The effect of numerical integration in the finite element method for nonmonotone nonlinear elliptic problems with application to numerical homogenization methods

Assyr Abdulle, Gilles Vilmart
Section de Mathématiques, École Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne, Switzerland.

Received *****; accepted after revision +++++

Abstract

A finite element method with numerical quadrature is considered for the solution of a class of second-order quasilinear elliptic problems of nonmonotone type. Optimal a-priori error estimates for the $H^1$ and the $L^2$ norms are derived. The uniqueness of the finite element solution is established for a sufficiently fine mesh. Our results permit the analysis of numerical homogenization methods. To cite this article: A. Abdulle, G. Vilmart, C. R. Acad. Sci. Paris, Ser. I *** (20**).

Résumé


Version française abrégée

Pour des problèmes elliptiques linéaires ou monotones, l’effet de l’intégration numérique sur la méthode des éléments finis est analysé dans [7,15] et [12]. Cependant, il n’existe à notre connaissance aucune analyse de vitesses de convergence pour des problèmes nonlinéaires de type nonmonotones. Dans [11], la convergence $H^1$ de la solution numérique est établie, mais sans vitesse de convergence et seulement pour des éléments finis linéaires par morceaux. L’objet de cet article est d’analyser l’influence des erreurs de quadrature pour la méthode des éléments finis appliquée à la classe d’équations elliptiques quasi-linéaires non-monotones (1). Sous des hypothèses usuelles sur le maillage pour des problèmes non-linéaires, sur la régularité des coefficients et des données, et sur les formules de quadrature (Q1), (Q2), également usuelles tant pour des problèmes avec intégration numérique (voir [7] ou [8, Sect. 29]) que pour des problèmes non-linéaires [11,16,10], nous prouvons des estimations optimales d’erreur pour les normes $H^1$ et $L^2$ de la méthode d’éléments finis (4), pour des éléments simpliciaux ou quadrilatéraux d’ordre arbitraire. Nous prouvons également l’unicité de la solution numérique.

Une application importante de notre étude est une analyse (avec discrétisation totale des échelles à la fois macroscopiques et microscopiques) d’une méthode d’homogénéisation numérique du type [1,2,3,10]

Email addresses: assyr.abdulle@epfl.ch (Assyr Abdulle), gilles.vilmart@epfl.ch (Gilles Vilmart).

Preprint submitted to the Académie des sciences 9 septembre 2011

1. Introduction

Nous étudions des discrétisations de deuxième ordre quasilinéaires elliptiques de la forme

$$-
abla \cdot (a(x,u(x))\nabla u(x)) = f(x) \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial \Omega,$$

(1)

où $\Omega$ est un polyèdre borné en $\mathbb{R}^d$ avec $d \leq 3$. Nous faisons les hypothèses suivantes sur le tenseur $a(x,s) = (a_{mn}(x,s))_{1 \leq m,n \leq d}$:

- les coefficients $a_{mn}(x,s)$ sont des fonctions continues de $\overline{\Xi} \times \mathbb{R}$ qui sont uniformément Lipschitz continues avec respect à $s$, i.e.,

$$\exists \Lambda_1 > 0, \quad |a_{mn}(x,s_1) - a_{mn}(x,s_2)| \leq \Lambda_1 |s_1 - s_2|, \quad \forall x \in \overline{\Xi}, \forall s_1, s_2 \in \mathbb{R}, \forall 1 \leq m,n \leq d.$$  

(2)

- $a(x,s)$ est uniformément coercive et borné, i.e.,

$$\exists \lambda, \Lambda_0 > 0, \quad \lambda \|\xi\|^2 \leq a(x,s)\xi \cdot \xi, \quad \|a(x,s)\xi\| \leq \Lambda_0 \|\xi\|, \quad \forall \xi \in \mathbb{R}^d, \forall x \in \overline{\Xi}, \forall s \in \mathbb{R}.$$  

(3)

Since (2)-(3) hold, it is known [13] that (1) has a unique solution $u \in H_0^1(\Omega)$ for all $f \in L^2(\Omega)$.

