Wave splitting
for time-dependent scattered field separation

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Joint work with Marcus J. Grote (Univ. Basel),
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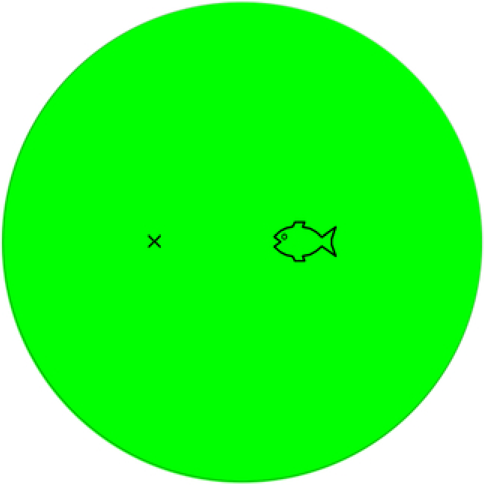
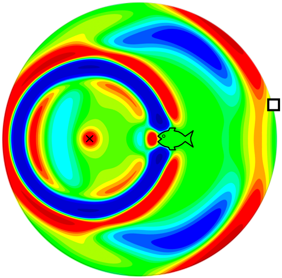
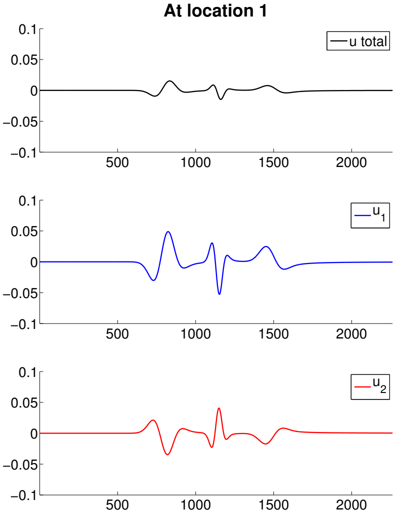
Content

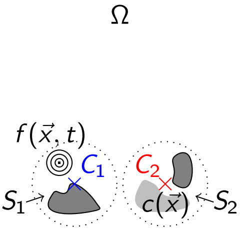


Time history of wave fields at one location: incident wave impinges on a sound-soft inclusion





Multiple scattering problem: $u = u_1 + u_2$, in $\Omega := \mathbb{R}^d \setminus (S_1 \cup S_2)$



u satifies:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \qquad \text{in } \Omega,\ t > 0.$$

Question: Given the measured total field u, can we recover u_1 and u_2 without knowing in advance either of them ?

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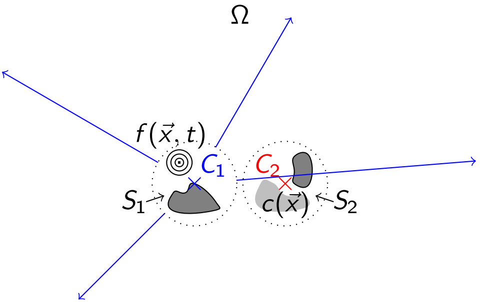

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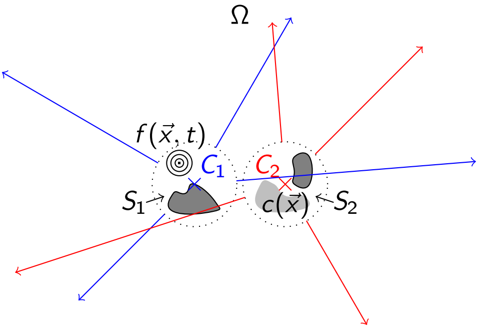

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Outside S_1 and S_2, u satisfies:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \qquad \text{in } \Omega,\ t > 0,$$

$c > 0$ constant.

At $t = 0$, no signal in Ω, then uniqueness of splitting[1]

$$u = {\color{blue}u_1} + {\color{red}u_2} \qquad \text{in } \Omega,\ t > 0$$

and u_k outgoing (3D):

$$u_k(t, r_k, \theta_k, \varphi_k) = \frac{1}{r_k} \sum_{i \geq 0} \frac{f_{k,i}(r_k - ct, \theta_k, \varphi_k)}{(r_k)^i}$$

$(r_k, \theta_k, \varphi_k)$ spherical coordinates centered at C_k.

[1] M. J. Grote and C. Kirsch. Nonreflecting boundary condition for time-dependent multiple scattering. J. Comput. Phys., 221(1):41–67, 2007.

