Ramsey Equilibrium with Liberal Borrowing*

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Abstract

This paper considers a multi-agent one-sector Ramsey equilibrium growth model with borrowing constraints. The extreme borrowing constraint used in the classical version of the model, surveyed in Becker (2006), and the limited form of borrowing constraint examined in Borissov and Dubey (2015) are relaxed to allow more liberal borrowing by the households. A perfect foresight equilibrium is shown to exist in this economy. We describe the steady state equilibria for the liberal borrowing regime and show that as the borrowing regime is progressively liberalized, the steady state wealth inequality increases. Unlike the case of a limited borrowing regime, an equilibrium path need not converge in the case of liberal borrowing regime. We show through an example that a two period cyclic equilibrium exists when agents are allowed to borrow against their two period future wage income. This result is similar to the possibility of non-convergent equilibrium capital stock sequences in the model with no borrowing.

Keywords: `Convergence, Existence, Gini Coefficient, Growth, Heterogeneous agent, Liberal borrowing, Turnpike property.`

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1 Introduction

In this paper we consider a discrete time economic growth model with one capital good and finitely many infinitely-lived heterogeneous households in an infinite time horizon framework and focus on the savings choices available to the households.

In the standard Ramsey economy, the households are not allowed to borrow against their future wage incomes. Becker (1980) formalized the discrete time version of the economy originally described in continuous time framework in Ramsey (1928). It led to a proliferation of research on the Ramsey model and continues to receive wide attention in macroeconomic growth literature[1]. Becker (2006) contains the main results of this literature. The borrowing constraints on the households' consumption - saving choices imply that the markets are not complete.

The no borrowing models based on Becker (1980) identify household savings and physical capital holdings. The capital stock must be non-negative at each time. In the liberal borrowing model proposed here savings include not only the capital stock, but also household debts. Therefore, the non-negativity of capital in the no borrowing model becomes a non-negativity constraint on the sum of each household's capital stock and its outstanding debt. The limit on a household's debt is determined endogenously in an equilibrium and it is always finite. We assume an exogenous maximum number of periods before outstanding loans must be repaid.

Alternative borrowing regimes help us understand the wealth distribution in the society driven solely by time preference and the borrowing constraints. It thus provides an opportunity to investigate the basic idea in Fisher (1930) that time preference based interactions with the loan market acts as a redistributive mechanism.

Incomplete markets economies potentially may generate inefficient aggregate allocations on equilibrium paths. However, Becker and Mitra (2012) show that if a Ramsey equilibrium path satisfies the *turnpike property*[2], then it satisfies the transversality condition of Malinvaud (1953) and is therefore inter-temporally *efficient* in terms of the aggregate consumption stream that it provides despite the presence of borrowing constraints. Becker (2006) points out that turnpike property does not hold in general but does hold only if the capital stock sequence converges to the steady state stock. An example by Michael L. Stern, reported in Becker (2006), shows that without additional assumptions about technology and / or preferences, this turnpike property does not hold[3].

The turnpike property is an asymptotic property by definition. It places restrictions on the Ramsey equilibrium itself, and is not easily verifiable, given the primitives of the model (i. e., the specification of production and utility functions, and the agents' discount factors). In order to overcome the difficulty in checking the turnpike property, efforts were made to explore the sufficient condi-

[1] See Sorger (1994), Le Van and Vailakis (2003), Le Van et al. (2007), Borissov (2011), Becker (2012), Becker and Mitra (2012), Nishimura et al. (2013), Mitra and Sorger (2013), Becker et al. (2014), Borissov and Dubey (2015), and Nishimura et al. (2015), for recent contributions.

[2] In the context of the Ramsey equilibria, the *turnpike property* states that eventually the most patient household owns the entire capital stock of the economy whereas remaining households eventually reach the zero capital stock ownership state and maintain that state thereafter.

[3] There is only one general result on the long-run behavior of Ramsey equilibria established in Becker and Foias (1987) known as the recurrence theorem: the zero capital state is recurrent for every household other than the most patient one.

tions on the equilibrium paths which would ensure that the turnpike property holds. In this regard, important results by Becker and Foias (1987) show that if the capital stock sequence converges in equilibrium, then the turnpike property holds. Sufficient conditions to ensure the convergence of the capital input stock sequence would therefore lead to the Ramsey equilibrium satisfying the turnpike property and efficiency with respect to the aggregate consumption stream. However, Becker and Foias (1987) proved by an example that the equilibrium capital stock sequence might not converge to the stationary equilibrium stock without further restrictions on their basic assumptions.

Becker and Foias (1987) came up with the first set of sufficient condition for the convergence of the capital stock sequence known as *Capital Income Monotonicity*. If the production function is such that the capital income is monotone increasing in the capital stock, then the turnpike property holds.

Monotonicity of the capital income turns out to be a rather strong assumption on the production function. Therefore, attempts were made in recent literature to seek other alternative / weaker conditions on the fundamentals of the model which guarantee the convergence of capital sequence along equilibrium path. In an unpublished paper, Borissov (2011) modifies the timing of wage payments (he considers a discrete-time model under the assumption that wages are paid *ante factum*) and proves the convergence property. Mitra and Sorger (2013) investigate a continuous time version of the Ramsey economy as was proposed originally by Ramsey (1928) and show that the turnpike property holds in every Ramsey equilibrium[4]. Becker et al. (2014) weaken the capital income monotonicity condition for the discrete-time Ramsey model to the monotonicity of the maximal income that any household can have. Borissov and Dubey (2015) relaxes the no borrowing condition by letting the households to be able to borrow against their next period wage income.

In this paper we take a more comprehensive view of the relaxed borrowing (we term it as liberal borrowing regime) by allowing the households to borrow against their future wage incomes for finitely many ($N \in \mathbb{N}$) time periods [5]. Thus our paper extends the line of enquiry initiated in Borissov and Dubey (2015). The borrowing constraint would take the form of

$$s_t^j + \frac{w_{t+1}}{\left(1 + r_{t+1}\right) J} + \cdots + \frac{w_{t+N}}{\left(1 + r_{t+1}\right) \cdots \left(1 + r_{t+N}\right) J} \geq 0.^6$$

We prove (a) the existence of an equilibrium in the Ramsey economy with liberal borrowing; (b) the existence of a unique stationary equilibrium, and (c) the existence of a period two equilibrium which is efficient. The question of what, if any additional restrictions might yield convergence of an equilibrium to the long-run steady state remains an open problem. We note that IF the equilibrium

[4]They also demonstrate that every Ramsey equilibrium in their model is inter-temporally efficient.

[5]This exercise could be one way of approaching the complete markets as the finite time horizon for the borrowing is extended to the borrowing against the life time incomes. Bewley (1982) considers such an economy with complete markets, where households can borrow against their life time wage incomes and shows that in equilibrium, the consumption of all households other than the most patient one is zero after some finite time in every equilibrium.

[6]In this expression, $J > 1$ is the number of households, s_j^t is savings for period t for household j, $\frac{w_t}{J}$ is the wage income for every household in period t, $\left(1 + r_t\right)$ is the rental rate in time period t and $\frac{w_{t+k}}{\left(1+r_{t+1}\right)\cdots\left(1+r_{t+k}\right)J}$ is the present (time t) value of the wage income in period $t + k$.

aggregate capital stock sequence converges to the steady state in our liberal borrowing regime, then the turnpike property holds in our liberal borrowing framework and that equilibrium is also efficient.

The proof of the existence of an equilibrium is in three steps and is along the lines of the existence result in the case of the limited borrowing regime in Borissov and Dubey (2015) [7]. First we devise a simultaneous move generalized game with a finite number of players to represent the finite time periods Ramsey economy with liberal borrowing and use a theorem by Debreu (1952) to show the existence of a Nash equilibrium in this game. In the second step we show that this Nash equilibrium is also an equilibrium in the finite time horizon economy. Finally, we apply a process similar to Cantor's diagonalization argument to the sequence of finite time periods equilibria to prove the existence of equilibrium in the infinite-horizon Ramsey economy [8].

Having established the existence of an equilibrium, we show that there is a unique stationary equilibrium in the Ramsey economy with liberal borrowing, in which all households except the most patient one are in the maximum borrowing state, whereas the most patient household owns entire capital stock and all debts of the other households.

Our analysis also provides an opportunity to investigate the role of different borrowing regimes on the consumption inequality in the society. While it is true that the aggregate steady state consumption is independent of N as it depends only upon technology and the most patient agent's pure rate of time preference, a feature which unifies the no-borrowing aggregate steady state consumption (see Becker (1980)) with the liberal borrowing case, the steady state consumption of the impatient households declines with increases in N. It implies that the patient household's consumption rises with N and approaches the aggregate consumption as N tends to infinity. Therefore, in the steady state the wealth is redistributed from the impatient households to the patient household as the credit regime is liberalized. In other words, the steady state consumption distribution in Becker (1980) Lorenz dominates the steady state distribution in our paper, implying a lower Gini coefficient in Becker (1980) for the distribution of consumption than in the current setup. Hence the steady state consumption Gini coefficient is increasing with N and approaches the long-run consumption distribution found in Bewley (1982) with complete markets. Our results corroborate Fisher's observation that the time preference based interactions with the loan market act as a redistributive mechanism[9].

Last, we turn our attention to dynamic properties of the Ramsey equilibria. Following techniques of proof in Becker and Foias (1987), we first show that in every equilibrium with a convergent capital stock sequence the following version of the turnpike property holds: from some time onward

[7] See Becker et al. (2015b) and Bosi et al. (2014) for an alternative finite horizon equilibrium approach and limiting argument in the no borrowing constraint case. Becker et al. (2015b) allow elastic labor supply in the Ramsey model and prove existence of equilibrium. The elastic labor supply assumption has received considerable attention in recent literature on Ramsey models, see, for example, Bosi and Seegmuller (2010), Kamihigashi (2015).

[8] Our existence result allows the possibility of the household's felicity function to be unbounded below and thus extends the results in Borissov and Dubey (2015). Thus we have established existence of equilibrium in the case of felicity function $u(c_t) = \ln c_t$. In the no - borrowing case, such possibilities have been taken care of in Becker et al. (1991).

[9] We consider the discount factors for households as parameters. If the discount factors are endogenous in the spirit of Borissov and Lambrecht (2009), then less stringent borrowing constraints and hence higher inequality will lead to more savings if initially inequality is low and to less savings if initially inequality is high.

the most patient household owns all the capital and the debts of all other households, whereas the latter eventually attain the maximum borrowing position and stay in that position thereafter. However, an example shows that a two-period cyclic equilibrium can exist. Moreover, that equilibrium is efficient and the turnpike property holds (by construction).

The rest of the paper is organized as follows. Section 2 introduces the model. In section 3 we define an equilibrium for this economy and state the existence of equilibrium result. Section 4 contains a description of the unique stationary equilibrium. In section 5, we prove the convergence of the capital stock sequence implies the limit point must be the stationary equilibrium and the turnpike property obtains. We also construct a two-period cyclic equilibrium counterexample to the general hypothesis that the equilibrium aggregate stocks form a convergent sequence. We conclude in section 6. The proof of the existence of equilibrium and results reported in Section 5 are contained in the Appendix[10].

2 Ramsey Economy with liberal borrowing

2.1 Firms

Firms produce output using one capital good and one unit of labor input. The production technology transforms labor and the capital goods into a composite good that can be either consumed or invested as the next period's capital goods input. The amount of labor is fixed in this economy. There is one unit of labor services in the aggregate in this economy and all labor services are assumed to be identical.

The technology is summarized by a *production function*, denoted by f. Let $y = f(k)$ denote the composite good y produced from one unit of labor (whose value is suppressed in the notation), together with the non-negative capital input k. Capital is assumed to depreciate completely within one period[11]. Hence, the model is formally one with circulating capital that is consumed within each production period. The output y is available for consumption or capital accumulation. The formal properties of f are:

Assumption 1. *The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, increasing and concave on $\mathbb{R}_+$ and satisfies $f(0) = 0$. Also, f is twice continuously differentiable on $\mathbb{R}_{++}$, with*

$$f'(\cdot) > 0,\ f''(\cdot) < 0 \text{ on } \mathbb{R}_{++},\ \lim_{k\to 0} f'(k) = \infty, \text{ and } \lim_{k\to\infty} f'(k) = 0.$$

This assumption implies the existence of a maximum sustainable capital stock, denoted by $\overline{K}$, satisfying $\overline{K} = f\left(\overline{K}\right) > 0$.

[10]The Appendix A is available in the working paper, Becker et al. (2015a, CAEPR Working Paper # 2015-003), online.

[11]It is a simplifying assumption in the standard Ramsey models as has been observed in Becker (2006, Footnote 30): "This assumption simplifies the presentation. Depreciated at a fixed rate is easily incorporated into the productive technology.". Similar observation holds true in our paper as well.

2.2 Households

There are $J > 1$ households indexed by $j = 1, \ldots, J$. Let c_t^j and s_t^j denote the consumption and savings of household j at time t respectively. Households' preferences assume time additively separable utility functions with fixed discount factors. Household j has *felicity function* $u_j(c)$ and discounts future utilities by the factor δ_j with $0 < \delta_j < 1$. Hence, the household's lifetime utility function is specified by $\sum_{t=0}^{\infty} \delta_j^t u_j\left(c_t^j\right)$. The felicity functions satisfy the following properties.

Assumption 2. *For each j, $u_j : \mathbb{R}_{++} \to \mathbb{R}$ is twice continuously differentiable and concave with*

$$u_j'(\cdot) > 0,\ u_j''(\cdot) < 0, \ \ \textit{and} \lim_{c \to 0} u_j'(c) = \infty.$$

We focus on the case where the first household's discount factor is larger than every other households' discount factors. Assumption 3 orders households from the most patient to the least patient.

Assumption 3. *For each household j, the discount factor $\delta_j \in (0, 1)$ is such that*

$$1 > \delta_1 > \delta_2 \geq \ldots \geq \delta_J > 0.$$

2.3 Liberal Borrowing Constraint

A general complete market competitive one-sector model treats household's budget constraints as restricting the present value of the household's consumption to be smaller than or equal to the initial wealth defined as the capitalized wage income plus the initial savings. This allows us to interpret the choice of a consumption stream as if the household is allowed to borrow and lend at market determined present value prices subject to repaying all loans. Markets are complete - any intertemporal trade satisfying the present value budget constraint is admissible at the individual household level.

In this paper, we introduce liberal borrowing in the following manner. Suppose that at time t, based on the time $t-1$ savings, which are not assumed to be non-negative, and the wage income accrued from working at time t, the households' total incomes are realized. Then, given their total incomes, households make the consumption - savings choices. Again, when making these choices, they are not prohibited from borrowing (equivalently having negative savings). They are allowed to borrow against the wage they will earn at times $t+1, t+2, \cdots, t+N$ where $N \in \mathbb{N}$. This borrowing can take place at the market determined rental rate. It is convenient to introduce following notation. For any $t+1, N \in \mathbb{N}$, we define

$$A(t+1, N) \equiv \frac{w_{t+1}}{\left(1 + r_{t+1}\right)} + \cdots + \frac{w_{t+N}}{\left(1 + r_{t+1}\right) \cdots \left(1 + r_{t+N}\right)} = \sum_{n=1}^{N} \frac{w_{t+n}}{\prod_{s=1}^{n} \left(1 + r_{t+s}\right)}.$$

It is possible to construct a trading institution for matching savings and aggregate capital as the households may not want to track individual debts in the sense of figuring out what each households owe to the other households in the economy. So, we imagine a costless clearinghouse (a very

Walrasian idea) that aggregates savings and matches them with aggregate capital. The clearinghouse also packages loans in the form of annuities due over the payback or loan horizon, N, with variable discount rates and "coupons". This "security" represents the maximum debt the economy can carry at any time. Its per capita expression is $\frac{A(t+1,N)}{J}$ which enters each agents borrowing constraint. That is, an agent can never be indebted to the clearinghouse by more than this amount. The clearinghouse passes through the repayment of this debt to the holders of the loans. It is important to note that there is a fairly simple expression for the maximum loan that can be outstanding at any time t given by the generic annuity term $\left(\frac{J-1}{J}\right) \cdot A\left(t+1, N\right)$. The expression $s_t^j + \frac{A(t+1,N)}{J} \geq 0$ describes the constraint on the savings of the household j in period t.

