AN ESTIMATE FROM ABOVE OF SPECTRAL RADII OF RANDOM WALKS ON SURFACE GROUPS

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Abstract. Using Gabber's Lemma, we get new estimates of the spectral radius of the simple random walk on the fundamental group of the orientable closed surface of genus $g, g \geq 2$. In order to get better numerical estimates we base our method on Cannon's classification of the group elements by their cone types. The method may as well be applied to many other groups and graphs with finite numbers of cone types

1. Introduction.

Our purpose here is to present a method which allows to estimate from above spectral radii of simple random walks on infinite locally finite graphs with finite numbers of cone types. We obtain estimates of spectral radii as real positive solutions of systems of finite number of algebraic equations. The method is applied to the Cayley graphs Γ_g of the fundamental groups of oriented surfaces of genus $g, g \geq 2$

$$G_g \;=\; \langle a_1, ..., a_g, b_1, ..., b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$$

with respect to the standard system of generators S_g. To our knowledge, the numerical estimates that we get for the spectral radii of the surface groups of small genus are best available:

$$\rho_2 \;<\; 0.662816 \;;$$
$$\rho_3 \;<\; 0.552792 \;.$$

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an infinite locally finite connected graph. A random walk on Γ is given by a stochastic transition matrix $\big(p(x,y)\big)_{x,y \in V(\Gamma)}$. The simple random walk on Γ is defined by

$$p(x,y) \;=\; \begin{cases} 1/d(x) & \text{if } y \sim x \;, \\ 0 & \text{otherwise} \;, \end{cases}$$

where $d(x)$ denotes the degree of the vertex x and $y \sim x$ means that y and x are connected by an edge. The corresponding Markov operator M is a selfadjoint operator on the space $l^2(V(\Gamma))$ with the inner product

$$\langle \phi, \psi \rangle \;=\; \sum_{x \in V(\Gamma)} \phi(x)\psi(x)d(x) \;.$$

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It is of the form

$$M\xi(x) \ = \ \frac{1}{d(x)} \sum_{y\sim x} \xi(y)$$

for any $\xi \in l^2(V(\Gamma))$ and $x \in V(\Gamma)$. The *spectral radius* of the simple random walk is

$$\rho \ = \ \limsup_{n\to\infty} \left(P^{(n)}_{x,x}\right)^{\frac{1}{n}} \ ,$$

where $P^{(n)}_{x,x}$ denotes the probability to return to a vertex x for the first time after n steps. It is known (see for example [4]) that

$$\rho \ = \ \|M\| \ .$$

Interesting estimates of spectral radii of surface groups have been obtained by L. Bartholdi, S. Cantat, T. Ceccherini-Silberstein and P. de la Harpe in [1]. They give also a detailed introduction to the subject. A slightly better estimate from above was obtained by A. Zuk in [5]. His result is based on the following statement.

Let $\mathbb{E}$ denote the set of all oriented edges of Γ, (to each edge in $E(\Gamma)$ correspond two oriented edges, $e = (x,y), \bar{e} = (y,x)$), and, for an oriented edge e, let e_- and e_+ denote respectively the beginning and the end of e.

Lemma (Gabber). *Let Γ be an infinite locally finite graph. Suppose that a function $F : \mathbb{E} \to \mathbb{R}_+$ satisfies*

$$F(e) \ = \ \frac{1}{F(\bar{e})}$$

for each edge $e \in \mathbb{E}$. If there exists a constant $C_F > 0$ such that for each vertex x of Γ

$$\frac{1}{d(x)} \sum_{e_+ = x} F(e) \ \leq \ C_F \ ,$$

then

$$\|M\| \ \leq \ C_F \ .$$

Proof. Let $\xi \in l^2(V(\Gamma))$. As $F(e) = 1/F(\bar{e})$, we have

$$\xi(x)\xi(y) = \xi(x)\sqrt{F(x,y)}\sqrt{F(y,x)}\xi(y) \leq \frac{1}{2}F(y,x)\xi(x)^2 + \frac{1}{2}F(x,y)\xi(y)^2.$$

