Technological Choice Generating Normalized CES Production Functions

Andrei Matveenko

CERGE-EI, Prague

amatveen@cerge-ei.cz

Vladimir Matveenko

National Research University Higher School of Economics, St. Petersburg

vmatveenko@hse.ru

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Different specifications of the constant elasticity of substitution (CES) production functions essentially differ in their properties; this leads to a non-robustness with respect to functional form in well-known models of production, economic growth and international trade. Klump and de La Grandville (20000 introduced a concept of normalized CES production function. This class of functions insures robustness and now is used intensively in various economic research. However, such contraction of the class of production function needs more profound motivation. The work on developing microfoundations and studying general properties of this class of production function is just begun. In our paper, for the analysis of the class of normalized CES functions, we use an approach based on representation of a 'global' neoclassical production function as a solution of a problem of optimal choice of a 'local' technology from a technological menu. We introduce a concept of a family of normalized technological menus and use it to derive a series of new properties of normalized CES production functions. Special attention is devoted to families of functions which are induced by some other production function and, symmetrically, to families of technological menus which are induced by some other technological menu. We study duality of these objects and demonstrate their relationship to some results known in the literature on production functions: the production function with "variable" elasticity of substitution introduced by Antony (2009) and the representation of the Cobb-Douglas function used by Jones (2005).

Contents

1. Introduction

Constant elasticity of substitution (CES) production function is an important tool in analysis of production and efficiency. However, different specifications of the CES function essentially differ in their properties. In particular, two-factor function $Y=(K^p+L^p)^{1/p}$ under fixed (K,L) decreases in p on each of the intervals $(-\infty,0),(0,1)$, while function $Y=(\alpha K^p+(1-\alpha)L^p)^{1/p}$, $0<\alpha<1$ – increases in p (see Martemyanov and Matveenko, 2014). This leads to unrobustness with respect to functional form in several well-known models of production, economic growth and international trade.

Klump and de La Gandville (2000) and Klump and Preissler (2000) introduced a concept of normalized CES production function; one of its equivalent representations is

$$Y=Y_0\left(\pi\left(\frac{K}{K_0}\right)^p+(1-\pi)\left(\frac{L}{L_0}\right)^p\right)^{\frac{1}{p}},$$

where K_0, L_0, Y_0, π are some fixed benchmark (baseline) values, $0<\pi<1$. This function is an immediate generalization of the CES function $A(\alpha K^p+(1-\alpha)L^p)^{\frac{1}{p}}$ introduced by Arrow et al. (1961).

Being a special case of the general class of CES production function, the family of normalized production functions has attractive properties. As it is expressed by Klump and Saam (2008), normalization is necessary to avoid “arbitrary and inconsistent results”.

Due to this “consistency”, the normalized CES production functions have become a popular tool in a variety of economic research (see Klump et al., 2012). Here are several examples of usage of normalized CES production functions. Klump and de La Gandville (2000), Klump and Preissler (2000), Papageorgiou and Saam (2008), Mallick (2012), Xue and Yip (2012) investigate the effect of the elasticity of substitution on the transitional dynamics in models of economic growth. Miyagiwa and Papageorgiou (2003) study the effect of change of the elasticity of substitution on wages. Saam (2004) studies the role of a rise in the elasticity of substitution in distributional effects of reforms aimed to promote economic growth; she shows that the choice of the point of normalization influences the existence of a trade-of between growth and equality. Guo and Lansing 2009, Wong and Yip 2010 and Photphisutthiphong and Weder, 2012 examine the role of the elasticity of substitution under conditions in which there is uncertainty in the local equilibrium model of the business cycle. Growiec et al. (2015) use a model with normalized CES production functions for analysis of dynamics of labor share in different countries.

Though normalized CES production functions are actively used in research for fifteen years, work on the construction of microfoundations and theoretical analysis of this class of functions has only just begun. The usage of normalized CES production functions meets criticism which is connected, in particular, with the absence of interpretation of the benchmark point (see, for example, Temple, 2012).

In the present paper for analysis of the class of normalized CES functions we use an approach which is connected with representation of any 'global' neoclassical production function as a solution of a problem of optimal choice of a 'local' technology from a technological menu: $F(K,L)=\max_{l\in\Lambda}\phi(l_K K, l_L L)$, where Λ is technological menu, l_K, l_L are factor efficiencies ('local' technology), and φ is 'local' production function. A universal 'local' production function which can be used for representation of any 'global' production function is the Leontief function $\phi(l_K K, l_L L)=\min\{l_K K, l_L L\}$ which possesses a zero elasticity of substitution.