Markets continue to be incomplete, however households are relatively less debt constrained. Further, the continued operation of a partial borrowing constraint in the households' problems hinders the possibility of an equilibrium allocation arising as the economy's Pareto optimal allocation.

3 Equilibrium in Ramsey Economy with liberal borrowing

We consider the Ramsey model with liberal borrowing described in Section 2 and define an equilibrium for this economy. The optimizing behaviors of the agents in this economy are as follows.

3.1 Households lifetime utility maximization

Let $\{1+r_t, w_t \,:\, t = 0, 1, \dots\}$ be sequences of one period rental returns and wage rates, respectively. The sequences $\{1 + r_t, w_t\}$ are always taken to be positive. Households are competitive agents and perfectly anticipate the profile of factor returns $\{1 + r_t, w_t\}$. At time t, each household can borrow against the wage earned at times $t+1, t+2, \cdots, t+N$, i. e., a household could have negative savings at any time t which is bounded below by the present value of the prevailing wage in time periods $t + 1, t + 2, \cdots, t + N$. Hence, for $j = 1, \dots, J$,

$$-s_t^j \leq \frac{A\left(t+1, N\right)}{J}, \text{ or } s_t^j + \frac{A\left(t+1, N\right)}{J} \geq 0.$$

At time $t = 0$, we are given s_{-1}^j, $j = 1, \dots, J$, such that

$$\sum_{j=1}^{J} s_{-1}^j = \kappa_0 > 0. \tag{1}$$

This formulation of the initial condition imposes no constraints on the households entering the economy with non-negative savings $s_{-1}^j \geq 0$. It could also accommodate some households with limited initial debt. Given $\{1 + r_t, w_t\}$, household j solves

$$\mathscr{P}(j): \quad \sup \sum_{t=0}^{\infty} \delta_j^t u_j\left(c_t^j\right),$$

by choice of a non-negative consumption sequence $\{c_t^j : t = 0, 1, 2, \ldots\}$, and a savings sequence $\{s_t^j : t = 0, 1, 2, \ldots\}$ such that

$$c_t^j + s_t^j \leq \left(1 + r_t\right) s_{t-1}^j + \frac{w_t}{J}, \text{ and } s_t^j + \frac{A(t+1, N)}{J} \geq 0; \quad t = 0, 1, 2, \ldots. \tag{2}$$

The first-order conditions of optimality[12] (*the Ramsey-Euler inequalities*) for problem $\mathscr{P}(j)$ are

$$\frac{u_j'\left(c_t^j\right)}{u_j'\left(c_{t-1}^j\right)} \leq \frac{1}{\delta_j\left(1 + r_t\right)} \quad t = 1, 2, \ldots. \tag{3}$$

In case,

$$s_t^j + \frac{A(t+1, N)}{J} > 0$$

for some $t \geq 1$, the Ramsey - Euler equality holds, i.e.,

$$\frac{u_j'\left(c_t^j\right)}{u_j'\left(c_{t-1}^j\right)} = \frac{1}{\delta_j\left(1 + r_t\right)}. \tag{4}$$

Observe that all inter-temporal decisions are taken by the households.

3.2 Firms' profit maximization

In contrast to the households, firms are not engaged in inter-temporal decision making and are completely myopic. They take the rental rate as given and solve the following profit maximization problem $\mathbf{P}(F)$ at each t:

$$\mathbf{P}(F): \quad \sup[f(K) - \left(1 + r_t\right) K],$$

by choice of $K \geq 0$. The residual profit is treated as the wage bill. It is shared equally among the households as wage income. If $0 < 1 + r_t < \infty$, then Assumption 1 implies that there exists a unique positive stock K_t which solves $\mathbf{P}(F)$ at each t. The first order condition yields

$$f'(K_t) = 1 + r_t; \tag{5}$$

[12] In addition, the *transversality condition* must hold on the optimal path.

furthermore, the corresponding w_t is positive and is defined by

$$w_t = f\left(K_t\right) - \left(1 + r_t\right) K_t. \tag{6}$$

3.3 The Ramsey Equilibrium Concept

A collection $\mathscr{E} = \left(f, \kappa_0, \left\{u_j, \delta_j, s^j_{-1}\right\}, j = 1, 2, \ldots, J\right)$ satisfying Assumptions 1 - 3 and the restrictions in (1) on initial savings is said to be an *economy*. For the borrowing constraint to be liberal, we need to assume that $N \geq 1$. The economy always has a positive aggregate capital stock and at least one agent has positive savings at $t = -1$.

The equilibrium concept is *perfect foresight*. Households perfectly anticipate the sequences of rental and wage rates. They solve their optimization problems for their planned consumption demand and saving sequences. The firms calculate the capital demand at each time and the corresponding total output supply. Rents are paid to the households for capital supplied and the residual profits are paid out as the total wage bill.

An equilibrium occurs when the households' savings supply equals the production firms' capital demand at every point of time. A form of Walras' law implies that the total consumption demand plus supply of savings for the next period equals current output. Thus, in equilibrium, every agent maximizes its objective function and planned supplies equal planned demands in every market.

Definition 1. *Sequences* $\left\{1 + r_t, w_t, K_t, \left(c^j_t, s^j_t\right),\ j = 1, \ldots, J; t = 0, 1, \ldots\right\}$ *constitute a* Ramsey equilibrium *for a given economy* $\mathscr{E}$ *provided:*

(E1) For each j, $\left\{\left(c^j_t, s^j_t\right) : t = 0, 1, \ldots\right\}$ solves $\mathscr{P}(j)$ given $\{1 + r_t, w_t : t = 0, 1, \ldots\}$.

(E2) For each t, K_t solves $\mathbf{P}(F)$ given $1 + r_t$.

(E3) $w_t = f\left(K_t\right) - \left(1 + r_t\right) K_t$ for $t = 0, 1, \ldots$.

(E4) $\sum_{j=1}^{J} s^j_{t-1} = K_t$ for $t = 1, 2, \ldots$, and $0 < \kappa_0 = K_0 \leq \overline{K}$.

The output market balance follows by combining (E1) - (E4):

$$\sum_{j=1}^{J} c^j_t + \sum_{j=1}^{J} s^j_t = f\left(K_t\right),\ t = 0, 1, \ldots . \tag{7}$$

Note that the equilibrium consumption, savings and capital sequences are bounded from above by the maximum sustainable stock $\overline{K}$. The assumed conditions for households and the firms imply that in an equilibrium, $c^j_t > 0,\ j = 1, \ldots, J$, and $K_t > 0$ for each t, given that κ_0 is positive. At least one agent's total income, $\left(1 + r_t\right) s^j_{t-1} + \frac{w_t}{J}$ is positive. However, it is possible for an agent to have negative income.

We also observe that even though non-negative initial savings for each household is a realistic description of the economy, the equilibrium concept defined above is not inconsistent with some of the households having negative initial savings. This aspect is further clarified when we describe the equilibrium for the case of stationary economy. In this case, all but the most patient household have negative savings on the equilibrium path.

3.4 Existence of Ramsey Equilibrium

In this sub-section we establish the existence of equilibrium in the Ramsey economy with liberal borrowing. The formal statement of the existence result is contained in the following theorem.

Theorem 1. *Consider a Ramsey economy with liberal borrowing, $\mathscr{E}$, with $\kappa_0 > 0$ and $s^j_{-1} + \frac{w_0}{J(1+r_0)} \geq 0$ for each household. There exists an equilibrium for the Ramsey economy with liberal borrowing for all $N \in \mathbb{N}$.*

The proof of this theorem consists of several steps. In the first step we restrict the economy to finite number of time periods (instead of the general infinite time horizon economy) and define a modified version of equilibrium contained in Definition 1. Next we show that an equilibrium exists in such Ramsey economy model with finite number of time periods. In the last step, we show that equilibrium in the infinite time horizon Ramsey economy model with liberal borrowing can be obtained by applying a process similar to Cantor diagonalization process to the finite time periods equilibria. The detailed proof is contained in Becker et al. (2015a, Appendix A).

4 Stationary Ramsey Equilibrium

We define the stationary equilibrium for the Ramsey economy with liberal borrowing.

Definition 2 (Stationary Ramsey Equilibrium)**.** *A tuple* $\{1+r^{**}, w^{**}, K^{**}, \left(c^{j**}, s^{j**}\right), \ j = 1, \dots, J\}$ *is called a* stationary Ramsey equilibrium *if the sequences* $\{1 + r_t, w_t, K_t, \left(c^j_t, s^j_t\right), \ j = 1, \dots, J \ : t = 0, 1, \dots\}$ *given by*

$$1 + r_t = 1 + r^{**}, w_t = w^{**}, K_t = K^{**}, c^j_t = c^{j**}, s^j_t = s^{j**} \ : \ t = 0, 1, \dots$$

represent an equilibrium for the Ramsey economy with liberal borrowing with $\kappa_0 = K^{**}$ and $s^j_{-1} = s^{j**}$, $j = 1, \dots, J$.

The following proposition shows that there is a unique stationary equilibrium and describes its structure. Its proof is along the lines of the proof of the main result in Becker (1980) and follows closely the proof of Proposition 3 in Borissov and Dubey (2015) and a sketch is provided here.

Proposition 1. *There is a unique stationary Ramsey equilibrium*

$$\left\{1 + r^{**}, w^{**}, K^{**}, \left(c^{j**}, s^{j**}\right), \ j = 1, \dots, J\right\},$$

which is determined as follows:

$$\left.\begin{array}{l} 1+r^{**}=\frac{1}{\delta_1}=f'\left(K^{**}\right),\ w^{**}=f\left(K^{**}\right)-K^{**}\cdot f'\left(K^{**}\right); \\ s^{1^{**}}=K^{**}+\left(\frac{J-1}{J}\right)\cdot\left(\frac{\delta_1\left(1-\delta_1^N\right)}{1-\delta_1}\right)\cdot w^{**},\ c^{1^{**}}=f\left(K^{**}\right)-K^{**}-\left(\frac{J-1}{J}\right)\cdot w^{**}\delta_1^N \\ s^{j^{**}}=-\left(\frac{1}{J}\right)\cdot\left(\frac{\delta_1\left(1-\delta_1^N\right)}{\left(1-\delta_1\right)}\right)\cdot w^{**},\ c^{j^{**}}=\frac{\delta_1^N}{J}\cdot w^{**},\ j=2,\dots,J. \end{array}\right\} \tag{8}$$

Proof. Consider problem $\mathscr{P}(j)$ under the assumption that there are $1+r>0$ and $w>0$ such that for all t, $1+r_t=1+r$ and $w_t=w$ and denote it by $\mathscr{P}^s(j)$. We call a pair $\left(s^j,c^j\right)$ a stationary solution to problem $\mathscr{P}^s(j)$ if the sequence $\left\{\left(c_t^j,s_t^j\right)\,:\,t=0,1,\dots\right\}$ given by

$$c_t^j=c^j,\ \text{and}\ s_t^j=s^j,\ t=0,1,\dots, \tag{9}$$

represents its solution at $s_{-1}^j=s^j$. Clearly, a tuple $\left\{1+r,w,K,\left(c^j,s^j\right),\ j=1,\dots,J\right\}$ is a stationary equilibrium if and only if it satisfies the following properties:

$$\left\{\begin{array}{ll} 1+r & =f'(K);\ w=f(K)-f'(K)K;\ K=\sum_j s^j;\text{and} \\ \left(s^j,c^j\right) & \text{is a stationary solution to problem } \mathscr{P}^s(j) \text{ for every } j=1,\dots,J. \end{array}\right\}. \tag{10}$$

A stationary solution to $\mathscr{P}^s(j)$ exists only if $\delta_j(1+r)\le 1$ because in the case where $\delta_j(1+r)>1$ no consumption stream which is constant over time can satisfy the first order conditions. Therefore, on any stationary equilibrium $\left\{1+r,w,K,\left(c^j,s^j\right),\ j=1,\dots,J\right\}$, we have $1+r\le\frac{1}{\delta_1}$.

Let $\delta_j(1+r)<1$. Then a pair $\left(s^j,c^j\right)$ such that

$$s^j>-\left(\frac{1}{J}\right)\cdot\left(\frac{\delta_1\left(1-\delta_1^N\right)}{\left(1-\delta_1\right)}\right)\cdot w$$

cannot be a stationary solution because otherwise, by (4), it would satisfy

$$1=\frac{u_j'\left(c^j\right)}{u_j'\left(c^j\right)}=\frac{1}{\delta_j(1+r)}>1,$$

which is impossible. At the same time, the sequence $\left\{\left(c_t^j,s_t^j\right)\,:\,t=0,1,\dots\right\}$ given for all $t=0,1,\dots,$ by

$$s_t^j=-\left(\frac{\delta_1\left(1-\delta_1^N\right)}{J\left(1-\delta_1\right)}\right)\cdot w\ \text{and}\ c_t^j=\frac{w\delta_1^N}{J}$$

is feasible for problem $\mathscr{P}^s(j)$ at

$$s_{-1}^{j}=-\left(\frac{\delta_1\left(1-\delta_1^N\right)}{J\left(1-\delta_1\right)}\right)\cdot w$$

and satisfies the first-order conditions and the transversality condition[13]. Therefore, the pair $\left(s^j,c^j\right)$ determined by

$$s^{j}=-\left(\frac{\delta_1\left(1-\delta_1^N\right)}{J\left(1-\delta_1\right)}\right)\cdot w \text{ and } c^j=\frac{w\delta_1^N}{J}$$

is the only stationary solution to $\mathscr{P}^s(j)$ and hence on any stationary equilibrium $\{1+r, w, K, \left(c^j, s^j\right),\ j=1,\ldots,J\}$, we have $1+r\geq\frac{1}{\delta_1}$ because otherwise we would have

$$K=-\left(\frac{\delta_1\left(1-\delta_1^N\right)}{1-\delta_1}\right)\cdot w<0.$$

Thus, on any stationary equilibrium $\left\{1+r, w, K, \left(c^j, s^j\right),\ j=1,\ldots,J\right\}$, we have

$$1+r=\frac{1}{\delta_1} \text{ and } s^{j}=-\left(\frac{\delta_1\left(1-\delta_1^N\right)}{J\left(1-\delta_1\right)}\right)\cdot w \text{ for all } j\geq 2.$$

It remains to note that if $\delta_j(1+r)=1$, then any pair $\left(s^j,c^j\right)$ such that

$$s^{j}\geq-\left(\frac{\delta_1\left(1-\delta_1^N\right)}{J\left(1-\delta_1\right)}\right)\cdot w \text{ and } c^j=rs^j+\frac{w}{J}>0$$

is a stationary solution to $\mathscr{P}^s(j)$ and to use (10).

