NOT EVERY UNIFORM TREE COVERS RAMANUJAN GRAPHS

Alexander Lubotzky and Tatiana Nagnibeda

Abstract. The notion of Ramanujan graph has been extended to not necessarily regular graphs by Y. Greenberg. We construct infinite trees with infinitely many finite quotients, none of which is Ramanujan. We give a sufficient condition for a finite graph to be covered by such a tree.

1. Introduction.

Let X be a finite (undirected) simple k-regular graph and δ its adjacency matrix. The graph X is called *Ramanujan* if, for every eigenvalue λ of δ, either $\lambda = \pm k$ or $|\lambda| \leq 2\sqrt{k-1}$. It is a well known result of Alon and Boppana (see [LPS, Proposition 4.2]) that the bound $|\lambda| \leq 2\sqrt{k-1}$ is the best possible if one seeks an infinite family of regular graphs of a fixed degree k.

Various constructions of families of Ramanujan graphs are known, all based on number theory (see [Lu1], [Va] and references therein). The most general one is due to M. Morgenstern [Mo] who constructed infinitely many k-regular Ramanujan graphs for every k of the form $p^\alpha + 1$ when p is a prime and α a positive integer. If k is not of this form, then only finitely many k-regular Ramanujan graphs are known.

Problem 1. *Let $k \geq 3$ be an integer. Are there infinitely many k-regular Ramanujan graphs?*

Problem 1 is open for every k which is not of the form $p^\alpha + 1$. The smallest open case is $k = 7$. One is tempted to conjecture that there should be infinitely many k-regular Ramanujan graphs for every $k \geq 3$. From the combinatorial point of view, there seems to be no difference between k of the form $p^\alpha + 1$ and others. Moreover, the results of J. Friedman, J. Kahn and E. Szemerédi show that, for every k, "almost every" k-regular graph is "almost Ramanujan" (see [Fr], [FKS] for precise definitions).

The purpose of this note is to illustrate that one should be more cautious in making such a conjecture.

In [Gr] Y. Greenberg introduced the notion of Ramanujan graph for a general finite graph (not necessarily regular). Namely, a finite graph X is called *Ramanujan* if for every non-trivial eigenvalue λ of X (i.e., except the Perron-Frobenius's one

---

Key words and phrases. Ramanujan graph, covering tree, spectral radius, minimal graph.

The first author acknowledges support from the "Fonds National Suisse de la Recherche Scientifique".

and its negative) $|\lambda| \le \rho(\tilde{X})$, where $\rho(\tilde{X})$ is the spectral radius of the universal covering tree $\tilde{X}$ of X. (Note that if X is k-regular $\rho(\tilde{X}) = 2\sqrt{k-1}$.) Moreover, Greenberg extended the Alon-Boppana's result to the general case. One can now extend Problem 1 and ask the following question:

Problem 2. *Does every infinite uniform tree cover a Ramanujan graph? infinitely many such graphs?*

An infinite tree is called *uniform* if it covers some (and hence infinitely many) finite graphs. (A necessary and sufficient condition for a tree to be uniform is given in [BK].)

Problems 1 and 2 were stated in [Lu2], see also [Lu1, Problem 10.4.4, page 129]. In this note we show that the answer to Problem 2 is *No*! In fact, we present many examples of graphs X such that $\tilde{X}$ covers no finite Ramanujan graph!

Remark. Observe however, that whenever X is a finite graph with a non-abelian fundamental group, its universal cover $\tilde{X}$ covers an infinite family of expanders. Indeed, choose in $\Gamma = \pi_1(X)$ an infinite family of normal subgroups $\{N_i\}$, such that $\cap_{i=1}^{\infty} N_i = \{1\}$, and Γ has property (τ) with respect to $\{N_i\}$. It follows from [Lu3] that the covers $\{X_i\}$ of X corresponding to $\{N_i\}$ form a family of expanders.

The paper is organized as follows. In Section 2 we overview the work of Greenberg defining general Ramanujan graphs and bring in some simple lemmas estimating the spectral radius of a tree and eigenvalues of a graph. In Section 3 we describe, following [BT], minimal graphs and give a sufficient condition for the covering tree of a minimal graph to have no Ramanujan quotient. In Section 4 we exhibit examples satisfying this sufficient condition. Thus the answer to Problem 2 is negative.