This approach was initially proposed by Rubinov and Glover (1998) for abstract increasing positively homogeneous function, used by Matveenko (1997) for production functions (see also Rubinov, 2000), and was later reopened by Jones (2005) for the special case of 'global' Cobb-Douglas production function. The approach is developed in Growiec, 2008a, Growiec, 2008b, Matveenko, 2010, Matveenko, 2011, Growiec, 2013, Hrendash and Matveenko, 2015.

As a primal object, we consider not production function, but the technological menu which generates it. Technological menu is a dual object in relation to the production function, and is associated with the idempotent duality, not with the usual linear. We introduce a concept of family of normalized technological menus, study its properties, and derive a series of new properties of normalized production functions. Such approach allows to use a direct description of changes of production technologies. As result, we obtain a possibility to evaluate, from the point of view of technological changes, assumptions concerning normalized production functions which were not motivated by previous authors; in particular, we provide economic interpretation for the benchmark point. The use of technological menus as a primary object, which generates production function, opens broad possibilities to study models of technological changes. In particular, it is more natural and simple to speak about creation of new technologies in terms of technological menus.

Despite the fact that normalized production functions are broadly used to-day, many authors (among them Grossman, Maggi, 2000, Bentolila and Saint-Paul, 2003, Fan, 2005, Jones, 2011, Ngienthi et al., 2013) use the CES production function of type $A(K^p + L^p)^{1/p}$, which cannot be normalized. We study differences in properties of normalized and non-normalized technological menus and, on this base, explain differences in properties of various types of production functions, and identify situations, in which it is more natural to use one or another specification of production function. We also discuss, in what way parameters of the CES production functions are determined by parameters of probability distributions in models of ideas flow, which serve probability-theoretic foundations of production functions.

2. Normalized CES production functions

Families of normalized CES production functions can be defined as possessing the following properties:

(a) Function in the family for different values of the parameter p (i.e. for different values of the elasticity of substitution $\sigma = 1/(1-p)$) has the prototype form

$$Y = A(\alpha K^p + (1-\alpha)L^p)^{\frac{1}{p}}, \quad (1)$$

where A and α, generally speaking, depend on p.

(b) For a family of normalized CES production functions *a point of normalization (benchmark point)* is specified. At this point all the functions of the family for any p have the same *benchmark values* $K_0,\ L_0,\ Y_0,\ MRTS = \mu_0$ [1].

The *intensive form* of the prototype form (1) is obtained by dividing both sides of equation (1) by L and transition to the variables $y = Y/L,\ k = K/L$:

$$y = A(\alpha k^p + 1 - \alpha)^{\frac{1}{p}}. \quad (2)$$

[1] The $MRTS$ - marginal rate of technical substitution for production function function $F(K,L)$ - is defined as $MRTS = -dK/dL = (\partial F/\partial L)/(\partial F/\partial K)$.

Building normalized CES functions

Let the benchmark values K_0, L_0, Y_0, $\mu_0 = MRTS = -dK/dL$ be given. To build a family of the normalized CES function, we can use the intensive prototype form (2). Then

$$\mu_0 = \frac{1-\alpha}{\alpha} k_0^{1-p},$$

where $k_0 = K_0 / L_0$. Hence,

$$\alpha = \frac{1}{1+\mu_0 k_0^{p-1}}.$$

Substituting into (2), we obtain

$$y = A\left(\frac{k^p + \mu_0 k_0^{p-1}}{1+\mu_0 k_0^{p-1}}\right)^{\frac{1}{p}}. \quad (3)$$

In particular, in the benchmark point,

$$y_0 = A\left(\frac{k_0^p + \mu_0 k_0^{p-1}}{1+\mu_0 k_0^{p-1}}\right)^{\frac{1}{p}}. \quad (4)$$

Now extract *A* from (4) and substitute into (3):

$$y = y_0\left(\frac{1+\mu_0 k_0^{p-1}}{k_0^{p-1}(k_0+\mu_0)}\right)^{\frac{1}{p}}\left(\frac{k^p + \mu_0 k_0^{p-1}}{1+\mu_0 k_0^{p-1}}\right)^{\frac{1}{p}} = y_0\left(\frac{k_0}{k_0+\mu_0}\left(\frac{k}{k_0}\right)^p + \frac{\mu_0}{k_0+\mu_0}\right)^{\frac{1}{p}}.$$

