An Adaptive Local Basis for Elliptic Problems with Complicated Discontinuous Coefficients

Monika Weymuth

Universität Zürich, Institut für Mathematik

joint work with Stefan Sauter

Swiss Numerical Analysis Day
University of Geneva, 17th April 2015

Model Problem

$\Omega \subset \mathbb{R}^d$ ($d \in \{2,3\}$) a bounded Lipschitz domain with piecewise analytic boundary

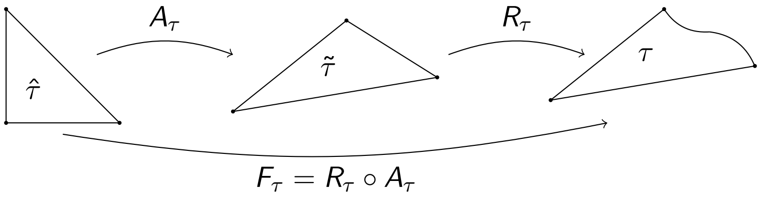
For a given function $f \in L^2(\Omega)$, we are seeking $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \langle A\nabla u, \nabla v\rangle = \int_\Omega fv =: F(v) \qquad \forall v \in H_0^1(\Omega).$$

The diffusion matrix $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$ is uniformly elliptic, i.e.

$$0 < \alpha(A,\Omega) := \operatorname*{ess\,inf}_{x\in\Omega} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle A(x)v,v\rangle}{\langle v,v\rangle}$$
$$\infty > \beta(A,\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle A(x)v,v\rangle}{\langle v,v\rangle}.$$

The Triangulation $\mathcal{G}$



Assumption (quasi-uniform regular triangulation)

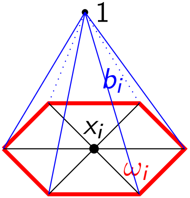
Each element map F_τ can be written as $F_\tau = R_\tau \circ A_\tau$, where A_τ is an affine map (containing the scaling by h_τ) and R_τ is an h_τ-independent analytic map. Let $\tilde{\tau} := A_\tau(\hat{\tau})$. The maps R_τ and A_τ satisfy for shape regularity constants C_{affine}, C_{metric}, $\gamma > 0$ independent of H:

$$\|A'_\tau\|_{L^\infty(\hat{\tau})} \leq C_{affine} H, \qquad \|(A'_\tau)^{-1}\|_{L^\infty(\hat{\tau})} \leq C_{affine} H^{-1}$$

$$\|(R'_\tau)^{-1}\|_{L^\infty(\tilde{\tau})} \leq C_{metric}, \qquad \|\nabla^n R_\tau\|_{L^\infty(\tilde{\tau})} \leq C_{metric} \gamma^n n! \quad \forall n \in \mathbb{N}_0.$$

Approximation Space and Basis Functions

$S := \{u \in H_0^1(\Omega) \mid \forall \tau \in \mathcal{G} : u|_\tau \circ F_\tau \in \mathbb{P}_1\}$



$(b_i)_{i=1}^n$ usual $\mathbb{P}_1$ basis

$\omega_i := \operatorname{supp} b_i$

conforming Galerkin method: Find $u_S \in S$ such that

$$a(u_S, v) = F(v) \quad \forall v \in S.$$

If A, f and Ω are sufficiently smooth such that the problem is H^2-regular, then

$$\|u - u_S\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)}.$$

Goal:

Construct $V_{AL} \subset H_0^1(\Omega)$ such that the linear convergence rate is preserved for heterogeneous and/or highly oscillatory coefficients.

