

Méthodes numériques d'homogénéisation pour des problèmes multi-échelles quasi-linéaires.

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en collaboration avec

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**UNIVERSITÉ
DE GENÈVE**

FACULTÉ DES SCIENCES
Section de mathématiques

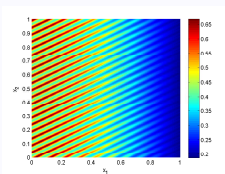
Collège de France, 5 décembre 2014

Example: nonlinear multiscale problems

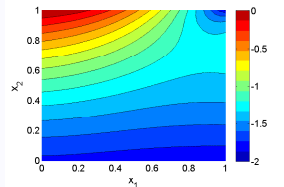
Water infiltration in unsaturated porous media (Richards model)



Porous media

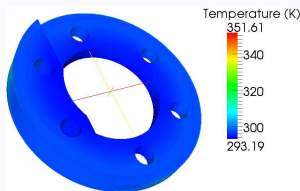


permeability tensor



Pressure simulation.

Nonlinear model of a disk brake (composite material)

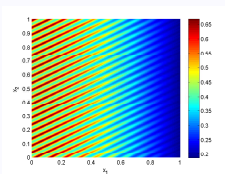


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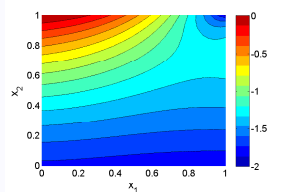
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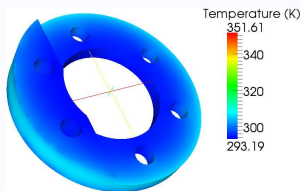


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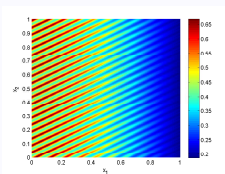


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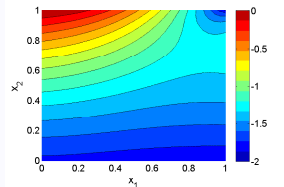
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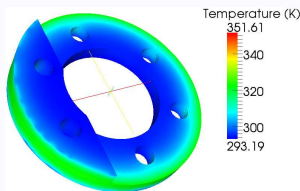


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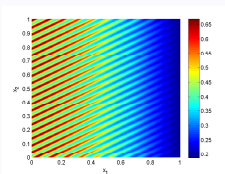


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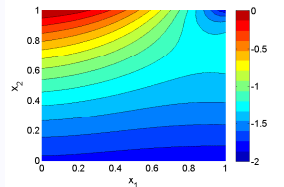
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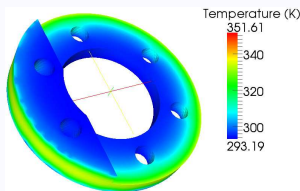


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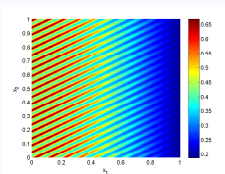


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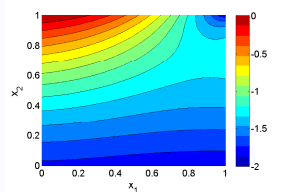
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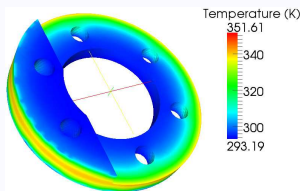


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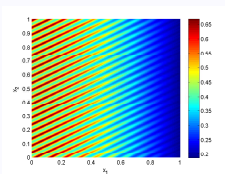


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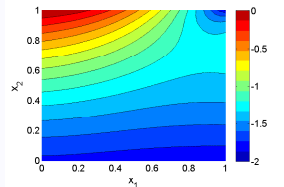
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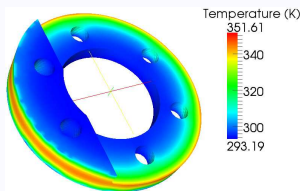


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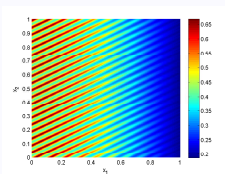


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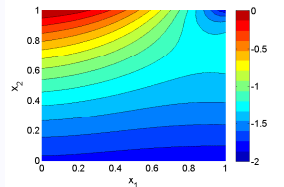
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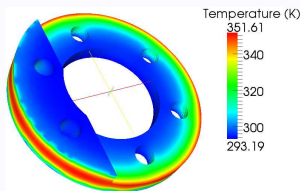


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1D elliptic illustrative example on $\Omega = (0, 1)$

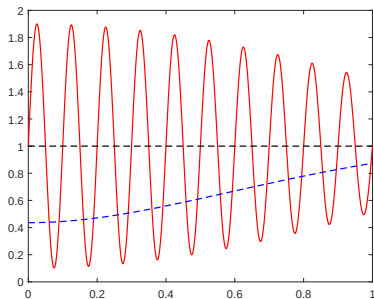
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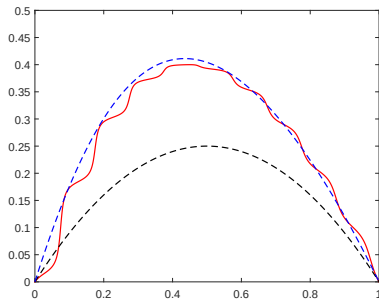
Illustration of $u_\varepsilon \rightarrow u_0$ for $\varepsilon \rightarrow 0$.

$\varepsilon = 1/10$

$$-\nabla \cdot (a^\varepsilon(x) \nabla u_\varepsilon) = f, \quad -\nabla \cdot (a^0(x) \nabla u_\varepsilon) = f, \quad -\nabla \cdot (\bar{a}(x) \nabla u_\varepsilon) = f.$$