□

Assumption 1 implies that K^{**} is unique. This proposition maintains that in the stationary Ramsey equilibrium all households except the most patient one are indebted and all their wage incomes are spent for the payment of their debts. Observe that in a steady state, the annuity $A(t+1,N)$ collapses to an ordinary one of the form

$$A\left(t+1,N\right)=\frac{w}{1+r}+\cdots+\frac{w}{\left(1+r\right)^N}=w\cdot\delta_1+\cdots+w\cdot\delta_1^N=w\left[\frac{\delta_1\left(1-\delta_1^N\right)}{\left(1-\delta_1\right)}\right],$$

[13]Here the transversality condition is as follows:

$$\lim_{t\to\infty}\delta_j^t u_j'\left(c_t^j\right)\left(s_t^j+\frac{w}{J(1+r)}+\frac{w}{J\left(1+r\right)^2}+\cdots+\frac{w}{J\left(1+r\right)^N}\right)=\lim_{t\to\infty}\delta_j^t u_j'\left(c_t^j\right)\left(s_t^j+\left(\frac{\delta_1\left(1-\delta_1^N\right)}{J\left(1-\delta_1\right)}\right)\cdot w\right)=0.$$

as $\frac{1}{(1+r)} = \delta_1$. This formula is the tip-off for the steady state borrowing at each time by the impatient agents. The steady state initial borrowings for the impatient agents is their per capita share of the above annuity evaluated at the steady state wages w^{**}, and the most patient agent's discount factor δ_1. Now shift time forward one unit. The previous first payment has been extinguished and to maintain a steady state, a new, final period payment factor must be loaded that corresponds to the borrowing made by that agent in the new period (say, time 1) — this is precisely the amount $\delta_1^N \left(\frac{w^{**}}{J}\right)$. That is, this is the incremental or new borrowing in each period that maintains the payout / loan structure of the fixed annuity $A(t+1, N)$ in the stationary state. It is easy to note that all the remaining expressions in the steady state follow from this observation. Thus, it is only necessary to prove that in each period the impatient agents take the maximum borrowing allowed – which entails taking just that increment to their existing debt which maintains the original savings (deficit) with which they entered the economy in the steady state.

In the steady state, in every period, the impatient agents enjoy consumption equal to the present value of their wage received N periods later discounted at the most patient agents discount factor. Therefore, their consumption is positive as the borrowing limit is positive in every period. As for the most patient household, it owns all capital and all debts of the other households. It is clear that the stationary-equilibrium capital stock and output in our economy are the same as in the no-borrowing economy. However, the output is distributed among the households in a somewhat different way: the consumption of the most patient household is higher and the consumption of every other household is lower than in the no-borrowing economy.

One can make the following observation using a plot of the Lorenz curve for consumption (plotting the share of the population on the horizontal axis and the share of consumption on the vertical in the usual manner) and use linear interpolation to draw the graph (i.e., the domain would need to be split into equally spaced subintervals of length $1/J$). Then the consumption of all the impatient households as a fraction of aggregate consumption (denoted by $\phi(N)$) would be

$$\phi(N) = \delta_1^{N-1} \left(\frac{J-1}{J}\right) \cdot \left(\frac{\delta_1 f(K^{**}) - K^{**}}{f(K^{**}) - K^{**}}\right), N = 0, 1, \cdots,$$

where the two terms in the brackets are independent of N and only the first term δ_1^{N-1} depends on N. In other words,

$$\phi(N) = \delta_1^{N-1} \cdot CF, N = 0, 1, \cdots, \text{ where } CF = \left(\frac{J-1}{J}\right) \cdot \left(\frac{\delta_1 f(K^{**}) - K^{**}}{f(K^{**}) - K^{**}}\right),$$

where CF denotes the constant term.

In this setting the steady state in Becker (1980) Lorenz dominates the steady state in our paper. This evidently implies a lower Gini coefficient in Becker (1980) for the distribution of consumption than in the current setup. This is the precise sense in which stationary state inequality increases in the newer model compared to Becker (1980).

We further notice that the aggregate steady state consumption is independent of N as it depends only upon technology and the most patient agents pure rate of time preference. It is also clear that for the $J-1$ impatient agents, their steady state consumption declines with increases in N. In particular, this implies the first agents consumption rises with N and approaches the aggregate consumption as N tends to infinity. The Gini coefficient for the distribution of consumption is

$$\text{Gini Coefficient} = \frac{J-1}{J} - \delta_1^{N-1} CF.$$

We infer that the steady state consumption Gini coefficient is increasing with N (reinforced by the Figure 1 for the case of $J = 5$ households) and approaches the long-run consumption distribution found in Bewley (1982) with complete markets. The more liberal the borrowing constraints, the higher the Gini coefficient. Fishers notion that the market for loans acts as a redistribution mechanism from impatient to patient individuals is reflected in our steady state results taken together with the long-run convergence property found in Section 5.

It is generally recognized (see, e.g., Alesina and Perotti (1996)) that income inequality increases sociopolitical instability and causes social tension, which in turn reduces investment incentives and affects the security of property rights. Thus, income inequality can have a negative impact on economic growth. Borissov and Lambrecht (2009) propose to model this impact of inequality on growth by assuming that inequality increases the impatience of all economic agents. Making this assumption in our model would imply that the more liberal the borrowing constraints, the lower the steady-state capital stock, output and aggregate consumption.

It is noteworthy that the total income for agent 1 is

$$\text{Income}^1 = f\left(K^{**}\right) + \left(\frac{J-1}{J}\right) \cdot \left(\frac{\delta_1\left(1-\delta_1^{N-1}\right)}{\left(1-\delta_1\right)}\right) \cdot w^{**},$$

and the income of the impatient agent is

$$\text{Income}^j = -\left(\frac{1}{J}\right) \cdot \left(\frac{\delta_1\left(1-\delta_1^{N-1}\right)}{\left(1-\delta_1\right)}\right) \cdot w^{**},$$

for $j = 2, \cdots, J$. Observe that the income for the impatient households in a stationary economy depends on the discount factor of only the first (most patient) household. The sum of total incomes over all agents is equal to the total output in the economy $f\left(K^{**}\right)$.

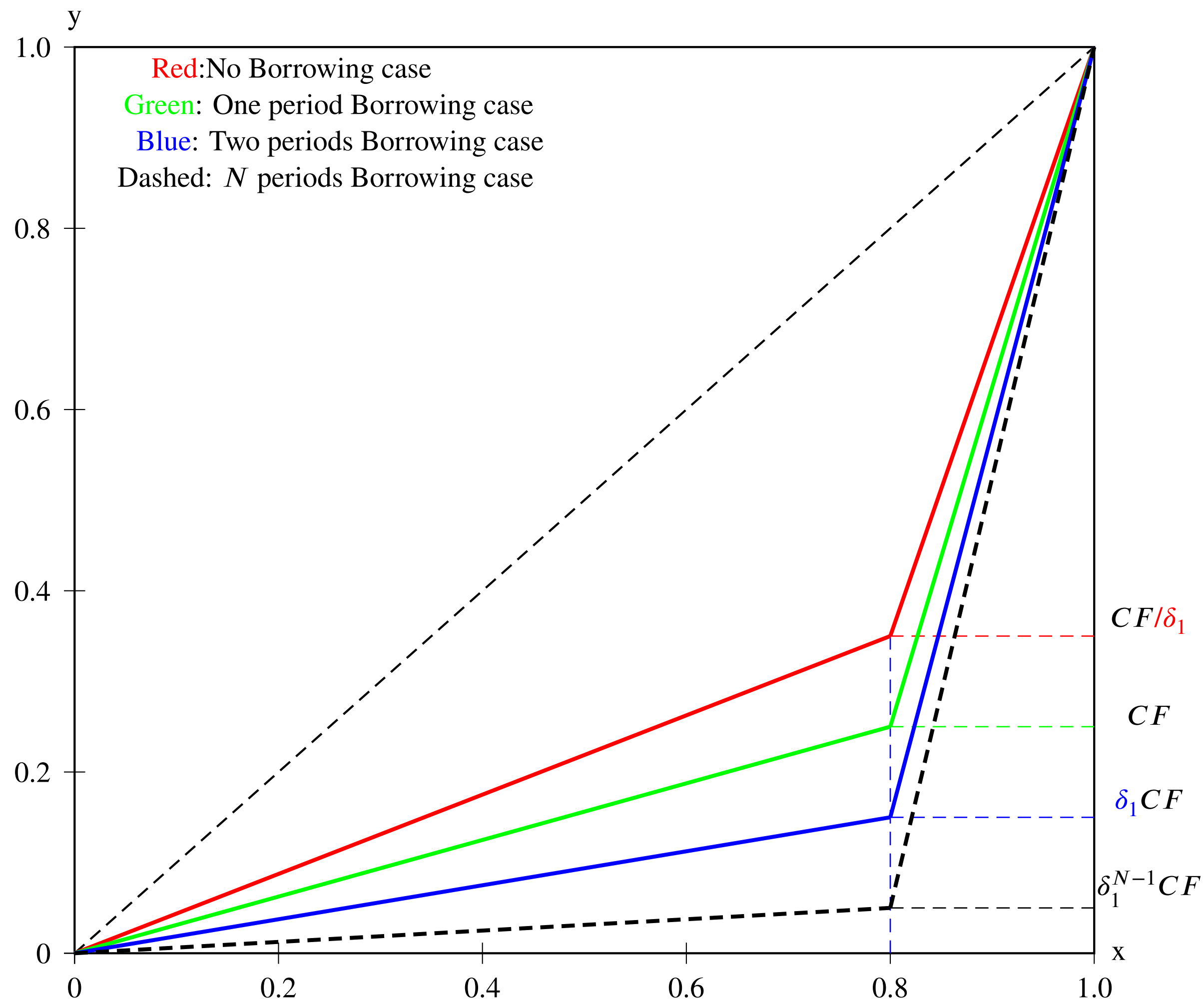

Figure 1: Lorentz Curves in Stationary Ramsey Equilibria for $J = 5$ households

5 Turnpike property and equilibrium dynamics

In this section, we examine the dynamic properties of the Ramsey equilibria for the economy with borrowing horizon $N \geq 1$. First, we consider convergent equilibria. We show that the following turnpike property is obtained on any convergent equilibrium path: starting from some time period t onward, for every household other than the most patient one, the borrowing at time $t-1$ equals the present value (at time $t-1$) of the wage income in period t. In addition, the capital sequence converges to the unique stationary capital stock. The approach adopted in the proof is essentially the same as the proof of the main result in Becker and Foias (1987). Let $\{1+r_t, w_t, K_t, \left(c_t^j, s_t^j\right),\ j = 1, \ldots, J; t = 0, 1, \ldots\}$ be a Ramsey equilibrium.

Proposition 2. *If $K_\infty = \lim_{t\to\infty} K_t$ exists, then $K_\infty = K^{**}$ and*

$$s_t^j + \frac{A(t+1, N)}{J} = 0, \text{ for } j \geq 2, \textit{ for all } t \textit{ large enough.}$$

Next we consider Ramsey equilibria where the capital stock sequence is bounded above by K^{**} from some time period t on. In the following Lemma it is shown that all such equilibria are convergent.

Proposition 3. *If $K_t \leq K^{**}$ for all t large enough, then $K_t \to K^{**}$ as $t \to \infty$.*

However in the general case the equilibrium dynamics is more complicated and convergence need not be observed in every case. We report some partial results on this line. Following claim is useful to describe the equilibrium dynamics in the general case. It shows the conditions under which the sum of capital stock in any period $n+1$ and the maximum possible debt created in the terminal period of borrowing regime $n+N$ is monotone increasing.

Claim 1. *Suppose that for some n, $K_n \geq K_{n-1}$ and $K_n > K^{**}$. Let*

$$\Gamma_n \equiv \left\{ j \in \{1, \ldots, J\} : s_{n-1}^j + \frac{A(n, N)}{J} > 0 \right\}.$$

Denote the cardinality of Γ_n by γ_n. Then

$$K_{n+1} + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{w_{n+N}}{\left(1+r_{n+1}\right) \cdots \left(1+r_{n+N}\right)} \geq K_n + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{w_{n+N-1}}{\left(1+r_n\right) \cdots \left(1+r_{n+N-1}\right)} \quad (11)$$

holds.

The inequality (11) implies $K_{n+1} > K_n$ when $N = 1$ as it must, in view of Borissov and Dubey (2015, Lemma 6). For $N = 1$,

$$K_{n+1} + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{w_{n+1}}{\left(1+r_{n+1}\right)} > K_n + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{w_n}{\left(1+r_n\right)} \Rightarrow K_{n+1} > K_n.$$

and therefore if $K_0 < K^{**}$, then $K_t < K^{**}$ for all t because otherwise we would have $\lim K_t > K^{**}$, which is impossible. However, if $N \geq 2$, (11) does not necessarily imply $K_{n+1} > K_n$. This could potentially lead to emergence of equilibrium cycles as the following example shows for the case of $N = 2$.

5.1 Cyclic Equilibrium: An Example

In this sub-section, for the case of liberal borrowing with $N = 2$, we construct an equilibrium exhibiting a cycle of period two, i.e. for $K_0 \neq K_1$, the sequence $K_0, K_1, K_0, K_1, \ldots$ may arise in equilibrium. Only the most patient household saves in this example and all other agents take the maximum borrowing allowed. Our construction is similar to the one in Becker and Foias (1987).

Example 1.

A cyclic equilibrium in which all households except the most patient one are in a maximum borrowing position is fully determined by K_0, K_1, c_0^1, s_0^1, c_1^1, s_1^1 such that for $1 + r_i = 1 + r(K_i)$, $i = 0, 1$, and $w_i = w(K_i)$, $i = 0, 1$,

$$\delta_1(1 + r_1)u_1'(c_1^1) = u_1'(c_0^1), \ \delta_1(1 + r_0)u_1'(c_0^1) = u_1'(c_1^1), \tag{12}$$

$$s_0^1 + c_0^1 = (1 + r_0)s_1^1 + \frac{w_0}{J}, \ s_1^1 + c_1^1 = (1 + r_1)s_0^1 + \frac{w_1}{J}, \tag{13}$$

$$\begin{aligned} s_0^1 &= K_1 + \left(\frac{J-1}{J}\right)\frac{w_1}{1+r_1} + \frac{J-1}{J}\frac{w_0}{(1+r_0)(1+r_1)} \\ s_1^1 &= K_0 + \left(\frac{J-1}{J}\right)\frac{w_0}{1+r_0} + \frac{J-1}{J}\frac{w_1}{(1+r_0)(1+r_1)} \end{aligned} \tag{14}$$

$$\delta_j(1 + r_1)u_j'\left(\frac{w_1}{(1+r_0)(1+r_1)}\right) \leq u_j'\left(\frac{w_0}{(1+r_0)(1+r_1)}\right), \tag{15}$$

$$\delta_j(1 + r_0)u_j'\left(\frac{w_0}{(1+r_0)(1+r_1)}\right) \leq u_j'\left(\frac{w_1}{(1+r_0)(1+r_1)}\right), \ j = 2, \ldots, J. \tag{16}$$

A standard argument implies (12) - (13) are sufficient for the sequence

$$\left\{\left(c_0^1, s_0^1\right), \left(c_1^1, s_1^1\right), \left(c_0^1, s_0^1\right), \left(c_1^1, s_1^1\right), \ldots\right\}$$

to solve the first household's optimization problem since the transversality condition is obtained. Condition (15) can also be shown to be sufficient for the impatient households problems provide $\delta_j > 0$, $j = 2, \ldots, J$, are sufficiently small.

For determinacy we assume that $K_0 < K_1$ and hence $1 + r_0 > 1 + r_1$. Therefore, taking account of (12), we obtain

$$c_0^1 > c_1^1 > 0. \tag{17}$$

Note that (12) implies

$$(1 + r_0)(1 + r_1) = \frac{1}{\delta_1^2}. \tag{18}$$

and that (13)-(14) can be aggregated to

$$K_1 + c_0^1 = f(K_0) - \left(\frac{J-1}{J}\right)\delta_1^2 w_0,\ K_0 + c_1^1 = f(K_1) - \left(\frac{J-1}{J}\right)\delta_1^2 w_1. \tag{19}$$

To construct the desired example, it is sufficient to find a function f and numbers δ_1^2, K_0, K_1, c_0^1, c_1^1 such that (17)-(19) hold true and to notice that (i) because of (17), it is not difficult to construct a function u_0 satisfying Assumption 2 and (12); and (ii) for any functions u_j, $j = 2, \ldots, J$, satisfying Assumption 2, (15) holds true if δ_j, $j = 2, \ldots, J$, are sufficiently small.