Pour les problèmes elliptiques linéaires ou monotones, l’effet de quadrature numérique en FEM a été analysé dans [7,15] et [12]. À notre connaissance, il n’existe pas d’analyse de la convergence de FEM avec quadrature numérique appliquée aux problèmes paraboliques ou non monotones, comme considéré dans ce papier. En [11], la convergence dans $H^1$ de l’application est montrée pour des éléments finis polyédriques, mais sans prendre en compte une convergence en $L^2$. En l’absence de quadrature numérique, les estimations a-priori sont optimales dans les normes $H^1$ et $L^2$ pour les méthodes FEM (FEMs) étaient d’abord effectuées en [9].

Les équations (1) sont utilisées pour modéliser de nombreux problèmes importants, mentionnons la filtration de l’eau dans un milieu poreux, l’étude de l’électricité dans un matériau ou la diffusion thermique. L’intégration numérique exacte des FEM est rarement possible, il est important de quantifier l’effet de quadrature numérique. Les taux de convergence-optimaux dans les normes $H^1$ et $L^2$ sont prouvés dans ce cas. La mise en œuvre pratique de la méthode non-linéaire FEM nécessite une méthode de Newton. Nous avons aussi établi la convergence de cette méthode numérique (crucial in applications) et la unicité de l’application de l’équation discrète pour un maillage polygonal et pour des problèmes de diffusion linéaire (cas polygonal complet).

Application de la méthode d’homogénéisation numérique est ensuite discutée. En contraste avec les résultats précédents [10] obtenus pour des problèmes nonmonotones d’homogénéisation en dimension $d \leq 2$ basés sur la fonction de Green logarithmique, nous obtenons les estimations de convergence optimales dans les normes $H^1$ et pour une formulation semi-discrète, nous obtenons des taux de convergence optimaux pour des dimensions $d \leq 3$ et pour une méthode discrète complète, dont l’application numérique des FEM discrétisations (voir [1,2,3]) dans le cas de problèmes linéaires). Dans l’analyse, nos résultats sont également valides pour des taux de convergence-optimaux d’éléments simpliciels ou quadrilatéraux, des estimations d’erreur optimaux sont obtenues pour
the $L^2$ norm, and improved estimates are obtained for the resonance error. More details on the results and the analysis presented here are given in [4] (one-scale problems) and [5] (multi-scale problems).

2. Finite element method with numerical quadrature

We consider a conformal shape regular family of partitions $T_h$ of $\Omega$ in simplicial or quadrilateral elements $K$ of diameter $h_K$ and denote $h := \max_{K \in T_h} h_K$. We consider the family of FE spaces $S_0^m(\Omega, T_h) := \{v^h \in H^1_0(\Omega); \ v^h|_K \in \mathcal{R}^m(K), \ \forall K \in T_h\}$, where $\mathcal{R}^m(K)$ is the space of polynomials on $K$ of total degree at most $\ell$ if $K$ is a simplicial FE, or the space $\mathcal{Q}^m(K)$ of polynomials on $K$ of degree at most $\ell$ in each variable if $K$ is a quadrilateral FE. We define a quadrature formula $\{\hat{x}_j, \hat{\omega}_j\}_{j=1}^J$ on a reference element $\hat{K}$, where $\hat{x}_j$ are integration points and $\hat{\omega}_j$ are quadrature weights. The quadrature formula $\{x_K, \omega_K\}_{j=1}^J$ is then defined as usual on any element $K$ of the triangulation using a $C^1$-diffeomorphism. We make the following assumptions, which are similar to the case of linear elliptic problems (see [7] or [8, Sect. 29]):

(Q1) $\hat{\omega}_j > 0, \ j = 1, \ldots, J$, \quad $\sum_{j=1}^J \hat{\omega}_j |\nabla \hat{p}(\hat{x}_j)|^2 \geq \lambda \|\nabla \hat{p}\|_{L^2(\hat{K})}^2, \ \forall \hat{p}(\hat{x}) \in \mathcal{R}^\ell(\hat{K}),$ \quad with $\lambda > 0$;
(Q2) $\int_{\hat{K}} \hat{p}(\hat{x}) d\hat{x} = \sum_{j=1}^J \hat{\omega}_j \hat{p}(\hat{x}_j)$, \quad $\forall \hat{p}(\hat{x}) \in \mathcal{R}^\ell(\hat{K})$, where $\sigma = \max(2\ell - 2, \ell)$ if $\hat{K}$ is a simplicial FE, or $\sigma = \max(2\ell - 1, \ell + 1)$ if $\hat{K}$ is a rectangular FE.