Acknowledgements. The first author acknowledges the warm hospitality of the University of Geneva where this work was carried on. We are also grateful to Hyman Bass, Pierre de la Harpe, and Alain Valette for valuable remarks.

2. General Ramanujan graphs.

Let $X = (V, E)$ be a connected undirected graph in which the degree $deg(x)$, i.e., the number of incident edges of each vertex x, is at most k. Let $l^2(X)$ denote the space of functions f on $V(X)$ with $\sum_{x\in V} |f(x)|^2 < \infty$ and $\delta : l^2(X) \to l^2(X)$ be the *adjacency operator*, i.e., $(\delta f)(x) = \sum_y \delta_{x,y}(f(y))$, where $\delta_{x,y}$ denotes the number of edges connecting x and y.

Denote by $\rho(X)$ the *spectral radius* of δ

$$\rho(X) \; = \; \sup\{|\lambda| \mid \lambda \in \text{ spectrum of } \delta\} \; .$$

It is well known (see [Lu1, Chapter 4] and references therein) that

$$\rho(X) \; = \; \limsup_{n\to\infty} a_n^{1/n} \; ,$$

where a_n is the number of closed paths of length n in X starting from a fixed vertex x_0 of X.

If Y_1, Y_2 are two graphs, a morphism $\pi : Y_1 \to Y_2$ is called a *cover map* if it is surjective and locally an isomorphism, namely, for every $y \in Y_1$ it induces an isomorphism from $st(y)$ to $st(\pi(y))$, where $st(y)$ denotes the set of vertices at distance at most 1 from y. It is not difficult to see ([Pa, Prop 2.1]) that if $\pi : Y_1 \to Y_2$ is a cover map, then $\rho(Y_1) \leq \rho(Y_2)$. If Y_1 is finite then $\rho(Y_1) = \rho(Y_2)$. A theorem of F. Leighton [Le] asserts that any two finite graphs Y_1 and Y_2 with the same universal covering tree have a common finite cover Y. One can now deduce

Proposition 2.1 (Greenberg [Gr]**).** *Let X be a connected locally finite graph and let $\Omega_f(X)$ denote the family of finite graphs covered by X. If $Y_1, Y_2 \in \Omega_f(X)$ then $\rho(Y_1) = \rho(Y_2)$. This common value is denoted by $\chi(X)$.*

Example 2.2. Let T_k be the infinite k-regular tree. Then $\chi(T_k) = k$ while $\rho(T_k) = 2\sqrt{k-1}$ (see [Lu1, Chapter 4]).

Let X be a fixed, connected, infinite, locally finite graph. For $Y \in \Omega_f(X)$ of order n, denote by $\lambda_0(Y) > \lambda_1(Y) \geq \lambda_2(Y) \geq \ldots \geq \lambda_{n-1}(Y)$ the eigenvalues of Y and $\text{spec}(Y) = \{\lambda_0(Y), \ldots, \lambda_{n-1}(Y)\}$. It follows from the Perron-Frobenius' Theorem that $\lambda_0(Y) = \chi(X)$.

Theorem 2.3 (Greenberg [Gr]**).** *Given $\epsilon > 0$ there exists $c = c(X, \epsilon) \in (0,1)$, such that for every $Y \in \Omega_f(X)$,*

$$|\{\lambda \in spec(Y) \mid \lambda \leq \rho(X) - \epsilon\}| \ < \ c|Y|$$

and

$$|\{\lambda \in spec(Y) \mid \lambda \geq -\rho(X) + \epsilon\}| \ < \ c|Y| \ ,$$

i.e., at least a $(1-c)$ fraction of the eigenvalues of Y is greater than $\rho(X) - \epsilon$, and similarly at least a $(1-c)$ fraction of the eigenvalues of Y is smaller than $-\rho(X) + \epsilon$.

Theorem 2.3 is a far reaching generalization of the following well known result of Alon and Boppana. (It also extends some unpublished results of M. Burger and J.-P. Serre, see [Li, Chapter 9, Theorem 13].)

Theorem 2.4. *If (X_n) is an infinite family of k-regular graphs (with k fixed) then $\liminf_{n\to\infty} \lambda_1(X_n) \geq 2\sqrt{k-1}$.*

Theorem 2.4 has been the motivation for the definition of Ramanujan k-regular graphs. Namely, a finite k-regular graph X is called *Ramanujan* if for every eigenvalue λ of X, either $\lambda = \pm k$ or $|\lambda| \leq 2\sqrt{k-1}$.