data: $a^\varepsilon(x)$, $a^0(x)$, $\bar{a}(x)$



solutions: $u_\varepsilon(x)$, $u_0(x)$, $\bar{u}(x)$

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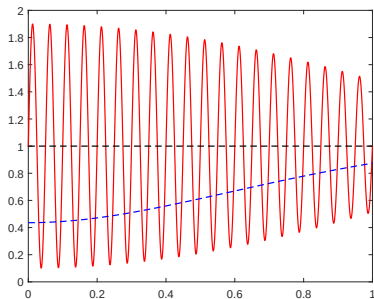
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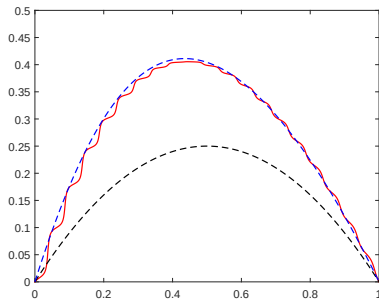
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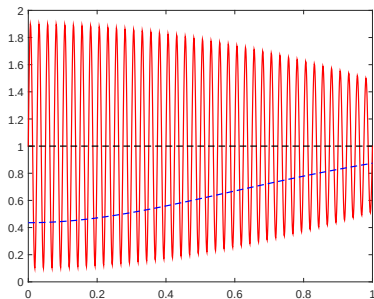
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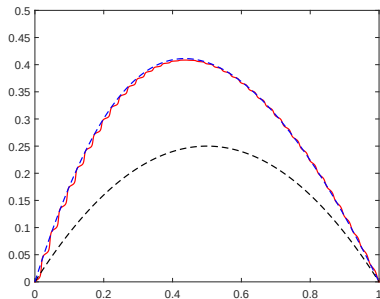
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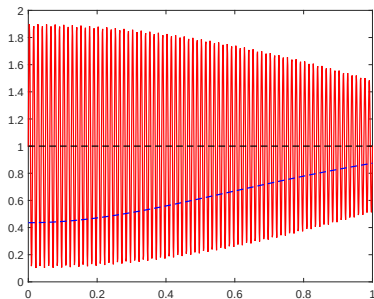
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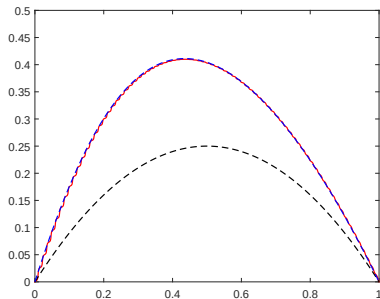
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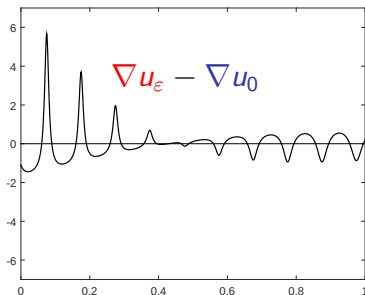
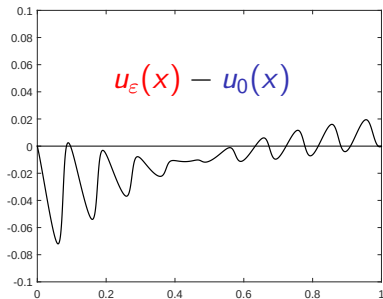
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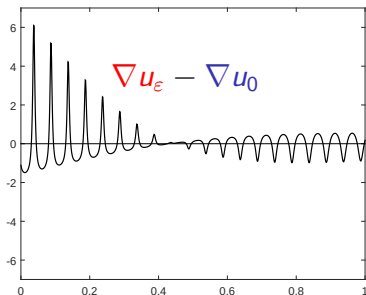
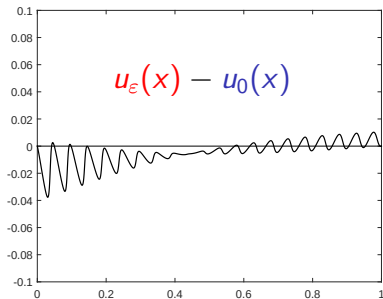
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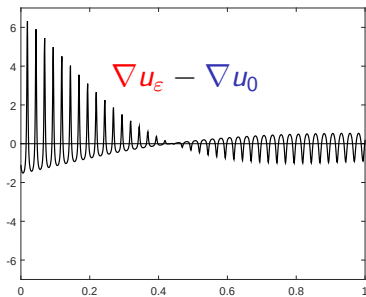
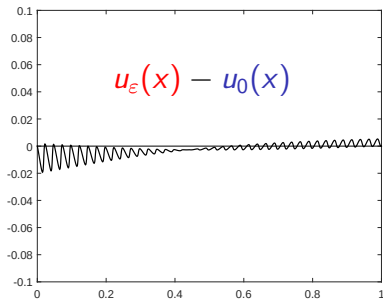
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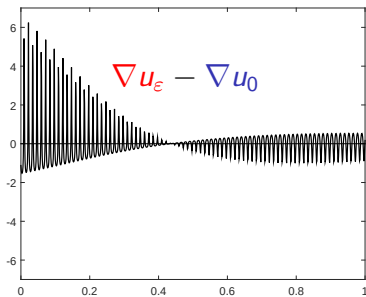
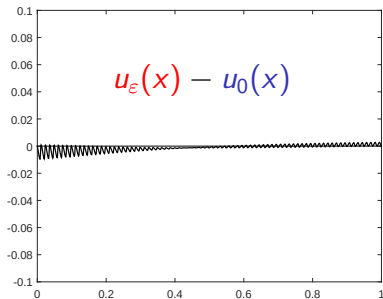
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Analytical framework: homogenization

Macroscopic behaviour of multiple scale problems (Bakhvalov, Babuska, Bensoussan, Lions, Papanicolaou, Tartar, Sanchez-Palencia, Jikov, Kozlov, Oleinik, Allaire, Nguetseng, Fusco, Moscarrello, Boccardo, Murat, Pankov, Svanstedt...)

Elliptic example. $-\nabla \cdot \underbrace{(a^\varepsilon \nabla u_\varepsilon)}_{\xi_\varepsilon} = f$, on Ω , $u_\varepsilon = 0$ on $\partial\Omega$.

where the tensor $a^\varepsilon(x)$ varies rapidly in space (at the scale ε).

Question: $u_\varepsilon \rightarrow u_0$ for $\varepsilon \rightarrow 0$? equation for u_0 ?

Assuming a^ε uniformly elliptic and bounded, we have:

$$u_\varepsilon \xrightarrow{H^1} u_0, \quad \xi_\varepsilon \xrightarrow{L^2} \xi_0, \quad \text{for } \varepsilon \rightarrow 0, \quad (\text{weak convergences}).$$

Homogenization problem: find $a^0 \in L^\infty(\Omega)^{d \times d}$ such that

$$-\nabla \cdot \underbrace{(a^0 \nabla u_0)}_{\xi_0} = f, \text{ on } \Omega, \quad u_0 = 0 \text{ on } \partial\Omega.$$

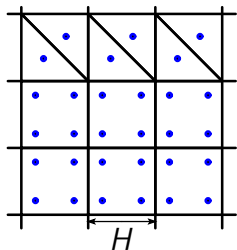
In general a^0 is not a “simple average” (no explicit formula).

Multiscale methods: a few numerical strategies

- ➊ **Generalized FEM** (Babuska and Osborn 1983): modify the FE space to take into account explicitly the microstructure of the problem.
 - ▶ Increasing FE space by adding functions with microstructures (residual-free bubble FEM (...; Brezzi,Marini 2002; Sangalli 2003;...), variational multiscale method (Hughes 1995; Hughes et al.; Ramm et al. 2006; Nolen,Papanicolaou,Pironneau 2008;...)
 - ▶ Changing the basis function of the FE space (MsFEM (Hou,Wu,Cai 1999;...;Allaire,Brizzi 2005;...;Efendiev,Hou 2009;...) two-scale FEM (Matache,Schwab 2002;...), corrector techniques (Gloria 2006, 2008), harmonic coordinates (Owhadi, Zhang, Berlyand 2008, ...), domain dec. techniques (Graham, Scheichl 2007, ...), LOD (Petersheim,Malqvist, Henning 2012,2013, ...)
- ➋ **Constrained macro simulations** (macro-to-micro methods)
 - ▶ Representative volume (averaging) elements, iterative coupling (Fish et al. 1997;...; Feyel,Chaboche 2000; Terada,Kikuchi 2001; Miehe,Schröder,Bayreuther 2002;...; Geers,Kouznetsova,Brekelmans 2010;...)
 - ▶ Equation free framework (Kevrekidis, Gear et al. 2003, Samaey, Dirk, Kevrekidis 2005, ... Gear, Kevrekidis 2010)
 - ▶ Heterogenous multiscale method (E,Engquist 2003; Abdulle,E 2003; Abdulle 2005; Abdulle,Schwab 2005; E,Ming,Zhang 2005; Ming,Yue 2007, Ohlberger 2005,...; Abdulle,E,Engquist,Vanden-Eijnden 2012, Abdulle, Bai, Huber, V.,2011,2012,2013)

Finite Element Heterogeneous Multiscale Method (E, Engquist 2003)