Consider $v, w$ scalar or vector functions that are piecewise continuous with respect to the partition $\mathcal{T}_h$ of $\Omega$, the semi-definite inner product $(u, v)_h := \sum_{K \in \mathcal{T}_h} \sum_{j=1}^J \omega_K \langle u(x_K), v(x_K) \rangle_\sigma(x_K)$ of (1) with numerical integration reads: find $u^h \in S_0^0(\Omega, T_h)$ such that

$$(a(\cdot, u^h) \nabla u^h, \nabla w^h)_h = F_h(w^h) \quad \forall w^h \in S_0^0(\Omega, T_h),$$

(4)

where the linear form $F_h(w^h)$ is an approximation of $\int_\Omega f(x) u^h(x) dx$ obtained for example by using a quadrature formula. If $f \in W^{r, r}(\Omega)$ with $1 \leq r \leq \infty$ and $\ell > d/r$, then $f$ is continuous on $\overline{\Omega}$ and one can take $F_h(w^h) := (f, w^h)_h$. The existence of the FE solution $u^h \in S_0^0(\Omega, T_h)$ in (4) can be shown for all $h > 0$ using the Brouwer fixed point theorem. Details can be found for example in [9].

3. Convergence rates for FEM with numerical quadrature for nonlinear problems

**Theorem 3.1** [4] Consider $u$ the solution of problem (1). Let $\ell \geq 1$. Let $d/\ell < r \leq \infty$. Let $\mu = 0$ or 1. Assume (Q1), (Q2), that the family of triangulations is quasi-uniform, and

$$u \in H^{\ell+1}(\Omega) \cap W^{1, \infty}(\Omega), \quad a \in (W^{\ell+\mu, \infty}(\Omega \times \mathbb{R}))^{d \times d}, \quad f \in W^{\ell+\mu, r}(\Omega).$$

In addition to (2), (3), assume that the operator $L^* \varphi = -\nabla \cdot (a(\cdot, u)^T \nabla \varphi) + \partial_m a(\cdot, u) \nabla u \cdot \nabla \varphi$ satisfies

$$\|\varphi\|_{H^2(\Omega)} \leq C(\|L^* \varphi\|_{L^2(\Omega)} + \|\varphi\|_{H^2(\Omega)}), \quad \text{for all } \varphi \in H^2(\Omega) \cap H^1_0(\Omega).$$

(5)

Assume further that $\partial_m a_{mn}(x, s)$ are twice differentiable with respect to $s$, with the first and second order derivatives continuous and bounded on $\overline{\Omega} \times \mathbb{R}$, for all $m, n = 1 \ldots d$.

1. Except for the $W^{1, \infty}$ assumption on $u$ and the smoothness of $s \mapsto a(x, s)$ assumed to treat the non-linearity (as in [9]), the smoothness assumptions of Thm. 3.1 are identical to those classically assumed for linear problems [7,8, Sect. 29].

2. The assumption (5) on the adjoint $L^*$ of the linearized operator $L$ associated to (1) is also required for $L^2$ estimates in the case of linear problems [7]. Using classical $H^2$ regularity results, it is automatically satisfied—owing to the assumptions on the coecients of $L^*$—if the domain $\Omega$ is a convex polyhedron.
Then there exists \( h_0 > 0 \) such that for all \( h \leq h_0 \), the solution \( u_h \) of (4) is unique and the following \( H^1 \) and \( L^2 \) error estimates hold,

\[
\begin{align*}
&\text{if } \mu = 0, 1, \quad \| u - u_h \|_{H^1(\Omega)} \leq C h^\mu \quad \text{for all } h \leq h_0, \\
&\text{if } \mu = 1, \quad \| u - u_h \|_{L^2(\Omega)} \leq C h^{\mu + 1} \quad \text{for all } h \leq h_0,
\end{align*}
\]

where the constant \( C \) is independent of \( h \).

Inspired by [9], the proof of Theorem 3.1 is conducted in three main steps.

