ISOPERIMETRIC CONSTANT OF A k-REGULAR GRAPH (A NOTE ON A THEOREM OF BIGGS, MOHAR, AND SHAWE-TAYLOR)

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Let Γ be an infinite k-regular graph, $k \geq 3$. Let $\rho \in (0,1]$ denote the spectral radius of Γ, i.e., the norm of the adjacency matrix B of Γ divided by k. The isoperimetric constant i of Γ is defined by

$$i \;:=\; \inf_{A\subset V(\Gamma), A \text{ finite}} \frac{|\partial A|}{|A|}\ ,$$

where ∂A is the subset of the edge set of Γ, which consists of the edges with one extremity in A and the other one in the complement of A.

Theorem. *We have the following inequalities connecting ρ and i:*

$$i \;\geq\; \frac{k^2(1-\rho)}{k-1} \;\;\Big(\Leftrightarrow\;\rho\;\geq\;1\;-\;\frac{i(k-1)}{k^2}\Big) \quad (1)$$

and

$$i \;\leq\; k\sqrt{1-\rho^2} \;\;\Big(\Leftrightarrow\;\rho\;\leq\;\sqrt{1-\frac{i^2}{k^2}}\Big) \quad (2)\ .$$

Remark. *In Theorems 3.1 and 3.2 in the paper* [BMS] *of Biggs, Mohar, and Shawe-Taylor we have instead of (1) and (2):*

$$i \;\geq\; 4(1-\rho) \;\;\Big(\Leftrightarrow\;\rho\;\geq\;1\;-\;\frac{i}{4}\Big) \quad (1')$$

and

$$i \;\leq\; \sqrt{2}k\sqrt{1-\rho} \;\;\Big(\Leftrightarrow\;\rho\;\leq\;1\;-\;\frac{i^2}{2k^2}\Big) \quad (2')\ .$$

The estimates (1) and (2) are better than estimates (1') and (2') respectively. Moreover, the estimate (2) is exact for regular trees.

Proof. The proof of Theorem follows the idea of the proof in [BMS]. To prove (1), let us consider a finite subset of vertices A and a function $x = (x_v)_{v\in V(\Gamma)}$ on the vertices of Γ defined by

$$x_v \;=\; \begin{cases} 1, & \text{if } v \in A,\\ c\in[0,1], & \text{if } v\notin A \text{ but } v \text{ adjacent with a vertex from } A,\\ 0, & \text{otherwise.}\end{cases}$$

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If d denote the coboundary operator on $l^2(V(\Gamma))$, we have

$$\begin{aligned}\|d^*x\|^2 &= \sum_{e\in E(\Gamma)} (x_{e_+} - x_{e_-})^2 \\ &\leq (1-c)^2|\partial A| + c^2(k-1)|\partial A|\end{aligned}$$

Therefore

$$\begin{aligned}\langle x, Bx\rangle &= \langle x, (kI - dd^*)x\rangle \\ &= k\|x\|^2 - \|d^*x\|^2 \\ &\geq k\|x\|^2 - \frac{(1-c)^2|\partial A|}{|A|}\|x\|^2 - \frac{c^2(k-1)|\partial A|}{|A|}\|x\|^2 \ .\end{aligned}$$

Hence we have

$$\rho \ \geq \ 1 - \frac{(1-2c+kc^2)i}{k} \ .$$

In order to make this estimate best possible we have to minimize $f(c) = 1-2c+kc^2$. It has a unique local minimum, with coordinates $\big(1/k, (k-1)/k\big)$. (1) follows.

The estimate $(2')$ can be improved, and (2) can be obtained, if we change one of two estimates of in the proof in [BMS]. More precisely, choose a finite subgraph of Γ with the adjacency matrix B of norm λ, and let x be a positive unit eigenvector for λ. We can extend it to $l^2(\Gamma)$ by 0. Define

$$\Delta^2 \ := \ \Big(\sum_{e\in E(\Gamma)} |x_{e_+}^2 - x_{e_-}^2|\Big)^2 \ .$$

It is shown in [BMS] that $\Delta \ \geq \ i$. Here is an estimate of Δ^2 which gives (2):

$$\begin{aligned}\Delta^2 &= \Big(\sum_{uv\in E(\Gamma)} |x_u + x_v||x_u - x_v|\Big)^2 \\ &\leq \sum_{uv\in E(\Gamma)} (x_v + x_u)^2 \sum_{uv\in E(\Gamma)} (x_v - x_u)^2 \\ &= \Big(\sum_{uv\in E(\Gamma)} (x_v^2 + x_u^2) + 2\sum_{uv\in E(\Gamma)} x_v x_u\Big)\Big(\sum_{uv\in E(\Gamma)} (x_v^2 + x_u^2) - 2\sum_{uv\in E(\Gamma)} x_v x_u\Big) \\ &= \Big(k\sum_{v\in V(\Gamma)} x_v^2 + \langle Bx, x\rangle\Big)\Big(k\sum_{v\in V(\Gamma)} x_v^2 - \langle Bx, x\rangle\Big) \\ &= (k+\lambda)(k-\lambda) \ .\end{aligned}$$

It follows that $\rho \ \leq \ \sqrt{1 - \frac{i^2}{k^2}}$.

References

BMS. N. Biggs, B. Mohar, J. Shawe-Taylor, *The spectral radius of infinite graphs*, Bull. London Math. Soc. **20** (1988), 116-120.