Therefore

$$\begin{aligned}
\langle M\xi, \xi\rangle &= \sum_{x\in V(\Gamma)} \Big(\frac{1}{d(x)} \sum_{y\sim x} \xi(y)\Big)\xi(x)d(x) = \sum_x \sum_{y\sim x} \xi(y)\xi(x) \\
&\leq \frac{1}{2}\sum_x \sum_{y\sim x} \frac{1}{d(x)} F(y,x)\xi(x)^2 d(x) \\
&\quad + \frac{1}{2}\sum_y \sum_{x\sim y} \frac{1}{d(y)} F(x,y)\xi(y)^2 d(y) \\
&\leq C_F \|\xi\|^2 \ .
\end{aligned}$$

□

Observation. For a suitable choice of function F on the edges of the graph, Gabber's Lemma gives the exact estimate of the spectral radius. Indeed, it is a well known fact (see for example [3]) that there exists a positive function $f : V(\Gamma) \to \mathbb{R}$ which satisfies $Mf = \rho f$. Define a function $\tilde{F} : \mathbb{E} \to \mathbb{R}_+$ by

$$\tilde{F}(e) \; = \; f(e_-)/f(e_+) \; .$$

Obviously, $\tilde{F}(e) = 1/\tilde{F}(\bar{e})$ for any $e \in \mathbb{E}$. For a vertex x of Γ we compute

$$1/d(x) \sum_{e_+=x} \tilde{F}(e) \; = \; 1/f(x)\Big(1/d(x) \sum_{y \sim x} f(y)\Big) \; = \; 1/f(x) M f(x) \; = \; \rho \; .$$

Thus, in notations of Gabber's Lemma,

$$C_{\tilde{F}} \; = \; \|M\| \; .$$

In practice, however, this observation does not help any in estimating the spectral radius. Indeed, if we were able to produce a ρ-harmonic function, we would have found the spectral radius.

2. Graphs with finite numbers of cone types.

Let Γ be as above an infinite, locally finite connedted graph. A function $t : V(\Gamma) \to \mathbb{N}$ is called a *type function on the vertices*, and its value $t(x)$ at a vertex x of Γ is called a *type of* x.

An orientation of the graph Γ is a subset $\mathcal{E} \subset \mathbb{E}$ such that $e \in \mathcal{E}$ if and only if $\bar{e} \notin \mathcal{E}$. A *star* $St_{\mathcal{E}}(x)$ of a vertex x with respect to an orientation $\mathcal{E}$ is the set of edges from $\mathcal{E}$ incident to x.

We say that an orientation $\mathcal{E}$ of Γ and a type function t are *compatible* if $t(x) = t(y)$ implies $\sharp\{e \in St_{\mathcal{E}}(x) \mid t(e_+) = k\} = \sharp\{e \in St_{\mathcal{E}}(y) \mid t(e_+) = k\}$ for all $k \in \mathbb{N}$.

For each $c = (c_k)_{k \geq 1}$, $c_k > 0 \; \forall k \in \mathbb{N}$, we can consider a function $F_c : \mathbb{E} \to \mathbb{R}_+$ defined by

$$F_c(e) \; = \; \begin{cases} c_k & \text{if } e \in \mathcal{E}, t(e_+) = k \; , \\ 1/c_k & \text{if } \bar{e} \in \mathcal{E}, t(\bar{e}_+) = k \; . \end{cases}$$

For compatible orientation and type function we can define

$$f_i(c) \; = \; \sum_{e \in \mathbb{E}, e_+ = x} F_c(e) \; ,$$

where x is a vertex of type i.

Suppose that a type function t takes values in a finite set $\{1, ..., K\}$ and that the vertices of the same type have the same degree. Denote by d_i the degree of a vertex of type i. Then Gabber's Lemma leads to the following statement:

Proposition. *For each* $(c_1, ..., c_K) \in \mathbb{R}_+^K$

$$\|M\| \; \leq \; \max_{i=1,\dots,K} \frac{1}{d_i} f_i(c_1, ..., c_K) \; .$$

Corollary 1. *Proposition gives us infinitely many estimates of $\|M\|$, one for each $(c_1, ..., c_K) \in \mathbb{R}_+^K$. We can then choose the best of them:*

$$\|M\| \;\leq\; \inf_{(c_1,...,c_k)\in\mathbb{R}_+^K} \max_{i=1,...,K} \frac{1}{d_i} f_i(c_1, ..., c_K) \;.$$

In [5] the type function on the vertices of $\Gamma_g = Cay(G_g, S_g)$ is defined using an embedding of Γ_g in the hyperbolic plane. This leads to the estimate

$$\rho_g \;\leq\; \frac{1}{\sqrt{g}} \;,$$

and even better, in particular,

$$\rho_2 \;<\; 0,688 \;.$$

Here we will use the combinatorial structure of the graph, namely the cone types. This allows to get better numerical estimates.

