COMPLETE GROWTH FUNCTIONS OF HYPERBOLIC GROUPS

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Abstract. We study a generalization of the growth functions of finitely generated groups, namely the growth functions $\Sigma_{g \in G} g z^{|g|}$ with coefficients in the group ring $\mathbb{Z}[G]$. Rationality and methods of computation of such functions are discussed, in particular for hyperbolic groups. The complete growth functions of surface groups are explicitly computed. The operator geodesic growth functions are also studied.

1. Introduction

Consider a finitely generated discrete group G with a finite system of semigroup generators $S = \{s_1, ..., s_k\}, 1 \notin S$, and the corresponding length metric. The *growth function of* G with respect to S is the power series

$$f(z) = \sum_{g \in G} z^{|g|} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Z}[[z]] ,$$

where $|g|$ is the length of g, and a_n is the number of elements of G of length n. The growth functions of discrete groups have been studied by many authors. Let us mention the computations of the growth functions of Coxeter groups ([39], [6], [32], [41], [7]), surface groups and Fuchsian groups ([7], [9], [41], [13]), Heisenberg and Nil groups ([35], [5]). In all these examples the growth functions are rational. They are also rational for virtually abelian groups ([4], [26]) and for hyperbolic groups ([8], [17], [14]). There exist however nilpotent groups and systems of generators in them, such that the corresponding growth functions are not rational ([18], [37]). On different problems related to growth in groups see also a recent survey [16].

In this paper we study the rationality and the methods of computation of a natural generalization of the growth function. We define the *complete growth function* (or *complete growth series*) of the group G with respect to the system of generators S as

$$F(z) = \sum_{g \in G} g z^{|g|} = \sum_{n=0}^{\infty} A_n z^n \in \mathbb{Z}[G][[z]] ,$$

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where A_n is the sum of all elements of G of length n viewed as an element of the group ring, i.e.,

$$A_n \;=\; \sum_{g\in G, |g|=n} g \quad \in \; \mathbb{Z}[G] \; ,$$

and $M\big[[z]\big]$ denotes as usual the ring of formal power series in z with coefficients in a ring M. The operator A_1 denoted A below is the sum of generators.

The augmentation mapping $h : \mathbb{Z}[G] \to \mathbb{Z}$, $g \mapsto 1$, induces the morphism $\tilde{h} : \mathbb{Z}[G]\big[[z]\big] \to \mathbb{Z}\big[[z]\big]$, such that $\tilde{h}(F(z)) = f(z)$. Thus the rationality of the complete growth function implies the rationality of the ordinary growth function. The notion of rationality of complete growth functions will be made precise in Section 3.

We state now our main result.

Let F_g denote the complete growth function of the fundamental group of the orientable closed surface of genus $g, g \geq 2$

$$G_g \;=\; \langle a_1, ..., a_g, b_1, ..., b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$$

with the standard system of generators

$$S_g \;=\; \{a_1^{\pm 1}, ..., a_g^{\pm 1}, b_1^{\pm 1}, ..., b_g^{\pm 1}\} \; .$$

We denote by $[a, b]$ the commutator $aba^{-1}b^{-1}$. A *cyclic word*, as usual in combinatorial group theory, is a word written on a circle (see [29]).

Theorem 1. *The following relation holds in* $\mathbb{Z}[G_g]\big[[z]\big]$*:*

$$(*) \qquad F_g(z) \;=\; \frac{(1 - z^2)(1 - z^{4g})}{\Delta} \; ,$$

with

$$\Delta \;=\; 1 + (4g-1)z^2 + (4g-1)z^{4g} + z^{4g+2} - Az(1 + z^{4g}) - \mathcal{P}_{2g-1}z^{2g+1} + \mathcal{P}^b_{2g}z^{2g}(1 + z^2) \; ,$$

where $A \in \mathbb{Z}[G_g]$ *is the sum of all elements of* S_g*;* $\mathcal{P}^b_{2g} \in \mathbb{Z}[G_g]$ *is the sum of all subwords of length* $2g$ *with first letters in* $\{b_1, b_1^{-1}, ..., b_g, b_g^{-1}\}$ *of the cyclic words* $r = \prod_{i=1}^{g}[a_i, b_i]$ *and* r^{-1}*; and* $\mathcal{P}_{2g-1} \in \mathbb{Z}[G_g]$ *is the sum of all subwords of length* $2g-1$ *of the cyclic words* r *and* r^{-1}*.*

The "classical" growth functions of these groups are rational and were computed for the first time by J. Cannon. It is shown in [9] (see also [7], [41]) that the growth function of the fundamental group G_g of the surface of genus g with respect to the system of generators S_g is

$$\begin{aligned} f_g(z) \;&=\; \frac{(1 + z)(1 - z^{2g})}{1 - (4g-1)z + (4g-1)z^{2g} - z^{2g+1}} \\ &=\; \frac{1 + 2z + 2z^2 + ... + 2z^{2g-1} + z^{2g}}{1 - (4g-2)z - (4g-2)z^2 - ... - (4g-2)z^{2g-1} + z^{2g}} \; . \end{aligned}$$

Our proof of Theorem 1 is based on a method of computation of growth functions in a wider context of formal languages. Theorem 2 below is analogous to the formula for the ordinary growth function of a formal language determined by a set of forbidden words, which was obtained by the first author in [15]. It is related to the formula for dimensions of homology groups of a complex, called now the

Anick-Groves-Squier complex, given in [1]. (For the history of the question see [15].)

Let $X = \{x_1, ..., x_l\}$ be an alphabet, X^* denote the set of all finite words in X, $\Omega \subset X^*$ a subset of X^* called a *language*. The formal power series

$$f_\Omega(z) \;=\; \sum_{n=0}^{\infty} \omega_n z^n \;\in\; \mathbb{Z}[[z]]\ ,$$

where ω_n denotes the number of words of length n in Ω, is called the *growth function*, and the formal power series

$$F_\Omega(z) \;=\; \sum_{w\in\Omega} w z^{|w|} \;=\; \sum_{n=0}^{\infty} \Omega_n z^n \;\in\; \mathbb{Z}[X^*][[z]]\ ,$$

where $\Omega_n = \sum_{|w|=n} w \;\in\; \mathbb{Z}[X^*]$, the *complete growth function of the language* Ω.

We will be particularly interested in the languages that are determined by a set of words, as follows. Let $U \subset X^*$ be a subset of X^* called the set of *forbidden words*. A language Ω is determined by U if it consists of all words from X^* that do not contain forbidden subwords. The words from Ω are called *reduced*. We will suppose that the set of forbidden words satisfies two natural conditions:

1. $U \cap X = \emptyset$;
2. any forbidden word contains no proper forbidden subword.

A word $w \in X^*$ is called a *chain* if, for some positive integer m, it has an "arc presentation" $(u_1, ..., u_m)$

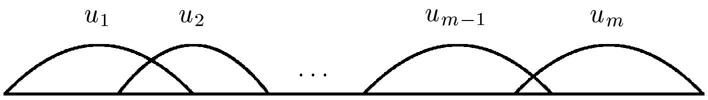

Figure 1.

where each arc corresponds to an occurence in w of a forbidden subword $u_i \in U, i \in \{1, ..., m\}$, and the intersection of two subwords u_i and u_j in w is non empty if and only if $|i - j| = 1$.

A *rigid chain of rank* m is defined by induction on m. Any forbidden word is a rigid chain of rank 1. A chain of rank 2 is rigid if it contains no forbidden subwords except the arcs u_1 and u_2. Suppose that the rigid chains of rank $m-1$ are already defined. Then a chain w is rigid of rank m if the following conditions are satisfied:

1. the prefix of w consisting of the first $m-1$ arcs is a rigid chain of rank $m-1$;
2. the suffix of w following the $(m-2)$-nd arc contains no forbidden subword except the arc u_m.

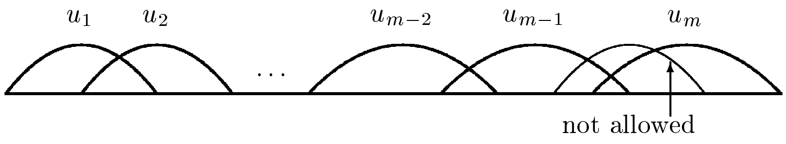

Figure 2.

We give one more equivalent definition of rigid chains. A word $w \in X^*$ is a rigid chain of rank m if $w = w_0 w_1 ... w_m$, so that

1. $w_0 \in X$, $w_i \in X^*$, $1 \le i \le m$;

2. $w_i w_{i+1}$ contains a forbidden subword, $i = 0, ..., m-1$;
3. any proper prefix of $w_i w_{i+1}$, $i = 0, ..., m-1$, is reduced.

The notions of chain and rigid chain are equivalent respectively to the notions of prechain and chain from [1].

Let $A = \Omega_1 = x_1 + ... + x_l \in \mathbb{Z}[X^*]$, and let $Q_n^{(m)} \in \mathbb{Z}[X^*]$ denote the sum of all words of X^* of length n which are rigid chains of rank m.

Theorem 2. *The complete growth function $F_\Omega = \sum_{n=0}^{\infty} \Omega_n z^n$ of the language Ω satisfies the following relation in the ring $\mathbb{Z}[X^*][[z]]$:*

$$(**) \qquad F_\Omega(z) = \frac{1}{1 - Az + \sum_{m=1}^{\infty}(-1)^{m+1} \sum_{n=1}^{\infty} Q_n^{(m)} z^n} .$$

Further results of independent interest are formulated in Sections 3,5,6. The paper is organized as follows.

In Section 2 we prove Theorem 2 and explain when and how this result may be applied to the computing of the complete growth function of a finitely generated group.