Thus,

$$y = y_0\left(\pi\left(\frac{k}{k_0}\right)^p + 1 - \pi\right)^{\frac{1}{p}}, \quad (5)$$

where the coefficient $\pi = \frac{k_0}{k_0+\mu_0} \in (0,1)$ does not depend on p. Equation (5) describes the family of the normalized CES function in the intensive form. The same family in the extensive form is:

$$Y = Y_0\left(\pi\left(\frac{K}{K_0}\right)^p + (1-\pi)\left(\frac{L}{L_0}\right)^p\right)^{\frac{1}{p}}. \quad (6)$$

PROPOSITION 1. *For any function of normalized family (i.e. for any p), in the benchmark point* k_0 *factors receive shares which are equal to* π *and* $1-\pi$. *In any point* $k>(<)k_0$ *ratio of factor shares,* $\lambda = \theta_K / \theta_L$, *is increasing(decreasing) in elasticity of substitution between factors.*

Proof: See the Appendix.

This result is consistent with empirically supported fact that for developed countries (i.e., for sufficiently large k) elasticity of substitution increases together with an increase in the capital share (e.g., Duffy and Papageorgiou, 2000, Karabarbounis and Neiman, 2014).

A differential equations approach to the CES functions

For production function with constant returns to scale

$$Y = F(K, L) = Lf(k), \quad k = \frac{K}{L}$$

elasticity of substitution satisfies the equation[2]

$$\sigma = \frac{1-e_f}{r_f}, \qquad (7)$$

where $e_f = \frac{f'\cdot k}{f}$ is elasticity of function f, $r_f = -\frac{f''\cdot k}{f'}$ is its "convexity"[3]. Given $\sigma \neq 1$, a solution of the second order differential equation (7) is the intensive form CES function

$$f(k) = \gamma_1\left[k^{\frac{\sigma-1}{\sigma}} + \gamma_2\right]^{\frac{\sigma}{\sigma-1}}, \quad (8)$$

[2] In the literature this identity occurs in a cumbersome form with usage of function f and its derivatives, f' and f''.

[3] The function r_f is known as the Arrow-Pratt coefficient of relative risk aversion.

where γ_1 and γ_2 are constants which specify partial solutions of the differential equation. They may depend on σ. Forms (2) and (8) are equivalent, with $\gamma_1 = A\alpha^{1/p}$, $\gamma_2 = (1-\alpha)/\alpha$.

For $\sigma = 1$ the solution of (7) is Ak^{α}, i.e. the intensive form Cobb-Douglas function. For $\gamma_2 = 0$ (8) is linear function. When $\sigma \to 0$, (8) turns into Leontief function.

We have to note that (8) describes various families of the CES functions, not only normalized ones. For example, for $\gamma_1 = 1, \gamma_2 = 1$ we come to the family of the CES functions $(K^p + L^p)^{\frac{1}{p}}$ which cannot be written as normalized.

The normalized CES functions families are those partial solutions of equation (7) which are defined by fixed benchmark values for each σ.

PROPOSITION 2. *For each* σ, *there exists a unique partial solution of the differential equation (7) which corresponds to given fixed benchmark values,* k_0, y_0, μ_0. *This solution defines a normalized CES function in the intensive form (5).*

Proof: See the Appendix.

Induced family of normalized CES functions

Till now the benchmark values K_0, L_0, Y_0 and μ_0 have been chosen arbitrarily and were not connected with each other. However, it is natural to consider the case when the benchmark values are generated by a function or are solution of an optimization problem.

Let a function $F(K, L)$ be given. Let us assume that this function has CRS and other standard neoclassical properties of production function. Let us take arbitrarily (K_0, L_0), $Y_0 = F(K_0, L_0)$ and $\mu_0 = \dfrac{\partial F / \partial L}{\partial F / \partial K}(K_0, L_0)$ as benchmark values. In such case we say that the family of normalized CES production functions *is induced* by function F at point (K_0, L_0).

LEMMA 1. *For the induced family, coefficients* π *and* $1-\pi$ *in equation (6) are the factor shares of the inducing function* F *at point* (K_0, L_0).

Proof: See the Appendix.

By Lemma 1, the family of normalized CES functions, which is induced by the function F at point (K_0, L_0), has the form

$$Y = F(K_0, L_0)\left[\frac{\partial F}{\partial K}(K_0, L_0)\frac{K_0}{F_0}\left(\frac{K}{K_0}\right)^p + \frac{\partial F}{\partial L}(K_0, L_0)\frac{L_0}{F_0}\left(\frac{L}{L_0}\right)^p\right]^{\frac{1}{p}}. \quad (9)$$

Because of homogeneity of function F, of importance for the induced CES function is the capital to labor ratio, $k_0 = K_0 / L_0$, but not the precise values K_0, L_0. Function (9) can be rewritten as

$$Y = f(k_0)\left[\frac{f'(k_0)k_0^{1-p}}{f(k_0)}K^p + \left(1 - \frac{f'(k_0)k_0}{f(k_0)}\right)L^p\right]^{\frac{1}{p}},$$

where $f(k_0) = F(k_0, 1)$.