Since

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$(r_k, \theta_k, \varphi_k)$ spherical coordinates centered at C_k,

m^{th}-order absorbing boundary condition[2] on any Γ in Ω

$$B_k[u_k] = O\left(\frac{1}{r_k^{2m+1}}\right), \qquad k = 1, 2$$

[2] A. Bayliss and E. Turkel. Radiation boundary conditions for wave-like equations. Comm. Pure Appl. Math., 33(6):707–725, 1980.

Neglecting the error term:

$$B_j[u_k] \; = B_j[u_k + u_j] = \; B_j[u], \qquad j = 1, 2, \quad k \neq j$$

Recover u_1 and u_2 by solving:

$$\begin{cases} B_2[u_1] = B_2[u] & (1) \\ B_1[u_2] = B_1[u] & (2) \end{cases}$$

where u is known (measurements on Γ)

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Difficulty: integration of partial differential equation (1)-(2) on the submanifold Γ

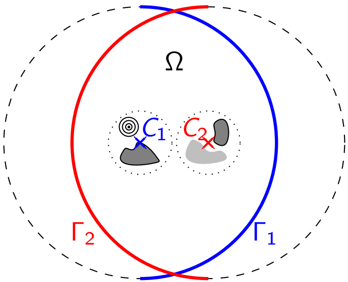
- Find adequate initial and boundary conditions
- Change of coordinates from $(r_k, \theta_k, \varphi_k)$ to $(r_j, \theta_j, \varphi_j)$
- Remove normal/radial derivatives (equation on Γ involving only (t, θ_j, φ_j))

In 2D, Bayliss-Turkel first order absorbing boundary condition

$$B_j[u] \;=\; \frac{1}{c}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r_j} + \frac{u}{2r_j}$$

For simplicity, let $\Gamma := \Gamma_1 \cup \Gamma_2$ with

$$\Gamma_k \;=\; \text{semi-circle centered at } C_k$$



E.g. to recover u_1 on Γ_1 (semi-circle centered at C_1)

$$\begin{aligned} B_2[u_1] &= B_2[u] \\ \frac{1}{c}\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial r_2} + \frac{u_1}{2r_2} &= \frac{1}{c}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r_2} + \frac{u}{2r_2} \end{aligned}$$

How to solve this PDE for u_1?

- need initial and boundary conditions
- remove the radial derivative! we solve on Γ_1
- derivatives in (r_2, θ_2), when domain in (r_1, θ_1)

$\Longrightarrow$ rewrite the PDE using only $\dfrac{\partial}{\partial t}$, $\dfrac{\partial}{\partial \theta_1}$ and 0^{th}-order term

Finally: PDE to recover $f_1 = \sqrt{r_1} u_1$ on Γ_1, $t > 0$

$$\left(\alpha_1(\theta_1)\frac{\partial}{\partial t} + \beta_1(\theta_1)\frac{\partial}{\partial \theta_1} + \gamma_1(\theta_1)\right) f_1 = \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{2r_2}\right) u,$$

with

$$\begin{aligned}
\alpha_1(\theta_1) &= \frac{\sqrt{r_1^2 + \ell^2 - 2r_1\ell\cos(\theta_1)} - r_1 + \ell\cos(\theta_1)}{c\sqrt{r_1}\sqrt{r_1^2 + \ell^2 - 2r_1\ell\cos(\theta_1)}}, \\
\beta_1(\theta_1) &= \frac{\ell\sin(\theta_1)}{r_1\sqrt{r_1}\sqrt{r_1^2 + \ell^2 - 2r_1\ell\cos(\theta_1)}}, \\
\gamma_1(\theta_1) &= \frac{\ell\cos(\theta_1)}{2r_1\sqrt{r_1}\sqrt{r_1^2 + \ell^2 - 2r_1\ell\cos(\theta_1)}},
\end{aligned}$$

and ℓ the signed distance between C_1 and C_2.

We want to recover $f_1 = \sqrt{r_1}u_1$ which satisfies on Γ, $t > 0$

$$\left(\alpha_1(\theta_1)\frac{\partial}{\partial t} + \beta_1(\theta_1)\frac{\partial}{\partial \theta_1} + \gamma_1(\theta_1)\right) f_1 = \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{2r_2}\right) u,$$

Initial condition?
At $t = 0$, no signal in Ω: all sources in $S_1 \cup S_2$

$\Longrightarrow$ f_1 and f_2 vanish in Ω, thus on $\Gamma_1 \cup \Gamma_2$

the initial condition is:

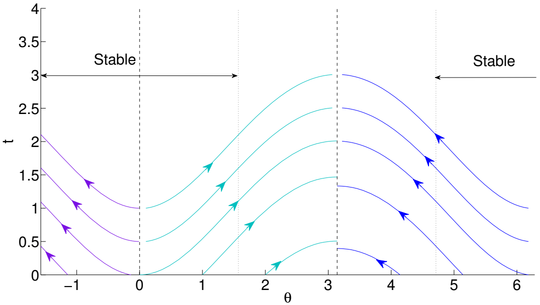
$$f_1 \;=\; 0\,, \qquad \text{on } \Gamma_1, \text{ at } t = 0.$$

Hyperbolic PDE

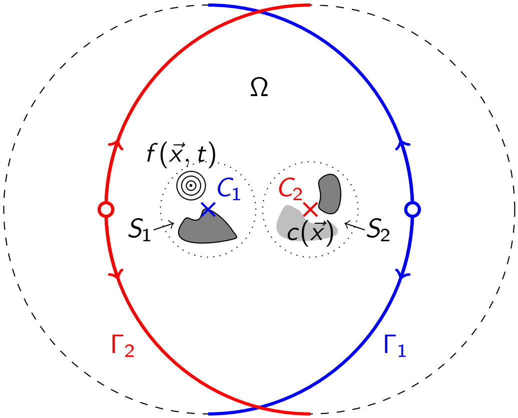
$$\left(\alpha_1(\theta_1)\frac{\partial}{\partial t} + \beta_1(\theta_1)\frac{\partial}{\partial \theta_1} + \gamma_1(\theta_1)\right) f_1 = \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{2r_2}\right) u$$

trivial at $\theta_1 = 0$ or π modulo 2π, since $\alpha_1(\theta_1) = 0, \beta_1(\theta_1) = 0$

$\Longrightarrow$ Dirichlet boundary condition: $f_1 = \dfrac{B_2[u]}{\gamma_1(0)}$ at $\theta_1 = 0$

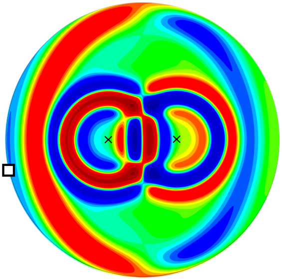


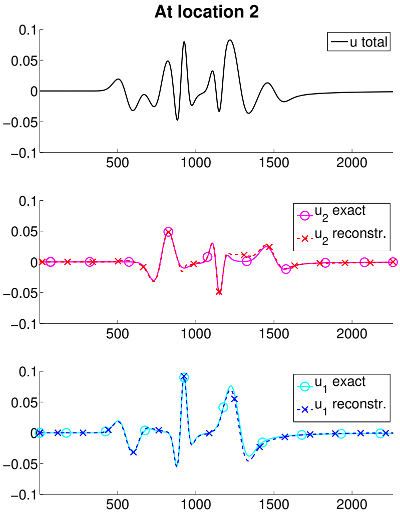
Reconstruction of f_1 (resp. f_2) on Γ_1 (resp. Γ_2)



by subtraction, f_2 (resp. f_1) can be reconstructed on Γ_1 (resp. Γ_2)

Time history of wave fields at one location: two purely radial wave fields generated by point-sources





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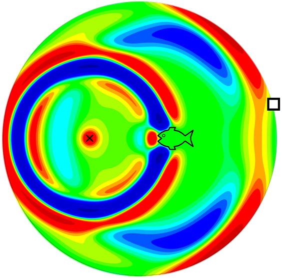

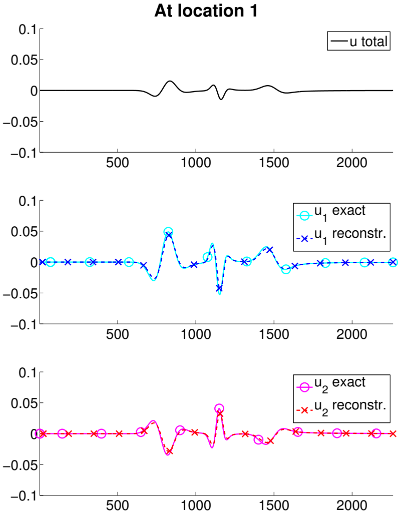

New partial differential equation

- on a submanifold Γ
- local in space and time
- in the time-dependent domain

Method extendable to:

- 2 or more scatterers
- vector-valued wave equations from electromagnetics and elasticity
- improved accuracy with higher order absorbing boundary condition (more terms in the progressive wave expansion)

M.J. Grote, M. Kray, F. Nataf and F. Assous.
Wave splitting for time-dependent scattered field separation. *C. R. Acad. Sci. Paris, Serie I* (2015)