Partition of Unity Finite Element Method (PUM)

[Babuška, Melenk 1997]

1. Let $\omega_1, \ldots, \omega_N$ be open sets satisfying
   - $\Omega = \bigcup_{i=1}^{N} \omega_i$
   - $\exists M \in \mathbb{N}$ s.t. $\forall x \in \Omega$ we have $\#\{i \mid x \in \omega_i\} \leq M$
2. Construct local approximation spaces V_i, i.e. on each patch ω_i let $V_i \subset H_0^1(\Omega)|_{\omega_i}$ be a space of functions by which the solution $u|_{\omega_i}$ can be approximated well.
3. Choose partition of unity functions $(\varphi_i)_{i=1}^{N}$ such that
   - $\operatorname{supp} \varphi_i \subset \overline{\omega_i}$
   - $\sum_i \varphi_i = 1$ on Ω
   - $\|\varphi_i\|_{L^\infty(\Omega)} \leq C$
   - $\|\nabla \varphi_i\|_{L^\infty(\Omega)} \leq \frac{C}{\operatorname{diam} \omega_i}$
4. Define the global finite element space V by

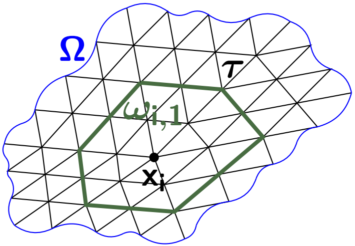
$$V := \sum_i \varphi_i V_i.$$

Let $L_\Omega^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ denote the solution operator: Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \langle A\nabla u, \nabla v\rangle = \int_\Omega fv =: F(v).$$

1. $V_{AL} := \operatorname{span}\{L_\Omega^{-1}(\chi_\tau) \mid \tau \in \mathcal{G}\}$
   – global basis functions
   – very large overlap
   – not computable

2. $V_i := \operatorname{span}\{b_i L_\Omega^{-1}(\chi_\tau) \mid \tau \in \mathcal{G}\}$
   $V_{AL} := V_1 + V_2 + \cdots + V_n$
   + local basis functions
   – very large overlap
   – not computable



$V_i^{near} := \operatorname{span}\{b_i L_\Omega^{-1}(\chi_\tau) \mid \tau \subset \omega_{i,1}\}$
$X_i^{far} := \operatorname{span}\{L_\Omega^{-1}(\chi_\tau)|_{\omega_{i,1}} \mid \tau \subset \Omega \backslash \omega_{i,1}\}$
$\tilde{V}_i^{far}$ low dim. approximation of X_i^{far}
$V_i^{far} := \{b_i v \mid v \in \tilde{V}_i^{far}\}$

3. $V_i := V_i^{near} + V_i^{far}$
   $V_{AL} := V_1 + V_2 + \cdots + V_n$

+ local basis functions
+ small overlap
– not computable

4. $L_\Omega^{-1} \leftarrow L_{\Omega,h}^{-1}$

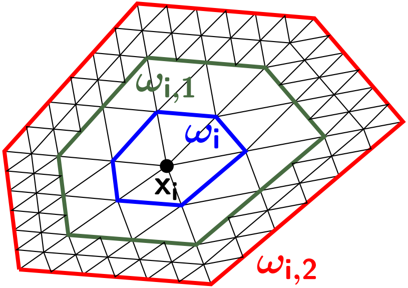
+ local basis functions
+ small overlap
+ computable
– not efficient

The Mesh and Local Solution Operator

$\omega_{i,0} := \omega_i = \text{supp}\, b_i$
$\omega_{i,j+1} := \bigcup\{\bar{\tau} \mid \tau \in \mathcal{G} \text{ and } \omega_{i,j} \cap \bar{\tau} \neq \emptyset\}$
$\omega_i^{far} := \omega_{i,2} \backslash \overline{\omega_{i,1}}$

$\mathcal{G}_{i,j} := \{\tau \in \mathcal{G} \;:\; \tau \subset \omega_{i,j}\}$
$\mathcal{G}_i^{far} := \mathcal{G}_{i,2} \backslash \mathcal{G}_{i,1}$

$\mathcal{R}^1(\mathcal{G}_i^{far})$ refinement of $\mathcal{G}_i^{far}$
$\mathcal{R}^t(\mathcal{G}_i^{far}) := \mathcal{R}^1(\mathcal{R}^{t-1}(\mathcal{G}_i^{far}))$