In Section 3 we make precise the notion of rationality of complete growth function and prove that a complete growth function of a group with a finite number of extended cone types, in particular, of a hyperbolic group with respect to any finite system of generators, is rational.

In Section 4 we compute the complete growth functions of the surface groups with respect to the standard systems of generators using the method described in Section 2. Other systems of generators of G_g are also discussed.

In Section 5 we introduce the *operator growth functions* and discuss the questions of its rationality and of convergence of growth series with coefficients in operator algebras.

In Section 6 we discuss the *complete* and *operator geodesic growth functions.* The operator geodesic growth functions of surface groups are explicitly computed.

We hope that the results and methods developed in this work will find further applications in the theory of growth functions, as well as in the theory of random walks on discrete groups, the theory of formal languages, and the theory of operator algebras.

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2. Complete Growth Functions of Languages Determined by Sets of Forbidden Words; Rewriting Systems in Groups

Theorem 2 will be proven in this Section. We need first to prove some preliminary results.

It may be shown (see e.g. [1]) that if $w \in X^*$ may be presented in the form of rigid chain, then such a presentation is unique. The following is a stronger property of rigid chains.

Lemma 1. *Suppose that a rigid chain $w = (u_1, ..., u_m)$ of rank m (given here in its arc presentation) possesses a prefix w' which is also a rigid chain of rank m. Then $w = w'$.*

Proof. We prove this by induction. The assertion holds for a chain of rank 1 because a forbidden word contains no proper forbidden subword. Suppose it is true for the chains of rank $m-1$. Let now w be of rank m and let $w' = (v_1, ..., v_m)$ be the arc presentation of w'. Consider the subchains $\tilde{w} = (u_1, ..., u_{m-1})$ of w and $\tilde{w}' = (v_1, ..., v_{m-1})$ of w'. The word $\tilde{w}'$ is a prefix of $\tilde{w}$ because otherwise v_m would be a forbidden subword of u_m, which is impossible. Both $\tilde{w}$ and $\tilde{w}'$ are rigid chains of rank $m-1$, hence $\tilde{w}' = \tilde{w}$ by induction hypothesis. Moreover, $u_m = v_m$ because otherwise v_m would be a forbidden subword of the suffix of w following u_{m-2}, which is impossible by definition of rigid chain. Hence $w = w'$. $\square$

Lemma 2. *For each $n, m \geq 1$, denote by W_n^m the set of words of length n from X^* which can be written in the form $w = w_1 w_2$ such that w_1 is a rigid chain of rank m and w_2 is reduced. Then*

$$\Omega_n = A\Omega_{n-1} - \sum_{w \in W_n^1 \setminus W_n^2} w , \quad n \geq 1 ; \tag{1}$$

$$\sum_{w \in W_n^m \setminus W_n^{m+1}} w = \sum_{j=1}^{n} Q_j^{(m)} \Omega_{n-j} - \sum_{w \in W_n^{m+1} \setminus W_n^{m+2}} w , \quad n, m \geq 1 , \tag{2}$$

where $Q_j^m \in \mathbb{Z}[X^*]$ denotes as before the sum of all rigid chains of rank m and of length j.

Proof. To prove (1) observe that the multiplication on the left of a reduced word of length $n-1$ by a letter s from the alphabet X gives a word of length n. Some of the words obtained in this way have a forbidden prefix, as shown on Figure 3. These words are in W_n^1 but not in W_n^2 because their suffixes of length $n-1$ are reduced.

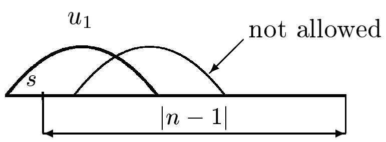

Figure 3.

In order to prove the second relation we prove the following equality of sets:

$$W_n^m = (W_n^m \setminus W_n^{m+1}) \sqcup (W_n^{m+1} \setminus W_n^{m+2}),$$

where $\sqcup$ stands as usual for the disjoint union. It is equivalent to (2) by Lemma 1.

Obviously $W_n^m \setminus W_n^{m+1} \subset W_n^m$. Also $W_n^{m+1} \setminus W_n^{m+2} \subset W_n^m$. Indeed, suppose $w \in W_n^{m+1} \setminus W_n^{m+2}$. Then $w = w_1 w_2$ where w_1 is a rigid chain of rank $m+1$ and w_2 is reduced. Moreover, the suffix of w following the m-th arc of w_1 is also reduced because $w \notin W_n^{m+2}$, as shown in Figure 4. Thus $w \in W_n^m$. This proves the inclusion "$\supset$".

Suppose now that $w \in W_n^m \cap W_n^{m+1}$. Then $w \notin W_n^{m+2}$ because its suffix following the m-th arc is reduced, see Figure 4 again. Thus $w \in W_n^{m+1} \setminus W_n^{m+2}$. This proves the inclusion "$\subset$".

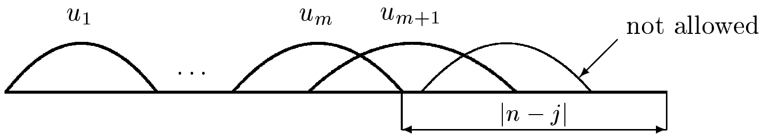

Figure 4.

$\square$

Proof of Theorem 2. It follows from Lemma 2 that

$$\Omega_n \;=\; A\Omega_{n-1} + \sum_{m=1}^{n-1} (-1)^m \sum_{j=1}^{n} Q_j^{(m)} \Omega_{n-j} \ .$$

Multiplying this relation by z^n, taking sum over $n = 1, 2, \ldots$, and changing the order of summation on the right-hand side, one gets

$$F_\Omega(z) - 1 = AzF_\Omega(z) + \sum_{m=1}^{\infty} (-1)^m \sum_{n=0}^{\infty} Q_n^{(m)} z^n F_\Omega(z)$$

which leads to $(**)$. $\square$

We are going to explain now how Theorem 2 may be used for computing the complete growth function of a group. We refer to [11] for a good exposition on rewriting systems for groups; here we just repeat several necessary definitions.

Let G be a finitely generated group with a finite symmetric system of generators $S = \{s_1, \ldots, s_k\}$, $1 \notin S$. Let us view S as an alphabet and denote by S^* the set of all finite words in the alphabet S including the empty word. A *rewriting system* $\mathcal{R}$ on (G, S) is a subset of $S^* \times S^*$. We write $l \to r$ for an element $(l, r) \in \mathcal{R}$, and call this a *rewriting rule*. Given a rewriting system, one has a process of rewriting in the group, such that for any words $w, w' \in S^*$ the word wlw' is rewritten as the word wrw', $wlw' \to wrw'$. A word x is *irreducible* if there is no word y such that $x \to y$.

A rewriting system is *terminating* if there is no infinite sequence $w_1 \to w_2 \to \ldots \to w_n \to \ldots$. Obviously, this property holds for the systems in which $|l| \geq |r|$ for each $(l, r) \in \mathcal{R}$ and, in the case $|l| = |r|$, r precedes l in the lexicographical order induced by some order on S. A rewriting system is *confluent* if the result of reduction of a word does not depend on the way of reducing. Finally, a system which is terminating and confluent is *complete.*

Let G be a finitely generated group, and suppose that G may be given by a presentation $G = \langle S \mid u_i = v_i, i \in I\rangle$, such that the system of rules $\mathcal{R} = \{u_i \to v_i, i \in I\}$ is complete and $|u_i| \geq |v_i|, i \in I$. Define the set U of forbidden words as the set of all left-hand parts of the rewriting rules. Suppose that U satisfies the conditions 1),2) from Section 1. The language Ω of irreducible words is then exactly the set of normal forms of group elements, i.e., there exists a bijection $g \mapsto w_g$ between the elements of G and the words from Ω, such that $g = w_g$ in G. Moreover this normal form is geodesic minimal in the sense that $|g| = |w_g|$. Thus the complete growth function F_Ω of the language Ω as defined in Section 1, becomes the complete growth function of G with respect to S if we consider the words from S^* in the expression for F_Ω as elements of the group. Therefore it can be computed using Theorem 2.

If the set of forbidden words is finite or regular, then the language Ω is regular and therefore the complete growth function F_Ω is rational.

3. Rationality of Complete Growth Functions

Let M be a ring (not necessarily commutative), and denote by $M[[z]]$ the ring of formal power series with the variable z and coefficients in M. We refer to [33] for a good exposition on formal power series.

An element $q = \sum_{n=0}^{\infty} q_n z^n \in M[[z]]$ is called *quasi-regular* if $q_0 = 0$. The following operation defined on the set of quasi-regular elements is called *quasi-inversion*. For a quasi-regular q define

$$q^+ = \sum_{n=1}^{\infty} q^n = \lim_{N\to\infty} \sum_{n=1}^{N} q^n .$$

The quasi-inverse q^+ is uniquely determined by the equality

$$q + q \cdot q^+ = q + q^+ \cdot q = q^+ .$$

A series from $M[[z]]$ is called *rational* if it may be obtained from a finite number of polynomials in z with coefficients in M by a finite number of operations of addition, multiplication, and quasi-inversion.

Note that in the situation when $M = \mathbb{C}$, this definition becomes the usual definition of rational function in $\mathbb{C}$.

The complete growth function of a group G with respect to the system of generators S is *rational* if it is rational as an element of $\mathbb{Z}[G][[z]]$.

For a finitely generated group G and a finite system of generators S, let $Cay(G, S)$ denote the Cayley graph of G with respect to S. For a word γ over the alphabet S denote by $\bar{\gamma}$ the element of G represented by γ. A word γ over S is called *geodesic string* if it determines a geodesic path beginning in the identity element and ending in $\bar{\gamma}$ in $Cay(G, S)$.