Let Γ be an infinite graph as above. For $x, y \in V(\Gamma)$ a *path* of length n from x to y is a sequence $x = x_0, x_1, ..., x_n = y$ of vertices such that $x_{i-1} \sim x_i$. The *distance* between two vertices $d(x, y)$ is the minimal length of a path from x to y. A path of minimal length is called *geodesic*.

From now on let us fix a base point $x_0 \in V(\Gamma)$. We define the *cone* $\mathcal{C}(x)$ of a vertex x of Γ as a subgraph of Γ rooted in x, such that a vertex y of Γ is a vertex of $\mathcal{C}(x)$ if and only if there exists a geodesic in Γ joining x_0 with y and passing through x. Two vertices of Γ are said to be of the same *cone type* if their cones are isomorphic as rooted graphs.

If Γ has a finite number of cone types, say $K + 1$, we can label the vertices of Γ by a function $t_0 : V \to \{0, 1, ..., K\}$ called *cone types labelling.*

The *norm* of a vertex x is $|x| = d(x_0, x)$. Let us say that a vertex x is a *predecessor* of a vertex y and y is a *successor* of x, if x and y are neighbors (i.e. connected by an edge), and $|x| < |y|$. Suppose that all cycles in Γ are of even length. This means that for each edge in $\mathbb{E}(\Gamma)$ either $|e_+| > |e_-|$ or $|e_-| > |e_+|$. We choose an orientation $\mathcal{E}_0$ of Γ in such a way that $|e_+| > |e_-|$ for each edge $e \in \mathcal{E}_0$ and call this orientation $\mathcal{E}_0$ *canonical.*

Lemma 1. *Let Γ be a bipartite infinite graph. Suppose that the number of predecessors of a vertex in Γ depends only on its cone type. Then the canonical orientation and the cone types labelling in Γ are compatible.*

Proof. Suppose Γ is bipartite. Then, for each edge in $\mathbb{E}(\Gamma)$, either $|e_+| > |e_-|$ or $|e_-| > |e_+|$. Denote by $r(x)$ the number of predecessors of x, and by $q_j(x)$ the number of successors of x of cone type j. If two vertices $x, y \in V(\Gamma)$ are of the same cone type then $r(x) = r(y)$ by the assumption and $q_j(x) = q_j(y)$ for each j by definition of the cone. By definition of the canonical orientation we have

$$\sharp\{e \in St_{\mathcal{E}_0}(x) \mid t(e_+) = j\} \;=\; \begin{cases} q_j(x) & \text{if } j \neq t(x) \;, \\ r(x) + q_{t(x)}(x) & \text{if } j = t(x) \;. \end{cases}$$

$\square$

Let Γ satisfy the conditions of Lemma 1. For a vertex x of type i in Γ, $i \in \{0,...,K\}$, denote by r_i the number of the predecessors of x, and by $q_{i,j}$ the number of successors of x of type j, $j \in \{0,...,K\}$.

The functions f_i from Proposition have then the following form:

$$f_i \;=\; r_i c_i + \sum_{j=0}^{K} \frac{q_{i,j}}{c_j} \;, \quad i = 0, 1, ..., K \;.$$

Observe that the cone type of the basepoint x_0 is different from the cone types of all other vertices. Suppose for simplicity that $t_0(x_0) = 0$. We have $r_0 = 0$ and $q_{i,0} = 0$ for every $i \in \{0,...,K\}$. In fact this cone type can be excluded.

Lemma 2. *Suppose there exists a sequence of automorphisms $\underline{h} = \{h_i\}_{i\geq 1}$ of Γ preserving the orientation $\mathcal{E}_0$ and such that $|h_i(x_0)| > i$ for each $i \in \mathbb{N}$. Then there exists a labelling $t : V(\Gamma) \to \{1,...,K\}$ of the vertices of Γ and an orientation $\mathcal{E}$ of Γ compatible with t, such that for every $i \in \{1,...,K\}$ we have $f_i^{(t_0,\mathcal{E}_0)}(c_0, c_1..., c_K) = f_i^{(t,\mathcal{E})}(c_1,...,c_K)$ for all positive $c_0,...,c_K$.*

Proof. Define a sequence of labellings $(t_n)_{n\geq 1}$ by

$$t_n(y) = t_0(h_n^{-1}(y)), \quad n \geq 1$$

for each vertex y of Γ.