In particular, let us find families of normalized CES production functions which are induced by Cobb-Douglas function and CES function.

In the case when the inducing function is the Cobb-Douglas function $F(K, L) = AK^{\alpha}L^{1-\alpha}$, where $A > 0,\ 0 < \alpha < 1$, because of Lemma 1, $\pi = \alpha, 1 - \pi = 1 - \alpha$; therefore, the induced family of CES functions (9) is

$$Y = AK_0^{\alpha}L_0^{1-\alpha}\left(\alpha\left(\frac{K}{K_0}\right)^p + (1-\alpha)\left(\frac{L}{L_0}\right)^p\right)^{\frac{1}{p}}. \quad (10)$$

In the limit, when $p \to 0$, function (10) is the initial inducing Cobb-Douglas function

$$F(K, L) = AK^{\alpha}L^{1-\alpha}.$$

When the inducing function is the CES function $F(K, L) = A(aK^q + bL^q)^{\frac{1}{q}}$, where $A, a, b > 0,\ q \in (-\infty, 0) \cup (0, 1)$, by Lemma 1, $\pi = aK_0^q / (aK_0^q + bL_0^q),\ 1 - \pi = bL_0^q / (aK_0^q + bL_0^q)$; therefore, the induced family of normalized CES functions is

$$Y = A(aK_0^q + bL_0^q)^{\frac{1}{q} - \frac{1}{p}}\left(aK_0^{q-p}K^p + bL_0^{q-p}L^p\right)^{\frac{1}{p}}. \quad (11)$$

If $p = q$ then the induced function (11) coincides with the inducing function

$$F(K,\ L) = A(aK^q + bL^q)^{\frac{1}{q}}.$$

Function (11), after simple transformation, can be rewritten in the intense form as

$$y = A(ak_0^q + b)^{\frac{1}{q}-\frac{1}{p}}\left(a\left(\frac{k}{k_0}\right)^p k_0^q + b\right)^{\frac{1}{p}}. \quad (12)$$

Antony (2009) took into consideration production function (12), suggesting that it has a variable elasticity of substitution and used this function in a number of economic studies supposing this property. It is hard to agree with that since our derivation of function (12) shows that it is actually usual NCESF. The fact that under given p function (12) at benchmark point $k = k_0$ takes benchmark value $y = A(ak_0^q + b)^{\frac{1}{q}}$ seems to us as natural property of NCESF; and this fact does not mean, as Antony suggests, that the elasticity of substitution of function (12) at point k_0 differs from $\sigma = 1/(1-p)$.

Coefficients before K^p and L^p in formulas (10) and (11) depend on *p*, except the case $K_0 = L_0$. When $K_0 = L_0$ (10) is

$$Y = A\left(\alpha K^p + (1-\alpha)L^p\right)^{\frac{1}{p}}, \quad (13)$$

and (11) is

$$Y = A(a+b)^{\frac{1}{q}}\left(\frac{a}{a+b}K^p + \frac{b}{a+b}L^p\right)^{\frac{1}{p}}. \quad (14)$$

If not only $K_0 = L_0$, but also $a + b = 1$, then forms of normalized CES functions (13) and (14) coincide; (14) does not depend on *q*. Vice versa, when $K_0 = L_0$ and $a = b = 1$, (14) depends on *q*:

$$Y = 2^{\frac{1}{q}-\frac{1}{p}} A\left(K^p + L^p\right)^{\frac{1}{p}}.$$

Such approach makes it possible to consider the preferred benchmark point. Let some basket of factors (K, L) be given and fixed. Let an inducing function F be given, but let a freedom be in the choice of benchmark point. Which point of normalization k_0 should be chosen in order to maximize output of the induced normalized CES production function at the point

(K, L)? We answer this question for the case when inducing functions are Cobb-Douglas and CES functions.

PROPOSITION 3. *If the inducing function is the Cobb-Douglas function* $F(K, L) = AK^{\alpha}L^{1-\alpha}$, *then, under given basket of factors* (K, L), *the maximum of the induced normalized CES production function for any* $p < 0$ *is achieved when* $k_0 = K / L$.