Let

$$f(K) = \begin{cases} \underline{w} + aK & K \le 1 \\ \overline{w} + bK & K > 1 \end{cases},\ a > 1,\ a > b,\ ab = \frac{1}{\delta_1^2},\ \underline{w} > 0,\ \overline{w} = \underline{w} + a - b. \tag{20}$$

We define

$$\gamma = \left(\frac{J-1}{J}\right)\delta_1^2.$$

It is easy to verify that the following inequality holds for all $J > 1$:

$$(1-\gamma)\frac{a-b}{1+b} + 1 < a - \frac{1-\delta_1^2}{\delta_1^2(1+b)}.$$

Let us take $K_0 < 1$ such that

$$(1-\gamma)\frac{a-b}{1+b} + 1 < \left[a - \frac{1-\delta_1^2}{\delta_1^2(1+b)}\right] K_0$$

and hence

$$(1-\gamma)\frac{(1+b)\underline{w} + a - b}{1+b} + \frac{1-\delta_1^2}{\delta_1^2(1+b)} K_0 < (1-\gamma)\underline{w} + aK_0 - 1. \tag{21}$$

Let us further take c_0^1 such that

$$\begin{aligned} (1-\gamma)\frac{(1+b)\underline{w} + a - b}{1+b} + \frac{1-\delta_1^2}{\delta_1^2(1+b)} K_0 &< c_0^1 \\ &< \min\left\{(1-\gamma)\underline{w} + aK_0 - 1,\ (1-\gamma)\frac{(1+b)\underline{w} + a - b}{b} + \frac{1-\delta_1^2}{b\delta_1^2} K_0\right\}. \end{aligned} \tag{22}$$

The existence of such c_0^1 follows from (21) and inequality $b + 1 > b$.

Finally, let

$$K_1 = (1-\gamma)\underline{w} + aK_0 - c_0^1,$$

$$c_1^1 = (1-\gamma)[(1+b)\underline{w} + a - b] + \frac{1-\delta_1^2}{\delta_1^2}K_0 - bc_0^1.$$

Now we show that, with the function f defined by (20), the chosen numbers $\delta_1, K_0, K_1, c_0^1, c_1^1$ satisfy (17)-(19). First, notice that since $K_0 < 1$, $w_0 = \underline{w}$ and $1 + r_0 = a$. Therefore,

$$f(K_0) - \left(\frac{J-1}{J}\right)\delta_1^2 w_0 = (1-\gamma)\underline{w} + aK_0$$

and hence the first equality in (19) holds true.

By the second inequality in (22), $K_1 > 1$ and hence $w_1 = \overline{w}$ and $1 + r_1 = b$. Therefore, taking into account the equalities $\overline{w} = \underline{w} + a - b$ and $ab = 1/\delta_1^2$, we obtain

$$\begin{aligned}
f(K_1) - \left(\frac{J-1}{J}\right)\delta_1^2 w_1 - c_1^1 &= (1-\gamma)\overline{w} + bK_1 - c_1^1 \\
&= (1-\gamma)[\underline{w} + a - b] + b[(1-\gamma)\underline{w} + aK_0 - c_0^1] - (1-\gamma)[(1+b)\underline{w} + a - b] \\
&- \frac{1-\delta_1^2}{\delta_1^2}K_0 + bc_0^1 = \frac{1}{\delta_1^2}K_0 - \frac{1-\delta_1^2}{\delta_1^2}K_0 = K_0.
\end{aligned}$$

This means that the second equality in (19) also holds true.

Observe that (18) follows from the the choice of a, b and δ_1 and from the equalities $1 + r_0 = a$ and $1 + r_1 = b$, and note that (22) and the choice of c_1^1 imply (17). It remains to smooth f over sufficiently small vicinities of 0 and 1.

Following parameter values satisfy all conditions of the example. Take

$$a = 8, b = 0.5, J = 2, \delta_1 = 0.5, \text{ and } \underline{\mathrm{w}} = 2.$$

Then, $\overline{w} = \frac{19}{2}$, $\gamma = \frac{1}{8}$, $\frac{a-b}{1+b} = 5$, $\frac{1-\delta_1^2}{\delta_1^2} = 3$ and $\frac{1-\delta_1^2}{\delta_1^2(1+b)} = 2$. Let $K_0 = \frac{15}{16}$, then

$$(1-\gamma)\frac{a-b}{1+b} + 1 = \frac{7}{8}\cdot 5 + 1 = \frac{43}{8} < \left[a - \frac{1-\delta_1^2}{\delta_1^2(1+b)}\right]K_0 = (8-2)\cdot\frac{15}{16} = \frac{45}{8}.$$

Take $c_0^1 = \frac{65}{8}$ which satisfies (22) as

$$\frac{7}{8}\cdot\frac{(1.5)(2) + 8 - 0.5}{1.5} + \frac{15}{8} = \frac{64}{8} < \frac{65}{8} < \min\left\{\frac{33}{4}, \frac{96}{4}\right\}.$$

Then $K_1 = \frac{7}{8}\cdot 2 + 8\cdot\frac{15}{16} - \frac{65}{8} = \frac{9}{8}$, and $c_1^1 = \frac{7}{8}\cdot(4 + 8 - 0.5) + 4\cdot\frac{15}{16} - (0.5)\cdot\frac{65}{8} = \frac{127}{16}$ satisfy remaining conditions.

6 Conclusions

In this paper we have devised a method to approach the classical complete markets economy from the Ramsey model by liberalizing each household's borrowing constraint. The turnpike property holds on every convergent equilibrium path independent of the production technology and the preferences of the households. However, not all equilibrium path are convergent as is shown via Example 1. Establishing the dynamic properties of equilibria needs further research.

The study also provides a formal model to examine the effect of various credit regimes on the inequality in the society and shows that the relationship is monotone in the following precise sense. The steady state consumption Gini coefficient of the liberal borrowing economy exceeds that of the borrowing constrained economy. Steady state consumption is, in this sense, more unequally distributed when the borrowing constraint becomes more liberal.

7 Appendix: Proofs

7.1 Proof of Theorem 1

7.1.1 Equilibrium for the finite time horizon Ramsey Economy

Let us define a finite T (with $T > N$) periods equilibrium in the Ramsey economy with liberal borrowing along the lines of the Definition 1. We consider here the case where agents are allowed to borrow against their future wage incomes of *two* periods. The general case where households are allowed to borrow against their future wage incomes of N periods can be proved on similar lines.

Definition 3. *Sequences* $\left\{1 + r_t, w_t, K_t, \left(c_t^j, s_t^j : j = 1, \dots, J\right) : t = 0, 1, \dots, T\right\}$, *constitute a finite T periods equilibrium in the Ramsey economy with liberal borrowing provided:*

(ET1) Given $\{1 + r_t, w_t : t = 0, 1, \dots, T\}$, *each household j, solves*

$$\mathcal{P}_T(j) : \quad \sup \sum_{t=0}^{T} \delta_j^t u_j\left(c_t^j\right)$$

by choice of a non-negative consumption and savings sequences $\left(c_t^j, s_t^j : t = 0, 1, \dots, T\right)$ such that

$$\left.\begin{array}{rl} c_t^j + s_t^j & \leq \left(1 + r_t\right) s_{t-1}^j + \frac{w_t}{J}, \quad t = 0, 1, \dots, T; \\ s_t^j + \frac{A(t+1,2)}{J} & \geq 0;\ t = 0, 1, \dots, T-2;\ s_{T-1}^j + \frac{A(T,1)}{J} \geq 0;\ \text{and } s_T^j \geq 0. \end{array}\right\} \tag{23}$$

(ET2) For each $t = 0, 1, \dots, T$, K_t *solves* $\boldsymbol{P}(F)$ *given* $1 + r_t$.

(ET3) $w_t = f(K_t) - (1 + r_t)K_t$ *for* $t = 0, 1, \ldots, T$.

(ET4) $\sum_{j=1}^{J} s_{t-1}^{j} = K_t$ *for* $t = 1, 2, \ldots, T$, *and* $0 < \kappa_0 = K_0 \leq \overline{K}$.

It is clear that $\left(c_t^{j*}, s_t^{j*} \,:\, t = 0, 1, \ldots, T\right)$ is a solution to $\mathcal{P}_T(j)$ if and only if the feasibility constraint (23) holds and the Ramsey - Euler inequality / equality (3) - (4) hold for $t = 1, \ldots, T$. Therefore, $\left\{1 + r_t^*, w_t^*, K_t^*, \left(c_t^{j*}, s_t^{j*} \,:\, j = 1, \ldots, J\right); t = 0, 1, \ldots, T\right\}$ is an equilibrium if and only if $s_T^{j*} = 0$,

$$1 + r_t^* = f'(K_t^*);\; w_t^* = f(K_t^*) - (1 + r_t^*)K_t^*;\; \sum_{j=1}^{J} s_{t-1}^{j*} = K_t^* > 0 \,:\; t = 1, \ldots, T, \tag{24}$$

(3)-(4) hold for all $j = 1, \ldots, J$, and $t = 1, \ldots, T$ and (23) hold for $t = 0, \ldots, T$.

The existence of equilibrium for the finite time horizon Ramsey economy with liberal borrowing is shown via the following steps.

STEP 1 We reduce the finite time horizon Ramsey economy model to a generalized game $\Gamma = (X_k, \psi_k, G_k)_{k \in I}$. Recall that to specify a game, we need to describe the set of players, I; and for each player $k \in I$,

(a) the strategy set X_k,

(b) the strategy correspondence

$$\psi_k \,:\, \prod_{i \in I} X_i \to X_k, \text{ and}$$

(c) the loss function

$$G_k \,:\, \prod_{i \in I} X_i \to \mathbb{R}.$$

Elements of $\prod_{i \in I} X_i$ are called *multistrategies*. The equilibrium of the game Γ is defined as follows.

Definition 4. A multistrategy $\left(x_1^*, \ldots, x_{|I|}^*\right)$ is called a Nash equilibrium of game Γ if for each $k \in I$, x_k^* is a solution to

$$\begin{aligned} \min_{x_k} \quad & G_k\left(x_1^*, \ldots, x_{k-1}^*, x_k, x_{k+1}^*, \ldots, x_{|I|}^*\right) \\ \text{subject to} \quad & x_k \in \psi_k\left(x_1^*, \ldots, x_{k-1}^*, x_k^*, x_{k+1}^*, \ldots, x_{|I|}^*\right). \end{aligned} \tag{25}$$

The sufficient conditions for the existence of Nash equilibrium of this game are specified in the following theorem.

Debreu's Theorem. *(Debreu (1952, p. 888)) Suppose that for each $k \in I$, the set X_k is a convex and compact subset of a finite dimensional space, ψ_k is a continuous correspondence with nonempty compact convex values and the function $G_k\left(x_1, \ldots, x_k, \ldots, x_{|I|}\right)$ is continuous in all variables and convex in x_k. Then a Nash equilibrium exists.*

In order to specify the game representing the Ramsey economy with liberal borrowing, we proceed as follows. We use notation

$$1 + r(K) = f'(K) \text{ and } w(K) = f(K) - K \cdot f'(K)$$

for the competitive rental rate and wage for the capital stock K.

STEP 1(a) Bounds of the capital sequence:
We first note that the upper bounds of the capital sequence $\{K_t\}$ is $\overline{K}$. The upper bound $\overline{c}$ of the consumption sequence is defined as

$$\overline{c} \equiv f\left(\overline{K}\right) + \frac{\overline{w}}{\left(1+\overline{r}\right)} + \frac{\overline{w}}{\left(1+\overline{r}\right)^2},$$

where

$$\overline{w} \equiv w\left(\overline{K}\right) \text{ and } 1 + \overline{r} \equiv 1 + r\left(\overline{K}\right)$$

denote the wage and rental rate for capital stock $\overline{K}$. We construct the sequence of lower bounds for the capital sequence, $\{\widetilde{K}_t\}$ recursively. Let $\widetilde{K}_0$ be such that $0 < \widetilde{K}_0 < \kappa_0$. Suppose we are given $\widetilde{K}_{t-1} > 0$. To construct $\widetilde{K}_t > 0$, it is sufficient to observe that

$$u_j'\left(\frac{f\left(\widetilde{K}_{t-1}\right) - K}{J}\right) \text{ is increasing in } K$$

for each j and choose $\widetilde{K}_t > 0$ as the (unique) solution to the following equation in K:

$$\max_j \left\{ u_j'\left(\frac{f\left(\widetilde{K}_{t-1}\right) - K}{J}\right)\right\} = \theta f'\left(\overline{K}\right), \text{ where } \theta \equiv \min_j \left\{ \delta_j u_j'\left(\overline{c}\right) \right\}.$$

It is noteworthy that

$$u_j'\left(\frac{f\left(\widetilde{K}_{t-1}\right) - \widetilde{K}_t}{J}\right) \leq \theta f'\left(\overline{K}\right), \text{ and } \theta \leq \delta_j u_j'\left(\overline{c}\right), \; j = 1, \ldots, J.$$

Therefore,

$$u_j'\left(\frac{f\left(\widetilde{K}_{t-1}\right)-\widetilde{K}_t}{J}\right)\le \delta_j u_j'\left(\overline{c}\right) f'\left(\overline{K}\right),\ j=1,\dots,J. \tag{26}$$

STEP 1(b) Multistrategies: Consider the following game with $T+(2T+1)J$ players where,

(i) for each household $j=1,\dots,J$,

(a) T players determine $s_t^j,\ t=0,1,\dots,T-1$, by solving

$$\min_s\ s\left(\frac{1}{\delta_j\left(1+r\left(K_{t+1}\right)\right)u_j'\left(c_{t+1}^j\right)}-\frac{1}{u_j'\left(c_t^j\right)}\right) \tag{27}$$

$$\text{subject to } -\frac{w\left(K_{t+1}\right)}{\left(1+r\left(K_{t+1}\right)\right)J}-\frac{w\left(K_{t+2}\right)}{\left(1+r\left(K_{t+1}\right)\right)\left(1+r\left(K_{t+2}\right)\right)J}\le s$$

$$\le \overline{K}+\left(\frac{J-1}{J}\right)\cdot\frac{\overline{w}}{\left(1+\overline{r}\right)}\ (t\le T-2)$$

$$\text{or subject to } -\frac{w\left(K_{t+1}\right)}{\left(1+r\left(K_{t+1}\right)\right)J}\le s\le \overline{K}\ (t=T-1).$$

(b) $T+1$ players determine c_t^j, $t=0,1,\dots,T$, by solving

$$\min_c\ \left|c-\left[\left(1+r\left(K_t\right)\right)s_{t-1}^j+\frac{w\left(K_t\right)}{J}-s_t^j\right]\right| \tag{28}$$

$$\text{subject to } 0\le c\le\overline{c};\ \text{where } s_{-1}^j \text{ is given, and } s_T^j=0.$$

(ii) T players determine $K_t,\ t=1,\dots,T$, by solving

$$\min_K\ \left|K-\sum_j s_{t-1}^j\right|\ \text{subject to } \widetilde{K}_t\le K\le\overline{K} \tag{29}$$

STEP 1(c) Nash Equilibrium: The existence of a Nash equilibrium of this game, which we denote by Γ_T, is established in the following lemma.