Theorem 2.3 justifies the following definition.

Definition 2.5. *A finite connected graph X is called Ramanujan if for every eigenvalue λ of X either $\lambda = \pm\chi(\tilde{X})$ or $|\lambda| \leq \rho(\tilde{X})$, where $\rho(\tilde{X})$ is the spectral radius of the covering tree $\tilde{X}$, and $\chi(\tilde{X})$ as in Proposition 2.1.*

Note that the largest eigenvalue of X is always equal to $\chi(\tilde{X})$ (it is of multiplicity one since X is connected). The smallest one is equal to $-\chi(\tilde{X})$ if and only if X is bipartite.

We end this section with three simple lemmas estimating the spectral radius and eigenvalues.

Lemma 2.6. *Let T be an infinite tree with $deg(x) \leq k$ for every vertex x of T. Then*

$$\rho(T) \;\leq\; \rho(T_k) \;=\; 2\sqrt{k-1}\ ,$$

where T_k is the k-regular tree.

Proof. As mentioned above, $\rho(T) = \liminf a_n(T)^{1/n}$, where $a_n(T)$ is the number of closed paths of length n in T starting from a fixed vertex x_0. Since the degree of each vertex of T is less than k, it is clear that $a_n(T) \leq a_n(T_k)$ for each n. Hence $\rho(T) \leq \rho(T_k)$. □

Lemma 2.7 (Interlacing of eigenvalues). *Let X be a finite graph with n vertices and eigenvalues $\theta_1 \geq \theta_2 \geq ... \geq \theta_n$. Let x_0 be a vertex of X and $Y = X \setminus \{x_0\}$ the graph obtained from X by deleting x_0 and all edges incident to it. Let $\tau_1 \geq \tau_2 \geq ... \geq \tau_{n-1}$ be the eigenvalues of Y. Then*

$$\theta_1 \;\geq\; \tau_1 \;\geq\; \theta_2 \;\geq\; \tau_2 \;\geq\; ... \;\geq\; \theta_{n-1} \;\geq\; \tau_{n-1} \;\geq\; \theta_n\ .$$

Proof. See [Go, Theorem 5.3, page 29]. □

Lemma 2.8. *The largest eigenvalue of a finite graph X is bounded from below by the average degree of X with the equality holding if and only if the graph is regular.*

Proof. See [Bi, page 54]. □

3. Minimal graphs.

Definition 3.1. *A finite graph X is called minimal if it is equal to the universal covering $\tilde{X}$ of X divided by the full automorphism group of $\tilde{X}$*

$$X \;=\; \tilde{X}/Aut(\tilde{X})\ .$$

In [BT] a method is developed which allows to establish whether a given finite graph $X = (V, E)$ (or more generally an "indexed graph") is minimal. For each vertex a of X and a set of vertices B, let $E(a, B)$ be the set of edges connecting a and B and let $i(a, B)$ denote $|E(a, B)|$.

Define a descending sequence R_n of equivalence relations on V (viewed as subsets of $V \times V$) as follows. The relation R_0 is defined to be the "egalitarion relation", i.e., all vertices are R_0-equivalent. Define R_{n+1} inductively by $aR_{n+1}b$ if and only if aR_nb and $i(a, C) = i(b, C)$ for any R_n-class C.

Note that aR_1b if and only if $deg(a) = deg(b)$. Also aR_2b if and only if $deg(a) = deg(b)$ and, for every $l \geq 1$, the number of neighbours of a of degree l is equal to the number of neighbours of b of the same degree. Now aR_3b if and only if aR_2b and, for all $l, m, n \geq 1$, $\big|\{x \in V \mid a \sim x; \deg(x) = l; x \text{ has } m \text{ neighbours of degree } n\}\big| = \big|\{x \in V \mid b \sim x; \deg(x) = l; x \text{ has } m \text{ neighbours of degree } n\}\big|$.

