estimating the spectral radius of the simple random walk on (G,S) in the situation when the corresponding Cayley graph has finitely many cone types, as for example, when G is hyperbolic, or when (G,S) is a Coxeter system.

To a pair (G,S) we associate its *tree of geodesics* $T_{G,S}$. This is an infinite tree with the vertex set $V(T_{G,S})$ which is in one-to-one correspondence with the set of geodesic segments of the form $\{[1_G,g]\}_{g\in G}$ in $Cay(G,S)$. Two vertices γ_1,γ_2 in $T_{G,S}$ are connected by an edge if the corresponding geodesic segment γ_1 in $Cay(G,S)$ is a one-step extension of the segment γ_2. The tree $T_{G,S}$ has naturally a base vertex γ_0 corresponding to the null geodesic segment $[1_G,1_G]$.

There is of course a natural projection of $T_{G,S}$ to $Cay(G,S)$:

$$\begin{array}{rccc}\theta: & V(T_{G,S}) & \to & G\\ & \gamma=[1_G,g] & \mapsto & g\;.\end{array}$$

This map is locally injective as the induced map on the set of edges is of the form

$$\big([1_G,g],[1_G,gs]\big)\mapsto(g,gs).$$

However, θ is not a covering map because it is not a local isomorphism.

Remark. We are going to "simulate" the simple random walk on $Cay(G,S)$ by a specially chosen random walk on $T_{G,S}$. This simulation is good only when $Cay(G,S)$ *is bipartite*, and we will only prove Theorem 3 below in this particular case. The same proof can be adopted for the general situation but the estimate becomes less precise. It is not hard to imagine a random walk on $T_{G,S}$ which would suit the non bipartite case better, the problem is in generalizing the result from [Nag] which is used in the proof.

So, suppose for the rest of this Section that $Cay(G,S)$ is bipartite.

Observe that the tree of geodesics $T_{G,S}$ has the same number of cone types as the Cayley graph of G with respect to S. Indeed, $\theta(\mathcal{C}(\gamma))=\mathcal{C}(\theta(\gamma))$ for every $\gamma\in V(T_{G,S})$.

Suppose the number of cone types in G with repsect to S is $K+1$. Denote by m the cardinality of S, and by r_i the number of predecessors of a vertex of the type i in $Cay(G,S)$. Consider a nearest neighbour random walk P_G on $T_{G,S}$ defined by the following probabilities:

$$\begin{aligned} p_{i,j} &= 1/m, \qquad i=0,...,K;\ j=1,...,K. \\ p_{-i} &= r_i/m \qquad\qquad i=1,...,K. \end{aligned}$$

Theorem 3. *The spectral radius ρ_G of the simple random walk on G with respect to S is bounded from above by the spectral radius ρ_T of the random walk P_G on $T_{G,S}$.*

Proof. For $i=0,...,K$, consider the functions $f_i:\mathbb{R}_+^K\to\mathbb{R}_+$ defined by

$$f_i(c_1,...,c_K) \;=\; r_i c_i + \sum_{j=1}^{K}\frac{s_{i,j}}{c_j}\ . \tag{4}$$

It is shown in [Nag] that

$$\rho_G \;\le\; \frac{1}{m}\min_{(c_1,\dots,c_K)\in\mathbb{R}_+^K}\ \max_{i=1,\dots,K} f_i(c_1,...,c_K)\ .$$

Therefore, it is enough to show that

$$\rho_T \;\ge\; \frac{1}{m}\min_{(c_1,\dots,c_K)\in\mathbb{R}_+^K}\ \max_{i=1,\dots,K} f_i(c_1,...,c_K)\ . \tag{5}$$

Consider the application $(f_1,...,f_K):\mathbb{R}_+^K\to\mathbb{R}_+^K$ defined by (4).

As established in Section 2, $1/\rho_T$ is equal to the minimum (over $i=1,...,K$) of the smallest real positive singularities of the functions $F_{-i}(z)$. These functions satisfy the following system of equations (Section 2, Lemma 3):

$$w_i \;=\; \frac{r_i}{m}z \;+\; \frac{z}{m}w_i\sum_{j=1}^{K}s_{i,j}w_j\ ,\quad i=1,...,K\ .$$

For a fixed z this system can be rewritten in the following form

$$\frac{m}{z} \;=\; \frac{r_i}{w\,i} \;+\; \sum_{j=1}^{K}s_{i,j}w_j\ ,\quad i=1,...,K\ .$$