Lemma 1. *There exists a Nash equilibrium in the game Γ_T with $T + (2T + 1)J$ players having the strategy sets, strategy correspondences and loss functions described by (27), (28), and (29).*

Proof. We need to show that all the conditions of Debreu's Theorem are satisfied. For the players making the consumption decision for the households and making the capital input stock decision, the strategy sets are closed intervals and the strategy correspondences assign to a multistrategy, the whole strategy set. For the players making the savings decisions at time $t = 0, 1, \ldots, T-1$, the strategy sets are the closed interval

$$\left[-\frac{\overline{w}}{(1+\overline{r})J} - \frac{\overline{w}}{(1+\overline{r})^2 J}, \overline{K} + \left(\frac{J-1}{J}\right) \cdot \left(\frac{\overline{w}}{1+\overline{r}}\right)\right]$$

and the strategy correspondences assign to a multistrategy

$$\left\{ \left(s_t^j\right)_{j=1,\ldots,J,\ t=0,1,\ldots,T-1}, \left(c_t^j\right)_{j=1,\ldots,J,\ t=0,1,\ldots,T}, \left(K_t\right)_{t=1,\ldots,T} \right\}$$

the interval

$$\left[-\frac{w(K_{t+1})}{(1+r(K_{t+1}))J} - \frac{w(K_{t+2})}{(1+r(K_{t+1}))(1+r(K_{t+2}))J}, \overline{K} + \left(\frac{J-1}{J}\right) \cdot \left(\frac{\overline{w}}{1+\overline{r}}\right)\right]$$

if $t \leq T-2$ or the interval

$$\left[-\frac{w(K_{t+1})}{(1+r(K_{t+1}))J}, \overline{K}\right]$$

if $t = T-1$, which contains 0. This last correspondence is upper- and lower- semi-continuous because the expression $\frac{w(K)}{(1+r(K))}$ is a continuous function of K on the interval $\left[\widetilde{K}_{t+1}, \overline{K}\right]$. Finally, for each player, the loss function is continuous in all variables and convex in the player's own strategy variable.

□

STEP 2 Equilibrium for finite periods Ramsey economy: In the next lemma, we show that the Nash equilibrium of the game Γ_T in Lemma 1 is an equilibrium for the T periods Ramsey economy with liberal borrowing.

Lemma 2. *Let $\left\{ \left(s_t^{j*}\right)_{j=1,\ldots,J,\ t=0,1,\ldots,T-1}, \left(c_t^{j*}\right)_{j=1,\ldots,J,\ t=0,1,\ldots,T}, \left(K_t^*\right)_{t=1,\ldots,T} \right\}$ be a Nash equilibrium of the game Γ_T. Let $K_0^* = \kappa_0$, and $s_T^{j*} = 0$ for each household j. Also let $1 + r_t^* = 1 + r\left(K_t^*\right)$ and $w_t^* = w\left(K_t^*\right)$. Then $\left\{ 1 + r_t^*, w_t^*, K_t^*, \left(c_t^{j*}, s_t^{j*}\right),\ j = 1, \ldots, J; t = 0, 1, \ldots, T \right\}$ is a T periods equilibrium of the Ramsey economy with liberal borrowing.*

First, observe that

- if

$$\frac{1}{\delta_j(1+r(K_{t+1}))u_j'(c_{t+1}^j)} > \frac{1}{u_j'(c_t^j)},$$

  then the only solution to problem (27) is

$$s = \begin{cases} -\frac{w(K_{t+1})}{(1+r(K_{t+1}))J} - \frac{w(K_{t+2})}{(1+r(K_{t+1}))(1+r(K_{t+2}))J},\ t \leq T-2, \\ -\frac{w(K_{t+1})}{(1+r(K_{t+1}))J},\ t = T-1, \end{cases}$$

- if

$$\frac{1}{\delta_j(1+r(K_{t+1}))u_j'(c_{t+1}^j)} = \frac{1}{u_j'(c_t^j)},$$

  then any element of the interval

$$\begin{cases} \left[-\frac{w(K_{t+1})}{(1+r(K_{t+1}))J} - \frac{w(K_{t+2})}{(1+r(K_{t+1}))(1+r(K_{t+2}))J}, \overline{K} + \left(\frac{J-1}{J}\right)\left(\frac{\overline{w}}{1+\overline{r}}\right)\right],\ \ t \leq T-2, \\ \left[-\frac{w(K_{t+1})}{(1+r(K_{t+1}))J}, \overline{K}\right],\ \ t = T-1, \end{cases}$$

  is a solution to (27); and
- if

$$\frac{1}{\delta_j(1+r(K_{t+1}))u_j'(c_{t+1}^j)} < \frac{1}{u_j'(c_t^j)},$$

  then the only solution to problem (27) is

$$s = \overline{K} + \left(\frac{J-1}{J}\right)\left(\frac{\overline{w}}{1+\overline{r}}\right).$$

Second, notice that minimization problems (28) and (29) are of the form

$$\min_x \ \ |x - \hat{x}| \ \text{ subject to } \ a_1 \leq x \leq a_2.$$

The unique solution to this problem, x^*, is given by

$$x^* = \begin{cases} a_1 & \text{if } \hat{x} < a_1; \\ a_2 & \text{if } \hat{x} > a_2; \\ \hat{x} & \text{if } a_1 \le \hat{x} \le a_2. \end{cases}$$

Remark 1. *In the case $\hat{x} \le a_2$, we have $\hat{x} \le x^*$.*

Let $\left\{ \left(s_t^{j*} \right)_{j=1,\ldots,J,\ t=0,1,\ldots,T-1}, \left(c_t^{j*} \right)_{j=1,\ldots,J,\ t=0,1,\ldots,T}, \left(K_t^* \right)_{t=1,\ldots,T} \right\}$ be a Nash equilibrium of the game Γ_T. Note that for all $t = 0, 1, \ldots, T$, $K_t^* \ge \widetilde{K}_t > 0$. We establish Lemma 2 via the following set of claims.

Claim 2. *For each household $j = 1, \ldots, J$,*

$$\left(1 + r_t^*\right) s_{t-1}^{j*} + \frac{w_t^*}{J} - s_t^{j*} \ge c_t^{j*} > 0, \ t = 0, 1, \ldots, T. \tag{30}$$

Proof. Assume the converse. Then, by the structure of problem (28), there are j and $0 \le \tau \le T$ such that

$$\left(1 + r_t^*\right) s_{t-1}^{j*} + \frac{w_t^*}{J} - s_t^{j*} \ge c_t^{j*} > 0, \ t = 0, 1, \ldots, \tau - 1,$$

and

$$0 = c_\tau^{j*} \ge (1 + r_\tau^*) s_{\tau-1}^{j*} + \frac{w_\tau^*}{J} - s_\tau^{j*}. \tag{31}$$

Now we fix this j and omit it for ease of notation. We consider three sub-cases.

(a) $\tau \le T - 2$: By (31),

$$s_\tau^* \ge (1 + r_\tau^*) s_{\tau-1}^* + \frac{w_\tau^*}{J}.$$

Since $(1 + r_\tau^*) s_{\tau-1}^* + \frac{w_\tau^*}{J} + \frac{w_{\tau+1}^*}{(1+r_{\tau+1}^*)J} \ge 0$, and $\frac{w_{\tau+2}^*}{(1+r_{\tau+1}^*)(1+r_{\tau+2}^*)J} > 0$, we have

$$\begin{aligned} s_\tau^* + \frac{w_{\tau+1}^*}{(1 + r_{\tau+1}^*)J} + \frac{w_{\tau+2}^*}{(1 + r_{\tau+1}^*)(1 + r_{\tau+2}^*)J} &\ge (1 + r_\tau^*) s_{\tau-1}^* + \frac{w_\tau^*}{J} + \frac{w_{\tau+1}^*}{(1 + r_{\tau+1}^*)J} \\ &+ \frac{w_{\tau+2}^*}{(1 + r_{\tau+1}^*)(1 + r_{\tau+2}^*)J} \ge \frac{w_{\tau+2}^*}{(1 + r_{\tau+1}^*)(1 + r_{\tau+2}^*)J} > 0. \end{aligned} \tag{32}$$

Taking into account the structure of problem (27), we have

$$\frac{1}{\delta(1+r^*_{\tau+1})u'(c^*_{\tau+1})} - \frac{1}{u'(c^*_\tau)} \leq 0,$$

because otherwise we would have

$$s^*_\tau + \frac{w^*_{\tau+1}}{(1+r^*_{\tau+1})J} + \frac{w^*_{\tau+2}}{(1+r^*_{\tau+1})(1+r^*_{\tau+2})J} = 0$$

a contradiction of (32). Therefore, $\frac{1}{u'(c^*_{\tau+1})} = 0$ and hence

$$0 = c^{j*}_{\tau+1} \geq (1+r^*_{\tau+1})s^{j*}_\tau + \frac{w^*_{\tau+1}}{J} - s^{j*}_{\tau+1}.$$

Repeating the argument, which take into account the structure of problem (27), for $t = \tau, \ldots, T-2$

$$s^*_{t-1} + \frac{w^*_t}{(1+r^*_t)J} + \frac{w^*_{t+1}}{(1+r^*_t)(1+r^*_{t+1})J} > 0 \text{ and } c^*_t = 0.$$

Moreover, $s^*_{T-1} + \frac{w^*_T}{(1+r^*_T)J} > 0$ and $c^*_T = 0$, which is impossible, because using the structure of problem (28) for $t = T$ and $s^*_T = 0$, we get

$$0 = c^*_T = c^*_T + s^*_T \geq (1+r^*_T)s^*_{T-1} + \frac{w^*_T}{J} > 0, \text{ a contradiction.}$$

(b) $\tau = T-1$: Since $c^*_{T-2} > 0$ and $c^*_{T-1} = 0$, we have

$$\frac{1}{\delta(1+r^*_{T-1})u'(c^*_{T-1})} - \frac{1}{u'(c^*_{T-2})} = -\frac{1}{u'(c^*_{T-2})} < 0$$

and, by the structure of problem (27),

$$s^*_{T-2} = \overline{K} + \left(\frac{J-1}{J}\right)\frac{\overline{w}}{1+\overline{r}}.$$

At the same time, by (31),

$$s^*_{T-1} \geq (1+r^*_{T-1})s^*_{T-2} + \frac{w^*_{T-1}}{J}.$$

Therefore,

$$s_{T-1}^* + \frac{w_T^*}{(1+r_T^*)J} > 0. \tag{33}$$

Taking into account the structure of problem (27), we have

$$\frac{1}{\delta(1+r_T^*)u'(c_T^*)} - \frac{1}{u'(c_{T-1}^*)} \leq 0,$$

because otherwise we would have

$$s_{T-1}^* + \frac{w_T^*}{(1+r_T^*)J} = 0,$$

a contradiction of (33). Therefore, $\frac{1}{u'(c_T^*)} = 0$ and hence, using the structure of problem (28) for $t = T$ and $s_T^* = 0$,

$$0 = c_T^{j*} \geq (1+r_T^*)s_{T-1}^{j*} + \frac{w_T^*}{J} - s_T^{j*} = (1+r_T^*)s_{T-1}^{j*} + \frac{w_T^*}{J} > 0, \text{ a contradiction.}$$

(c) $t = T$: Since $c_{T-1}^* > 0$ and $c_T^* = 0$, we have

$$\frac{1}{\delta(1+r_T^*)u'(c_T^*)} - \frac{1}{u'(c_{T-1}^*)} = -\frac{1}{u'(c_{T-1}^*)} < 0$$

and, by the structure of problem (27),

$$s_{T-1}^* = \overline{K} + \left(\frac{J-1}{J}\right)\frac{\overline{w}}{1+\overline{r}}.$$

Therefore, because using the structure of problem (28) for $t = T$ and $s_T^* = 0$, we get

$$0 = c_T^* = c_T^* + s_T^* \geq (1+r_t^*)s_{T-1}^* + \frac{w_T^*}{J} > 0, \text{ a contradiction.}$$

□

Claim 3.

$$\sum_{j=1}^{J} s_{t-1}^{j*} \leq K_t^*, \; t = 0, 1, \ldots. \tag{34}$$

Proof. It follows from Claim 2 that

$$c_t^{j*} + s_t^{j*} \leq (1 + r_t^*) s_{t-1}^{j*} + \frac{w_t^*}{J}, \ t = 0, 1, \dots, T.$$

Also, we have

$$\sum_{j=1}^{J} \left[(1 + r_0^*) s_{-1}^{j*} + \frac{w_0^*}{J} \right] = f(K_0^*) < \overline{K}.$$

Therefore,

$$s_0^{j*} \leq c_0^{j*} + s_0^{j*} < \overline{K}.$$

Taking account of the structure of problem (29) and Remark 1, we obtain

$$\sum_{j=1}^{J} s_0^{j*} \leq K_1^*$$

Repeating the argument, we obtain (34).

□

Claim 4. *For each household $j = 1, \dots, J$ and for $t = 0, 1, \dots, T$,*

$$\left(1 + r_t^*\right) s_{t-1}^{j*} + \frac{w_t^*}{J} \leq \begin{cases} f\left(K_t^*\right) + \left(\frac{J-1}{J}\right) \cdot \frac{w_{t+1}^*}{1+r_{t+1}^*}, \ t = 0, 1, \dots, T-1, \\ f\left(K_t^*\right), \ t = T. \end{cases}$$

Proof. The constraints in (27) imply that for each household $j = 1, \dots, J$,

$$s_{t-1}^{j*} + \frac{w_t^*}{(1 + r_t^*)J} + \frac{w_{t+1}^*}{(1 + r_t^*)(1 + r_{t+1}^*)J} \geq 0, \ t = 0, \dots, T-1, \text{ and } s_{T-1}^{j*} + \frac{w_T^*}{(1 + r_T^*)J} \geq 0.$$

Therefore, taking account of Claim 3, for each $j = 1, \dots, J$, $t = 0, 1, \dots, T-1$

$$\begin{aligned} (1 + r_t^*) s_{t-1}^{j*} + \frac{w_t^*}{J} + \frac{w_{t+1}^*}{(1 + r_{t+1}^*)J} &\leq \sum_{i=1}^{J} \left[(1 + r_t^*) s_{t-1}^{i*} + \frac{w_t^*}{J} + \frac{w_{t+1}^*}{(1 + r_{t+1}^*)J} \right] \\ &\leq (1 + r_t^*) K_t^* + w_t^* + \frac{w_{t+1}^*}{(1 + r_{t+1}^*)} = f(K_t^*) + \frac{w_{t+1}^*}{(1 + r_{t+1}^*)}, \end{aligned}$$

$$(1+r_T^*)s_{T-1}^{j*}+\frac{w_T^*}{J}\leq\sum_{i=1}^{J}\left[(1+r_T^*)s_{T-1}^{i*}+\frac{w_T^*}{J}\right]\leq(1+r_T^*)K_T^*+w_T^*=f(K_T^*).$$

□

Claim 5. *For each $j=1,\dots,J$,*

$$c_t^{j*}+s_t^{j*}=(1+r_t^*)s_{t-1}^{j*}+\frac{w_t^*}{J},\ t=0,\dots,T.$$

Proof. By Claim 4 and the constraints in (27) and (29), for $t=0,1,\dots,T-2$,

$$\begin{aligned}(1+r_t^*)s_{t-1}^{j*}+\frac{w_t^*}{J}-s_t^{j*}&\leq f(K_t^*)+\left(\frac{J-1}{J}\right)\cdot\frac{w_{t+1}^*}{1+r_{t+1}^*}+\frac{w_{t+1}^*}{(1+r_{t+1}^*)J}+\frac{w_{t+2}^*}{(1+r_{t+1}^*)(1+r_{t+2}^*)J}\\&\leq f(K_t^*)+\frac{w_{t+1}^*}{1+r_{t+1}^*}+\frac{w_{t+2}^*}{(1+r_{t+1}^*)(1+r_{t+2}^*)}\\&\leq f(\overline{K})+\frac{\overline{w}}{(1+\overline{r})}+\frac{\overline{w}}{(1+\overline{r})^2}=\overline{c},\ \text{and}\\(1+r_{T-1}^*)s_{T-2}^{j*}+\frac{w_{T-1}^*}{J}-s_{T-1}^{j*}&\leq f(K_{T-1}^*)+\left(\frac{J-1}{J}\right)\cdot\frac{w_T^*}{1+r_T^*}+\frac{w_T^*}{(1+r_T^*)J}\\&\leq f(K_{T-1}^*)+\frac{w_T^*}{1+r_T^*}\leq f(\overline{K})+\frac{\overline{w}}{(1+\overline{r})}+\frac{\overline{w}}{(1+\overline{r})^2}=\overline{c}.\end{aligned}$$