There is only a finite number of possible labellings of vertices inside the ball of radius 1. So there exists an infinite subsequence $\underline{h}^1 = \{h_{i_1},..,h_{i_n},...\}$ such that for each y of norm less or equal to 1, all $t_{i_n}(y)$'s coincide and are different from 0. We can now extract a subsequence $\underline{h}^2$ from $\underline{h}^1$ so that the labelling of the elements in the ball of radius 2 with numbers $1,...,K$ is stabilized. Hence, by repeating this procedure, we will be able to get a labelling $t : V(\Gamma) \to \{1,...,K\}$ of the vertices of Γ.

Let us consider orientations $\mathcal{E}_n = h_n(\mathcal{E}_0)$ of Γ. In the limit of the procedure described above we will get an orientation $\mathcal{E}$ of Γ. It is compatible with the new labelling. Indeed, for each n, the condition $t_n(x) = t_n(y)$ means that $t_0(h_n^{-1}(x)) = t_0(h_n^{-1}(y))$. On the other hand, the condition $\sharp\{e \in St_{\mathcal{E}_n}(x) \mid t_n(e_+) = k\} = \sharp\{e \in St_{\mathcal{E}_n}(y) \mid t_n(e_+) = k\}$ $\forall k \in \mathbb{N}$ may be rewritten as $\sharp\{e \in St_{\mathcal{E}_0}(h_n^{-1}(x)) \mid t_0(h_n^{-1}(e_+)) = k\} = \sharp\{e \in St_{\mathcal{E}_0}(y) \mid t_0(h_n^{-1}(e_+)) = k\}$ $\forall k \in \mathbb{N}$. Thus the new orientation and labelling are compatible.

It is clear from the construction of t and $\mathcal{E}$ that $f_i^{(t_0,\mathcal{E}_0)}(c_0, c_1..., c_K) = f_i^{(t,\mathcal{E})}$ $(c_1,...,c_K)$ for all positive $c_0,...,c_K$ and for every $i \in \{1,...,K\}$. □

3. Surface groups.

It was established in [2] that there are $2g+1$ cone types in the Cayley graph Γ_g of (G_g, S_g). Let us take the identity element for the base point. The corresponding stars in Γ_g with respect to the canonical orientation may be described in the following way.

(1) the identity element is of type 0, it has $4g$ successors of type 1;
(2) vertices of type 1 have one predecessor and $4g-1$ successors: $4g-3$ of type 1 and two of type 2;
(3) vertices of types $i = 2,...,2g-1$ have one predecessor and $4g-1$ successors: $4g-3$ of type 1, one of type 2 and one of type $i+1$;
(4) vertices of type $2g$ have two predecessors and $4g-2$ successors: $4g-4$ of type 1 and two of type 2.

The procedure described in the proof of Lemma 2 allows us to label the vertices of Γ_g by numbers $1, ..., 2g$ and to put an orientation on Γ_g compatible with this labelling. We have then

$$f_i(c_1, ..., c_{2g}) = \begin{cases} c_i + 1/c_2 + 1/c_{i+1} + (4g-3)/c_1 & \text{if } i = 1, ..., 2g-1 \ , \\ 2c_{2g} + 2/c_2 + (4g-4)/c_1 & \text{if } i = 2g \ . \end{cases}$$

The corresponding Jacobi matrix is of the form

$$J = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_{2g} \end{pmatrix} = \begin{pmatrix} 1-(4g-3)/c_1^2 & -2/c_2^2 & 0 & \cdots & 0 & 0 \\ -(4g-3)/c_1^2 & 1-1/c_2^2 & -1/c_3^2 & \cdots & 0 & 0 \\ -(4g-3)/c_1^2 & -1/c_2^2 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ -(4g-3)/c_1^2 & -1/c_2^2 & 0 & \cdots & 1 & -1/c_{2g}^2 \\ -(4g-4)/c_1^2 & -2/c_2^2 & 0 & \cdots & 0 & 2 \end{pmatrix}$$

with the determinant

$$\begin{aligned} \det J = & -2\big(c_1^2c_2^2...c_{2g}^2 - c_1^2c_3^2...c_{2g}^2 - ... - c_1^2c_{2g}^2 - c_1^2 - (4g-3)c_2^2c_3^2...c_{2g}^2 \\ & - (4g-3)c_3^2...c_{2g}^2 - ... - (4g-3)c_{2g}^2 + 1\big)/c_1^2...c_{2g}^2 \ . \end{aligned}$$