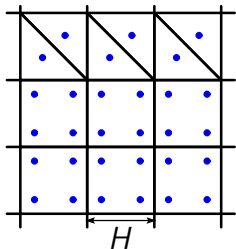
$$A(u^H, v^H) := \int_{\Omega} a^0(x) \nabla u^H \cdot \nabla v^H dx = \int_{\Omega} f v^H dx, \quad \forall v^H$$



standard FEM
with quadrature formulas

Finite Element Heterogeneous Multiscale Method (E, Engquist 2003)

$$A_H(u^H, v^H) := \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \omega_{K_j} a^0(x_{K_j}) \nabla u^H(x_{K_j}) \cdot \nabla v^H(x_{K_j}), \quad \forall u^H, v^H$$



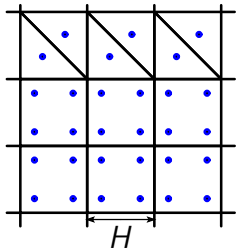
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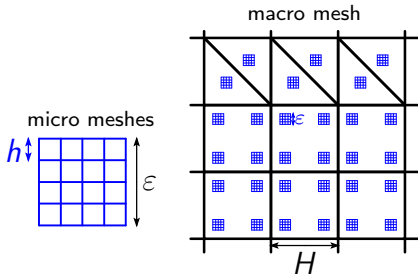
$$B_H(u^H, v^H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a^\varepsilon(x) \nabla u_{K_j}^h(x) \cdot v_{K_j}^h(x) dx, \quad \forall u^H, v^H \in S_0^\ell(\Omega, \mathcal{T}_H),$$

where $v_{K_j}^h$ is the solution of the micro problem $v_{K_j}^h - v_{lin}^H \in S^q(K_{\delta_j}, \mathcal{T}_h)$,

$$\int_{K_{\delta_j}} a^\varepsilon(x) \nabla v_{K_j}^h(x) \cdot \nabla z^h(x) dx = 0, \quad \forall z^h \in S^q(K_{\delta_j}, \mathcal{T}_h).$$



standard FEM
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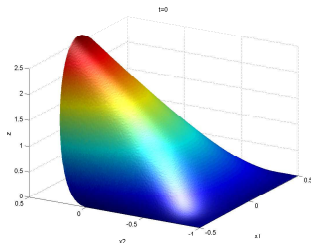
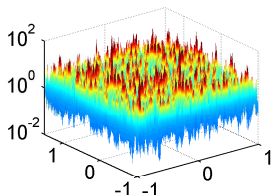
Heterogeneous Multiscale Method (HMM)
(micro meshes: FEs of size h)

Homogenization in space & stabilization in time (Abdulle and V., M3AS, 2012)

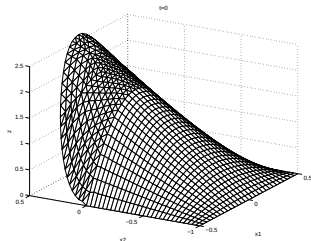
$$\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon(x, t) \nabla u_\varepsilon) = f \text{ in } \Omega \times (0, T)$$

+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.0$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.0$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

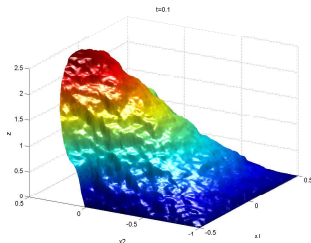
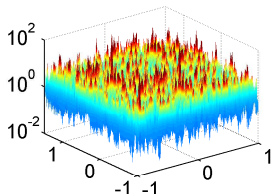
Multiscale space-time analysis: coupling with efficient time integrators: Radau (implicit), Chebyshev (explicit stabilized).

Homogenization in space & stabilization in time (Abdulle and V., M3AS, 2012)

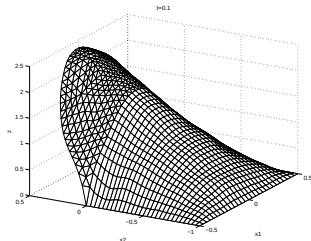
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Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.1$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

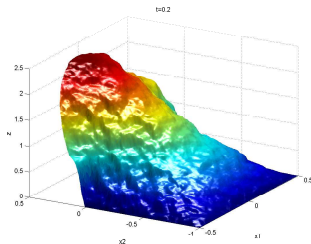
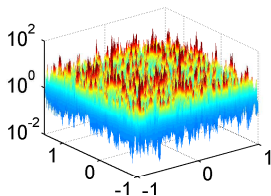
Multiscale space-time analysis: coupling with efficient time integrators: Radau (implicit), Chebyshev (explicit stabilized).

Homogenization in space & stabilization in time (Abdulle and V., M3AS, 2012)

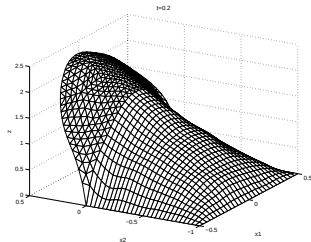
$$\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon(x, t) \nabla u_\varepsilon) = f \text{ in } \Omega \times (0, T)$$

+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.2$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.2$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

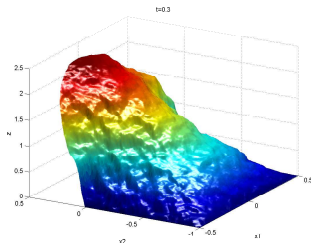
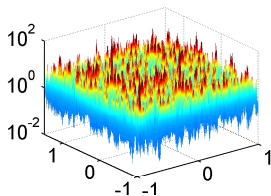
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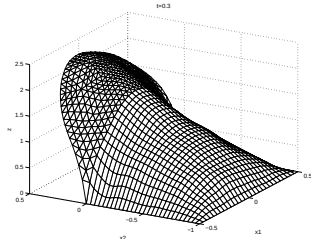
$$\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon(x, t) \nabla u_\varepsilon) = f \text{ in } \Omega \times (0, T)$$

+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.3$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.3$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

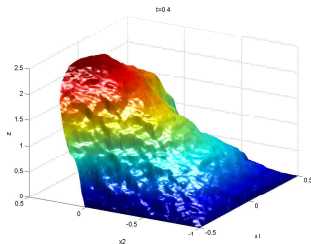
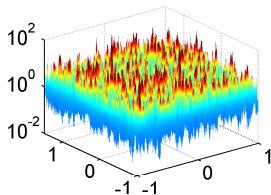
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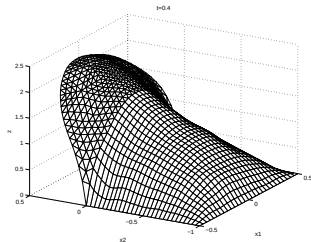
$$\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon(x, t) \nabla u_\varepsilon) = f \text{ in } \Omega \times (0, T)$$

+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.4$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.4$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

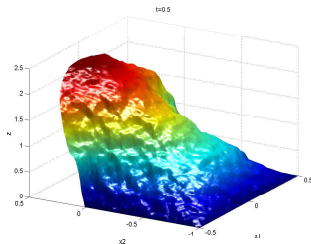
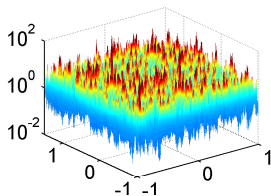
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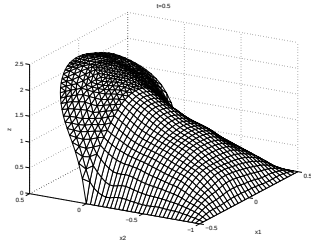
$$\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon(x, t) \nabla u_\varepsilon) = f \text{ in } \Omega \times (0, T)$$

+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.5$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.5$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