It follows from the structure of problem (28) that for all $j=1,\dots,J$,

$$c_t^{j*}\geq(1+r_t^*)s_{t-1}^{j*}+\frac{w_t^*}{J}-s_t^{j*},\ t=0,1,\dots,T.$$

Taking account of Claim 2, we obtain (5). □

Claim 6. *For all $t=0,1,\dots,T$, $K_t^*>\widetilde{K}_t$ and hence $\sum_{j=1}^{J}s_{t-1}^{j*}=K_t^*$.*

Proof. Note that, by the choice of $\widetilde{K}_0$, $\sum_{j=1}^{J}s_{-1}^{j*}=K_0^*>\widetilde{K}_0$ and assume that for some $t=1,\dots,T$,

$$\sum_j s_{t-2}^{j*}=K_{t-1}^*>\widetilde{K}_{t-1}\ \text{and}\ \sum_{j=1}^{J}s_{t-1}^{j*}\leq K_t^*\leq\widetilde{K}_t.$$

By Claim 5,

$$\sum_{j=1}^{J}\left(c_{t-1}^{j*}+s_{t-1}^{j*}\right)\geq\sum_{j=1}^{J}\left[(1+r_{t-1}^{*})s_{t-2}^{j*}+\frac{w_{t-1}^{*}}{J}\right]=(1+r_{t-1}^{*})K_{t-1}^{*}+w_{t-1}^{*}=f(K_{t-1}^{*})>f(\widetilde{K}_{t-1}).$$

Therefore,

$$\sum_{j=1}^{J}c_{t-1}^{j*}>f(\widetilde{K}_{t-1})-\sum_{j=1}^{J}s_{t-1}^{j*}\geq f(\widetilde{K}_{t-1})-\widetilde{K}_{t}.$$

It follows that there is some household j such that

$$c_{t-1}^{j*}>\frac{f(\widetilde{K}_{t-1})-\widetilde{K}_{t}}{J}>0 \tag{35}$$

and hence, by (26) and the constraints in (29),

$$\frac{1}{u_j'(c_{t-1}^{j*})}>\frac{1}{u_j'\left(\frac{f(\widetilde{K}_{t-1})-\widetilde{K}_{t}}{J}\right)}\geq\frac{1}{\delta_j f'(\overline{K})u_j'(\overline{c}_t)}\geq\frac{1}{\delta_j f'(K_t^{*})u_j'(c_t^{j*})}=\frac{1}{\delta_j(1+r_t^{*})u_j'(c_t^{j*})}.$$

By the structure of problem (27), for j satisfying (35), we have

$$s_{t-1}^{j*}=\overline{K}+\left(\frac{J-1}{J}\right)\cdot\left(\frac{\overline{w}}{1+\overline{r}}\right).$$

Hence, by Claim 4 and the constraints in (29),

$$\begin{aligned}(1+r_{t-1}^{*})s_{t-2}^{j*}+\frac{w_{t-1}^{*}}{J}-s_{t-1}^{j*}&\leq f(K_{t-1}^{*})+\left(\frac{J-1}{J}\right)\frac{w_t^{*}}{1+r_t^{*}}-s_{t-1}^{j*}\\&\leq\overline{K}+\left(\frac{J-1}{J}\right)\frac{\overline{w}}{1+\overline{r}}-\left[\overline{K}+\left(\frac{J-1}{J}\right)\frac{\overline{w}}{1+\overline{r}}\right]=0,\end{aligned}$$

which implies $c_{t-1}^{j*}=0$, a contradiction of (35).

□

Claim 7. *For all households $j=1,\ldots,J$, the Ramsey - Euler inequalities / equalities (3) - (4) hold.*

Proof. To prove (3), assume that for some j and $t < T$,

$$\frac{1}{u_j'(c_t^{j*})} > \frac{1}{\delta_j(1+r_{t+1}^*)u_j'(c_{t+1}^{j*})}$$

and therefore, by the structure of problem (27), $s_t^{j*} = \overline{K} + \left(\frac{J-1}{J}\right)\frac{\overline{w}}{1+\overline{r}}$. This implies that $(1+r_t^*)s_{t-1}^{j*} + \frac{w_t^*}{J} - s_t^{j*} \le 0$, a contradiction of Claim 2. This contradiction proves that

$$\frac{1}{u_j'(c_t^{j*})} \le \frac{1}{\delta_j(1+r_{t+1}^*)u_j'(c_{t+1}^{j*})}.$$

It remains to note that if this inequality fulfills as a strict inequality, then, by the structure of problem (27),

$$s_t^{j*} = -\frac{w_{t+1}^*}{(1+r_{t+1}^*)J} - \frac{w_{t+2}^*}{(1+r_{t+1}^*)(1+r_{t+2}^*)J}.$$

□

The Claims 2-7 complete the proof of the Lemma 2.

STEP 3 Equilibrium in the finite time horizon Ramsey economy:

Using Lemma 1 and 2 we have established the existence of an equilibrium in the finite time horizon (T periods) Ramsey economy with liberal borrowing for the case where borrowing horizon $N = 2$. The proof for the case of $N > 2$ is similar in nature. Also, in the no borrowing case, $N = 0$, the savings for each household would need to be non-negative in each period. It would restrict the savings for households in period $t-1$ to be in the interval $[0, K_t]$. We can show the existence of an equilibrium in the finite time horizon (T periods) Ramsey economy with no borrowing, $N = 0$ following similar arguments.

Proposition 4. *Consider a Ramsey economy with liberal borrowing, $\mathscr{E}$, with $\kappa_0 > 0$ and $s_{-1}^j + \frac{w_0}{J(1+r_0)} \ge 0$ for each household. For any $T \in \mathbb{N}$ with $T > N$, there exists a T periods equilibrium.*

7.1.2 Equilibrium for the infinite time horizon Ramsey Economy

STEP A Candidate Equilibrium path: Let for $T = 1, 2, \ldots,$

$$\mathbb{P}_T = \left\{1 + r_t^*(T), w_t^*(T), K_t^*(T), \left(c_t^{j*}(T), s_t^{j*}(T)\right),\ j = 1, \ldots, J; t = 0, 1, \ldots, T\right\}$$

be a finite T periods equilibrium path. We can apply the following process to the sequence $\{\mathbb{P}_T\}_{T=1,2,\ldots}$.

(a) At the first step of this process we take a cluster point of the sequence

$$\left\{1 + r_0^*(T), w_0^*(T), K_0^*(T), \left(c_0^{j*}(T), s_0^{j*}(T)\right),\ j = 1, \dots, J\right\}_{T=1,2,\dots},$$

denote it as $\left\{1 + r_0^*, w_0^*, K_0^*, \left(c_0^{j*}, s_0^{j*}\right),\ j = 1, \dots, J\right\}$, and extract a subsequence $\{T_{0n}\}_{n=1,2,\dots}$ from the sequence $\{T\}_{T=1,2,\dots}$ such that

$$\left\{1 + r_0^*\left(T_{0n}\right), w_0^*\left(T_{0n}\right), K_0^*\left(T_{0n}\right), \left(c_0^{j*}\left(T_{0n}\right), s_0^{j*}\left(T_{0n}\right)\right),\ j = 1, \dots, J\right\}_{n=1,2,\dots}$$

converges to $\{1 + r_0^*, w_0^*, K_0^*, \left(c_0^{j*}, s_0^{j*}\right),\ j = 1, \dots, J\}$.

(b) At the second step we take a cluster point of the sequence

$$\left\{1 + r_1^*\left(T_{0n}\right), w_1^*\left(T_{0n}\right), K_1^*\left(T_{0n}\right), \left(c_1^{j*}\left(T_{0n}\right), s_1^{j*}\left(T_{0n}\right)\right),\ j = 1, \dots, J\right\}_{n=1,2,\dots}$$

denoting it as $\left\{1 + r_1^*, w_1^*, K_1^*, \left(c_1^{j*}, s_1^{j*}\right),\ j = 1, \dots, J\right\}$, and extract a subsequence $\{T_{1n}\}_{n=1,2,\dots}$ from the sequence $\{T_{0n}\}_{n=1,2,\dots}$ such that $T_{11} > 1$ and

$$\left\{1 + r_1^*\left(T_{1n}\right), w_1^*\left(T_{1n}\right), K_1^*\left(T_{1n}\right), \left(c_1^{j*}\left(T_{1n}\right), s_1^{j*}\left(T_{1n}\right)\right),\ j = 1, \dots, J\right\}_{n=1,2,\dots}$$

converges to $\left\{1 + r_1^*, w_1^*, K_1^*, \left(c_1^{j*}, s_1^{j*}\right),\ j = 1, \dots, J\right\}$, …, and so on, *ad infinitum*.

As a result, we obtain an infinite path

$$\mathbb{P}_\infty = \left\{1 + r_t^*, w_t^*, K_t^*, \left(c_t^{j*}, s_t^{j*}\right),\ j = 1, \dots, J; t = 0, 1, \dots\right\}.$$

We claim that $\mathbb{P}_\infty$ is an equilibrium path for the Ramsey economy with liberal borrowing. It is clear that $\mathbb{P}_\infty$ satisfies conditions (E2)-(E4). Thus, to prove that $\mathbb{P}_\infty$ is an equilibrium, it is sufficient to show that it satisfies (E1).

STEP B Lower bound for K on tail of any finite T period equilibrium: Let $0 < K' < \kappa_0$ be such that $(1 + r')\delta_J > 1$, where $1 + r' = 1 + r(K')$, and let $w' = w(K')$.

Claim 8. *There is a T' such that, for any finite T period equilibrium and any $t \leq T - T'$,*

$$K_t > K', w_t > w', 1 + r_t < 1 + r'.$$

Proof. Let τ be such that $K_\tau > K'$ and $K_{\tau+1} \leq K'$. We have

$$\sum_{j=1}^{J} c_\tau^j = f(K_\tau) - K_{\tau+1} > f(K') - K'.$$

Therefore, there is j_0 such that $c_\tau^{j_0} > \frac{f(K')-K'}{J}$. Also, for all j,

$$u_j'\left(c_{\tau+1}^j\right) < \left(1+r'\right)\delta_J u_j'\left(c_{\tau+1}^j\right) \leq \left(1+r_{\tau+1}\right)\delta_J u_j'\left(c_{\tau+1}^j\right) \leq \left(1+r_{\tau+1}\right)\delta_j u_j'\left(c_{\tau+1}^j\right) \leq u_j'\left(c_\tau^j\right)$$

and hence $c_{\tau+1}^j > c_\tau^j$. It follows that

$$f(K_{\tau+1}) - K_{\tau+2} = \sum_{j=1}^{J} c_{\tau+1}^j > \sum_{j=1}^{J} c_\tau^j = f(K_\tau) - K_{\tau+1}.$$

Thence $K_{\tau+2} < K_{\tau+1}$ and therefore, for any j,

$$u_j'\left(c_{\tau+2}^j\right) < \left(1+r'\right)\delta_J u_j'\left(c_{\tau+2}^j\right) < \left(1+r_{\tau+2}\right)\delta_J u_j'\left(c_{\tau+2}^j\right) \leq \left(1+r_{\tau+2}\right)\delta_j u_j'\left(c_{\tau+2}^j\right) \leq u_j'\left(c_{\tau+1}^j\right).$$

Repeating the argument, we obtain for all j, $\left(1+r'\right)\delta_J u_j'\left(c_{\tau+1}^j\right) \leq u_j'\left(c_\tau^j\right)$ and

$$\left(1+r'\right)\delta_J u_j'\left(c_{t+1}^j\right) < u_j'\left(c_t^j\right), \quad t = \tau+1, \tau+2, \ldots, T-1.$$

Since, clearly, $c_t^j < \overline{K}$ for all j,

$$\begin{aligned}\left(\left(1+r'\right)\delta_J\right)^{t-\tau} u_{j_0}'\left(\overline{K}\right) &< \left(\left(1+r'\right)\delta_J\right)^{t-\tau} u_{j_0}'\left(c_t^{j_0}\right) \\ &\leq u_{j_0}'\left(c_\tau^{j_0}\right) < u_{j_0}'\left(\frac{f(K')-K'}{J}\right), t = \tau+1, \ldots, T\end{aligned}$$

and thus

$$\left(\left(1+r'\right)\delta_J\right)^{t-\tau} \leq \frac{u_{j_0}'\left(\frac{f(K')-K'}{J}\right)}{u_{j_0}'(\overline{K})} \leq \max_j \frac{u_j'\left(\frac{f(K')-K'}{J}\right)}{u_j'(\overline{K})},$$

where the RHS is a finite number. This along with the fact that $\left(1+r'\right)\delta_J > 1$ shows that $t-\tau$ cannot be arbitrarily large. □

Let $c' > 0$ be such that

$$\left(1 + (1 + r') + (1 + r')^2\right) c' < \frac{w'}{J}, \text{ and,} \tag{36}$$

$$\left(1 + \frac{1}{1 + \bar{r}} + \ldots + \frac{1}{(1 + \bar{r})^{T'}}\right) c' < \frac{w'}{J}. \tag{37}$$

The following claim will be useful in the proof of Lemma 3.

Claim 9. *Suppose that $F_r(x, y)$, $r = 1, \ldots, R$, are continuous and concave in y functions defined on $X \times Y$, where X and Y are convex compact subsets of finite dimensional spaces. If there exists $\hat{y} \in Y$ such that $F_r(x, \hat{y}) > 0$ for all $x \in X$, $r = 1, \ldots, R$, then the correspondence*

$$x \to \bigcap_{r=1}^{R} \{y \in Y \mid F_r(x, y) \geq 0\}$$

is upper and lower semi-continuous, and all sets

$$\bigcap_{r=1}^{R} \{y \in Y \mid F_r(x, y) \geq 0\}$$

are non-empty, convex and closed.