Theorem. *Consider the system of equations*

$$\left.\begin{aligned} 1/c_1 - 1/c_2 + c_i + 1/c_{i+1} - 2c_{2g} &= 0 \ , \quad i = 1, ..., 2g-1 \\ \det J &= 0 \ . \end{aligned}\right\} (\ast)$$

(1) *There exists a real positive solution of the system* $(\ast)$.
(2) *Let* $(c_1^{(1)}, ..., c_{2g}^{(1)}),(c_1^{(2)}, ..., c_{2g}^{(2)}),...,(c_1^{(n)}, ..., c_{2g}^{(n)}) \in \mathbb{R}_+^{2g}$ *be all real positive solutions of the system* $(\ast)$. *Then*

$$\rho_g \le \max_{j=1,\dots,n} f_1(c_1^{(j)}, ..., c_{2g}^{(j)}) \ .$$

Corollary 2.

$$\rho_2 < 0.662816 \ .$$

Proof of Corollary 2. In case of $g = 2$, we get the following expressions for $\{f_i\}$:

$$\begin{aligned} f_1 &= c_1 + 5/c_1 + 2/c_2 \\ f_2 &= 5/c_1 + c_2 + 1/c_2 + 1/c_3 \\ f_3 &= 5/c_1 + 1/c_2 + c_3 + 1/c_4 \\ f_4 &= 4/c_1 + 2/c_2 + 2c_4. \end{aligned}$$

The determinant of the corresponding Jacobian is

$$\det J = \frac{-2}{c_1^2c_2^2c_3^2c_4^2}(c_1^2c_2^2c_3^2c_4^2 - c_1^2c_3^2c_4^2 - c_1^2c_4^2 - c_1^2 - 5c_2^2c_3^2c_4^2 - 5c_3^2c_4^2 - 5c_4^2 + 1) \ .$$

The system $(*)$ has the following form:

$$\left.\begin{array}{rcl} c_1^2+1-2c_4c_1 &=& 0 \\ c_3c_2-c_3c_1+c_2^2c_1c_3+c_2c_1-2c_1c_2c_3c_4 &=& 0 \\ c_2c_4-c_1c_4+c_1c_2c_3c_4+c_1c_2-2c_4^2c_1c_2 &=& 0 \\ c_1^2c_2^2c_3^2c_4^2-c_1^2c_3^2c_4^2-c_1^2c_4^2-c_1^2-5c_2^2c_3^2c_4^2-5c_3^2c_4^2-5c_4^2+1 &=& 0\ . \end{array}\right\}$$

It can be solved numerically. We eliminate variables c_3, c_2, c_4 (we used CoCoA package at this point) and get the polynomial $p(t)$ of degree 11 in $t = c_1^2$:

$$\begin{aligned} p(t) = 4t^{11} &+ 49t^{10} - 13t^9 - 2128t^8 - 8392t^7 - 19078t^6 - 43530t^5 - 87016t^4 \\ &- 94604t^3 - 65235t^2 - 24225t - 4000, \end{aligned}$$

which has a unique real positive solution $\tilde{c}_1^2 \approx 0.71242$. From the first equation we get $\tilde{c}_4 \approx 1.52188$. Now we are left with a system of equations in c_2 and c_3. By eliminating c_3 we get the polynomial $p'(t)$ of degree 3 in $w = c_2^2$:

$$p'(t) = 10w^3 - 9w^2 + 9w - 1.$$

It has a unique real positive solution $\tilde{c}_2^2 \approx 6.92285$. Finally we are able to find $\tilde{c}_3 \approx 2.39209$. Thus the only real positive solution of the system $(*)$ is

$$\{\tilde{c}_1 \approx 0.26691, \tilde{c}_2 \approx 2.63113, \tilde{c}_3 \approx 2.39209, \tilde{c}_4 \approx 1.52188\}\ .$$