In general, the definition of the relation R_n for an arbitrary n can be restated as follows. For a vertex $a \in V(X)$ of degree k_0, let $deg_{k_0,k_1,...,k_n}(a)$ denote the number of paths $(a = a_0, a_1, ..., a_n)$ in X, such that $deg(a_i) = k_i$ (note that back-tracking is allowed in a path). Vertices a and b are R_n-equivalent, aR_nb, if and only if $deg_{k_0,k_1,...,k_m}(a) = deg_{k_0,k_1,...,k_m}(b)$ for every $m \leq n$ and for every (m+1)-tuple

$k_0, ..., k_m$. (Remark that each k_i takes values in the finite interval $\big[\min\{deg(x) \mid x \in V(X)\}, \max\{deg(x) \mid x \in V(X)\}\big]$.)

These relations stabilize at some relation $R_n = R$ for all n large enough. One can form the quotient graph X^* whose vertices are the R-classes of vertices of V. Two vertices a^* and b^* of X^* are connected by an edge if and only if $i(a, b^*) > 0$ (note that this number depends only on a^* and not on a representant a of a^* since the relation R is stable). The following is shown in [BT]:

Theorem 3.2. *The graph X^* is isomorphic to $\tilde{X}/Aut(\tilde{X})$.*

If $X = X^*$ then each class a^* consists of one vertex a, hence it is impossible to make a proper quotient of X and $X = \tilde{X}/\mathrm{Aut}(\tilde{X})$. On the other hand, note that if X is k-regular then X^* contains a single vertex.

We can now state the main result.

Theorem 3.3. *Let X be a finite minimal graph with a cut vertex x_0, that is, if we delete x_0 and the edges incident to it we are left with two disjoint non-empty subgraphs, say Y and Z. Assume that*

(1) *$deg(x) \leq k$ for every vertex x of X;*
(2) *the average degrees of both Y and Z are strictly greater than $2\sqrt{k-1}$.*

Then the universal cover $\tilde{X}$ of X is a locally finite uniform tree which covers no Ramanujan graph.

Proof. The universal cover $\tilde{X}$ of X is a tree with the degree of every vertex at most k. Thus by Lemma 2.6, $\rho(\tilde{X}) \leq 2\sqrt{k-1}$. Let X_0 be the graph obtained from X by deleting x_0. Then the two largest eigenvalues of X_0 denoted by τ_1 and τ_2 satisfy $\tau_i > 2\sqrt{k-1}$ for $i = 1, 2$ (the values τ_1 and τ_2 might coincide). Indeed, $\rho(Y)$ and $\rho(Z)$ are both eigenvalues of X_0, and by assumption (2) and Lemma 2.8, $\rho(Y)$ and $\rho(Z)$ are both strictly greater than $2\sqrt{k-1}$, hence $\tau_i > 2\sqrt{k-1}$ for $i = 1, 2$. By Lemma 2.7 the second largest eigenvalue of X is strictly greater than $2\sqrt{k-1}$, hence also strictly greater than $\rho(\tilde{X})$. This proves that X is not Ramanujan.

If X' is any finite quotient of $\tilde{X}$, then X is a quotient of X' since X is minimal. Hence every eigenvalue of X is also an eigenvalue of X', and X' is not Ramanujan either. This proves that $\tilde{X}$ has no Ramanujan quotients, as claimed. □

4. Examples.

We construct now a series of examples of graphs which satisfy all the assumptions of Theorem 3.3 and therefore their universal covers cover no Ramanujan graph.

Example 1. The first class of examples (see Figure 1) is parametrized by two integers $m > l > 1$. Each graph $X_{m,l}$ is formed of two subgraphs X_m, X_l connected by a bridge of length 2. There are $2m + 1$ vertices in the graph X_m: vertices $u_1, ..., u_m$; vertices $\tilde{u}_1, ..., \tilde{u}_{m-1}$; a vertex U which is adjacent to all the vertices of the graph X_m and is therefore of degree $2m$; a vertex u_0 which is adjacent to the vertices $U, u_1, ..., u_m$ and is also an extremity of the "bridge" connecting X_m with X_l in $X_{m,l}$ (thus u_0 is of degree $m + 1$ in X_m but of degree $m + 2$ in $X_{m,l}$). The vertices $u_1, ..., u_m$ are adjacent to u_0, U and to the vertices $\tilde{u}_j$ with $j < i$. Thus a vertex u_i is of degree $i + 1$, $i = 1, ..., m$. A vertex $\tilde{u}_j$, $j = 1, ..., m - 1$ is adjacent to the vertex U and to the vertices u_i with $i > j$, hence is of degree

$m-j+1$. Similarly, the graph X_l has the vertices $v_1, \ldots, v_l$; $\tilde{v}_1, \ldots, \tilde{v}_{l-1}$; v_0; V, and the adjacency structure analogous to that of X_m. The graph $X_{m,l}$ is formed by connecting u_0 and v_0 to a vertex c, thus constructing a "bridge" between X_m and X_l.