Proof. It is trivial. □

Lemma 3. *For each $j = 1, \ldots, J$, $\left\{\left(c_t^{j*}, s_t^{j*}\right),\ t = 0, 1, \ldots\right\}$ is a solution to $\mathscr{P}(j)$ with $w_t = w_t^*$ and $r_t = r_t^*$, $t = 0, 1, \ldots$.*

Proof. Without loss of generality we assume that $u_j(c') = 0$, $j = 1, \ldots, J$. Assume the converse. Then for some j (we fix this j and drop its reference for ease of notation in the remaining part of the proof of Lemma 3) there is a feasible sequence $\{(\widehat{c}_t, \widehat{s}_t),\ t = 0, 1, \ldots\}$ such that

$$\widehat{V} > V^*, \text{ where } \widehat{V} \equiv \sum_{t=0}^{\infty} \delta^t u\left(\widehat{c}_t\right), \text{ and } V^* \equiv \sum_{t=0}^{\infty} \delta^t u\left(c_t^*\right).$$

Choose $0 < \epsilon < \widehat{V} - V^*$ and let Θ be such that

$$\sum_{t=\Theta+1}^{\infty} \delta^t u\left(\overline{K}\right) < \frac{\epsilon}{2}.$$

Further, let

$$V^{*\Theta} \equiv \sum_{t=0}^{\Theta} \delta^t u\left(c_t^*\right), \ \widehat{V}^{\Theta} \equiv \sum_{t=0}^{\Theta} \delta^t u\left(\widehat{c}_t\right);$$

$$V^*(T) \equiv \sum_{t=0}^{T} \delta^t u\left(c_t^*(T)\right), \text{ and } V^{*\Theta}(T) \equiv \sum_{t=0}^{\Theta} \delta^t u\left(c_t^*(T)\right), \ T = \Theta + T' + 2, \Theta + T' + 3, \dots.$$

Also, we define $W^{*\Theta}$ as the Θ period maximum feasible utility given the wage w_t^* and rental rate $1 + r_t^*$ (i.e., on the candidate infinite horizon equilibrium path) as follows:

$$W^{*\Theta} \equiv \begin{cases} \max & \sum_{t=0}^{\Theta} \delta^t u\left(c_t\right); \\ \text{subject to} & c_t + s_t \leq (1 + r_t^*) s_{t-1} + \frac{w_t^*}{J}, t = 0, 1, \dots, \Theta, \\ \text{and} & s_t + \frac{w_{t+1}^*}{(1+r_{t+1}^*)J} + \frac{w_{t+2}^*}{(1+r_{t+1}^*)(1+r_{t+2}^*)J} \geq 0, t = 0, 1, \dots, \Theta, \\ \text{where} & s_{-1} \text{ is given,} \end{cases}$$

and $W^{*\Theta}(T)$ as the Θ period maximum feasible utility given the wage $w_t^*(T)$ and rental rate $1 + r_t^*(T)$ (i.e., on the finite T period equilibrium path) as follows:

$$W^{*\Theta}(T) \equiv \begin{cases} \max & \sum_{t=0}^{\Theta} \delta^t u\left(c_t\right) \\ \text{subject to} & c_t + s_t \leq (1 + r_t^*(T)) s_{t-1} + \frac{w_t^*(T)}{J}, t = 0, 1, \dots, \Theta, \\ \text{and} & s_t + \frac{w_{t+1}^*(T)}{(1+r_{t+1}^*(T))J} + \frac{w_{t+2}^*}{(1+r_{t+1}^*)(1+r_{t+2}^*)J} \geq 0, t = 0, 1, \dots, \Theta, \\ \text{where} & s_{-1} \text{ is given,} \end{cases} \tag{38}$$

for $T = \Theta + T' + 4, \Theta + T' + 5, \dots$. Since $\mathbb{P}_\infty$ is obtained as a result of the application of the process described in sub-section 7.1.2 to the sequence $\{\mathbb{P}_T\}_{T=1,2,\dots}$, we have for $t = 0, 1, \dots, \Theta$

$$\lim_{n\to\infty} K_t^*(T_{\Theta n}) = K_t^*, \ \lim_{n\to\infty} w_t^*(T_{\Theta n}) = w_t^*, \ \lim_{n\to\infty} 1 + r_t^*(T_{\Theta n}) = 1 + r_t^*,$$

$$\lim_{n\to\infty} c_t^*(T_{\Theta n}) = c_t^* \text{ and } \lim_{n\to\infty} s_t^*(T_{\Theta n}) = s_t^*, \ j = 1, \dots, J.$$

With no loss of generality we suppose that $T_{\Theta n} > \Theta + T' + 4$ for any n.

(a) We show that $W^{*\Theta}(T_{\Theta n}) \underset{n\to\infty}{\rightarrow} W^{*\Theta}$.

It is sufficient to note that, by Claim 9, the correspondence that takes to each

$$\left\{\left(1+r_0, w_0\right), \ldots, \left(1+r_{\Theta+1}, w_{\Theta+1}\right)\right\} \in \prod_{t=0}^{\Theta+1}\left(\left[1+r\left(\overline{K}_t\right), 1+r\left(\widetilde{K}_t\right)\right] \times \left[w\left(\widetilde{K}_t\right), w\left(\overline{K}_t\right)\right]\right)$$

the set $\left(\left(s_0, c_0\right), \ldots, \left(s_\Theta, c_\Theta\right)\right) \in \mathbb{R}^{2(\Theta+1)}$ is such that, with s_{-1} given,

$$\left(1+r_t^*(T)\right) s_{t-1} + \frac{w_t^*(T)}{J} - c_t - s_t \geq 0, \text{ and}$$
$$s_t + \frac{w_{t+1}^*(T)}{\left(1+r_{t+1}^*(T)\right) J} + \frac{w_{t+2}^*(T)}{\left(1+r_{t+1}^*(T)\right)\left(1+r_{t+2}^*(T)\right) J} \geq 0, \text{ hold } \forall t = 0, \ldots, \Theta,$$

is lower- and upper- semicontinuous, and to apply the Maximum Theorem.

(b) Let, for some $T > \Theta + T' + 4$, $((\check{s}_0, \check{c}_0), \ldots, (\check{s}_\Theta, \check{c}_\Theta))$ be a solution to (38). Let further for $t = \Theta+1, \ldots, T$, $(\check{s}_t, \check{c}_t)$ be defined recursively by

$$\check{c}_t = c', \; \check{s}_t = (1+r_t^*(T))\check{s}_{t-1} + \frac{w_t^*(T)}{J} - \check{c}_t.$$

We show that the sequence $\left(\left(\check{s}_0, \check{c}_0\right), \ldots, \left(\check{s}_\Theta, \check{c}_\Theta\right), \left(\check{s}_{\Theta+1}, \check{c}_{\Theta+1}\right), \ldots, \left(\check{s}_T, \check{c}_T\right)\right)$ is feasible for the problem:

$$\begin{cases} \max & \sum_{t=0}^{T} \delta^t u\left(c_t\right), \\ \textit{subject to} & c_t + s_t \leq \left(1+r_t^*(T)\right) s_{t-1} + \frac{w_t^*(T)}{J}, t = 0, 1, \ldots, T, \\ \textit{and} & s_t + \frac{w_{t+1}^*(T)}{\left(1+r_{t+1}^*(T)\right)J} + \frac{w_{t+2}^*(T)}{(1+r_{t+1}^*(T))(1+r_{t+2}^*(T))J} \geq 0, t = 0, 1, \ldots, T-2, \\ & s_{T-1} + \frac{w_T^*(T)}{(1+r_T^*(T))J} \geq 0, \; s_T \geq 0, \; \textit{where } s_{-1} \textit{ is given.} \end{cases} \tag{39}$$

We have

$$\check{s}_\Theta \geq -\frac{w_{\Theta+1}^*(T)}{\left(1+r_{\Theta+1}^*(T)\right) J} - \frac{w_{\Theta+2}^*(T)}{(1+r_{\Theta+1}^*(T))(1+r_{\Theta+2}^*(T))J}. \tag{40}$$

Therefore,

$$\check{s}_{\Theta+1} = (1+r_{\Theta+1}^*(T))\check{s}_\Theta + \frac{w_{\Theta+1}^*(T)}{J} - \check{c}_{\Theta+1} \geq -\frac{w_{\Theta+2}^*(T)}{\left(1+r_{\Theta+2}^*(T)\right) J} - c'.$$

By Claim 8 and (36), we have

$$c' < \frac{w'}{(1+r')^2 J} < \frac{w^*_{\Theta+3}(T)}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))J}.$$

Therefore,

$$\check{s}_{\Theta+1} \geq -\frac{w^*_{\Theta+2}(T)}{\left(1+r^*_{\Theta+2}(T)\right)J} - \frac{w^*_{\Theta+3}(T)}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))J}.$$

Repeating the argument we obtain

$$\check{s}_{\Theta+2} \geq -\frac{w^*_{\Theta+3}(T)}{\left(1+r^*_{\Theta+3}(T)\right)J} - \frac{w^*_{\Theta+4}(T)}{(1+r^*_{\Theta+3}(T))(1+r^*_{\Theta+4}(T))J}.$$

Now we show that

$$\check{s}_{\Theta+3} > 0. \tag{41}$$

Indeed, taking account of the choice of $\check{s}_t$ for $t = \Theta+1, \Theta+2, \Theta+3$ and (40), we get

$$\begin{aligned}
& c' + \frac{c'}{\left(1+r^*_{\Theta+2}(T)\right)} + \frac{c'}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))} + \frac{\check{s}_{\Theta+3}}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))} \\
&= \check{c}_{\Theta+1} + \frac{\check{c}_{\Theta+2}}{\left(1+r^*_{\Theta+2}(T)\right)} + \frac{\check{c}_{\Theta+3}}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))} + \frac{\check{s}_{\Theta+3}}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))} \\
&= (1+r^*_{\Theta+1}(T))\check{s}_{\Theta} + \frac{w^*_{\Theta+1}(T)}{J} + \frac{w^*_{\Theta+2}(T)}{\left(1+r^*_{\Theta+2}(T)\right)J} + \frac{w^*_{\Theta+3}(T)}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))J} \\
&\geq \frac{w^*_{\Theta+3}(T)}{(1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))J}.
\end{aligned}$$

Therefore, by Claim 8 and (36)

$$\begin{aligned}
\check{s}_{\Theta+3} &\geq \frac{w^*_{\Theta+3}(T)}{J} - \left(c' + (1+r^*_{\Theta+2}(T))c' + (1+r^*_{\Theta+2}(T))(1+r^*_{\Theta+3}(T))c'\right) \\
&> \frac{w'}{J} - \left(1 + (1+r') + (1+r')^2\right) c' > 0.
\end{aligned}$$

This proves (41). To complete the proof it is sufficient to check that $\check{s}_t \geq 0,\ t = \Theta+3, \Theta+4, \ldots, T$. We have proved this inequality for $t = \Theta+3$ and prove it for $t = \Theta+3, \Theta+4, \ldots, T-T'-1$ by induction. Suppose we have proved that $\check{s}_t > 0$ for

$\Theta + 3 \le t < T - T' - 1$. Then, by Claim 8 and the inequality $c' < \frac{w'}{J}$, which follows from (36), we have

$$\check{s}_{t+1} = (1 + r^*_{t+1}(T))\check{s}_t + \frac{w^*_{t+1}(T)}{J} - \check{c}_{t+1} > \frac{w'}{J} - c' > 0.$$

Thus, $\check{s}_t > 0,\ t = \Theta + 3, \Theta + 4, \dots, T - T' - 1$. In particular, $\check{s}_{T-T'-1} > 0$. Hence, by (37),

$$\begin{aligned}\check{s}_{T-T'} &= (1 + r^*_{T-T'}(T))\check{s}_{T-T'-1} + \frac{w^*_{T-T'}(T)}{J} - \check{c}_{T-T'} > \frac{w'}{J} - c' \\ &> \left(\frac{1}{1+\bar{r}} + \dots + \frac{1}{(1+\bar{r})^{T'}}\right) c' > 0,\end{aligned}$$

$$\begin{aligned}\check{s}_{T-T'+1} &= (1 + r^*_{T-T'+1}(T))\check{s}_{T-T'} + \frac{w^*_{T-T'+1}(T)}{J} - \check{c}_{T-T'+1} > (1+\bar{r})\check{s}_{T-T'} - c' \\ &> \left(1 + \frac{1}{1+\bar{r}} + \dots + \frac{1}{(1+\bar{r})^{T'-1}}\right) c' - c' = \left(\frac{1}{1+\bar{r}} + \dots + \frac{1}{(1+\bar{r})^{T'-1}}\right) c' > 0,\end{aligned}$$

$$\begin{aligned}\check{s}_{T-T'+2} &= (1 + r^*_{T-T'+2}(T))\check{s}_{T-T'+1} + \frac{w^*_{T-T'+2}(T)}{J} - \check{c}_{T-T'+2} > (1+\bar{r})\check{s}_{T-T'+1} - c' \\ &> \left(1 + \frac{1}{1+\bar{r}} + \dots + \frac{1}{(1+\bar{r})^{T'-2}}\right) c' - c' = \left(\frac{1}{1+\bar{r}} + \dots + \frac{1}{(1+\bar{r})^{T'-2}}\right) c' > 0,\end{aligned}$$

$$\dots$$

$$\begin{aligned}\check{s}_{T-1} &= (1 + r^*_{T-1}(T))\check{s}_{T-2} + \frac{w^*_{T-1}(T)}{J} - \check{c}_{T-1} > (1+\bar{r})\check{s}_{T-1} - c' \\ &> \left(1 + \frac{1}{1+\bar{r}}\right) c' - c' = \frac{1}{1+\bar{r}} c' > 0, \text{ and} \\ \check{s}_T &= (1 + r^*_T(T))\check{s}_{T-1} + \frac{w^*_T(T)}{J} - \check{c}_T > (1+\bar{r})\check{s}_{T-1} - c' > c' - c' = 0.\end{aligned}$$

(c) We show that $V^*(T) \ge W^{*\Theta}(T),\ T = \Theta + T' + 4, \Theta + T' + 5, \dots$.
The sequence $((\check{s}_0, \check{c}_0), \dots, (\check{s}_\Theta, \check{c}_\Theta), (\check{s}_{\Theta+1}, \check{c}_{\Theta+1}) \dots, (\check{s}_T, \check{c}_T))$ is feasible for problem (39) whereas $((s^*_0(T), c^*_0(T)), \dots, (s^*_T(T), c^*_T(T))$ is a solution to this problem. Therefore,

$$V^*(T) = \sum_{t=0}^{T} \delta^t u\left(c_t^*(T)\right) \geq \sum_{t=0}^{\Theta} \delta^t u\left(\check{c}_t\right) + \sum_{t=\Theta+1}^{T} \delta^t u\left(c'\right) = \sum_{t=0}^{\Theta} \delta^t u\left(\check{c}_t\right) = W^{*\Theta}(T).$$

By the choice of Θ, we have

$$V^{*\Theta}(T) \geq V^*(T) - \frac{\epsilon}{2},\ T = \Theta + 1, \Theta + 2, \ldots,\ \text{and},\ W^{*\Theta} \geq \widehat{V}^{\Theta} > \widehat{V} - \frac{\epsilon}{2}. \tag{42}$$

Also we clearly have

$$V^* \geq V^{*\Theta},\ \text{and},\ V^{*\Theta}(T_{\Theta n}) \to V^{*\Theta} \text{ as } n \to \infty. \tag{43}$$

Combining (42) - (43) and using results in (a) and (c), we obtain

$$V^* \geq V^{*\Theta} = \lim_{n\to\infty} V^{*\Theta}(T_{\Theta n}) \geq \lim_{n\to\infty} W^{*\Theta}(T_{\Theta n}) - \frac{\epsilon}{2} = W^{*\Theta} - \frac{\epsilon}{2} \geq \widehat{V} - \epsilon,$$

which contradicts the choice of ϵ. This contradiction completes the proof of the Lemma 3. □

STEP C Having established Lemma 3, we have proved the following proposition.

Proposition 5. *The sequences* $\mathbb{P}_\infty = \left\{1 + r_t^*, w_t^*, K_t^*, \left(c_t^{j*}, s_t^{j*}\right),\ j = 1, \ldots, J; t = 0, 1, \ldots\right\}$ *constitute an equilibrium for the Ramsey economy with liberal borrowing.*

Also Proposition 5 completes the proof of Theorem 1 for the case where borrowing horizon $N = 2$. The proof for the case of $N \neq 2$ is similar in nature.

7.2 Proofs of results in Section 4

7.2.1 Proof of Proposition 2

Let $\{1 + r_t, w_t, K_t, \left(c_t^j, s_t^j\right),\ j = 1, \ldots, J;\ t = 0, 1, \ldots\}$ be a Ramsey equilibrium such that $K_\infty = \lim_{t\to\infty} K_t$ exists. It follows that $1 + r_t \to 1 + r\left(K_\infty\right) \leq \infty$ as $t \to \infty$.