**Step 1.** Using the compact injection \( H^1(\Omega) \subset L^2(\Omega) \), the boundedness of a numerical solution in \( H^1_0(\Omega) \) and the uniqueness in \( H^1_0(\Omega) \) of the exact solution of (1), we show,

\[
\| u - u_h \|_{L^2(\Omega)} \to 0 \quad \text{for } h \to 0.
\]

**Step 2.** We derive the following \( H^1 \) a-priori error bound

\[
\| u - u_h \|_{H^1(\Omega)} \leq C(h^\mu + \| u - u_h \|_{L^2(\Omega)}), \quad \text{for all } h > 0.
\]

The additional term \( \| u - u_h \|_{L^2(\Omega)} \) in the right-hand side is due to the non-monotonicity of the differential operator of (1). The proof of (9) relies on an estimate for \( (a(u_h) \nabla u_h, \nabla w_h) - (a(u) \nabla u_h, \nabla w_h)_h \) (obtained by using the Bramble-Hilbert lemma), uniform bounds for the semi-definite inner product \( (v, w)_{T_h} := \sum_{K \in T_h} \sum_{j=1}^J \omega_K v(x_{K,j}) \cdot w(x_{K,j}) \) (defined for piecewise continuous functions \( v, w \)) and the use of the Gagliardo-Niremberg inequality \( \| v \|_{L^p(\Omega)} \leq C \| v \|_{L^2(\Omega)}^{\frac{p}{2}} \| v \|_{H^1(\Omega)}^{\frac{p-2}{2}} \) for all \( v \in H^1(\Omega) \), for \( d \leq 3 \).

**Step 3.** Using an Aubin-Nitsche duality argument and (5), we show that there exists \( h_1 > 0 \) such that

\[
\| u - u_h \|_{L^2(\Omega)} \leq C(h^{\mu + 1} + \| u - u_h \|_{H^1(\Omega)}^2), \quad \text{for all } h \leq h_1.
\]

We consider the FEM solution with numerical quadrature associated to the indefinite linear elliptic problem \( L^* \). We first show that \( L^* \) is an isomorphism and then derive error estimates generalizing a compactness result of Schatz [14] to FEM with numerical quadrature.

**Proof of the \( H^1 \) and \( L^2 \) estimates.** Substituting (9) into (10) (with \( \mu = 0 \)), we obtain

\[
\| u - u_h \|_{H^1(\Omega)} \leq C(h^{\mu} + \| u - u_h \|_{L^2(\Omega)}), \quad \text{for all } h \leq h_1.
\]

Substituting (8) into (9), we obtain \( \| u - u_h \|_{H^1(\Omega)} \to 0 \) for \( h \to 0 \). We deduce in the above inequality \( 1 - C \| u - u_h \|_{H^1(\Omega)} \geq \delta > 0 \) for all \( h \leq h_2 \), with \( h_2 \) small enough (but independent of the particular solution \( u_h \)) hence, (6) is established for all \( h \leq \min\{h_1, h_2\} \). The estimate (7) is deduced by substituting (6) into (10) with \( \mu = 1 \). The uniqueness of the FEM solution follows from Theorem 3.2. \( \Box \)

**Theorem 3.2** Consider \( u_h \) a solution of (4). Under the assumptions of Theorem 3.1, there exist \( h_0, \delta > 0 \) such that if \( h \leq h_0 \) and \( \sigma_h \| z_h - u_h \|_{H^1(\Omega)} \leq \delta \), then the sequence \( \{ z_h \} \) for the Newton method

\[
N_h(z_h; z_h, v_h) = F_h(v_h) - (a(z_h^k) \nabla z_h, \nabla v_h)_h, \quad \forall v_h \in S^h_0(\Omega, T_h),
\]

is well defined, and

\[
\| z_{h+1} - u_h \|_{H^1(\Omega)} \leq C \sigma_h \| z_h - u_h \|_{H^1(\Omega)}^2,
\]

where \( C \) is a constant independent of \( h, k \).

In the above theorem, \( \sigma_h := \sup_{v_h \in S^h_0(\Omega, T_h)} \| v_h \|_{L^\infty(\Omega)} / \| v_h \|_{H^1(\Omega)} \). Using the quasi-uniformity of the family of triangulations, one can show the standard estimates \( \sigma_h \leq C(1 + |\ln h|)^{1/2} \) for \( d = 2 \), and \( \sigma_h \leq C h^{-1/2} \) for \( d = 3 \), where \( C \) is independent of \( h \).