For a subdomain $\omega_{i,2} \subset \Omega$ let $L^{-1}_{\omega_{i,2}} : L^2(\omega_{i,2}) \to H^1_0(\omega_{i,2})$ denote the local solution operator: Given $g \in L^2(\omega_{i,2})$, find $u \in H^1_0(\omega_{i,2})$ such that

$$a(u,v) := \int_{\omega_{i,2}} \langle A\nabla u, \nabla v\rangle = \int_{\omega_{i,2}} gv =: G(v).$$

Construction of Local PUM Spaces I

$S_H := \mathrm{span}\{\chi_\tau \mid \tau \in \mathcal{G}\}, \qquad S_H \subset S_h \subset H_0^1(\Omega)$
$S_{i,2,h} := \{u|_{\omega_{i,2}} \mid u \in S_h \wedge \mathrm{supp}\, u \subset \omega_{i,2}\}$

Nearfield ($\tau \in \mathcal{G}_{i,1}$):
Find $\tilde{B}_{i,\tau} \in S_{i,2,h}$ such that

$$\int_{\omega_{i,2}} \langle A\nabla \tilde{B}_{i,\tau}, \nabla v\rangle = \int_{\omega_{i,2}} \chi_\tau v \quad \forall v \in S_{i,2,h}.$$

$B_{i,\tau} := b_i \tilde{B}_{i,\tau} \qquad V_i^{near} := \mathrm{span}\{B_{i,\tau} \mid \tau \in \mathcal{G}_{i,1}\}$

Farfield ($\tau \in \mathcal{R}^t(\mathcal{G}_i^{far})$, $t \sim \log \frac{1}{H}$):
Find $\tilde{B}_{i,\tau} \in S_{i,2,h}$ such that

$$\int_{\omega_{i,2}} \langle A\nabla \tilde{B}_{i,\tau}, \nabla v\rangle = \int_{\omega_{i,2}} \chi_\tau v \quad \forall v \in S_{i,2,h}.$$

Construction of Local PUM Spaces II

$X_i^{far} := \text{span}\{\tilde{B}_{i,\tau}|_{\omega_{i,1}} \; : \; \tau \in \mathcal{R}^t(\mathcal{G}_i^{far})\}$

The functions in X_i^{far} are locally L-harmonic on $\omega_{i,1}$, i.e. any $v \in X_i^{far}$ satisfies

$$\int_{\omega_{i,1}} \langle A\nabla v, \nabla w\rangle = 0 \quad \forall w \in S_{i,1,h} := \{w \in S_h \; : \; \text{supp}\, w \subset \omega_{i,1}\}.$$

X_i^{far} can be approximated by a low dimensional space. Let $\tilde{V}_i^{far}$ be the low dimensional approximation of X_i^{far}. Then we set

$$V_i^{far} := \{b_i v \,|\, v \in \tilde{V}_i^{far}\}.$$

Finally we define

$$V_{AL} := (V_1^{near} + V_1^{far}) + \cdots + (V_n^{near} + V_n^{far}).$$

Approximation of X_i^{far}

Lemma (Bebendorf, Hackbusch 2003)

Let $D \subset \Omega$ and $X(D)$ the space of locally L-harmonic functions on D. Furthermore, let $D_2 \subset D$ be a convex domain such that

$$\operatorname{dist}(D_2, \partial D) \geq \operatorname{diam}(D_2) > 0.$$

Then for any $M > 1$ there is a subspace $W \subset X(D_2)$ so that

$$\inf_{w \in W} \|u - w\|_{L^2(D_2)} \leq \frac{1}{M} \|u\|_{L^2(D)} \quad \forall u \in X(D)$$

and

$$\dim W \leq c^d \lceil \log M \rceil^{d+1} \tag{1}$$

where d is the spatial dimension and c only depends on α and β.

Error Analysis I

1. Let $S := \operatorname{span}\{\chi_\tau \,|\, \tau \in \mathcal{G}\}$.

$$\|L_\Omega^{-1} f - L_\Omega^{-1} P_S f\|_{H^1(\Omega)} \leq \frac{C}{\alpha} \|f - P_S f\|_{H^{-1}(\Omega)} \leq \frac{C}{\alpha} H \|f\|_{L^2(\Omega)},$$

   where $P_S : L^2(\Omega) \to S$ is the L^2-orthogonal projection onto S.