Define the *cone type* of an element $g \in G$ to be the set of geodesic strings γ over the alphabet S such that $\alpha\gamma$ is a geodesic string, where α is any geodesic string with $\bar{\alpha} = g$.

The *extended cone type* of g is defined to be the union of the cone type and the set of words over S of the form γs, such that γ is in the cone type of g, $s \in S$, and $|\overline{\gamma s}| = |\bar{\gamma}|$.

The definition of cone type as given here may be found in [12]. In an earlier paper, J. Cannon [8] considered the extended cone types which he called order types.

Observe that if $Cay(G, S)$ is bipartite, then the cone type and the extended cone type coincide for every g.

Proposition 1. *Suppose that G has a finite number of extended cone types with respect to S. Then the complete growth function of G with respect to S is rational.*

Corollary 1. *The complete growth function of a hyperbolic group with respect to any finite system of generators is rational.*

Proof of Corollary 1. The finiteness of the number of extended cone types of a group of hyperbolic isometries acting discretely on a hyperbolic space with finite quotient was proved by J. Cannon in [8]. The result was extended to all hyperbolic groups in [12, Th. 3.4.5]. They prove the finiteness of the number of cone types, but their proof works without changes for the extended cone types.

To prove Proposition 1 we need the following statement ([33, Theorems 1.1 and 1.2]). Consider the ring of formal power series $M[[z]]$. Denote by I the unit element in $M[[z]]$.

Lemma 3. *Let* $Z = (z_1, ..., z_m)$, $P = (p_1, ..., p_m)$ *be two vectors with* $z_i, p_i \in M[[z]]$, $i = 1, ..., m$; *and* $Q = (q_{ij})_{m \times m}$ *be a matrix with quasi-regular coefficients* $q_{ij} \in M[[z]]$, $i, j = 1, ..., m$. *Then the system of equations*

$$Z = P + ZQ$$

has a unique solution $Y = (y_1, ..., y_m)$ *given by*

$$Y = P(I + Q^+)$$

with $y_i \in M[[z]], i = 1, ..., m$. *If, in addition, the components of* P *and* Q *are rational, then the components of* Y *are also rational.*

Proof of Proposition 1. Denote by $K+1$ the number of extended cone types $0, ..., K$ in G with respect to S, so that the identity element is of type 0. Then

$$F(z) \; = \; \sum_{i=0}^{K} F_i(z) \; ,$$

where

$$F_i(z) \; = \sum_{g \text{ of type } i} g z^{|g|} \; = \; \sum_{n=0}^{\infty} A_n^i z^n$$

with $A_n^i = \sum_{|g|=n, g \text{ of type } i} g$.

Note that

$$A_n^0 \; = \; \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \; ; \end{cases}$$

and

$$A_0^i \; = \; \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \; . \end{cases}$$

For each group element g of type i, $1 \le i \le K$, denote by $p_i \in \mathbb{Z}$ the number of predecessors at distance 1 of g (this number is uniquely determined by the extended cone type), and, for $1 \le j \le K$ and $0 \le i \le K$,

$$r_{ij} \; = \sum_{s \in S, |gs|=|g|+1, gs \text{ of type } j} s \; \in \; \mathbb{Z}[G] \; .$$

For $1 \le i \le K$, $n \ge 1$, the following recursion formula holds:

$$A_n^i \; = \; \frac{1}{p_i} \sum_{j=0}^{K} A_{n-1}^j r_{ji} \; .$$

Hence we have

$$F_0(z) \;=\; 1\;;$$

and for $1 \le i \le K$

$$F_i(z) \;=\; \sum_{n=0}^{\infty} A_n^i z^n \;=\; \sum_{n=1}^{\infty} \frac{1}{p_i} \sum_{j=0}^{K} A_{n-1}^j r_{ji} z^n$$
$$= \frac{z}{p_i} \sum_{j=0}^{K} F_j(z) r_{ji}\;.$$

Let $q_{ij} = zr_{ij}/p_j$ for $1 \le j \le K$, $0 \le i \le K$, and $q_{i0} = 0$ for $0 \le i \le K$. Each q_{ij} is a quasi-regular element of $\mathbb{Z}[G][z]$. Let $Q = (q_{ij})_{0 \le i,j \le K}$. Denote by $J = (1, 0, ..., 0)$ and $\mathcal{F} = (F_0, F_1, ..., F_K)$ the vectors with coefficients in $\mathbb{Z}[G]\big[[z]\big]$. Then $\mathcal{F}$ is the solution of the system

$$Z = J + ZQ,$$

and has rational components by Lemma 3. Thus each $F_i(z)$ is rational which implies the rationality of $F(z)$. □

An interesting problem in this context is the question about the rationality of complete growth functions of nilpotent and abelian groups. For instance, it is known that the growth function of the first Heisenberg group $\langle x, y \mid [y,[x,y]] = [x,[x,y]] = 1\rangle$ is rational. It seems to us that the complete growth function of this group is not rational. It is natural to conjecture the following:

Conjecture 1. *A nilpotent group has a rational complete growth function if and only if the group is virtually abelian.*

The rationality of complete growth functions of virtually abelian groups is studied in the Ph. D. thesis of F. Liardet (see [28]).

4. Complete Growth Functions of Surface Groups

Complete rewriting systems for the standard presentations of surface groups may be found in [38] and [21]. M. Stoll describes in [38] the complete rewriting systems for some class of presentations including the standard presentations of surface groups, and uses them for computing the growth series. These rewriting systems were also used in [30] for computing the growth functions of surface groups using the formula from [15], which is a weaker form of our Theorem 2. Let us mention that some finite complete systems of rewriting rules for the surface groups given by the presentations $\langle c_1, ..., c_{2g} \mid c_1...c_{2g}c_1^{-1}...c_{2g}^{-1} = 1\rangle$, $g \ge 2$, were considered in [27] in connection with algorithmic problems of the group theory.

We describe below a complete rewriting system $\mathcal{R}_g$ for the standard presentation of G_g.

Let us fix the order $a_1^{-1} < a_1 < ... < a_g^{-1} < a_g < b_1^{-1} < b_1 < ... < b_g^{-1} < b_g$ on the standard generators of G_g and consider the induced lexicographical order on the set of words S_g^* in the alphabet S_g.

There are two series of rewriting rules in each $\mathcal{R}_g$:

- the first consists of $4g$ rules of the form $ss^{-1} \to 1$ with $s \in S_g$;

- the second consists of $4g$ rules of the form $u_i \to v_i$ such that each u_i and v_i is a word of length $2g$, and either $u_i v_i^{-1}$ or $v_i u_i^{-1}$ coincides with one of the cyclic permutations of the relator $\prod_{i=1}^{g}[a_i, b_i]$. Each u_i begins with a letter from the set $\{b_1, b_1^{-1}, ..., b_g, b_g^{-1}\}$.

In accordance with the order chosen on S_g, the set of forbidden words $\{u_i\}_{i=1}^{8g}$ consists of the subwords of length $2g$ with the first letter in $\{b_1, b_1^{-1}, ...b_g, b_g^{-1}\}$ of the cyclic words $r = \prod_{i=1}^{g}[a_i, b_i]$ and r^{-1}, and of the words of the form ss^{-1}, $s \in S_g$.

For example, in the case of $g = 2$ the system $\mathcal{R}_2$ consists of the following 16 rewriting rules:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
b_1^{-1}a_1^{-1}b_2a_2 & \to & a_1^{-1}b_1^{-1}a_2b_2 & a_1a_1^{-1} & \to & 1\\
b_1^{-1}a_2b_2a_2^{-1} & \to & a_1b_1^{-1}a_1^{-1}b_2 & a_1^{-1}a_1 & \to & 1\\
b_1a_1^{-1}b_1^{-1}a_2 & \to & a_1^{-1}b_2a_2b_2^{-1} & a_2a_2^{-1} & \to & 1\\
b_1a_1b_1^{-1}a_1^{-1} & \to & a_2b_2a_2^{-1}b_2^{-1} & a_2^{-1}a_2 & \to & 1\\
b_2^{-1}a_1b_1a_1^{-1} & \to & a_2b_2^{-1}a_2^{-1}b_1 & b_1b_1^{-1} & \to & 1\\
b_2^{-1}a_2^{-1}b_1a_1 & \to & a_2^{-1}b_2^{-1}a_1b_1 & b_1^{-1}b_1 & \to & 1\\
b_2a_2^{-1}b_2^{-1}a_1 & \to & a_2^{-1}b_1a_1b_1^{-1} & b_2b_2^{-1} & \to & 1\\
b_2a_2b_2^{-1}a_2^{-1} & \to & a_1b_1a_1^{-1}b_1^{-1} & b_2^{-1}b_2 & \to & 1 \quad .
\end{array}$$

(Observe that 1 denotes the empty word in S^* as well as the identity element in the group.)

Proof of Theorem 1. In order to apply Theorem 2 for the computation of the complete growth function $F_g(z)$ let us compute the sum of rigid chains $Q_n^{(m)}$ for each n and m.

We begin with the case $g = 2$.

All chains are rigid in (G_2, S_2). Indeed, the property of a chain to be rigid means that the intersections of the arcs in the chain are maximal. But, if $g = 2$, for any two forbidden words in the system $\mathcal{R}_2$ described above, there exists a unique way of intersection (by two letters, if both forbidden words are of length 4, and by one letter, if at least one of two forbidden words is of length 2).

Consequently, there are four types of rigid chains:

1. all arcs in a rigid chain are of length 2 and all intersections are of length 1;

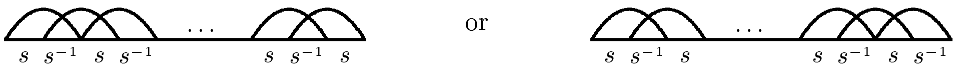

Figure 5.