We first prove that $X_{m,l}$ is minimal. The vertex U is not R_1-equivalent to any other vertex because it is the only vertex of degree $2m$. The vertex V is not R_1-equivalent to any vertex of X_l, and also not R_2-equivalent to any vertex in X_m because V and U are not adjacent. Therefore all vertices of X_m are not R_2-equivalent to the vertices of X_l (being all adjacent respectively to U and V). The vertex c of degree 2 in the middle of the bridge is not R_2-equivalent to any other vertex because it is the only vertex not adjacent to U or V.

Finally we check that no two vertices of X_m (and similarly for X_l) are R_2-equivalent. The vertex u_0 is the only vertex of degree $m+2$, thus not R_1-equivalent to any other vertex. The vertices $u_1, \ldots, u_m$ (and similarly the vertices $\tilde{u}_1, \ldots, \tilde{u}_{m-1}$) are not R_1-equivalent because they are all of different degrees. A vertex u_i is not R_2-equivalent to a vertex $\tilde{u}_j$ because u_i is adjacent to u_0 and $\tilde{u}_j$ is not.

Let us check now the other assumptions of Theorem 3.3. All vertices of $X_{m,l}$ are of degree at most $2m$. The average degrees of X_m and X_l are respectively $(m^2+5m)/(2m+1)$ and $(l^2+5l)/(2l+1)$. It is clear that when m is very big and l is not very different from m, both average degrees are bigger than $2\sqrt{2m-1}$ (and hence $X_{m,l}$ is not Ramanujan). The smallest possible values are $l=24$, $m=25$.

The graphs $X_{m,l}$ are very far from being regular, that is, the degrees of their vertices vary from 2 to $m+1$. In fact, one can also find examples of graphs satisfying all the assumptions of Theorem 3.3 with vertices of only two different degrees.

Example 2. All vertices of the graph X (see Figure 2) are of degree 5 or 6. The vertices of degree 5 are marked by "tildas" on the picture. Others are of degree 6. We will check the minimality of the graph X with help of the table below.

vertex of X	degree	adjacent vertices	number of neighbours of degree 6
u	6	$l_1, l_2, l_3, l_4, l_5, l_6$	6
l_1	6	$u, \tilde{l}_2, \tilde{l}_3, \tilde{l}_4, \tilde{l}_5, \tilde{l}_6$	1
l_2	6	$u, l_6, \tilde{l}_3, \tilde{l}_4, \tilde{l}_5, \tilde{u}$	2
l_3	6	$u, l_4, l_6, \tilde{l}_3, \tilde{l}_4, \tilde{u}$	3
l_4	6	$u, l_3, l_5, l_6, \tilde{l}_6, \tilde{u}$	4
l_5	6	$u, l_4, l_6, \tilde{l}_2, \tilde{l}_3, \tilde{u}$	3
l_6	6	$u, l_2, l_3, l_4, l_5, \tilde{u}$	5
$\tilde{u}$	5	l_2, l_3, l_4, l_5, l_6	5
$\tilde{l}_1$	5	$c, \tilde{l}_2, \tilde{l}_3, \tilde{l}_4, \tilde{l}_5$	1
$\tilde{l}_2$	5	$c, l_1, l_5, \tilde{l}_1, \tilde{l}_6$	3
$\tilde{l}_3$	5	$l_1, l_2, l_3, l_5, \tilde{l}_1$	4
$\tilde{l}_4$	5	$l_1, l_2, l_3, \tilde{l}_1, \tilde{l}_5$	3
$\tilde{l}_5$	5	$l_1, l_2, \tilde{l}_1, \tilde{l}_4, \tilde{l}_6$	2
$\tilde{l}_6$	5	$c, l_1, l_4, \tilde{l}_2, \tilde{l}_5$	3