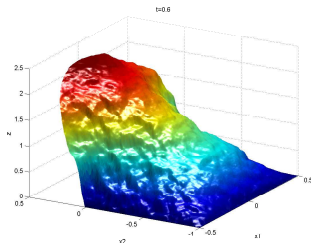
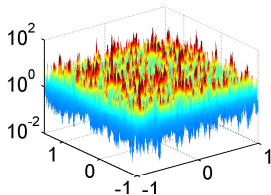
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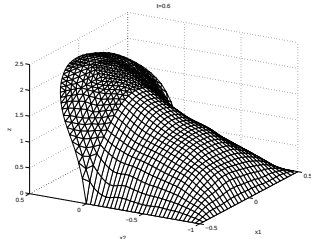
$$\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon(x, t) \nabla u_\varepsilon) = f \text{ in } \Omega \times (0, T)$$

+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.6$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.6$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

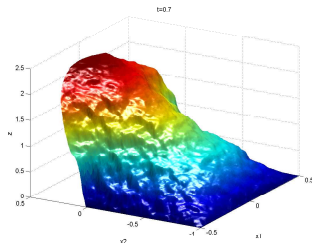
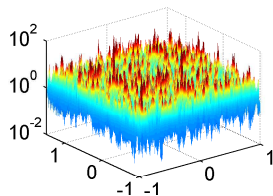
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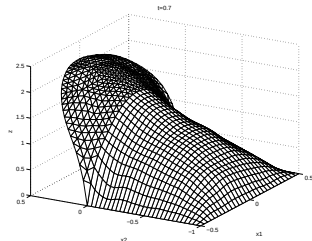
$$\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon(x, t) \nabla u_\varepsilon) = f \text{ in } \Omega \times (0, T)$$

+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.7$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.7$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

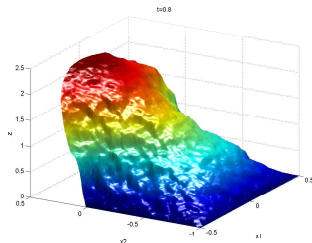
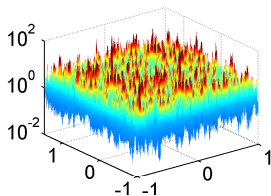
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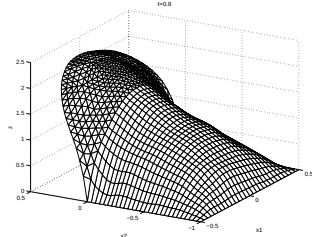
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+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.8$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



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(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

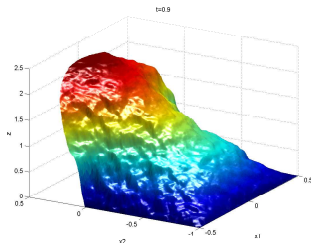
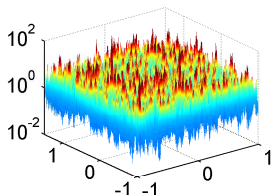
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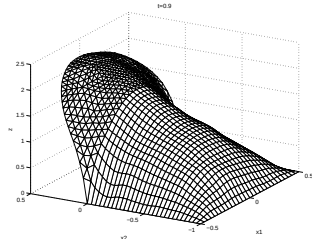
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+initial and boundary conditions

random tensor a^ε :



finescale solution u_ε at $t = 0.9$
(FE standard, 10^6 degrees of freedom)
Explicit Euler $\Delta t = 0.0025$



Homogenized solution u_0 at $t = 0.9$
(FE-HMM, 10^3 degrees of freedom)
Chebyshev method (explicit, $s = 8$): $\Delta t = 0.1$

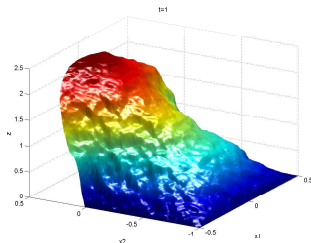
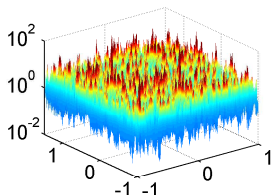
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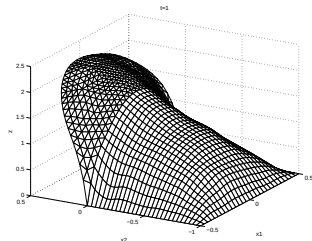
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+initial and boundary conditions

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(FE standard, 10^6 degrees of freedom)
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Multiscale space-time analysis: coupling with efficient time integrators: Radau (implicit), Chebyshev (explicit stabilized).

Nonlinear problems: numerical homogenization

Nonmonotone elliptic problems.

$$-\nabla \cdot (a^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon) = f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

We provide a fully discrete 3D analysis using compactness arguments.

First 2D results for the analysis of a numerical homogenization method by E, Ming, Zhang 2005 (HMM); Chen, Savchuk 2007 (MsFEM), using two-grid discretization techniques (Xu 1996).

Monotone parabolic problems.

$$\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot (a^\varepsilon(x, \nabla u_\varepsilon) \nabla u_\varepsilon) = f \quad \text{in } \Omega \quad + \text{ boundary and init. cond.}$$

Monotonicity condition: $A(u; v, w) = \int_\Omega a^\varepsilon(x, u) \nabla v \cdot \nabla w \, dx$ satisfies $A(v; v, v - w) - A(w; w, v - w) \geq C \|\nabla v - \nabla w\|_{L^2(\Omega)}^2, \forall v, w \in H_0^1$.
Efendiev, Parkov (2004), Efendiev, Hou (MsFEM, 2009), Henning, Ohlberger (MsFEM and HMM, 2012),...

Plan of the talk

- 1 One-scale nonmonotone problems
- 2 Homogenization nonmonotone problems (two scales)
- 3 Coupling with the reduced basis technique
- 4 Monotone parabolic problems

One-scale nonmonotone problems

$$\begin{aligned} -\nabla \cdot (a(x, u) \nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

- 1 One-scale nonmonotone problems
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Abdulle and V., *A priori error estimates for finite element methods with numerical quadrature for nonmonotone nonlinear elliptic problems*, *Numer. Math.*, 2012.

Nonlinear nonmonotone problem

Problem. $\nabla \cdot (a(x, u) \nabla u) = f$ in Ω , $u = 0$ on $\partial\Omega$.

Important problems: thermal diffusion in materials, water infiltration in porous medium (Richards), ...

A priori error analysis for FEM with numerical quadrature ?

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- Douglas, Dupont (1975) H^1 and L^2 estimates, no numerical quadrature
- Nitsche (1977) L^∞ estimates, no numerical quadrature
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A priori error analysis for FEM with numerical quadrature ?

- Douglas, Dupont (1975) H^1 and L^2 estimates, no numerical quadrature
- Nitsche (1977) L^∞ estimates, no numerical quadrature
- ...
- Feistauer, Ženíšek (1987) FEM with numerical integration, for **monotone problems** where the weak formulation form satisfies $A(v; v, v-w) - A(w; w, v-w) \geq C \|\nabla v - \nabla w\|_{L^2(\Omega)}^2, \forall v, w \in H_0^1$.
- Feistauer, Křížek, Sobotíková (1993) FEM with numerical quadrature. For nonmonotone problems, **but no convergence rates**).
- Korotov, Křížek (2000) FEM with numerical integration (domain approx. in 3D), no convergence rates

Basic assumptions on the tensor

Define $a(x, s) = (a_{mn}(x, s))_{1 \leq m, n \leq d}$.