---

3. We define \( N_h(z; v, w) := (a(\cdot, z) \nabla v, \nabla w)_h + (v \partial_a a(\cdot, z) \nabla z, \nabla w)_h. \)
Remark 1 Notice that the requirement of a quasi-uniform mesh for the family of triangulations is often assumed for the analysis of FEM for nonlinear problems [11, 16, 10]. In our proof, we need it in Step 3 to have an a-priori estimate in $W^{1,6}(\Omega)$ for the FEM solution (with numerical quadrature) associated to $L^\ast$. We further need this assumption in the uniqueness result below. However, if $\|u\|_{H^2(\Omega)}$ or the Lipschitz constant are small enough such that $C \Lambda^{-1} A_1 \|u\|_{H^2(\Omega)} < 1$, where $C$ depends only on $\Omega$ and the polynomial degree of the FE space, then (5) and $u \in W^{1,\infty}(\Omega)$ are not required to prove the uniqueness result, and removing in addition the assumptions of quasi-uniform meshes and $h \leq h_0$, the $H^1$ estimate (6) still holds.

4. Application to numerical homogenization.

We consider a class of nonlinear nonmonotone multiscale problems

\[- \nabla \cdot (a^\varepsilon(x, u_\varepsilon(x))) \nabla u_\varepsilon(x)) = f(x) \quad \text{in} \quad \Omega, \quad u_\varepsilon(x) = 0 \quad \text{on} \partial \Omega,\]

with a $d \times d$ tensor $a^\varepsilon(x, x)$ satisfying (2), (3) uniformly in $\varepsilon$. Here $\varepsilon$ represent a small scale in the problem. The following homogenization result is shown in [6, Theorem 3.6]: there exists a subsequence of $\{a^\varepsilon(\cdot, s)\}$ (again indexed by $\varepsilon$) such that the corresponding sequence of solutions $\{u_\varepsilon\}$ converges weakly to $u_0$ in $H^1(\Omega)$, where $u_0$ is solution of the so-called homogenized problem

\[- \nabla \cdot (a^0(x, u_0(x))) \nabla u_0(x)) = f(x) \quad \text{in} \quad \Omega, \quad u_0(x) = 0 \quad \text{on} \partial \Omega,\]

with a homogenized tensor $a^0(\cdot, s)$ which can be shown to have similar properties as assumed for $a^\varepsilon(x, s)$.

The FE-HMM method for computing a numerical approximation $u^H$ of $u_0$, essentially similar to the method proposed in [10] 4 reads as follows. It is based on a macroscopic FEM defined on $QF$ with a macro FE space $S^0_0(\Omega, T_H)$ (defined as in Sect. 2), and microscopic FEMs recovering the missing macroscopic tensor at the macroscopic quadrature points. For each macro element $K \in T_H$ and each integration point $x_{K_{\delta_j}} \in K$, $j = 1, \ldots, J$, we define the sampling domains $K_{\delta_j} = x_{K_j} + (-\delta, \delta)^d$, $\delta \geq \varepsilon$. For each $K_{\delta_j}$, we then define a micro FE space $S^0(K_{\delta_j}, T_h) \subset W(K_{\delta_j})$ with simplicial or quadrilateral FEs and a conformal and shape regular family of triangulations $T_h$. The space $W(K_{\delta_j})$ is either the Sobolev space $W^1(\Omega) = \{ \varepsilon \in H^1_{\text{per}}(K_{\delta_j}); \int_{K_{\delta_j}} z \, dx = 0 \}$ for a periodic coupling or $W(K_{\delta_j}) = H^1_0(K_{\delta_j})$ for a coupling through Dirichlet boundary conditions.

**FE-HMM** Find $u^H \in S^0_0(\Omega, T_H)$ such that $B_H(u^H; v^H, w^H) = F_H(v^H, w^H), \forall v^H, w^H \in S^0_0(\Omega, T_H)$, where

\[ B_H(u^H; v^H, w^H) := \sum_{K \in T_H} \sum_{j=1}^J \omega_{K_j} / |K_{\delta_j}| \int_{K_{\delta_j}} a^\varepsilon(x, u^H(x_{K_j})) \nabla v^{h,u^H(x_{K_j})}_j(x) \cdot \nabla w^{h,u^H(x_{K_j})}_j(x) \, dx,\]