c	6	$r_5, r_4, r_3, \tilde{l}_1, \tilde{l}_2, \tilde{l}_6$	3
r_1	6	$r_2, r_3, r_5, \tilde{r}_1, \tilde{r}_2, \tilde{r}_3$	3
r_2	6	$r_5, r_1, \tilde{r}_1, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5$	2
r_3	6	$c, r_1, r_4, r_5, \tilde{r}_1, \tilde{r}_3$	4
r_4	6	$c, r_3, r_5, \tilde{r}_1, \tilde{r}_4, \tilde{r}_5$	3
r_5	6	$c, r_1, r_2, r_3, r_4, \tilde{r}_2$	5
v	6	$\tilde{r}_1\tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6$	0
$\tilde{v}$	5	$\tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6$	0
$\tilde{r}_1$	5	v, r_1, r_2, r_3, r_4	5
$\tilde{r}_2$	5	$v, r_1, r_5, \tilde{v}, \tilde{r}_6$	3
$\tilde{r}_3$	5	$v, r_2, r_3, r_4, \tilde{v}$	4
$\tilde{r}_4$	5	$v, r_2, r_4, \tilde{v}, \tilde{r}_6$	3
$\tilde{r}_5$	5	$v, r_1, r_2, \tilde{v}, \tilde{r}_6$	3
$\tilde{r}_6$	5	$v, \tilde{v}, \tilde{r}_2, \tilde{r}_4, \tilde{r}_5$	1

The vertices of degree 5 are not R_1-equivalent to the vertices of degree 6. The vertices of a same degree which have different numbers of neighbours of degree 6 are not R_2-equivalent. The vertices l_i, $i = 1, ..., 6$, are not R_3-equivalent to the vetices r_j, $j = 1, ..., 5$, because each vertex l_i and no vertex r_j is adjacent with u which is the only vertex with 6 neighbours of degree 6. The vertices $\tilde{l}_i$, $i = 1, ..., 6$, are not R_3-equivalent to the vertices $\tilde{r}_j$, $j = 1, ..., 6$, because each vertex $\tilde{r}_j$ and no vertex $\tilde{l}_i$ is adjacent to one of two vertices $v, \tilde{v}$ which have 0 neighbours of degree 6.

Among the vertices $l_1, ..., l_6$ only l_3 and l_5 are R_2-equivalent. The neighbours of degree 5 of l_3 are $\tilde{l}_3, \tilde{l}_4, \tilde{u}$; the neighbours of degree 5 of l_5 are $\tilde{l}_3, \tilde{l}_2, \tilde{u}$. Both vertices $\tilde{l}_2$ and $\tilde{l}_4$ have the same number of neighbours of degree 6 (hence also of degree 5). Therefore, the vertices l_3 and l_5 are also R_3-equivalent. But, $deg_{6,5,5,6}(l_3) = 4$ while $deg_{6,5,5,6}(l_5) = 5$, thus l_3 and l_5 are not R_4-equivalent. The vertex c is R_2 but not R_3-equivalent to l_3 and l_5.

Analogously, looking at the adjacency table, one concludes that $\tilde{l}_2, \tilde{l}_4, \tilde{l}_5$ are R_2 but not R_3-equivalent.

The vertices c, r_1, r_4 are R_2 but not R_3-equivalent. The vertex $\tilde{r}_2$ is R_2 but not R_3-equivalent to the vertices $\tilde{r}_4, \tilde{r}_5$. The vertices $\tilde{r}_4$ and $\tilde{r}_5$ are R_3 but not R_4-equivalent because the sets of their neighbours of degree 6 are different, and all vertices of type r_i are not R_3-equivalent (for example, $deg_{5,6,6,6}(\tilde{r}_4) = 12$ while $deg_{5,6,6,6}(\tilde{r}_5) = 11$).

Thus X is a minimal graph.

The vertex c is a cut-vertex in the graph X. After deleting it we get two components: the "left" one and the "right" one. The average degree of the "left" component is $37/7$, the average degree of the "right" component is $68/13$. Both average degrees are thus strictly greater than $2\sqrt{5}$, and the graph X is minimal and not Ramanujan. Hence all the assumptions of Theorem 3.3 are fulfilled, and $\tilde{X}$ covers, therefore, no Ramanujan graph.

References

BK. H. Bass, R. Kulkarni, *Uniform tree lattices*, J. of A.M.S. **3** (1990), 843-902.