Claim 10. $\delta_1\left(1 + r\left(K_\infty\right)\right) \leq 1$ *and hence* $K_\infty \geq K^{**}$.

Proof. By applying the Ramsey-Euler inequalities (3) to household $j = 1$ and using the inequality $c_t^1 < \overline{K},\ t = 0, 1, \ldots$, we obtain

$$\prod_{t=1}^{T} \delta_1 \left(1 + r_t\right) \leq \frac{u_1' \left(c_0^1\right)}{u_1' \left(c_T^1\right)} \leq \frac{u_1' \left(c_0^1\right)}{u_1' \left(\overline{K}\right)},\ T = 1, 2, \ldots .$$

Therefore,

$$\limsup_{T \to \infty} \prod_{t=1}^{T} \delta_1 \left(1 + r_t\right) < \infty \text{ and hence } \lim_{t \to \infty} \delta_1 \left(1 + r_t\right) \leq 1.$$

□

Claim 11. *If $\delta_j \left(1 + r \left(K_\infty\right)\right) < 1$, then for any τ there is $t > \tau$ such that*

$$s_t^j + \frac{A\left(t+1, N\right)}{J} = 0.$$

Proof. Assume the converse. Then there is t_0 such that

$$s_t^j + \frac{A\left(t+1, N\right)}{J} > 0$$

for all $t \geq t_0$. By (4),

$$\frac{u_j' \left(c_{t_0-1}^j\right)}{u_j' \left(c_T^j\right)} = \prod_{t=t_0}^{T} \delta_j \left(1 + r_t\right) \to 0 \text{ as } T \to \infty.$$

Therefore, $\lim_{T \to \infty} u_j' \left(c_T^j\right) = \infty$ and hence $\lim_{T \to \infty} c_T^j = 0$. It follows that

$$c_t^j < \frac{w_{t+N}}{\left(1 + r_{t+1}\right) \cdots \left(1 + r_{t+N}\right) J}$$

for all t large enough, which is impossible because $\{\left(c_t^j, s_t^j\right) : t = 0, 1, \ldots\}$ solves problem $\mathscr{P}(j)$.

□

Claim 12. *If $\delta_j \left(1 + r \left(K_\infty\right)\right) < 1$, then*

$$s_t^j + \frac{A\left(t+1, N\right)}{J} = 0 \textit{ for all } t \textit{ large enough.}$$

Proof. First note that

$$\lim_{t\to\infty} \frac{w_t}{\left(1+r_t\right) J} = \frac{w\left(K_\infty\right)}{\left(1+r\left(K_\infty\right)\right) J} > 0,$$

there exists T such that for all $t \geq T$,

$$u_j'\left(\frac{w_{t+N-1}}{\left(1+r_t\right)\cdots\left(1+r_{t+N-1}\right) J}\right) > \delta_j\left(1+r_t\right) u_j'\left(\frac{w_{t+N}}{\left(1+r_{t+1}\right)\cdots\left(1+r_{t+N}\right) J}\right). \tag{44}$$

If the claim is not correct then by Claim 11, there are $t_1 > T$ and $t_2 > t_1$ such that

$$s^j_{t_1-1} + \frac{A\left(t_1, N\right)}{J} = 0;\ s^j_{t-1} + \frac{A\left(t, N\right)}{J} > 0,\ t = t_1+1, \ldots, t_2;\ \text{and}$$
$$s^j_{t_2} + \frac{A\left(t_2+1, N\right)}{J} = 0.$$

Therefore,

$$\begin{aligned}
c^j_{t_1} &= \left(1+r_{t_1}\right) s^j_{t_1-1} + \frac{w_{t_1}}{J} - s^j_{t_1} \\
&= \left(1+r_{t_1}\right)\left[s^j_{t_1-1} + \frac{A\left(t_1, N\right)}{J}\right] - \left[\frac{A\left(t_1+1, N-1\right)}{J}\right] - s^j_{t_1} \\
&= -\left[s^j_{t_1} + \frac{A\left(t_1+1, N-1\right)}{J}\right] < \frac{w_{t_1+N}}{\left(1+r_{t_1+1}\right)\cdots\left(1+r_{t_1+N-1}\right) J},\ \text{and} \\
c^j_{t_2} &= \left(1+r_{t_2}\right) s^j_{t_2-1} + \frac{w_{t_2}}{J} - s^j_{t_2} = \left(1+r_{t_2}\right) s^j_{t_2-1} + \frac{A\left(t_2, N+1\right)}{J} - \left[s^j_{t_2} + \frac{A\left(t_2+1, N\right)}{J}\right] \\
&= \left(1+r_{t_2}\right)\left[s^j_{t_2-1} + \frac{A\left(t_2, N\right)}{J}\right] + \frac{w_{t_2+N}}{\left(1+r_{t_2+1}\right)\cdots\left(1+r_{t_2+N}\right) J} > \frac{w_{t_2+N}}{\left(1+r_{t_2+1}\right)\cdots\left(1+r_{t_2+N}\right) J}.
\end{aligned}$$

These inequalities show that the value of consumption in period t_1 is less than the discounted values of wage income in period t_1+N, and the value of consumption in period t_2 is greater than the discounted values of wage income in period t_2+N. Since, by (4), $\delta_j\left(1+r_t\right) u_j'\left(c^j_t\right) = u_j'\left(c^j_{t-1}\right)$, $t = t_1+1, t_1+2, \ldots, t_2$, we get

$$u_j'\left(\frac{w_{t_1+N}}{\left(1+r_{t_1+1}\right)\cdots\left(1+r_{t_1+N-1}\right)J}\right) < u_j'\left(c_{t_1}^j\right) = \delta_j\left(1+r_{t_1+1}\right)u_j'\left(c_{t_1+1}^j\right) = \dots$$

$$= \delta_j^{t_2-t_1}\left(1+r_{t_1+1}\right)\dots\left(1+r_{t_2}\right)u_j'\left(c_{t_2}^j\right)$$

$$< \delta_j^{t_2-t_1}\left(1+r_{t_1+1}\right)\dots\left(1+r_{t_2}\right)u_j'\left(\frac{w_{t_2+N}}{\left(1+r_{t_2+1}\right)\cdots\left(1+r_{t_2+N}\right)J}\right).$$

At the same time, it follows from (44) that

$$u_j'\left(\frac{w_{t_1+N}}{\left(1+r_{t_1+1}\right)\cdots\left(1+r_{t_1+N-1}\right)J}\right) > \delta_j^{t_2-t_1}\left(1+r_{t_1+1}\right)\dots\left(1+r_{t_2}\right)u_j'\left(\frac{w_{t_2+N}}{\left(1+r_{t_2+1}\right)\cdots\left(1+r_{t_2+N}\right)J}\right).$$

We have obtained a contradiction which proves Claim 12. □

By Claim 10, $K_\infty \geq K^{**}$. Assume that $K_\infty > K^{**}$. By Claim 12,

$$f\left(K_t\right) = \sum_{j=1}^{J}\left[\left(1+r_t\right)s_{t-1}^j + \frac{w_t}{J}\right] = \left(1+r_t\right)\cdot\sum_{j=1}^{J}\left[s_{t-1}^j + \frac{w_t}{\left(1+r_t\right)J}\right] = 0$$

from some time onward, which is impossible. This proves that $K_\infty = K^{**}$. To complete the proof of Lemma **??**, it is sufficient to note that $\delta_j\left(1+r\left(K_\infty\right)\right) < 1$ for $j \geq 2$ and to use Claim 12.

7.2.2 Proof of Proposition 3

Proof. We have $\delta_1\left(1+r_t\right) \geq 1$ and thereby $K_t \leq K^{**}$ for all t large enough. Therefore, by (3), for all sufficiently large t we have

$$u_1'\left(c_t^1\right) \leq \delta_1\left(1+r_t\right)u_1'\left(c_t^1\right) \leq u_1'\left(c_{t-1}^1\right). \tag{45}$$

It follows that from some time onward the sequence $\{c_t^1\}$ is non-decreasing. Since this sequence is bounded from above, it converges as $t \to \infty$. Hence

$$\frac{u_1'\left(c_{t-1}^1\right)}{u_1'\left(c_t^1\right)} \to 1 \text{ as } t \to \infty.$$

Taking account of (45), for all t large enough we have

$$1 \leq \delta_1\left(1+r_t\right) \leq \frac{u_1'\left(c_{t-1}^1\right)}{u_1'\left(c_t^1\right)}.$$

Thus, $\delta_1\left(1+r_t\right) \to 1$ and hence $K_t \to K^{**}$ as $t \to \infty$. □

7.3 Proof of Claim 1

Proof. The budget constraint for each household $j \notin \Gamma_n$

$$s_n^j + c_n^j = \left(1 + r_n\right) s_{n-1}^j + \frac{w_n}{J} = -\frac{A\left(n+1, N-1\right)}{J},$$

since $s_{n-1}^j + \frac{A(n,N)}{J} = 0$. Also for each $j \notin \Gamma_n$, $s_n^j + \frac{A(n+1,N)}{J} \geq 0$, and

$$s_n^j + c_n^j + \frac{A\left(n+1, N\right)}{J} = \frac{A\left(n+1, N\right)}{J} - \frac{A\left(n+1, N-1\right)}{J} = \frac{w_{n+N}}{\left(1 + r_{n+1}\right) \cdots \left(1 + r_{n+N}\right) J}$$
$$\Rightarrow c_n^j \leq \frac{w_{n+N}}{\left(1 + r_{n+1}\right) \cdots \left(1 + r_{n+N}\right) J}.$$

The budget constraint for each household $j \in \Gamma_n$ can be used to infer

$$\begin{aligned}
\sum_{j\in\Gamma_n} s_n^j + \sum_{j\in\Gamma_n} c_n^j &= \left(1 + r_n\right) \sum_{j\in\Gamma_n} s_{n-1}^j + \left(\frac{\gamma_n}{J}\right) \cdot w_n \\
&= \left(1 + r_n\right) K_n + \left(\frac{\gamma_n}{J}\right) \cdot w_n + \left(\frac{J-\gamma_n}{J}\right) \cdot w_n + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{A\left(n+1, N-1\right)}{J} \\
&= f(K_n) + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{A\left(n+1, N-1\right)}{J}
\end{aligned}$$

since $\sum_{j\in\Gamma_n} s_{n-1}^j = K_n + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{A(n,N)}{J}$[14]. On similar lines, since $\sum_{j\in\Gamma_n} s_{n-2}^j \leq K_{n-1} + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{A(n-1,N)}{J}$, we get

$$\begin{aligned}
\sum_{j\in\Gamma_n} s_{n-1}^j + \sum_{j\in\Gamma_n} c_{n-1}^j &= \left(1 + r_{n-1}\right) \sum_{j\in\Gamma_n} s_{n-2}^j + \left(\frac{\gamma_n}{J}\right) \cdot w_{n-1} \\
&\leq \left(1 + r_{n-1}\right) K_{n-1} + \left(\frac{\gamma_n}{J}\right) \cdot w_{n-1} + \left(\frac{J-\gamma_n}{J}\right) \cdot w_{n-1} + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{A\left(n, N-1\right)}{J} \\
&= f(K_{n-1}) + \left(\frac{J-\gamma_n}{J}\right) \cdot \frac{A\left(n, N-1\right)}{J}.
\end{aligned}$$

We have $K_n > K^{**}$ and hence $\delta_j \left(1 + r_n\right) < 1$. Therefore by (4), for each $j \in \Gamma_n$, we get

[14] In the no - borrowing case, $\Gamma_n \equiv \left\{j \in \{1, \ldots, J\} : s_{n-1}^j > 0\right\}$. Then, since $\sum_{j\in\Gamma_n} s_{n-1}^j = K_n$

$$\sum_{j\in\Gamma_n} s_n^j + \sum_{j\in\Gamma_n} c_n^j = \left(1 + r_n\right) \sum_{j\in\Gamma_n} s_{n-1}^j + \left(\frac{\gamma_n}{J}\right) \cdot w_n = \left(1 + r_n\right) K_n + \left(\frac{\gamma_n}{J}\right) \cdot w_n.$$

Thus in the presence of borrowing, the income of these households increases by $\left(1 + r_n\right) \cdot \left(\frac{J-\gamma_n}{J}\right) \cdot A\left(n, N\right)$.

$$\frac{u_j'\left(c_n^j\right)}{u_j'\left(c_{n-1}^j\right)} = \frac{1}{\delta_j\left(1+r_n\right)} = \frac{1}{\delta_j f'\left(K_n\right)} > 1.$$

Hence, for each $j \in \Gamma_n$, we get $c_n^j < c_{n-1}^j$. Therefore,[15]

$$\sum_{j\in\Gamma_n} s_n^j - \sum_{j\in\Gamma_n} s_{n-1}^j \geq [f(K_n) - f(K_{n-1})] + \sum_{j\in\Gamma_n}[c_{n-1}^j - c_n^j] + \left(\frac{J-\gamma_n}{J}\right)\cdot[A(n+1,N-1) - A(n,N-1)]$$

Further

$$\sum_{j=1}^{J} s_n^j = \sum_{j\in\Gamma_n} s_n^j + \sum_{j\notin\Gamma_n} s_n^j = K_{n+1}$$

It implies

$$\sum_{j\in\Gamma_n} s_n^j \leq K_{n+1} + \left(\frac{J-\gamma_n}{J}\right)\cdot A(n+1,N).$$

Thus we get

$$\begin{aligned} K_{n+1} + \left(\frac{J-\gamma_n}{J}\right)\cdot A(n+1,N) - K_n - \left(\frac{J-\gamma_n}{J}\right)\cdot A(n,N) &\geq \sum_{j\in\Gamma_n} s_n^j - \sum_{j\in\Gamma_n} s_{n-1}^j \\ \geq [f(K_n) - f(K_{n-1})] + \sum_{j\in\Gamma_n}[c_{n-1}^j - c_n^j] + \left(\frac{J-\gamma_n}{J}\right)\cdot[A(n+1,N-1) - A(n,N-1)] \end{aligned}$$

It simplifies to

$$\begin{aligned} K_{n+1} + \left(\frac{J-\gamma_n}{J}\right)\cdot\frac{w_{n+N}}{\left(1+r_{n+1}\right)\cdots\left(1+r_{n+N}\right)} - K_n - \left(\frac{J-\gamma_n}{J}\right)\cdot\frac{w_{n+N-1}}{\left(1+r_n\right)\cdots\left(1+r_{n+N-1}\right)} \\ \geq [f(K_n) - f(K_{n-1})] + \sum_{j\in\Gamma_n}[c_{n-1}^j - c_n^j] > 0, \end{aligned}$$

which proves (11). □

[15] In the no-borrowing case,

$$\sum_{j\in\Gamma_n} s_n^j - \sum_{j\in\Gamma_n} s_{n-1}^j \geq \left(1+r_n\right)K_n + \left(\frac{\gamma_n}{J}\right)\cdot w_n - \left(1+r_{n-1}\right)K_{n-1} - \left(\frac{\gamma_n}{J}\right)\cdot w_{n-1} + \sum_{j\in\Gamma_n}[c_{n-1}^j - c_n^j], \text{ or}$$

$$K_{n+1} - K_n \geq \left(1+r_n\right)K_n + \left(\frac{\gamma_n}{J}\right)\cdot w_n - \left(1+r_{n-1}\right)K_{n-1} - \left(\frac{\gamma_n}{J}\right)\cdot w_{n-1} + \sum_{j\in\Gamma_n}[c_{n-1}^j - c_n^j],$$

$$K_{n+1} - K_n \geq [f(K_n) - f(K_{n-1})] + \sum_{j\in\Gamma_n}[c_{n-1}^j - c_n^j] - \left(\frac{J-\gamma_n}{J}\right)\cdot\left[w_n - w_{n-1}\right]$$

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