and $v^{h,u^H(x_{K_j})}_j$ (and similarly for $v^{h,w^H(x_{K_j})}_j$) denotes the solution of the following micro problem (16) with parameter $s = u^H(x_{K_j})$. Find $u^{h,s}_{K_j}$ such that $u^{h,s}_{K_j} - (u^H(x_{K_j}) + (x - x_{K_j}) \cdot \nabla w^H(x_{K_j})) \in S^0(K_{\delta_j}, T_h)$ and

\[ \int_{K_{\delta_j}} a^\varepsilon(x, s) \nabla w^{h,s}_{K_j}(x) \cdot \nabla z^h(x) \, dx = 0 \quad \forall z^h \in S^0(K_{\delta_j}, T_h).\]

We make the following smoothness and structure assumptions on the tensor.

---

4. In [10] (15) is based on exact micro functions $v_{K_j}, w_{K_j}$ instead of the FE micro functions $v^{h,s}_{K_j}, w^{h,s}_{K_j}$ and the micro-problems are nonlinear (see [10, equs. (5.3)-(5.4)]).
The same assumption also holds with the tensor domain \( H \) and the parameter \( s \) for all \( i = 1, \ldots, d \). Here, \( e_1, \ldots, e_d \) denotes the canonical basis of \( \mathbb{R}^d \). The same assumption also holds with the tensor \( a^\varepsilon \) replaced by \( (a^\varepsilon)^T \) in (17).

**Theorem 4.1** [5] Let \( \ell \geq 1, q \geq 1 \) and \( \mu = 0 \) or \( 1 \). In addition to the assumptions of Theorem 3.1 on problem (14), assume \( (H1), (H2) \), and assume that \( a^\varepsilon \) satisfies (2), (3). Then, there exist \( H_0 > 0 \) and \( r_0 > 0 \) such at if \( H \leq H_0 \) and \( h/\varepsilon \leq r_0 \) then

\[
\| u_0 - u^H \|_{H^{1-\mu}(\Omega)} \leq \begin{cases} 
C(H^{\ell+\mu} + (h/\varepsilon)^{2q} + \delta), & \text{if } W(K_{\dot{e}}) = W^{1}_{per}(K_{\dot{e}}) \text{ and } \delta \varepsilon \in \mathbb{N}, \\
C(H^{\ell+\mu} + (h/\varepsilon)^{2q}), & \text{if } W(K_{\dot{e}}) = W^{1}_{per}(K_{\dot{e}}) \text{ and } \frac{\delta}{\varepsilon} \in \mathbb{N}, \text{ and } a^\varepsilon(x, s) \text{ is replaced by } a(x_{K_{\dot{e}}}, x/\varepsilon, s) \text{ in (15), (16), (17)}, \\
C(H^{\ell+\mu} + (h/\varepsilon)^{2q} + \delta + \varepsilon/\delta), & \text{if } W(K_{\dot{e}}) = H^{1}_{0}(K_{\dot{e}}) \text{ (} \delta > \varepsilon), \end{cases}
\]

where we also assume \( \delta \leq r_0 \) or \( \delta + \varepsilon/\delta \leq r_0 \) in the first and third cases, respectively. We use the notation \( H^0(\Omega) = L^2(\Omega) \). The constants \( C \) are independent of \( H, h, \varepsilon, \delta \).

If we assume in addition to the assumptions of Theorem 4.1, the map \( s \in \mathbb{R} \mapsto a^\varepsilon(\cdot, s) \in (W^{1,\infty}(\Omega))^d \) is of class \( C^2 \) with first and second derivatives bounded by \( C \varepsilon^{-1} \), then for sufficiently fine meshes and modeling errors (e.g. in the second case of Theorem 4.1, for \( (h/\varepsilon)^{2q} \leq H \leq H_1 \)), one can show the convergence of a Newton method, and the uniqueness of the numerical solution \( u^H \). Notice that in the third case with non-periodic boundary conditions for the micro-macro coupling (i.e. \( W(K_{\dot{e}}) = H^{1}_{0}(K_{\dot{e}}) \)) we obtain the resonance error estimate \( r_{MOD} \leq C(\delta + \varepsilon/\delta) \) (similar as for linear problems), whereas \( r_{MOD} \leq C(\delta + (\varepsilon/\delta)^{1/2}) \) has been obtain in [10, Thm. 5.5].

References