BT. H. Bass, J. Tits, *Discreteness criteria for certain tree automorphism groups*, Preprint.

Bi. N. Biggs, *Algebraic Graph Theory, 2nd ed.*, Cambridge University Press, 1993.

Fr. J. Friedman, *On the second eigenvalue and random walk in random d-regular graph*, Combinatorica **11** (1991), 331-362.

FKS. J. Friedman, J. Kahn, E. Szemerédi, *On the second eigenvalue in random regular graphs* (1989), ACM Press, 587-598.

Go. C. Godsil, *Algebraic Combinatorics*, Chapman and Hall Mathematics Series, Chapman & Hall, New York, 1993.

Gr. Y. Greenberg, *Ph. D. Thesis*, Hebrew University of Jerusalem, 1995. (Hebrew)

Le. F. Leighton, *Finite common coverings of graphs*, J. of Comb. Th. B **33** (1982), 231-238.

Li. W. Li, *Number Theory With Applications*, World Scientific, 1996.

Lu1. A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Mathematics, Birkhäuser, Basel-Boston-Berlin, 1994.

Lu2. A. Lubotzky, *Cayley graphs: eigenvalues, expanders and random walks*, Surveys in Combinatorics (P. Rowlinson, ed.), London Math. Society Lecture Notes Series, 1995, pp. 155-189.

Lu3. A. Lubotzky, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, Annals of Mathematics **145** (1997), 441-452.

LPS. A. Lubotzky, R. Phillips, P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), 261-277.

Mo. M. Morgenstern, *Existence and explicit construction of $q+1$-regular Ramanujan graphs for every prime power q*, J. of Comb. Th. B **62** (1994), no. 1, 44-62.

Pa. W. Paschke, *Lower bound for the norm of a vertex-transitive graph*, Math. Zeit. **213** (1993), 225-239.

Va. A. Valette, *Graphes de Ramanujan et applications*, Séminaire Bourbaki, No 829, mars 1997.

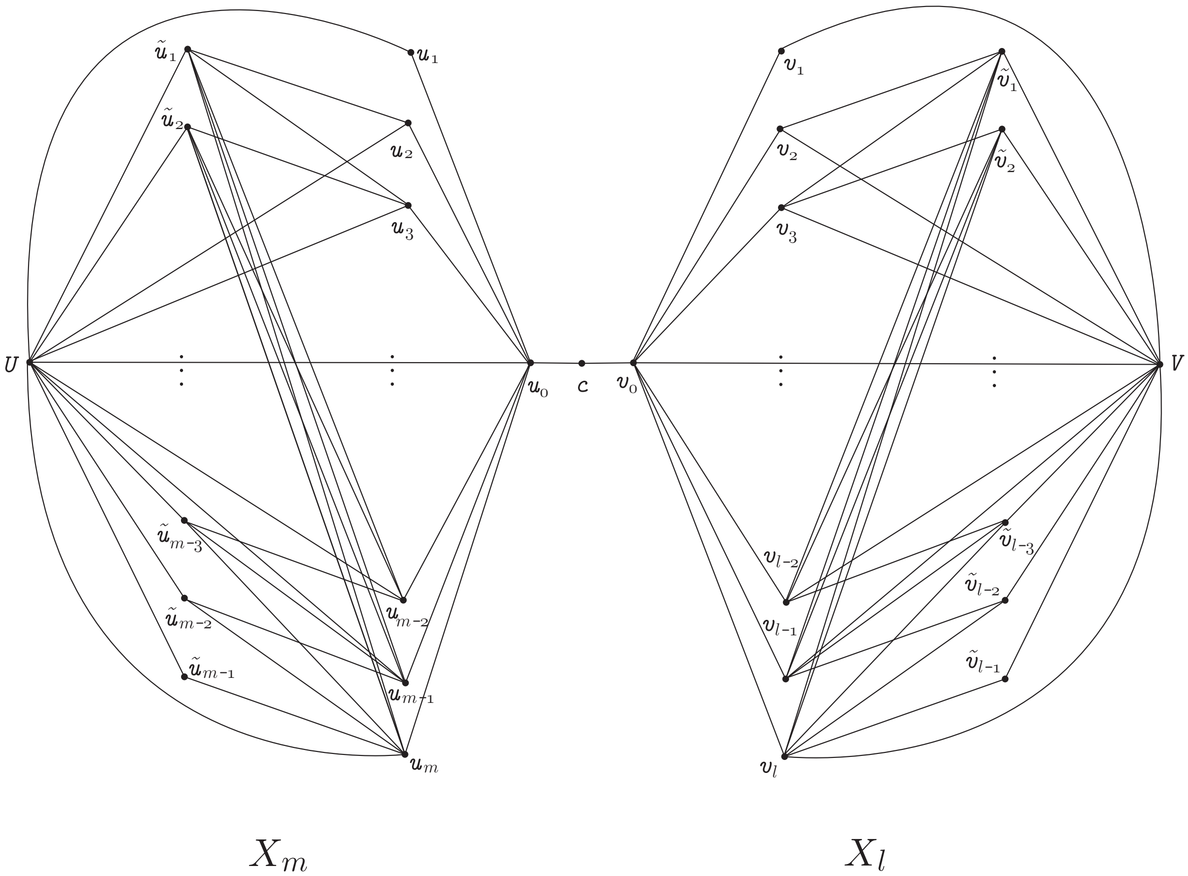
$\tilde{u}_1$
u_1
$\tilde{u}_2$
u_2
u_3
U
u_0
c
v_0
$\tilde{u}_{m-3}$
u_{m-2}
$\tilde{u}_{m-2}$
$\tilde{u}_{m-1}$
u_{m-1}
u_m
v_1
$\tilde{v}_1$
v_2
$\tilde{v}_2$
v_3
V
v_{l-2}
$\tilde{v}_{l-3}$
$\tilde{v}_{l-2}$
v_{l-1}
$\tilde{v}_{l-1}$
v_l
X_m
X_l