Assume

1. a_{mn} continuous on $\overline{\Omega} \times \mathbb{R}$,
2. $|a_{mn}(x, s_1) - a_{mn}(x, s_2)| \leq \Lambda_1 |s_1 - s_2|, \forall x \in \overline{\Omega}, s_1, s_2 \in \mathbb{R}$,
3. $\lambda \|\xi\|^2 \leq a(x, s) \xi \cdot \xi, \quad \|a(x, s) \xi\| \leq \Lambda_0 \|\xi\|, \forall \xi \in \mathbb{R}^d$.

(Q) standard assumptions on the quadrature formulas.

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1., 2., 3. $\Rightarrow$ nonlinear elliptic problem has **one and only one solution**, classical result, Douglas, Dupont, Serrin (1971) see Chipot (2009)

1., 3., (Q) $\Rightarrow$ for all $h > 0$ the nonlinear problem with numerical quadrature has **at least** one solution $u^h \in S_0^\ell(\Omega, \mathcal{T}_h)$ (Brouwer fixed point argument).

Theorem

u sol. of nonlinear pb., u^h sol. of nonlinear FEM with quadrature.

Assume 1., 2., 3., $(\mathbf{Q})$ and $h/h_K \leq C, \forall K \in \mathcal{T}_h$. Let $\ell \geq 1$ and

$$u \in H^{\ell+1}(\Omega) \cap W^{1,\infty}(\Omega),$$

$$a_{mn} \in W^{\ell+1,\infty}(\Omega \times \mathbb{R}), \quad \forall m, n = 1 \dots d.$$

Assume that $\partial_u a_{mn} \in W^{1,\infty}(\Omega \times \mathbb{R})$, and that $\partial_u a_{mn}(x, s)$ and $\partial_{uu} a_{mn}(x, s)$ are continuous and bounded on $\bar{\Omega} \times \mathbb{R}$.

Then, there exists $h_0 > 0$ s.t. for all $h \leq h_0$, u^h is unique and

$$\|u - u^h\|_{H^1(\Omega)} \leq Ch^\ell, \quad \|u - u^h\|_{L^2(\Omega)} \leq Ch^{\ell+1}.$$

$\Rightarrow$ For linear problems, we recover estimates of Ciarlet, Raviart (1972) with the **same assumptions** (excepted $u \in W^{1,\infty}(\Omega)$ and the inverse assumption $h/h_K \leq C, \forall K \in \mathcal{T}_h$).

Idea of the analysis: convergences rates

- **Step 1.** Using the boundedness of u^h in $H^1(\Omega)$, the compact injection $H^1(\Omega) \subset L^2(\Omega)$ and the uniqueness of u , we show

$$\|u - u^h\|_{L^2(\Omega)} \rightarrow 0 \quad \text{for } h \rightarrow 0.$$

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- **Step 2.** Using estimates for $A - A_H$ (using the Bramble-Hilbert lemma) and the Gagliardo-Nirenberg inequality

$\|v\|_{L^3(\Omega)}^2 \leq C \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}$ ($\dim \Omega \leq 3$), we derive

$$\|u - u^h\|_{H^1(\Omega)} \leq C(h^\ell + \|u - u^h\|_{L^2(\Omega)})$$

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$$\|u - u^h\|_{H^1(\Omega)} \leq C(h^\ell + \|u - u^h\|_{L^2(\Omega)})$$

- **Step 3.** Using the Aubin-Nitche duality and Schatz' compactness argument, we show

$$\|u - u^h\|_{L^2(\Omega)} \leq C(h^{\ell+1} + \|u - u^h\|_{H^1(\Omega)}^2)$$

The idea is to consider the adjoint L^* of the linearized differential operator, i.e. $L\varphi := -\nabla \cdot (a(\cdot, u)\nabla\varphi + \varphi\partial_u a(\cdot, u)\nabla u)$.

The Newton method and the uniqueness of u^h

Newton method for the non-linear FEM. Initial guess $z_0^h \approx u^h$.

$$N_h(z_k^h; z_{k+1}^h - z_k^h, v^h) = F_h(v^h) - (a(z_k^h) \nabla z_k^h, \nabla v^h)_h, \quad \forall v^h \in S_0^\ell(\Omega, \mathcal{T}_h),$$

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Theorem (Convergence of the Newton method)

Under assumptions of theorem, there exist $h_0, \delta > 0$ s.t. if $h \leq h_0$ and $\sigma_h \|z_0^h - u^h\|_{H^1(\Omega)} \leq \delta$, then $\{z_k^h\}$ is well defined, and

$$\|z_{k+1}^h - u^h\|_{H^1(\Omega)} \leq C \sigma_h \|z_k^h - u^h\|_{H^1(\Omega)}^2.$$

($\sigma_h \leq C(1 + |\log h|)^{1/2}$ for $d = 2$, $\sigma_h \leq Ch^{-1/2}$ for $d = 3$).

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($\sigma_h \leq C(1 + |\log h|)^{1/2}$ for $d = 2$, $\sigma_h \leq Ch^{-1/2}$ for $d = 3$).

Proof of the uniqueness of u^h . Given two solutions $u^h, \tilde{u}^h$, apply the Newton method convergence theorem with initial guess $z_0^h := \tilde{u}^h$.

Homogenization nonlinear problems (two scales)

$$\begin{aligned} -\nabla \cdot (a^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon) &= f && \text{in } \Omega, \\ u_\varepsilon &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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Abdulle and V., *Analysis of the finite element heterogeneous multiscale method for quasilinear elliptic homogenization problems*, *Math. Comp.*, 2014.

Fully discrete error analysis

Theorem

Consider u_0 solution of homogenized problem. Assume **(H)**. Then there exist $r_0 > 0$ and $H_0 > 0$ such that if $H \leq H_0$, $r_{HMM} \leq r_0$, any solution u^H of the FE-HMM satisfies

$$\begin{aligned}\|u_0 - u^H\|_{H^1(\Omega)} &\leq C(H^\ell + r_{HMM}) \\ \|u_0 - u^H\|_{L^2(\Omega)} &\leq C(H^{\ell+1} + r_{HMM})\end{aligned}$$

r_{HMM} is analyzed similarly as for linear problems:

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r_{HMM} is analyzed similarly as for linear problems:

$$r_{HMM} \leq C\left(\frac{h}{\varepsilon}\right)^{2q} + r_{MOD}$$

Remark. r_{MOD} modeling error. **We can reuse linear case results:**

If a^ε locally periodic and $\delta = \varepsilon \Rightarrow r_{MOD} = 0$.

Otherwise: a^ε locally periodic with $\delta/\varepsilon \notin \mathbb{N}^*$ with Dirichlet bound.
cond. $\Rightarrow$ resonance errors e.g. $r_{HMM} \leq \delta + \varepsilon/\delta$.