We get then

$$\frac{1}{8} f_i(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4) \ \approx\ 0.662815\ , \quad i = 1, ..., 4\ .$$

□

Lemma 3. *For any $g \geq 2$*

$$\inf_{(c_1,\ldots,c_{2g})\in\mathbb{R}_+^{2g}} \max_{i=1,\ldots,2g} f_i(c_1, ..., c_{2g}) \ =\ \min_{(c_1,\ldots,c_{2g})\in\mathbb{R}_+^{2g}} \max_{i=1,\ldots,2g} f_i(c_1, ..., c_{2g})\ .$$

Proof. Choose a natural number $N > 4g$. Consider the cube $Q = [1/N, N]^{2g} \subset \mathbb{R}_+^{2g}$. Observe that

(1) for any $j \in \{1, ..., 2g\}$ we have $f_j(1, 1, ..., 1) = \max_{i=1,\ldots,2g} f_i(1, 1, ..., 1) = 4g$; and $(1, 1, ..., 1) \in Q$;
(2) if $(\tilde{c}_1, ..., \tilde{c}_{2g}) \notin Q$, i.e., if $\tilde{c}_j > N$ or $\tilde{c}_j < 1/N$ for a certain $j \in \{1, ..., 2g\}$, then
$\max_{i=1,\ldots,2g} f_i(\tilde{c}_1, ..., \tilde{c}_{2g}) > N$.

The function $\max_{i=1,\ldots,2g} f_i(c_1, ..., c_{2g})$ is continuous, hence it must have its minimum in Q. □

Lemma 4. *Let $\tilde{c} = (\tilde{c}_1, ..., \tilde{c}_{2g}) \in \mathbb{R}_+^{2g}$ be a point of local minimum of the function $\max_{i\in\{1,...,2g\}} f_i$. Then $\tilde{c}$ is a solution of the system $(*)$.*

Proof. Suppose that $\tilde{c} = (\tilde{c}_1, ..., \tilde{c}_{2g}) \in \mathbb{R}_+^{2g}$ is so that

$$\max_{i=1,...,2g} f_i(\tilde{c}_1, ..., \tilde{c}_{2g}) = f_j(\tilde{c}_1, ..., \tilde{c}_{2g}) \neq f_k(\tilde{c}_1, ..., \tilde{c}_{2g})$$

for a certain $j \in \{1, ..., 2g\}$ and all $k \neq j$. Then $\nabla f_j(\tilde{c}) = 0$ is necessary for $\tilde{c}$ to be a point of local minimum of $\max_{i=1,...,2g} f_i$. But $\nabla f_j(c_1, ..., c_{2g})$ is never zero for any j and any positive $c_1, ..., c_{2g}$.

Suppose now that $\tilde{c}$ is so that

$$\max_{i=1,...,2g} f_i(\tilde{c}_1, ..., \tilde{c}_{2g}) = f_{i_1}(\tilde{c}_1, ..., \tilde{c}_{2g}) = ... = f_{i_k}(\tilde{c}_1, ..., \tilde{c}_{2g})$$

with $1 < k \leq 2g$. If $\tilde{c}$ is a point of local minimum of $\max_{i=1,...,2g} f_i$, then the gradients $\nabla f_{i_1}(\tilde{c}), ..., \nabla f_{i_k}(\tilde{c})$ are lineary dependent. We prove that it cannot happen if $k < 2g$. More precisely, we show that the Jacobi matrices of all collections $(f_{i_1}, ..., f_{i_k})$ are of maximal rank when $k < 2g$.

Indeed, let $k < 2g$. Suppose first that $i_k \neq 2g$. (We can suppose without lost of generality that $i_1 < i_2 < ... < i_k$.) Then the submatrix of the Jacobi matrix J which consists of the rows $i_1, ..., i_k$ of J and the columns $i_1 + 1, ..., i_k + 1$ of J, is a lower-triangular matrix

$$\begin{pmatrix} -\alpha/c_{i_1+1}^2 & 0 & \cdots & 0 & 0 \\ & -1/c_{i_2+1}^2 & \cdots & 0 & 0 \\ & & \ddots & & \\ & & & & -1/c_{i_k+1}^2 \end{pmatrix},$$

where $\alpha = 2$ if $i_1 = 1$ and $\alpha = 1$ otherwise. Its determinant is obviously non zero for all positive $c_1, ..., c_{2g}$.