Figure 1

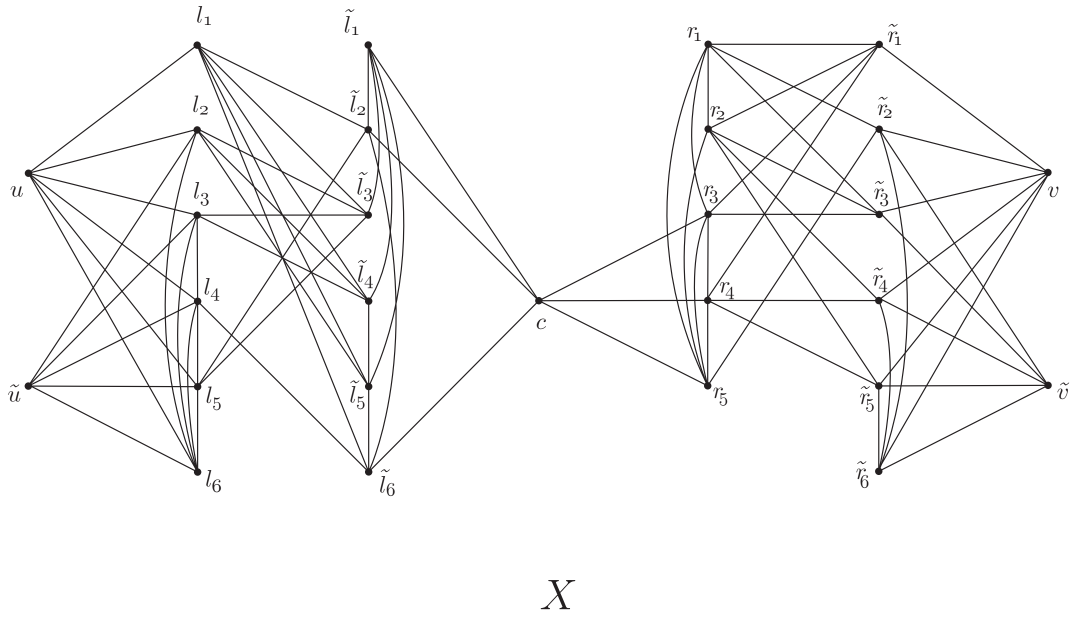

Figure 2

Institute of mathematics, Hebrew University, Jerusalem 91904 Israel

Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, C. P. 240, 1211 Genève 24, Switzerland

E-mail address: `alexlub@math.huji.ac.il` `Tatiana.Nagnibeda@math.unige.ch`