The Newton method: consider a sequence $\{u_k^H\}$ such that

$$\partial B_H(u_k^H; u_{k+1}^H - u_k^H, w^H) = F_H(w^H) - B_H(u_k^H; u_k^H, w^H), \quad \forall w^H \in S_0^\ell(\Omega, \mathcal{T}_H)$$

where the derivative $\partial B_H(z^H; \cdot, \cdot)$ is given by

$$\partial B_H(z^H; v^H, w^H) := B_H(z^H; v^H, w^H) + \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \omega_{K_j} a_{K_j}^0{}'(z^H(x_{K_j})) |v^H(x_{K_j}) \nabla z^H(x_{K_j}) \cdot \nabla w^H(x_{K_j})$$

Implementation: $z_k^H = \sum_{i=1}^{M_{macro}} U_k^i \phi_i^H$, $U_k = (U_k^1, \dots, U_k^{M_{macro}})^T$

$$(B(z_k^H) + B'(z_k^H))(U_{k+1} - U_k) = -B(z_k^H)U_k + F.$$

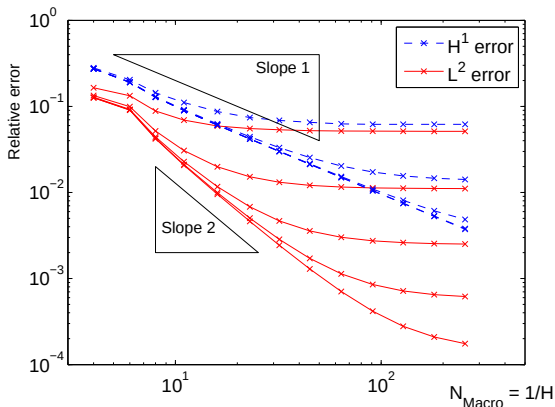
Local contribution to the stiffness matrix: need to consider $B_K(z_k^H)$ and $B'_K(z_k^H)$ which involves the computation of $B'_{K,j}(z^H(x_{K_j}))$

that can be approximated by $(B'_{K,j}(s) \approx \frac{B_{K,j}(s + \sqrt{\epsilon ps}) - B_{K,j}(s)}{\sqrt{\epsilon ps}}).$

Numerical experiment: convergence rates

Example (2D multiscale problem, $\ell = 1$, $q = 1$)

Macro mesh refinement, fixed micro mesh ($4 \times 4, 8 \times 8, 16 \times 16, \dots$)



$$a^\varepsilon(x, s) = \text{diag}\left((2 + \sin(2\pi x_1/\varepsilon))(1 + x_1 \sin(\pi s)), (2 + \sin(2\pi x_2/\varepsilon))(2 + \arctan(s))\right).$$

$$\|u_0 - u^H\|_{H^1(\Omega)} \leq C(H + (h/\varepsilon)^2) \quad \|u_0 - u^H\|_{L^2(\Omega)} \leq C(H^2 + (h/\varepsilon)^2).$$

Coupling with the reduced basis technique

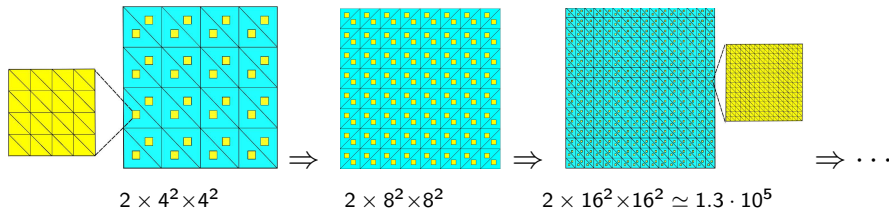
- 1 One-scale nonmonotone problems
- 2 Homogenization nonmonotone problems (two scales)
- 3 Coupling with the reduced basis technique
- 4 Monotone parabolic problems

- Abdulle, Bai and V., *Reduced basis finite element heterogeneous multiscale method for quasilinear elliptic homogenization problems*, *DCDS-S*, 2015.
- Abdulle, Bai and V., *An offline-online homogenization strategy to solve quasilinear two-scale problems at the cost of one-scale problems*, *Int. J. Numer. Meth. Engng*, 2014.

Nonlinear coupling with the RB technique (Abdulle, Bai, V., 2015)

FE-HMM

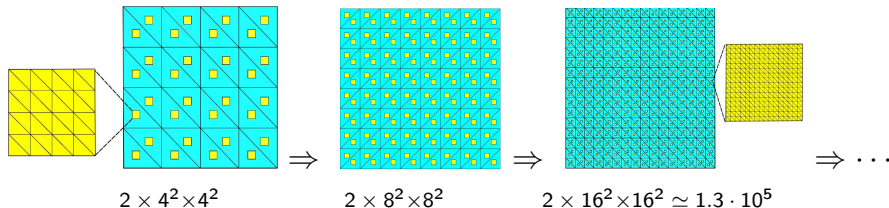
- **Issue 1:** Main cost $\rightarrow$ repeated computation of cell problems with different (x, s) .
- **Issue 2:** Micromesh $\rightarrow$ to be refined at the same rate as macromesh



Nonlinear coupling with the RB technique (Abdulle, Bai, V., 2015)

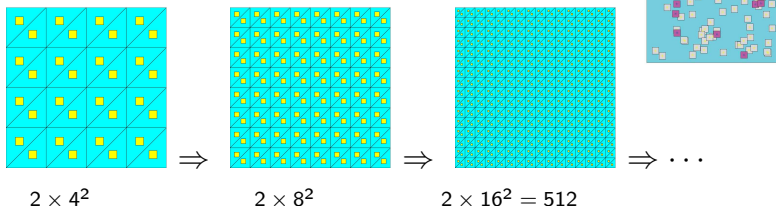
FE-HMM

- **Issue 1:** Main cost $\rightarrow$ repeated computation of cell problems with different (x, s) .
- **Issue 2:** Micromesh $\rightarrow$ to be refined at the same rate as macromesh



Reduced basis FE-HMM (RB-FE-HMM) with a new a posteriori error estimator

- **Offline stage:** Select the reduced basis $\{ \text{pink square}, \text{pink square}, \dots, \text{pink square} \}$
- **Online stage:** Solve standard FEM $\text{yellow square} = \alpha_1 \text{pink square} + \alpha_2 \text{pink square} + \dots + \alpha_n \text{pink square}$



Goal of reduced basis FE-HMM

- Find the best N -dimensional approximation (**Greedy algorithm**) of the space $\mathcal{M}^N$ of micro cell problem solutions,

$$\mathcal{S}_N(Y) = \{\hat{\xi}_{\mathcal{N}}^{i_n, x_n, s_n}, n = 1, \dots, N\}, \quad (x_n, s_n, i_n) \in \Theta_{RB}.$$

- Prove existence, uniqueness, convergence of the numerical homogenization scheme.

Some literature on RB:

- C. Prud'homme, . . . , Maday, Rozza, Patera, Turinici, . . . 2002
- Boyaval, 2008, **reduced-basis approach for homogenization problems**
- Buffa, Maday, Patera, Prud'homme, Turinici, 2012, **a priori error convergence for Greedy algorithm**
- Abdulle, Bai 2012, 2013, **a priori and a posteriori error analysis and goal oriented reduced basis FE-HMM for linear problems**

Setting for the reduced basis FE-HMM

- Parameter space: $\Theta_{RB} = \{(x, s, i)\} = \tilde{\Omega} \times E \times \{1, \dots, d\}$ with compact subsets $\hat{\Omega} \subset \Omega$, $E \subset \mathbb{R}$.
- Parametrized cell problem: For any $x \in \tilde{\Omega}$, consider the affine map $G_x(y) = x + \delta y$ mapping $Y = (-1/2, 1/2)^d$ to a micro sampling domain $K_\delta(x)$, and assume $a^\varepsilon(x, s) = a^\varepsilon(G_x(y), s) \simeq a_{x,s}(y) = \sum_{p=1}^P \Theta_p(x, s) a_p(y)$, $\forall y \in Y$.
- Offline basis function: for $(x, s, i) \in \Theta_{RB}$, consider $\psi^{i,x,s}, \partial_s \psi^{i,x,s}$ s.t.

$$\begin{aligned} \int_Y a_{x,s}(y) (e_i + \nabla \psi^{i,x,s}) \cdot \nabla z^{\hat{h}} dy &= 0, \quad \forall z^{\hat{h}} \in \mathcal{W}(Y) \\ \int_Y a_{x,s}(y) (e_i + \nabla \partial_s \psi^{i,x,s}) \cdot \nabla z^{\hat{h}} dy &= - \int_Y \partial_s a_{x,s}(y) \nabla \psi^{i,x,s} \cdot \nabla z^{\hat{h}} dy \end{aligned}$$

define $S(Y) := \{(\psi^{i,x,s}, \partial_s \psi^{i,x,s}); (x, s, i) \in \Theta_{RB}\} \subset \mathcal{W}(Y)^2$ (space of micro cell functions).