2. all arcs in a rigid chain are of length 4 and all intersections are of length 2;

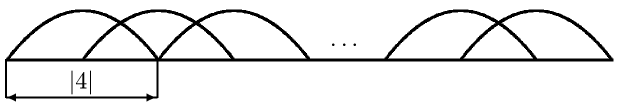

Figure 6.

3. arcs of length 2 are followed by arcs of length 4 or arcs of length 4 are followed by arcs of length 2 (here the intersections are of length 1 and 2);

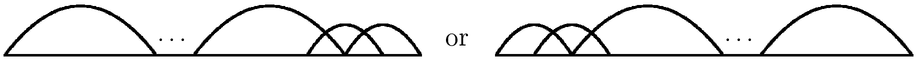

Figure 7.

4. arcs of length 4 are surrounded from both sides by arcs of length 2 (here again the intersections are of length 1 and 2).

Figure 8.

For $n, m \geq 1$; $1 \leq l \leq 4$, let $Q_{n,l}^{(m)}$ denote the sum of rigid chains of length n, type l and rank m. We are going to compute $Q_{n,l}^{(m)}$ separately for $l = 1, ..., 4$. All computations are performed in $\mathbb{Z}[G_2][[z]]$.

Type 1. A word w which is a chain of type 1 must be of the form $ss^{-1}ss^{-1}...ss^{-1}$, in which case w may be reduced to 1, or of the form $ss^{-1}...s^{-1}s$, in which case w may be reduced to $s \in S_2$. In both cases there exist 8 different chains of such form and given length. This implies that

$$Q_{n,1}^{(m)} = \begin{cases} 8 \cdot 1 & \text{if } m = n+1,\ n \text{ is even },\\ A & \text{if } m = n+1,\ n \text{ is odd },\\ 0 & \text{otherwise }, \end{cases}$$

where A denotes, as in Section 1, the sum of all elements of S_2. Thus,

$$\begin{aligned} \sum_{m=1}^{\infty}(-1)^{i+1}\sum_{n=2}^{\infty} Q_{n,1}^{(m)} z^n &= \sum_{n=2}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1} Q_{n,1}^{(m)} z^n = \sum_{n=2}^{\infty}(-1)^n Q_{n,1}^{(n+1)} z^n \\ &= 8\sum_{k=1}^{\infty} z^{2k} - A\sum_{k=1}^{\infty} z^{2k+1} \\ &= \frac{z^2(8-Az)}{1-z^2}\ . \end{aligned}$$

Type 2. As mentioned above, the only possibility, when all arcs in a rigid chain are of length 4, is that they intersect each other by subwords of length 2. Thus $Q_{n,2}^{(m)}$ is different from 0 if and only if n is even and $m = \frac{n}{2} - 1$. We introduce the following notations:

$$\begin{aligned}
A_a &= a_1 + a_1^{-1} + a_2 + a_2^{-1};\\
A_b &= b_1 + b_1^{-1} + b_2 + b_2^{-1};\\
A &= A_a + A_b;\\
B_a &= a_2b_2 + a_2b_2^{-1} + a_2^{-1}b_1 + a_2^{-1}b_2^{-1} + a_1b_1^{-1} + a_1b_1 + a_1^{-1}b_1^{-1} + a_1^{-1}b_2;\\
B_b &= b_1^{-1}a_1^{-1} + b_1^{-1}a_2 + b_1a_1^{-1} + b_1a_1 + b_2^{-1}a_1 + b_2^{-1}a_2^{-1} + b_2a_2^{-1} + b_2a_2;\\
B &= B_a + B_b;\\
C_a &= a_1^{-1}b_2a_2 + a_2b_2a_2^{-1} + a_1^{-1}b_1^{-1}a_2 + a_1b_1^{-1}a_1^{-1}\\
&\qquad + a_1b_1a_1^{-1} + a_2^{-1}b_1a_1 + a_2^{-1}b_2^{-1}a_1 + a_2b_2^{-1}a_2^{-1};
\end{aligned}$$

$$\begin{aligned}
C_b &= b_1a_1^{-1}b_1^{-1} + b_1^{-1}a_2b_2 + b_2a_2^{-1}b_2^{-1} + b_2^{-1}a_1b_1\\
&\qquad + b_2a_2b_2^{-1} + b_2^{-1}a_2^{-1}b_1 + b_1a_1b_1^{-1} + b_1^{-1}a_1^{-1}b_2;\\
C &= C_a + C_b;\\
D &= b_1^{-1}a_1^{-1}b_2a_2 + b_1^{-1}a_2b_2a_2^{-1} + b_1a_1^{-1}b_1^{-1}a_2 + b_1a_1b_1^{-1}a_1^{-1}\\
&\qquad + b_2^{-1}a_1b_1a_1^{-1} + b_2^{-1}a_2^{-1}b_1a_1 + b_2a_2^{-1}b_2^{-1}a_1 + b_2a_2b_2^{-1}a_2^{-1}.
\end{aligned}$$

Observe that A (respectively B, C, D) is the sum of all different subwords of length 1 (respectively 2,3,4) of the forbidden words of length 4. In these notations we have

$$Q_{n,2}^{(\frac{n}{2}-1)} = \begin{cases} 8\cdot 1 & \text{if } n \equiv 0(mod\ 8)\ ,\\ B_b & \text{if } n \equiv 2(mod\ 8)\ ,\\ D & \text{if } n \equiv 4(mod\ 8),\\ B_a & \text{if } n \equiv 6(mod\ 8)\ . \end{cases}$$

Thus

$$\begin{aligned}
\sum_{n=4}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}Q_{n,2}^{(m)}z^n &= \sum_{n=4}^{\infty}(-1)^{\frac{n}{2}}Q_{n,2}^{(\frac{n}{2}-1)}z^n = \sum_{k=2}^{\infty}(-1)^kQ_{2k,2}^{(k-1)}z^{2k}\\
&= 8\sum_{l=1}^{\infty}z^{8l} - B_b\sum_{l=1}^{\infty}z^{8l+2} + D\sum_{l=0}^{\infty}z^{8l+4} - B_a\sum_{l=0}^{\infty}z^{8l+6}\\
&= \frac{z^4(D - B_az^2 + 8z^4 - B_bz^6)}{1-z^8}\ .
\end{aligned}$$

Type 3. Let r be the rank of the subchain that consists only of arcs of length 2. Then the rank of the whole chain of type 3 of length n is $\frac{n+r}{2} - 1$, and $n + 2 \equiv 0(mod\ 2)$. Thus

$$\begin{aligned}
\sum_{n=5}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}Q_{n,3}^{(m)}z^n &= \sum_{r=1}^{\infty}\sum_{n=r+4}^{\infty}(-1)^{\frac{n+r}{2}}Q_{n,3}^{(\frac{n+r}{2}-1)}z^n\\
&= \sum_{l=1}^{\infty}\sum_{k=l+2}^{\infty}(-1)^{k+l}Q_{2k,3}^{(k+l-1)}z^{2k} + \sum_{l=0}^{\infty}\sum_{k=l+2}^{\infty}(-1)^{k+l+1}Q_{2k+1,3}^{(k+l)}z^{2k+1}\ .
\end{aligned}$$

When n and r are both even we have

$$Q_{n,3}^{(\frac{n+r}{2}-1)} = \begin{cases} 16\cdot 1 & \text{if } n-r \equiv 0(mod\ 8)\ , \\ 2B_b & \text{if } n-r \equiv 2(mod\ 8)\ , \\ 2D & \text{if } n-r \equiv 4(mod\ 8)\ , \\ 2B_a & \text{if } n-r \equiv 6(mod\ 8)\ . \end{cases}$$

When n and r are both odd we have

$$Q_{n,3}^{(\frac{n+r}{2}-1)} = \begin{cases} 2A & \text{if } n-r \equiv 0(mod\ 8)\ , \\ 2A & \text{if } n-r \equiv 2(mod\ 8)\ , \\ C & \text{if } n-r \equiv 4(mod\ 8)\ , \\ C & \text{if } n-r \equiv 6(mod\ 8)\ . \end{cases}$$

Consequently,

$$\sum_{n=5}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}Q_{n,3}^{(m)}z^n = \frac{2z^6(D-B_a z^2+8z^4-B_b z^6)+z^5(-C+Cz^2-2Az^4+2Az^6)}{(1-z^2)(1-z^8)}\ .$$

Type 4. Let $r = r_1 + r_2$ where r_1 and r_2 denote the ranks of the prefix and suffix consisting only of arcs of length 2. If r_1 and r_2 are both even then

$$Q_{n,4}^{(\frac{n+r}{2}-1)} = \begin{cases} 16\cdot 1 & \text{if } n-r \equiv 0(mod\ 8)\ , \\ 2B_b & \text{if } n-r \equiv 2(mod\ 8)\ , \\ 2D & \text{if } n-r \equiv 4(mod\ 8)\ , \\ 2B_a & \text{if } n-r \equiv 6(mod\ 8)\ . \end{cases}$$

If one of r_i, $i = 1, 2$, is even and another is odd, then

$$Q_{n,4}^{(\frac{n+r}{2}-1)} = \begin{cases} 2A & \text{if } n-r \equiv 0(mod\ 8)\ , \\ 2A & \text{if } n-r \equiv 2(mod\ 8)\ , \\ C & \text{if } n-r \equiv 4(mod\ 8)\ , \\ C & \text{if } n-r \equiv 6(mod\ 8)\ . \end{cases}$$

Finally, if r_1 and r_2 are both odd, then

$$Q_{n,4}^{(\frac{n+r}{2}-1)} = \begin{cases} B_b & \text{if } n-r \equiv 0(mod\ 8)\ , \\ 8\cdot 1 & \text{if } n-r \equiv 2(mod\ 8)\ , \\ B_a & \text{if } n-r \equiv 4(mod\ 8)\ , \\ D & \text{if } n-r \equiv 6(mod\ 8)\ . \end{cases}$$

Consequently we get

$$\sum_{n=6}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}Q_{n,4}^{(m)}z^n = \frac{\Delta}{(1-z^2)^2(1-z^8)}\ ,$$

where

$$\Delta = z^6(B_a - Dz^2 + B_b z^4 - 8z^6) + z^8(D - B_a z^2 + 8z^4 - B_b z^6) + z^7(-C + Cz^2 - 2Az^4 + 2Az^6)\ .$$

We apply now Theorem 2 and get the following formula for the *complete growth function of the surface group* (G_2, S_2):

$$F_2(z) = \frac{(1-z^2)(1-z^8)}{(1+7z^2+7z^8+z^{10}) - Az(1+z^8) - Cz^5 + Dz^4(1+z^2)}\ .$$

The same expression for $F_2(z)$ was obtained by L. Bartholdi, also using Theorem 1.