For each $\epsilon>0$, let $z_\epsilon = 1/\rho_T-\epsilon\in\mathbb{R}_+$. Then there exists a solution $\big(F_{-1}(z_\epsilon),...,F_{-K}(z_\epsilon)\big)\in\mathbb{R}_+^K$ of the system above. The functions $F_{-i}(z)$ are increasing in the intersection of their disc of convergence with $\mathbb{R}_+$, thus $F_{-i}(z_\epsilon)\neq 0$. We can put $c_i = 1/F_{-i}(z_\epsilon)$ in (4) and get

$$\begin{aligned} f_1\big(1/F_{-1}(z_\epsilon),...,1/F_{-K}(z_\epsilon)\big) \;&=\; ... \;=\; f_K\big(1/F_{-1}(z_\epsilon),...,1/F_{-K}(z_\epsilon)\big) \\ =\; \max_{i=1,\dots,K} f_i\big(1/F_{-1}(z_\epsilon),...,1/F_{-K}(z_\epsilon)\big) \;&=\; \frac{m}{z_\epsilon} \;=\; \frac{m}{1/\rho_T-\epsilon}\ . \end{aligned}$$

Consequently,

$$\min_{(c_1,\dots,c_K)\in\mathbb{R}_+^K}\ \max_{i=1,\dots,K} f_i(c_1,...,c_K) \;\le\; \frac{m}{1/\rho_T-\epsilon}$$

and

$$\rho_T \ \geq \ \frac{1}{m} \min_{(c_1,\dots,c_K)\in\mathbb{R}_+^K} \max_{i=1,\dots,K} f_i(c_1,\dots,c_K)\ . \qquad \square$$

Remark. It can be shown that the inequality (5) is in fact an equality.

It is clear that by choosing suitable "homesickness" parameters, i.e., $p(x, pr(x))$ for random walks on $T_{G,S}$ or on the universal covering, m-homogeneous tree T_m, one can get more precise estimates of ρ_G. The advantage of ρ_T is that it is computable.

Indeed, the tree of geodesics $T_{G,S}$ has finitely many cone types and the system of equations (2) can be written for the random walk P_G. The polynomials $\mathcal{Q}_i$ and thus the polynomial $\mathcal{Q}$ can be found algorithmically using the *elimination theory* (see [KSa]). The radius of convergence of the function $F(x_0, x_0 \mid z)$ is then found either among the zeroes of the first coefficient or among the zeroes of the resultant of $\mathcal{Q}$ (see [Ahl]).

This procedure leads to the following estimate of the spectral radius of the surface group G_2. The same estimate was obtained in [Nag] by computing the number

$$\frac{1}{m} \min_{(c_1,\dots,c_K)\in\mathbb{R}_+^K} \max_{i=1,\dots,K} f_i(c_1,\dots,c_K).$$

Example. Let ρ_g denote the spectral radius of the simple random walk on the Cayley graph of the fundamental group G_g of the orientable closed surface of genus g with respect to the standard system of generators. It improves previous existing estimates and is

$$\rho_2 \ < \ 0.662816\ ,$$

$$\rho_3 \ < \ 0.552792\ .$$

The first estimate is close to the best known estimate from below found by L. Bartholdi (see [BCec]):

$$\rho_2 \ \geq \ 0.6624\ .$$

References

Ahl. L. Ahlfors, *Complex Analysis - 3rd ed.*, McGraw-Hill Book Company, 1979.

BCec. L. Bartholdi, T. Ceccherini-Silberstein, *Growth series and random walks on hyperbolic graphs*, Preprint (1997).

BM. S. Bochner, W. Martin, *Several Complex Variables*, Princeton University Press, 1948.

Ca. P. Cartier, *Fonctions harmoniques sur un arbre*, Sympos. Math. 9, Academic Press, New York, 1972, pp. 203-270.

GW. P. Gerl, W. Woess, *Local limits and harmonic functions for nonisotropic random walks on free groups*, Prob. Theory Related Fields **71** (1986), 341-355.

KSa. Kuich, Salomaa, *Semirings, Automata, Languages*, Springer, 1985.

L. S. Lalley, *Finite range random walk on free groups and homogeneous trees*, Ann. Probab. **21** (1993), no. 4, 2087-2130.

Ly. R. Lyons, *Random walks and percolation on trees*, Ann. Probab. **18** (1990), no. 3, 931-958.

Nag. T. Nagnibeda, *An estimate from above of spectral radii of surface groups*, Zapiski Nauchnykh Seminarov POMI (to appear) (1997).

St. T. Steger, *Harmonic analysis for an anisotropic random walk on a homogeneous tree*, Ph.D. Thesis, Washington University, 1986.

Pr. W. Pruitt, *Eigenvalues of non-negative matrices*, Ann. Math. Statist. **35** (1964), 1797-1800.

Rem. R. Remmert, *Theory of Complex Functions*, Springer-Verlag, 1990.

T. Ch. Takacs, *Random walks on periodic trees*, Electron. J. Probab. **2** (1997), no. 1, 1-16.

W. W. Woess, *Context-free languages and random walks on groups*, Discr. Maths. **67** (1967), 81-87.