2.
$$u_{AL} = \sum_{i=1}^{n} b_i L_\Omega^{-1} P_S f = L_\Omega^{-1} P_S f$$

3.
$$f_i^{near} := \sum_{x_j \in \mathring{\omega}_{i,1}} (P_S f)(x_j) b_j \quad \text{and} \quad f_i^{far} := \sum_{x_j \in \mathring{\Omega} \setminus \mathring{\omega}_{i,1}} (P_S f)(x_j) b_j$$

$$L_\Omega^{-1} P_S f = \sum_{i=1}^{n} \underbrace{b_i L_\Omega^{-1} f_i^{near}}_{u_i^{near}} + \sum_{i=1}^{n} b_i \underbrace{L_\Omega^{-1} f_i^{far}}_{u_i^{far}}.$$

Error Analysis II

Lemma (Caccioppoli inequality)

Let $u \in X(D)$ and let $K \subseteq D$ be a domain with $\operatorname{dist}(K, \partial D) > 0$. *Then we have $u|_K \in H^1(K)$ and*

$$\|\nabla u\|_{L^2(K)} \leq \sqrt{\frac{\beta}{\alpha}} \frac{4}{\operatorname{dist}(K, \partial D)} \|u\|_{L^2(D)}.$$

There exists $\tilde{u}_i^{far} \in \tilde{V}_i^{far}$ such that

$$\|u_i^{far} - \tilde{u}_i^{far}\|_{H^m(\omega_i)} \leq CH^{3-m} \|\nabla L_\Omega^{-1} f_i^{far}\|_{L^2(\omega_{i,1})} \quad m = 0, 1$$

[cf. Bebendorf, Hackbusch 2003; Börm 2010; Grasedyck, Greff, Sauter 2012].

$$u_{AL} := \sum_{i=1}^{n} u_i^{near} + \sum_{i=1}^{n} b_i \tilde{u}_i^{far} \in V_{AL}$$

Error Analysis III

$$\begin{aligned}\|u - u_{AL}\|_{H^1(\Omega)} &\leq \|u - L_\Omega^{-1} P_S f\|_{H^1(\Omega)} + \|L_\Omega^{-1} P_S f - u_{AL}\|_{H^1(\Omega)} \\ &\leq CH\|f\|_{L^2(\Omega)} + \left\| \sum_{i=1}^{n} b_i (u_i^{far} - \tilde{u}_i^{far}) \right\|_{H^1(\Omega)}\end{aligned}$$

4. Replace L_Ω^{-1} by $L_{\Omega,h}^{-1}$.

Assumption

Let $f \in H^{p-1}(\Omega)$ for some $p \in \mathbb{N}$.

$$\sup_{f \in H^{p-1}(\Omega) \setminus \{0\}} \inf_{v \in S_h} \frac{\|L_\Omega^{-1} f - v\|_{H^1(\Omega)}}{\|f\|_{H^{p-1}(\Omega)}} \leq CH$$

Céa's lemma implies

$$\|L_\Omega^{-1} f - L_{\Omega,h}^{-1} f\|_{H^1(\Omega)} \leq CH\|f\|_{L^2(\Omega)}.$$

Main Result

Assumption

$$\sup_{f\in L^2(\omega_{i,2})\setminus\{0\}} \inf_{v\in S_{i,2,h}} \frac{\|L_{\omega_{i,2}}^{-1} f - v\|_{H^1(\omega_{i,2})}}{\|f\|_{L^2(\omega_{i,2})}} \leq CH_i^2,$$

where H_i denotes the mesh width of the mesh $\mathcal{G}_{i,2}$.

The control parameters for the AL-basis can be chosen in such a way that for every $f \in L^\infty(\Omega)$ and $\frac{\alpha}{\beta} \in [1 - C, 1)$ with $C = O(1)$ the estimate

$$\|u - u_{AL}\|_{H^1(\Omega)} \leq CH\|f\|_{L^\infty(\Omega)}$$

holds and for the dimension we have

$$\dim V_{AL} \leq CH^{-d} \log^{d+1} \frac{1}{H}.$$

Thank you for your attention!