Let us now turn to the case $g \geq 3$. In order to demonstrate the difference between the cases $g = 2$ and $g \geq 3$, we give here an example of a chain in $(G_g, S_g), g \geq 3$ which is not rigid. Indeed, the forbidden words of length $2g$ can intersect each other by subwords of length $2, 4, ..., 2g-2$, hence the number of different types of chains should increase while g increases. Nevertheless, as the intersections of the arcs in a rigid chain have to be maximal possible, only the chains with the intersections of length 1 and $2g-2$ are rigid. Thus there are still only few different types of rigid chains in G_g, essentially the same as in G_2. The difference is that the form of the chains consisting of arcs of length $2g$ depends on the rank of the chain when g is at least 3:

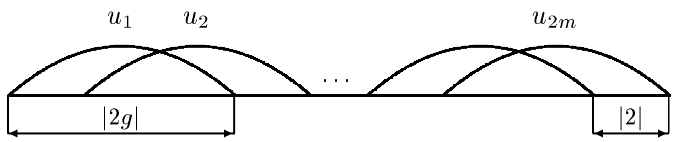

Figure 9.

- for even rank,

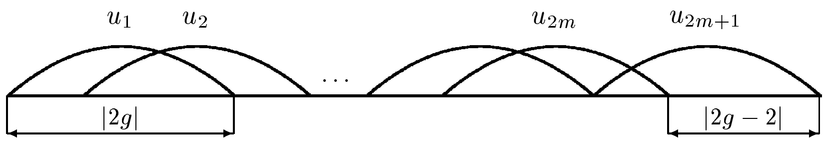

Figure 10.

- for odd rank.

We introduce the types $1, ..., 4$ of rigid chains, similar to the case $g = 2$. Let $Q_{n,l}^{(m)}$ denote as before the sum of rigid chains of length n, type l and rank m. We are going to compute $Q_{n,l}^{(m)}$ separately for $l = 1, ..., 4$.

We adopt following notations: A, (respectively A_a, A_b) will still denote the sum of elements of S_g, (respectively the sum of $a_i^{\pm 1}$'s, $b_i^{\pm 1}$'s). The sum of all subwords of length k of the cyclic words r and r^{-1} will be denoted by $\mathcal{P}_k$, respectively $\mathcal{P}_k^a, \mathcal{P}_k^b$ will denote the sums of all subwords of length k in the cyclic words r and r^{-1} which begin with an $a_i^{\pm 1}$, $b_i^{\pm 1}$. Remark that $\mathcal{P}_{2g}^b$ is the sum of all forbidden words of length $2g$.

Type 1. A word w which is a rigid chain of type 1 must be of the form $ss^{-1}...ss^{-1}$, in which case w may be reduced to 1, or of the form $ss^{-1}...s^{-1}s$, in which case w may be reduced to $s \in S$. The computations are similar to the case of $g = 2$ and we have

$$Q_{n,1}^{(m)} = \begin{cases} 4g \cdot 1 & \text{if } m = n+1, \ n \text{ is even }, \\ A & \text{if } m = n+1, \ n \text{ is odd }, \\ 0 & \text{otherwise }; \end{cases}$$

and

$$\sum_{m=1}^{\infty} (-1)^{i+1} \sum_{n=2}^{\infty} Q_{n,1}^{(m)} z^n = \frac{z^2(4g - Az)}{1 - z^2} .$$

Type 2. As mentioned above, one should distinguish the rigid chains of type 2 (i.e., consisiting only of arcs of length $2g$) of odd and even rank (see Figures 9 and 10). If rank is even, $m = 2k$, then $n = 2gk + 2$ and n can be congruent to 2 or $2g + 2$ modulo $4g$. If rank is odd, $m = 2k + 1$, then $n = 2gk + 2g$ and n can be congruent to $2g$ or 0 modulo $4g$. Consequently we get

$$Q^{(m)}_{n,2} = \begin{cases} 4g \cdot 1 & \text{if } n \equiv 0(mod\ 4g)\ , \\ \mathcal{P}^b_2 & \text{if } n \equiv 2(mod\ 4g)\ , \\ \mathcal{P}^b_{2g} & \text{if } n \equiv 2g(mod\ 4g), \\ \mathcal{P}^a_{2g-2} & \text{if } n \equiv 2g+2(mod\ 4g)\ . \end{cases}$$

Therefore

$$\sum_{n=2g}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} Q^{(m)}_{n,2} z^n = \frac{z^{2g}(4gz^{2g} + \mathcal{P}^b_{2g} - \mathcal{P}^a_{2g-2}z^2 - \mathcal{P}^b_2 z^{2g+2})}{1 - z^{4g}}.$$

Type 3. Let r be the rank of the subchain that consists only of arcs of length 2 and m be as usual the rank of the whole chain.

It will be convenient to split

$$\sum_{n=2g+1}^{\infty} \sum_{m=2}^{\infty} (-1)^{m+1} Q^{(m)}_{n,3} z^n = \Sigma_1 + \Sigma_2\ ,$$

where Σ_1 (respectively Σ_2) stands for the sum of chains with even (respectively odd) r.

Each chain of type 3 with even r represents the same element of G_g as its subchain consisting only of arcs of length $2g$. Thus, if r is even,

$$Q^{(m)}_{n,3} = \begin{cases} 4g \cdot 1 & \text{if } n-r \equiv 0(mod\ 4g)\ , \\ \mathcal{P}^b_2 & \text{if } n-r \equiv 2(mod\ 4g)\ , \\ \mathcal{P}^b_{2g} & \text{if } n-r \equiv 2g(mod\ 4g), \\ \mathcal{P}^a_{2g-2} & \text{if } n-r \equiv 2g+2(mod\ 4g)\ . \end{cases}$$

Consequently,

$$\Sigma_1 = 2\frac{z^{2g+2}(\mathcal{P}^b_{2g} - \mathcal{P}^a_{2g-2}z^2 + 4gz^{2g} - \mathcal{P}^b_2 z^{2g+2})}{(1 - z^{4g})(1 - z^2)}\ .$$

We are going to compute now the contribution of the chains with odd r. The case when the arcs of length 2 are in the end of a rigid chain should be considered separately from the case when the arcs of length 2 are in the beginning. For these two cases we get:

$$\begin{array}{c}\text{the sum of rigid chains of type 3}\\ \text{which } \textit{end} \text{ with an odd number}\\ \text{of arcs of length 2}\end{array} = \begin{cases} 2A_b & \text{if } m-r \equiv 0(mod\ 4)\ , \\ \mathcal{P}^b_{2g-1} & \text{if } m-r \equiv 1(mod\ 4)\ , \\ \mathcal{P}^a_{2g-1} & \text{if } m-r \equiv 2(mod\ 4)\ , \\ 2A_a & \text{if } m-r \equiv 3(mod\ 4)\ ; \end{cases}$$

and

$$\begin{array}{c}\text{the sum of rigid chains of type 3}\\ \text{which } \textit{begin} \text{ with an odd number}\\ \text{of arcs of length 2}\end{array} = \begin{cases} 2A_a & \text{if } m-r \equiv 0(mod\ 4)\ , \\ \mathcal{P}^a_{2g-1} & \text{if } m-r \equiv 1(mod\ 4)\ , \\ \mathcal{P}^b_{2g-1} & \text{if } m-r \equiv 2(mod\ 4)\ , \\ 2A_b & \text{if } m-r \equiv 3(mod\ 4)\ . \end{cases}$$

Observe now that the summands of Σ_2 corresponding to the chains with even $m-r$ (equivalently odd m) get positive signs, and the summands corresponding to the chains with odd $m-r$ (equivalently even m) get negative signs. Thus

$$\Sigma_2 = \frac{z^{2g+1}(\mathcal{P}^a_{2g-1}z^2 + 2A_b z^{2g+2} - \mathcal{P}^b_{2g-1} - 2A_a z^{2g})}{(1-z^2)(1-z^{4g})} + \frac{z^{2g+1}(\mathcal{P}^b_{2g-1}z^2 + 2A_a z^{2g+2} - \mathcal{P}^a_{2g-1} - 2A_b z^{2g})}{(1-z^2)(1-z^{4g})} = -\frac{z^{2g+1}(\mathcal{P}_{2g-1} + 2Az^{2g})}{1-z^{4g}} .$$