- Greedy selection: Find the best N -dimensional subspace $\mathcal{S}_N(Y)$ of $S(Y)$ (using appropriate a posteriori error estimates).

Offline stage: greedy algorithm and a posteriori error control

Computable Greedy: Find $\eta_{m+1} = (x_{m+1}, s_{m+1}, i_{m+1}) \in \Theta_{RB}$ such that

$$\begin{aligned}\eta_{m+1} &= \arg \max_{(x,s,i) \in \Theta_{RB}} \Delta_m^{x,s,i} \quad (\lambda_{LB} \simeq \lambda \text{ with } 0 < \lambda_{LB} \leq \lambda) \\ &\geq c(\lambda_{LB}) \arg \max_{(x,s,i) \in \Theta_{RB}} \|\psi^{x,s,i} - \Pi_m^{x,s,i}(\psi^{x,s,i}), \partial_s \psi^{x,s,i} - \Pi_m^{x,s,i}(\partial_s \psi^{x,s,i})\|_{\mathcal{W}^2}\end{aligned}$$

with a posteriori error estimator (using Riesz representation of the residual) given by

$$\Delta_I^{i,x,s} := \frac{\|\bar{e}_I^{i,x,s}\|_{\mathcal{W}}}{\sqrt{\lambda_{LB}}} + \frac{\|\partial_s \bar{e}_I^{i,x,s}\|_{\mathcal{W}}}{\sqrt{\lambda_{LB}}}$$

Outputs of interest (for online stage): $a_N^0(x, s)$ computed in $S_N(Y)$ and controlled by a posteriori error estimator.

To guaranty the Newton method convergence, need to control:

- $\|\psi^{x,s,i} - \Pi_m^{x,s,i}(\psi^{x,s,i})\|_{\mathcal{W}}, \|\partial_s \psi^{x,s,i} - \Pi_m^{x,s,i}(\partial_s \psi^{x,s,i})\|_{\mathcal{W}}$ (proj. error),
- $|a_m^0(x, s) - a^0(x, s)| + |\partial_s a_m^0(x, s) - \partial_s a^0(x, s)|$ (output error).

Theorem. (output error) $\leq 3(1 + \frac{\Lambda_1}{\lambda_{LB}}) \Delta_m^{x,s,i} \Delta_m^{x,s,j}$.

Online stage of RB-FE-HMM nonlinear multiscale

Theorem: a priori error estimates

u_0 solution of homogenized problem. Assume (H).

Assume that the parametrized cell solution space $S(Y) \subset \mathcal{W}(Y)^2$ has an exponentially small Kolmogorov N -width,

$$\inf_{\dim W_N = N} \sup_{u \in S(Y)} \inf_{v \in W_N} \|u - v\|_{\mathcal{W}(Y)^2} \leq C e^{-\alpha N}.$$

Then there exist $r_0 > 0$ and $H_0 > 0$ such that if $H \leq H_0$, $r_{HMM} \leq r_0$, any solution u^H of the FE-HMM satisfies

$$\begin{aligned} \|u_0 - u^H\|_{H^1(\Omega)} &\leq C(H^\ell + r_{HMM}) \\ \|u_0 - u^H\|_{L^2(\Omega)} &\leq C(H^{\ell+1} + r_{HMM}) \end{aligned}$$

$r_{HMM} \leq \text{micro err.} + \text{reduced basis err} + \text{mod err.}$

$$r_{HMM} \leq C_1 \left(\frac{h}{\varepsilon}\right)^{2q} + C_2 e^{-\beta N} + r_{MOD}$$

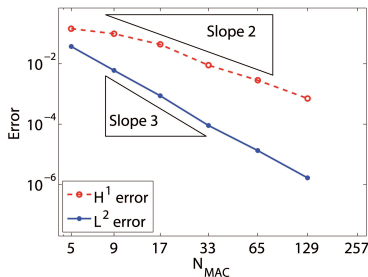
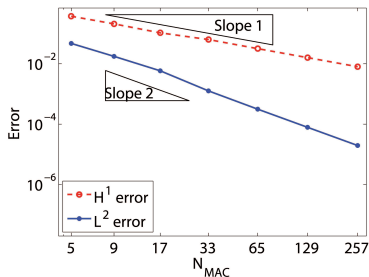
RB-FE-HMM numerical illustration

Mixed boundary cond. and locally periodic nonlinear tensor.

$$a^\varepsilon(x, s) = \begin{pmatrix} (2 + \sin(2\pi x_1/\varepsilon))(1 + x_1 \sin(\pi s)) & 0 \\ 0 & (2 + \sin(2\pi x_2/\varepsilon))(2 + \arctan(s)) \end{pmatrix}$$

Offline procedure

Parameter space	$[0, 1]^2 \times [0.9, 3.66]$
Training set size	4400
Solver	P1 FEM
Mesh	1500×1500
tol_{RB}	$1e-10$
Basis number	9
Offline CPU time	1300



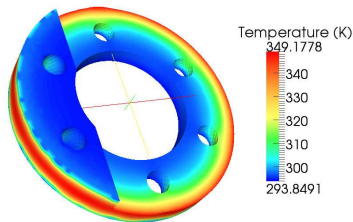
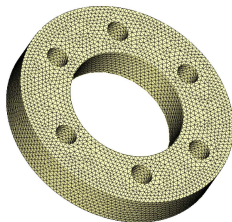
Comparison of FE-HMM and RB-FE-HMM

DOF	RB-FE-HMM offline:1300 (s)			FE-HMM		
	H^1 error	L^2 error	online time (s)	H^1 error	L^2 error	time cost (s)
4×4	3.73e-1	4.71e-2	0.08	3.72e-1	4.81e-2	0.26
8×8	2.09e-1	1.76e-2	0.23	2.08e-1	1.67e-2	1.59
16×16	1.05e-1	5.80e-3	0.90	1.05e-1	5.60e-3	11.49
32×32	6.32e-2	1.30e-3	3.82	6.31e-2	1.20e-3	160.20
64×64	3.16e-2	3.15e-4	19.83	3.16e-2	3.03e-4	2802.68
128×128	1.59e-2	7.89e-5	146.75	1.59e-2	7.61e-5	49260.89

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3D problem with RB-FE-HMM : a disk rotor brake model



Monotone parabolic problems (present work)

$$\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot (a^\varepsilon(x, \nabla u_\varepsilon) \nabla u_\varepsilon) = f \quad \text{in } \Omega \quad + \text{ boundary and init. cond.}$$

- 1 One-scale nonmonotone problems
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- Abdulle and Huber, *Finite element heterogeneous multiscale method for nonlinear monotone parabolic homogenization problems*, preprint, 2014.
- Abdulle, Huber and V., *Linearized numerical homogenization methods for nonlinear monotone parabolic multiscale problems*, preprint, 2014.

Nonlinear and linearized multiscale methods

Given: previous macro state u_n^H at $t = t_n$.