If $i_k = 2g$ but $i_{k-1} \neq 2g - 1$, then the submatrix of J consisting of the rows $i_1, ..., i_k$ and the columns $i_1 + 1, ..., i_{k-1} + 1, 2g$ of J is a lower-triangular matrix

$$\begin{pmatrix} -\alpha/c_{i_1+1}^2 & 0 & \cdots & 0 & 0 \\ & -1/c_{i_2+1}^2 & \cdots & 0 & 0 \\ & & \ddots & & \\ & & & & 2 \end{pmatrix},$$

and also has a non zero determinant for all positive $c_1, ..., c_{2g}$.

Finally we consider the case when $i_{k-1} = 2g - 1$ and $i_k = 2g$.

If $i_1 \neq 1$, then the submatrix of J consisting of the rows $i_1, ..., i_k$ and the columns $i_1 + 1, ..., i_{k-2} + 1, 2g$ of J is of the form

$$\begin{pmatrix} -(4g-3)/c_1^2 & -1/c_{i_1+1}^2 & 0 & \cdots & 0 & 0 \\ -(4g-3)/c_1^2 & \gamma & -1/c_{i_2+1}^2 & \cdots & 0 & 0 \\ -(4g-3)/c_1^2 & 0 & \beta & \cdots & 0 & 0 \\ & & & \ddots & \ddots & \\ -(4g-3)/c_1^2 & & & \cdots & \delta & -1/c_{2g}^2 \\ -(4g-4)/c_1^2 & & & & 0 & 2 \end{pmatrix},$$

where $\beta, \gamma, \delta \in \{1, 0\}$. It is easy to see that its determinant is non zero (develop the first line).

If $i_1 = 1, i_2 \neq 2$, then the submatrix of J consisting of the rows $i_1, ..., i_k$ and the columns $2, i_2, ..., i_k$ is of the form

$$\begin{pmatrix} -2/c_2^2 & 0 & 0 & \cdots & 0 & 0 \\ -1/c_2^2 & 1 & \gamma & \cdots & 0 & 0 \\ -1/c_2^2 & 0 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ -1/c_2^2 & 0 & 0 & \cdots & 1 & -1/c_{2g}^2 \\ -2/c_2^2 & 0 & 0 & \cdots & 0 & 2 \end{pmatrix},$$

and its determinant is equal to $-4/c_2^2$.

The last case to consider is when $i_1 = 1$, $i_2 = 2$, $i_{k-1} = 2g - 1$, $i_k = 2g$. There exists $l \geq 1$ such that $i_{k-l} = 2g-l$, but $i_{k-l-1} < 2g-l-1$. In this case the submatrix of J consisting of the rows $i_1, ..., i_k$ and the columns $2, 3, ..., i_{k-l-1} + 1, 2g - l, 2g$ is of the form

$$\begin{pmatrix} -2/c_2^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1-1/c_2^2 & -1/c_3^2 & 0 & \cdots & 0 & 0 & 0 \\ -1/c_2^2 & \alpha & \beta & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ -1/c_2^2 & 0 & 0 & \cdots & 1 & -1/c_{2g-1}^2 & 0 \\ -1/c_2^2 & 0 & 0 & \cdots & 0 & 1 & -1/c_{2g}^2 \\ -2/c_2^2 & 0 & 0 & \cdots & 0 & 0 & 2 \end{pmatrix}.$$

We can easily develop it with respect to the first line, then with respect to the last line. Repeating this procedure a certain number of times will lead inevitably to a diagonal matrix. The determinant up to the sign is equal to 4 divided by a product of certain number of c_i^2's, which is always different from 0.

Thus we have shown that the $\min_{\{c_1,...,c_{2g}\}} \max_{i=1,...,2g} f_i(c_1, ..., c_{2g})$ may be reached only in a point where all f_i's coincide and their gradients are lineary dependent, i.e., in a point which solves the following system of algebraic equations:

$$\left.\begin{aligned} f_1 - f_{2g} &= 0 \\ f_2 - f_{2g} &= 0 \\ &\vdots \\ f_{2g-1} - f_{2g} &= 0 \\ \det J &= 0, \end{aligned}\right\}$$

which is exactly the system $(*)$. □

Proof of Theorem. Part (1) of Theorem follows from Lemma 3. Part (2) follows from Corollary 1 and Lemma 4. □

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