Type 4. Let r_1 and r_2 denote the ranks of two subchains which consist only of arcs of length 2. The values of $Q^{(m)}_{n,4}$ may be found separately when r_1, r_2, and n are even or odd. Namely, when r_1 and r_2 are both even we have

$$Q^{(m)}_{n,4} = \begin{cases} 4g\cdot 1 & \text{if } n-r_1-r_2 \equiv 0(mod\ 4g) \ , \\ \mathcal{P}^b_2 & \text{if } n-r_1-r_2 \equiv 2(mod\ 4g) \ , \\ \mathcal{P}^b_{2g} & \text{if } n-r_1-r_2 \equiv 2g(mod\ 4g), \\ \mathcal{P}^a_{2g-2} & \text{if } n-r_1-r_2 \equiv 2g+2(mod\ 4g) \ . \end{cases}$$

When r_1 and r_2 are both odd

$$Q^{(m)}_{n,4} = \begin{cases} \mathcal{P}^b_2 & \text{if } n-r_1-r_2 \equiv 0(mod\ 4g) \ , \\ 4g\cdot 1 & \text{if } n-r_1-r_2 \equiv 2(mod\ 4g) \ , \\ \mathcal{P}^a_{2g-2} & \text{if } n-r_1-r_2 \equiv 2g(mod\ 4g), \\ \mathcal{P}^b_{2g} & \text{if } n-r_1-r_2 \equiv 2g+2(mod\ 4g) \ . \end{cases}$$

When r_1+r_2 is odd

$$Q^{(m)}_{n,4} = \begin{cases} 2A & \text{if } n-r_1-r_2 \equiv 0 \text{ or } 1(mod\ 4) \ , \\ \mathcal{P}_{2g-1} & \text{if } n-r_1-r_2 \equiv 2 \text{ or } 3(mod\ 4) \ . \end{cases}$$

Finally we get

$$\sum_{n=2g+2}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}Q^{(m)}_{n,4}z^n = \frac{z^{2g+2}\big((\mathcal{P}^a_{2g-2} + \mathcal{P}^b_2 z^{2g})(1+z^2) - z(\mathcal{P}_{2g-1} + 2Az^{2g})\big)}{(1-z^2)(1-z^{4g})} .$$

Putting together the results of the computations of $Q^{(m)}_{n,l}$ for $l = 1, ..., 4$, we get $(*)$. □

Among many different presentations of the group G_g as one-relator group

$$G_g = \langle c_1, ..., c_{2g} \mid r = 1\rangle \ ,$$

those are of most interest, for which the relator r is a quadratic word, i.e., such that, for each i, c_i and c_i^{-1} occur in r only once.

For example, for $g = 2$ there exist 4 essentially different presentations with quadratic relators:

$$G_2 = \langle c_1, c_2, c_3, c_4 \mid c_1c_2c_1^{-1}c_2^{-1}c_3c_4c_3^{-1}c_4^{-1} = 1\rangle \ ,$$
$$G_2 = \langle c_1, c_2, c_3, c_4 \mid c_1c_2c_3c_4c_1^{-1}c_2^{-1}c_3^{-1}c_4^{-1} = 1\rangle \ ,$$
$$G_2 = \langle c_1, c_2, c_3, c_4 \mid c_1c_2c_3c_1^{-1}c_2^{-1}c_4c_3^{-1}c_4^{-1} = 1\rangle \ ,$$
$$G_2 = \langle c_1, c_2, c_3, c_4 \mid c_1c_2c_1^{-1}c_3c_2^{-1}c_4c_3^{-1}c_4^{-1} = 1\rangle \ .$$

It is shown in [7] that corresponding to these presentations non oriented Cayley graphs without labeling of the edges by generators, are pairwise isomorphic. Therefore the ordinary growth functions corresponding to these four presentations coincide.

The corresponding complete growth functions are clearly different as elements of $\mathbb{Z}[G_g][[z]]$. It is interesting to observe however, that they look similar.

For example let $\tilde{S}_g = \{c_1, ..., c_{2g}, c_1^{-1}, ..., c_{2g}^{-1}\}$ denote the system of generators that leads to the second presentation

$$G_g \;=\; \langle c_1, ..., c_{2g} \mid c_1 ... c_{2g} c_1^{-1} ... c_{2g}^{-1} = 1\rangle \; .$$

One can check easily, using Nielsen transformations, that for $g = 2$ the generators $c_1^{\pm1}, c_2^{\pm1}, c_3^{\pm1}, c_4^{\pm1}$ of G_2 have the following form when written in standard generators $\{a_1^{\pm1}, a_2^{\pm1}, b_1^{\pm1}, b_2^{\pm1}\}$:

$$\begin{aligned} c_1 &= a_1 a_2 \; , \\ c_2 &= a_2^{-1} b_1 a_1^{-1} \; , \\ c_3 &= b_1^{-1} a_2 \; , \\ c_4 &= b_2 \; . \end{aligned}$$

The complete growth function $\tilde{F}_g(z)$ of G_g with respect to the system of generators $\tilde{S}_g$ is of the form:

$$\tilde{F}_g(z) \;=\; \frac{(1-z^2)(1-z^{4g})}{\Delta} \;\in\; \mathbb{Z}[G_g][[z]] \; ,$$

with

$$\Delta \;=\; 1+(4g-1)z^2+(4g-1)z^{4g}+z^{4g+2}-\tilde{A}z(1+z^{4g})-\tilde{\mathcal{P}}_{2g-1}z^{2g+1}+\tilde{\mathcal{P}}^b_{2g}z^{2g}(1+z^2) \; ,$$

where $\tilde{A} = c_1 + ... + c_{2g} + c_1^{-1} + ... + c_{2g}^{-1} \in \mathbb{Z}[G_g]$ is the sum of all elements of $\tilde{S}_g$; $\tilde{\mathcal{P}}^b_{2g} \in \mathbb{Z}[G_g]$ is the sum of all subwords of length $2g$ with first letter in $\{c_2, c_2^{-1}, c_4, c_4^{-1}, ..., c_{2g}, c_{2g}^{-1}\}$ of the cyclic words $\tilde{r} = c_1 ... c_{2g} c_1^{-1} ... c_{2g}^{-1}$ and $\tilde{r}^{-1}$; and $\tilde{\mathcal{P}}_{2g-1} \in \mathbb{Z}[G_g]$ is the sum of all subwords of length $2g-1$ of the cyclic words $\tilde{r}$ and $\tilde{r}^{-1}$.

5. Operator Growth Functions

Let G be a finitely generated group, S a finite system of generators, and let $F(z) = \sum_{n=0}^{\infty} A_n z^n$ be as in Section 1, the complete growth function of G with respect to S. For each $z \in \mathbb{C}$ with $|z|$ small enough, $F(z)$ may be considered as an element of a closure $\widehat{\mathbb{C}[G]}$ of the group algebra in some norm, for instance in l^1 norm. This leads us to the following definition.

If $T : G \to \mathcal{B}_*$ is a representation of the group G in the group of invertible elements of a Banach algebra $\mathcal{B}$, we define the *operator growth function* (or *operator growth series*) *with respect to the representation* T as

$$F^T(z) \;=\; \sum_{g\in G} T_g z^{|g|} \;=\; \sum_{n=0}^{\infty} A_n^T z^n \;\in\; \mathcal{B}[[z]] \; ,$$

where $A_n^T = \sum_{g\in G, |g|=n} T_g$.

Note that if the complete growth function $F(z)$ is rational one can compute the operator growth function with respect to any representation T for which the elements $T_s, s \in S$ are known explicitly, by replacing the generators s in the rational expression for $F(z)$ by the corresponding elements T_s.

The natural question about the radius of convergence of such operator growth series arises in this context.

The case of left regular representation in the reduced C^*-algebra $C^*_r(G)$ is of particular interest, and $F^{reg}(z)$ with values in $C^*_r(G)$ will be called the *operator growth function* of the group G. The corresponding coefficients A^{reg}_n may be given explicitly:

$$A^{reg}_n \ = \sum_{g \in G, |g|=n} g \ ,$$

viewed as operators of left convolution in the Hilbert space $l^2(G)$.

Though we are not aware of any study of the operator growth functions, they appear implicitly in some works. For example, in [2] the recursion formula for the operators A^T_n/a_n is obtained, in the case where G is the free group with two generators and T is its representation in the space $L_2(S^2)$ induced by two independent rotations of the sphere S^2. (Recall from Section 1 that a_n denotes the number of group elements of length n.) The corresponding formula for the operators A^T_n leads to the following expression for the operator growth function of the free group $\mathbb{F}_m$ of rank m with respect to the basis:

$$F^{reg}(z) \ = \ \frac{1 - z^2}{1 \ - \ Az \ + \ (2m-1)z^2} \ ,$$

where A is the operator on $l^2(\mathbb{F}_m)$ corresponding to the sum of generators.

An analogous formula holds for the Hecke operators on trees (see [34, Chap. 1.1, exercise 3]).

Note that $A/2m$ is the Markov operator of the simple random walk on $\mathbb{F}_m$, and consequently $A - 2m$ is the discrete Laplace operator on $\mathbb{F}_m$. It is interesting to observe that the spectral radius of the operator A may be computed using the above formula for $F^{reg}(z)$ and the value of the radius of convergence of $F^{reg}(z)$ (which can be easily computed: see Proposition 4 below). This gives us one more method of computation of the spectral radius of the simple random walk on the free group, which was first found in [24].

Consider a finitely generated group G with a finite symmetric system of generators S.

Proposition 2. *The radius of convergence of the series $\sum_{n=0}^{\infty} A_n z^n$ with $z \in \mathbb{C}$ and coefficients $A_n \in l^1(G)$ coincides with the radius of convergence of the ordinary growth series of G.*

Proof. By definition of the norm l^1, $\|A_n\|_1 = a_n$, where a_n is as above the number of elements of G of length n. Hence the radius of convergence of the series $F(z) = \sum_{n=0}^{\infty} A_n z^n$ coincides with the radius of convergence of the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. □

More interesting is the case of the operator growth functions with coefficients in the C^*-reduced algebra $C^*_r(G)$.