Unknown: New macro state u_{n+1}^H at $t = t_n + \Delta t$.

$$\text{(macro eq.)} \quad \int_{\Omega} \frac{u_{n+1}^H - u_n^H}{\Delta t} w^H dx + B_H(u_{n+1}^H; w^H) = \int_{\Omega} f w^H dx.$$

$$\text{(micro eqs.)} \quad \int_{K_{\delta}} \mathcal{A}^{\varepsilon}(x, \nabla u_{n+1,K}^h) \cdot \nabla q^h dx = 0, \quad \text{for every } K_{\delta}.$$

$\Rightarrow$ Coupled nonlinear problems on macro *and* micro scale;
e.g. nested macro-micro Newton method (Henning, Ohlberger 2012).

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Question: How to increase efficiency for practical problems?

Assumption: Map $\mathcal{A}^{\varepsilon}$ decomposed into $\mathcal{A}^{\varepsilon}(x, \xi) = a^{\varepsilon}(x, \xi)\xi$.

$$\text{(macro map)} \quad B_H = \sum_K \frac{|K|}{|K_{\delta}|} \int_{K_{\delta}} a^{\varepsilon}(\nabla u_{n+1,K}^h) \nabla u_{n+1,K}^h \cdot \nabla w^H dx.$$

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$$\text{(micro eqs.)} \quad \int_{K_{\delta}} a^{\varepsilon}(\nabla u_{n,K}^h) \nabla u_{n+1,K}^h \cdot \nabla q^h dx = 0.$$

$\Rightarrow$ **Simultaneous** linearization of macro and micro equations.

Linearized multiscale method

Space for the macro+micro FE-HMM solutions:

$$\hat{u}_n = (u_n^H, \{u_{n,K}^h\}) \in \mathcal{S}^{H,h} = S_0^1(\Omega, \mathcal{T}_H) \times \prod_{K \in \mathcal{T}_H} S^1(K_\delta, \mathcal{T}_h).$$

Denote $\hat{u}_{n,K}^h = u_n^H + u_{n,K}^h$ (macro + micro solutions).

Given $\hat{u}_n$, compute $\hat{u}_{n+1}$ (at time $t_{n+1} = t_n + \Delta t$) as follows:

- 1 Update the macroscopic state: find $u_{n+1}^H \in S_0^1(\Omega, \mathcal{T}_H)$ solution of the linear problem

$$\int_{\Omega} \frac{1}{\Delta t} (u_{n+1}^H - u_n^H) w^H dx + B_H(\hat{u}_n; u_{n+1}^H, w^H) = \int_{\Omega} f w^H dx;$$

with (macro map)

$$B_H(\hat{u}_n, u_{n+1}^H, w^H) = \sum_K \frac{|K|}{|K_\delta|} \int_{K_\delta} a^\varepsilon(\nabla \hat{u}_{n,K}^h) \nabla \hat{u}_{n+1,K}^h \cdot \nabla w^H dx.$$

- 2 Update micro states: For each $K \in \mathcal{T}_H$, find $u_{n+1,K}^h \in S^1(K_\delta, \mathcal{T}_h)$ solution of the linear micro problem

$$\int_{K_\delta} a^\varepsilon(\nabla \hat{u}_{n,K}^h) \nabla \hat{u}_{n+1,K}^h \cdot \nabla q^h dx = 0.$$

Analysis of the linearized multiscale method

Theorem (Abdulle, Huber, V., 2014)

If u^0 and micro solutions smooth, monotonicity and Lip. cont., and

$$CL_a \|u^0\|_{W^{1,\infty}(\Omega)} < \lambda_a,$$

where λ_a (ellipticity cst.) and L_a (Lipschitz cst. w.r.t. ξ) of $a^\varepsilon(x, \xi)$, then

$$\begin{aligned} (\Delta t \sum_{n=1}^N \|\nabla u_n^H - \nabla u^0(\cdot, t_n)\|_{L^2(\Omega)}^2)^{1/2} &\leq C(\Delta t + H + (\frac{h}{\varepsilon})^2 + e_{mod}), \\ \max_{1 \leq n \leq N} \|u_n^H - u^0(\cdot, t_n)\|_{L^2(\Omega)} &\leq C(\Delta t + H^2 + (\frac{h}{\varepsilon})^2 + e_{mod}). \end{aligned}$$

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$$\max_{1 \leq n \leq N} \|u_n^H - u^0(\cdot, t_n)\|_{L^2(\Omega)} \leq C(\Delta t + H^2 + (\frac{h}{\varepsilon})^2 + e_{mod}).$$

Results:

- well-defined scheme,
- rigorous a priori estimates,
- simple implementation.

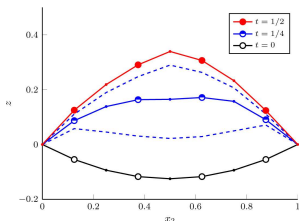
$L^\infty(L^2)$ err. for $a^0 \simeq 1 + (1 + |\xi|^2)^{-0.5}$

Δt	lin.	nonlin.	speed-up
1/16	0.0697	0.0646	$\times 7$
1/64	0.0174	0.0159	$\times 9.5$
1/256	0.0043	0.0040	$\times 10$

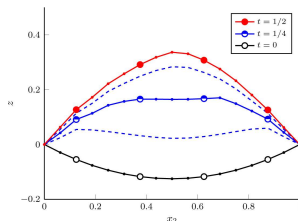
Example with a multiscale p -Laplacian

Domain $\Omega = (0, 1)^2$, $p = 3$. Plot of profiles at $x_1 = 1$ and $t = 0, 1/8, 1/4, 3/8, 1/2$, resp.

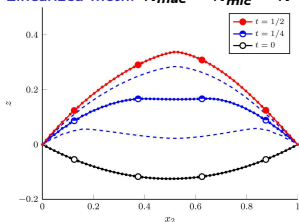
$$a^\varepsilon(x, \xi) = a\left(\frac{x}{\varepsilon}, \xi\right) = \left(\frac{11}{10} + \sin(2\pi(x_1 + x_2)) + \frac{\frac{9}{8} + \sin(2\pi\frac{x_1}{\varepsilon})}{\frac{9}{8} + \cos(2\pi\frac{x_2}{\varepsilon})} + \frac{\frac{9}{8} + \sin(2\pi\frac{x_2}{\varepsilon})}{\frac{9}{8} + \sin(2\pi\frac{x_1}{\varepsilon})} \right) |\xi|^{p-2} Id$$



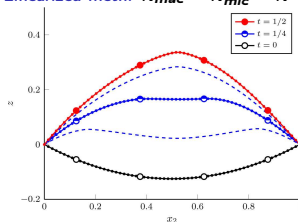
(a) Linearized meth. $N_{mac} = N_{mic} = N = 8$.



(b) Linearized meth. $N_{mac} = N_{mic} = N = 16$.



(c) Linearized meth. $N_{mac} = N_{mic} = N = 64$.



(d) Nonlinear meth. $N_{mac} = N_{mic} = 64$, $N = 256$.

Summary

One-scale nonmonotone elliptic problems:

- Optimal a priori H^1 and L^2 estimates on FEM.
- Newton method convergence and uniqueness of FEM solution (for a sufficiently fine mesh).

Homogenization nonmonotone problems (two-scales):

- Optimal fully discrete error analysis (H^1 and L^2 norms) where both the macro and micro errors are taken into account (mesh sizes H , h have to be refined simultaneously).
- Newton method convergence and uniqueness of FEM solution (for sufficiently fine macro and micro meshes).

Reduced basis: considerably improves the efficiency, reducing the number of DOF.

Linearized scheme for monotone problems: considerably improves the efficiency, avoiding Newton iterations.