The following proposition is a result of a conversation between the first author and Pierre Julg.

Denote by R the radius of convergence of the operator growth series $F^{reg}(z) = \sum_{n=0}^{\infty} A_n^{reg} z^n$ with coefficients $A_n^{reg} = \sum_{|g|=n} g \in C_r^*(G)$. Then R satisfies the relation

$$\frac{1}{R} = \limsup_{n\to\infty} \|A_n^{reg}\|^{\frac{1}{n}} ,$$

where $\|A_n^{reg}\|$ denotes the norm of A_n^{reg} in $C_r^*(G)$, i.e., the operator norm.

Proposition 3. *If G is amenable then the radius of convergence of the operator growth series with coefficients in $C_r^*(G)$ is equal to the radius of convergence of the ordinary growth series of G.*

Proof. For each $n \geq 1$, the operator A_n^{reg}/a_n is the Markov operator of the random walk on G with respect to the probability measure uniformly distributed on the elements of length n. As G is amenable, its subgroup generated by the elements of length n is amenable too. Hence Kesten's criterion of amenability [25] may be applied; it yields that $\|A_n^{reg}/a_n\| = 1$ and thus the radius of convergence of $F^{reg}(z)$ coincides with the radius of convergence of $f(z)$. □

We think that the converse should also be true. More precisely, we formulate the following conjecture.

Conjecture 2. *A finitely generated group G is amenable if and only if the radius of convergence of its operator growth series with coefficients in $C_r^*(G)$ with respect to any finite symmetric system of generators coincides with the radius of convergence of its ordinary growth series.*

The following computation of the radius of convergence of the operator growth series of hyperbolic groups may be regarded as the first step towards the conjecture.

Let G be a non elementary hyperbolic group with a finite symmetric system of generators S and let

$$\lambda = \limsup_{n\to\infty} (a_n)^{1/n}$$

be the rate of exponential growth of G. The radius of convergence of the growth series $f(z)$ of G with respect of S is equal to $1/\lambda$.

Consider now the operator growth series $F^{reg}(z) = \sum_{n=0}^{\infty} A_n^{reg} z^n$ with coefficients in $C_r^*(G)$.

Proposition 4. *The radius of convergence of $F^{reg}(z)$ is equal to $1/\sqrt{\lambda}$.*

It follows from Proposition 4 that, for a non elementary hyperbolic group, the growth series with operator coefficients converges in a strictly wider circle than the growth series with integral coefficients.

The proof of Proposition 4 is based on the following result.

For any group G, we denote by $\{\delta_g\}_{g\in G}$ the canonical orthonormal basis of the Hilbert space $l^2(G)$. Consider the linear map $\xi \mapsto \xi\delta_1$ which is an injection of the reduced C^*-algebra $C_r^*(G)$ into the Hilbert space $l^2(G)$.

Lemma 4. *Let G be a hyperbolic group. There exist the constants $C > 0$ and $k > 1$ such that*

$$\sum_{g\in G} |(\xi\delta_1)(g)|^2 \leq \|\xi\|^2 \leq C \sum_{g\in G} |(\xi\delta_1)(g)|^2 (1+|g|)^{2k}$$

for all $\xi \in C_r^*(G)$.

Proof. The first inequality is obvious because $\|\xi\|^2 = \sup\{\|\xi f\|_2^2 \mid f \in l^2(G), \|f\|_2 = 1\} \geq \|\xi\delta_1\|_2^2$. The second one is due to U. Haagerup for free groups ([19, Lemma 1.5]) and to P. Jolissaint and P. de la Harpe for hyperbolic groups (see [22] and [20]). □

As a corollary we get the following useful technical lemma.

Lemma 5. *Let G be a hyperbolic group. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of elements of $C_r^*(G)$, and suppose that for each $n \geq 1$ the support of the function $\xi_n\delta_1 \in l^2(G)$ is contained in the sphere of radius n in G. Then*

$$\limsup_{n\to\infty} \|\xi_n\|^{\frac{1}{n}} = \limsup_{n\to\infty} \|\xi_n\delta_1\|_2^{\frac{1}{n}} .$$

Proof. It follows from Lemma 4 that for each n

$$\|\xi_n\delta_1\|_2^2 \leq \|\xi_n\|^2 \leq C(1+n)^{2k}\|\xi_n\delta_1\|_2^2 .$$

Thus

$$\limsup_{n\to\infty} \|\xi_n\delta_1\|_2^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|\xi_n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|\xi_n\delta_1\|_2^{\frac{1}{n}} .$$

□

Proof of Proposition 4. Consider the power series $\sum_{n=0}^{\infty} A_n^{reg} z^n$. For each n the function $A_n^{reg}\delta_1 \in l^2(G)$ is supported on the shpere of radius n. Indeed,

$$\left|(A_n^{reg}\delta_1)(g)\right|^2 = \Big|\sum_{|h|=n} \delta_h(g)\Big|^2 = \begin{cases} 1 & \text{if } |g| = n , \\ 0 & \text{otherwise} . \end{cases}$$

Hence, by Lemma 5,

$$\limsup_{n\to\infty} \|A_n^{reg}\|^{\frac{1}{n}} = \sqrt{\limsup_{n\to\infty}(a_n)^{\frac{1}{n}}} ,$$

which leads to

$$R = \frac{1}{\sqrt{\lambda}} .$$

□

It was communicated to us by Alain Valette that Lemma 4 and thus Proposition 4 hold in fact for any *(RD)-group*, i.e., *group with the rapid decay property.* A group G is said to have the rapid decay property if there exist constants $C > 0$ and $k > 1$ such that for any function ϕ finitely supported on G

$$\|\Phi\|^2 \leq C\sum_{g\in G} |\phi(g)|^2(1+|g|)^{2k} ,$$

where Φ denotes the operator of convolution with ϕ on $l^2(G)$. All hyperbolic groups have rapid decay property. The class of (RD)-groups is closed under direct product. More properties of (RD)-groups are discussed in [23].

Interesting examples of (RD)-groups are groups acting on buildings. More precisely, it was proven in [10] that a group acting simply transitively on the vertices of an $\tilde{A}_2$-building is (RD). Let now G act simply transitively on the vertices of an $\tilde{A}_n$-building. Consider such a generating set in G that the corresponding Cayley graph coincides with the 1-skeleton of the building. It was proven by A. Valette [40] that the (RD)-inequality holds for the functions which are radial with respect to the corresponding length metric (i.e., constant on spheres). Thus our Proposition 4 holds for the groups acting simply transitively on the vertices of $\tilde{A}_n$-buildings.

6. Complete and Operator Geodesic Growth Functions

The *geodesic growth function* of a finitely generated group G with respect to a finite system of generators S is the power series

$$l(z) \;=\; \sum_{g\in G} \lambda(g) z^{|g|} \;\in\; \mathbb{Z}\left[[z]\right] \;,$$

where $\lambda(g)$ is the number of geodesics connecting g and 1 in $Cay(G,S)$. We have $l(z) = \sum_{n=0}^{\infty} \lambda_n z^n$, with $\lambda_n = \sum_{|g|=n} \lambda(g)$.

Up to now not much is known about the geodesic growth functions. Probably for the first time they were introduced in [31]. In [36] an algorithm is presented, which permits one to compute the number of geodesics joining 1 with any element of an automatic group.

The geodesic growth function may be viewed also as the growth function of the weighted Cayley graph, the weight of a vertex being the number of geodesics connecting it to 1. Similarly, the growth functions with other weights can be considered. We mention here only the function

$$m(z) \;=\; \sum_{g\in G} \lambda(g)^2 z^{|g|} \;=\; \sum_{n=0}^{\infty} \mu_n z^n \;,$$

where $\mu_n = \sum_{|g|=n} \lambda(g)^2$. In fact, μ_n counts the number of "unimodal loops", i.e., the number of closed geodesic paths beginning at the identity and returning to the identity after touching for the first time the sphere of radius n in the Cayley graph. The functions $l(z)$ and $m(z)$ of surface groups, as well as some other weighted growth functions, are computed in [3].

Each path in $Cay(G,S)$ may be viewed as a word in the alphabet S. Therefore, if S^* denotes the set of all finite words in the alphabet S, we can consider the language of geodesics

$$\begin{aligned}\mathcal{L} \;=\; \{w \in S^* \mid &\text{ the path in } Cay(G,S) \text{ beginning at } 1 \\ &\text{ and corresponding to } w \text{ is a geodesic}\} \;.\end{aligned}$$

It is known that this language is regular for a hyperbolic group with any finite system of generators (this is essentially due to J. Cannon, see [8], [12]). Therefore the ordinary geodesic growth functions of hyperbolic groups are rational. In [31] the rationality of the geodesic growth functions is proven for some other class of groups with special systems of generators. It is interesting to remark that the language of geodesics can be not regular for a virtually abelian group (an example of J. Cannon, see [12]).

Let us consider the ring $\mathbb{Z}\langle S\rangle$ of polynomials with noncommuting variables in the alphabet S and coefficients in $\mathbb{Z}$ and define the *complete geodesic growth function* of G with respect to S as:

$$L(z) \;=\; \sum_{w\in\mathcal{L}} w z^{|w|} \;=\; \sum_{n=0}^{\infty} \Lambda_n z^n \quad\in\; \mathbb{Z}\langle S\rangle\left[[z]\right] \;,$$

where $\Lambda_n = \sum_{w\in\mathcal{L},|w|=n} w \quad\in\; \mathbb{Z}\langle S\rangle$.

The function $L(z)$ is *rational* if it is rational as an element of $\mathbb{Z}\langle S\rangle\left[[z]\right]$.

The ordinary geodesic growth function may be obtained from the complete geodesic growth function via the augmentation mapping $\mathbb{Z}\langle S\rangle \to \mathbb{Z}$, $s \mapsto 1$.

Proposition 5. *Suppose that G has a finite number of cone types with respect to S. Then the complete geodesic growth function of G with respect to S is rational.*

In particular, this implies that the ordinary geodesic growth function of a group with a finite number of cone types is rational.

Note that (G,S) has finitely many cone types if and only if the language of geodesics $\mathcal{L}$ is regular. The regularity of the language of geodesics of a group G with a system of generators S implies (see [33, Theorem 5.1]) that the formal power series

$$\sum_{w\in\mathcal{L}} w \quad \in \ \mathbb{Z}\big\langle\langle S\rangle\big\rangle \ ,$$

is rational, where $\mathbb{Z}\big\langle\langle S\rangle\big\rangle$ denotes the ring of formal power series with noncommuting variables in the alphabet S and coefficients in $\mathbb{Z}$.
Proof of Proposition 5. The proof is similar to the proof of Proposition 1. If we denote by $K+1$ the number of cone types, then the recursion formula for Λ^i_n is:

$$\Lambda^i_n \ = \ \sum_{j=0}^{K} \Lambda^j_{n-1} r_{ji} \ ,$$

with

$$r_{ij} \ = \sum_{s\in S, ws\in\mathcal{L}, ws \text{ of type } j} s \ \in \ \mathbb{Z}\langle S\rangle \ ,$$

where $w \in \mathcal{L}$ is a geodesic of type i. Here the type of a $w\in\mathcal{L}$ is the cone type of the group element represented by w. □

We also define the *operator geodesic growth function* of G with respect to S:

$$L^{reg}(z) \ = \ \sum_{n=0}^{\infty} \Lambda^{reg}_n z^n \quad \in \ C^*_r(G)\big[[z]\big] \ ,$$

where $\Lambda^{reg}_n = \sum_{g\in G, |g|=n} \lambda(g) g$ viewed for each n as an operator of convolution in $l^2(G)$ acting by left multiplication.

Let τ denote the radius of convergence of the series of growth of unimodal loops $m(z) = \sum_{g\in G} \lambda(g)^2 g$.

Proposition 6. *The radius of convergence of $L^{reg}(z)$ is equal to $\sqrt{\tau}$.*

Proof. The proof runs similarly to the proof of Proposition 4. Indeed,

$$\big|(\Lambda^{reg}_n \delta_1)(g)\big|^2 \ = \ \big|\sum_{|h|=n} \lambda(h)\delta_h(g)\big|^2 \ = \ \begin{cases} \lambda(g)^2 & \text{if } |g| = n \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Hence, by Lemma 5 we have

$$\limsup_{n\to\infty} \|\Lambda^{reg}_n\|^{\frac{1}{n}} \ = \ \sqrt{\limsup_{n\to\infty} (\mu_n)^{\frac{1}{n}}} \ ,$$

which leads to

$$\Theta \ = \ \sqrt{\tau} \ .$$

□

Using the expression for $m(z)$ obtained in [3] for the surface groups, one can find τ and thus the radius of convergence of the operator geodesic growth series of surface groups.

In this section we need a different notion of cone types in a Cayley graph, weaker than what was given in Section 3. Let Γ be an infinite, locally finite connected graph, and let v_0 be a fixed vertex of Γ. We define the *cone* $\mathcal{C}(w)$ of a vertex w in Γ to be a subgraph of Γ rooted in w, such that a vertex v of Γ is a vertex of $\mathcal{C}(w)$ if and only if there exists a geodesic connecting v_0 and v in Γ and passing through w. Two vertices of Γ are said to be of the same *weak cone type* if their cones are isomorphic as rooted graphs.

Let (G_g, S_g) be as above the fundamental group of the orientable closed surface of genus $g, g \geq 2$ with the standard system of generators. There are $2g+1$ weak cone types in $Cay(G_g, S_g)$. (Observe that there are $1 + 8g(2g-1)$ cone types in $Cay(G_g, S_g)$, as defined in Section 3.) The weak cone types can be described as follows. The unit element is of type 0 and it is the only element of this type. Let us say that a vertex x is a *predecessor* of a vertex y and y is a *successor* of x, if x lies on a geodesic connecting 1 with y. For example, if x and y are adjacent and $|x| < |y|$, then x is a predecessor of y. Observe that, except the identity, each vertex in this graph has one or two predecessors among the adjacent vertices. We now define the vertices of type 2g to be precisely those which have two predecessors at distance 1. The vertices of type $2g-i$, $i = 1, ..., 2g-2$, are those which have the closest successor of type $2g$ at distance i. Finally the vertices of type 1 are those which have exactly one predecessor at distance 1 and the closest successor of type $2g$ at distance $2g-1$.

Consequently $l^2(G_g)$ can be naturally decomposed into orthogonal sum of $2g+1$ subspaces $l^2_i(G_g), i = 0, 1, ..., 2g$, which correspond to $2g+1$ weak cone types of vertices in the Cayley graph of (G_g, S_g). Among these subspaces there is one of dimension 1, it is generated by the characteristic function of the identity element.

According to [3], the ordinary geodesic growth function of G_g with respect to S_g is given by

$$l_g(z) \;=\; \frac{1 + 2z + ... + 2z^{2g-2} + z^{2g-1} + 2z^{2g}}{1 - (4g-2)z - ... - (4g-2)z^{2g-2} - (4g-1)z^{2g-1} + 2z^{2g}} \;.$$

Denote by $L^{reg}_g(z)$ the operator geodesic growth function of G_g with respect to S_g, and $\mathcal{B}(l^2(G_g))$ the space of bounded operators on $l^2(G_g)$.

Theorem 3. *The following relation holds in* $\mathcal{B}(l^2(G_g))$*:*

$$L^{reg}_g(z) \;=\; \big(1 \;+\; (\;B \;-\; 4g\;)\; z^2\big)\big(1 \;-\; A\; z \;+\; B\; z^2\big)^{-1} \;,$$

where $A = \Lambda^{reg}_1 \in C^*_r(G_g)$ *is as before the sum of generators acting on* $l^2(G_g)$*, and* $B \in \mathcal{B}(l^2(G_g))$ *is a diagonal operator defined by the matrix*

$$B \;=\; \begin{pmatrix} b_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{2g-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{2g} \end{pmatrix}$$

with $b_i : l^2_i(G_g) \to l^2_i(G_g), i = 0, ..., 2g$ *scalar operators taking values* $b_0 = 0; b_1 = b_2 = ... = b_{2g-2} = 4g-1; b_{2g-1} = 4g; b_{2g} = 4g-2.$

Hence we got a rational expression for $L_g^{reg}(z)$. Remark that the relation above does not hold in $C_r^*(G_g)$ because B does not belong to $C_r^*(G_g)$. In particular, the ordinary geodesic growth function cannot be obtained directly from this expression.

Proof of Theorem 3. In order to compute the function $L_g^{reg}(z) = \sum_{n=0}^{\infty} \Lambda_n^{reg} z^n$ let us show that

$$\begin{aligned}
\Lambda_0^{reg} &= 1 ; \\
\Lambda_1^{reg} &= A = \sum_{s\in S} s ; \\
\Lambda_2^{reg} &= \Lambda_1^{reg}\Lambda_1^{reg} - 4g ; \\
\Lambda_n^{reg} &= \Lambda_{n-1}^{reg}\Lambda_1^{reg} - B\Lambda_{n-2}^{reg} \quad \text{for} \quad n \geq 3 .
\end{aligned}$$

The first two formulas are obvious. Suppose $n \geq 3$. Then

$$\begin{aligned}
\Lambda_{n-1}^{reg}\Lambda_1^{reg} &= \sum_{\substack{\gamma\in G, s\in S\\ |\gamma|=n-1}} \lambda(\gamma)\gamma s = \sum_{|\gamma|=n} \lambda(\gamma)\gamma + \sum_{\substack{|\gamma|=n-2, s\in S\\ |\gamma s|=n-1}} \lambda(\gamma s)\gamma \\
&= \Lambda_n^{reg} + \sum_{\substack{|\gamma|=n-2\\ \gamma \text{ of type } 2g}} (4g-2)\lambda(\gamma)\gamma + \sum_{\substack{|\gamma|=n-2\\ \gamma \text{ of type } 2g-1}} (4g)\lambda(\gamma)\gamma \\
&\quad + \sum_{\substack{|\gamma|=n-2\\ \gamma \text{ of type } 1,\ldots,2g-2}} (4g-1)\lambda(\gamma)\gamma .
\end{aligned}$$

Thus

$$\Lambda_n^{reg} = \Lambda_{n-1}^{reg}\Lambda_1^{reg} - \Lambda_{n-2}^{reg}B .$$

Now we compute

$$\begin{aligned}
L_g^{reg}(z) = \sum_{n=0}^{\infty} \Lambda_n^{reg} z^n &= 1 + Az + A^2z^2 - 4gz^2 + \sum_{n=3}^{\infty} \Lambda_n^{reg} z^n \\
&= 1 - 4gz^2 + Az + A^2z^2 + \big(L_g^{reg}(z) - 1 - Az\big)Az \\
&\quad - \big(L_g^{reg}(z) - 1\big)Bz^2 ,
\end{aligned}$$

and after necessary simplifications we get

$$L_g^{reg}(z) = \big(1 + (B - 4g)z^2\big)\big(1 - Az + Bz^2\big)^{-1} .$$

□

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