Proof: See the Appendix.

PROPOSITION 4. *If the inducing function is the CES function* $F(K, L) = A(\beta K^q + (1-\beta)L^q)^{\frac{1}{q}}$, *then, under given basket of factors* (K, L), *the maximum of the induced normalized CES production function for any* $p < 0$ *is achieved when* $k_0 = K / L$.

Proof: See the Appendix.

Induced family of production functions with arbitrary MRTS

Again, let the Cobb-Douglas function $F(K, L) = AK^{\alpha}L^{1-\alpha}$, where $A > 0,\ 0 < \alpha < 1$ be inducing function; let us take as a benchmark point arbitrary point $(K_0, L_0) \in M$, where $M = \{(K, L):\ AK^{\alpha}L^{1-\alpha} = 1\}$. Above, while building induced family of normalized CES functions, we used μ_0 equal to the MRTS in point (K_0, L_0), i.e. $\mu_0 = \frac{1-\alpha}{\alpha}k_0$. Now let us expand our opportunities and while defining μ_0, let us take arbitrary positive number $\frac{1-\chi}{\chi}$ instead of $\frac{1-\alpha}{\alpha}$, i.e., take $\mu_0 = \frac{1-\chi}{\chi}k_0$. Then

$$\pi = \frac{k_0}{k_0 + \mu_0} = \frac{1}{1 + (1-\chi)/\chi} = \chi,$$

and family of normalized CES functions has the form

$$\tilde{g}_{p,(K_o,L_0)}(K, L) = \left(\chi \left(\frac{K}{K_0} \right)^p + (1-\chi) \left(\frac{L}{L_0} \right)^p \right)^{\frac{1}{p}}.$$

Again, let us consider the problem of optimal choice of point (K_0, L_0).

PROPOSITION 5. *For any* $K, L > 0$,

$$AK^{\alpha}L^{1-\alpha} = \max_{(K_0,L_0)\in M} \tilde{g}_{p,(K_0,L_0)}(K,\ L), \quad (15)$$

and in point of maximum, $K_0 / L_0 = K / L$.

Proof: See the Appendix.

3. Production function as a result of optimal choice of technology from a technological menu

Representations of production functions

Let us consider production function $AF(K,L)$, where *A* is TFP, *K* is capital, *L* is labor. It is assumed that the function *F* possesses the standard neoclassical properties (has constant return to scale – CRS, increases, has decreasing returns). The contribution of individual factors can be identified on the basis of their marginal or average productivity. Accordingly, there are two representations of the production function. Well-known is the representation of the production function by use of the Euler theorem:

$$AF(K,L) = A\left(\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L\right).$$

Elasticities $\theta_K = \frac{\partial F}{\partial K}\frac{K}{F}$, $\theta_L = \frac{\partial F}{\partial L}\frac{L}{F}$ (where $\theta_K, \theta_L \in (0,1)$; $\theta_K + \theta_L = 1$) are, correspondingly, *capital share* and *labor share* in the income. For a set of factor prices $\Pi = \{(p_K,\ p_L)\}$ the production function can be represented as a result of solution of the choice problem[4]:

$$AF(K,L) = \min_{(p_K,p_L)\in\Pi}(p_K K + p_L L). \quad (16)$$

A parallel in many respects representation is the following representation[5]

$$AF(K,L) = \max_{(l_K,l_L)\in\Lambda} \min\{l_K K,\ l_L L\}. \quad (17)$$

[4] A model providing microfoundations for this equation is proposed by Matveenko (2013).

[5] This representation was first introduced by Rubinov and Glover (1998) and Matveenko (1997) for general type n-factor production function and later by Jones (2005) for the special case of two-factor Cobb-Douglas function. See also Rubinov, 2000 and Matveenko, 2010.

Here $\min\{l_K K, l_L L\}$ is the Leontief production function[6].

Representation (17) is a model of choice of technology. A firm (country) has available a set of Leontief 'local' technologies – a *technological menu* Λ. Given bundle (K, L) of production factors and searching for the maximum output, the firm (country) chooses a Leontief technology (l_K, l_L) from the technological menu Λ. In result of this choice, the 'global' production function $AF(K, L)$ is obtained. Such approach corresponds to the known view that only one 'local' technology can be effective under given ratio of production factors[7].

Following mathematical notion of the conjugate function (Rubinov, Glover, 1998), it is possible to associate to any neoclassical production function $AF(K, L)$ a unique technological menu as the following set of Leontief technologies:

$$\Lambda = \{(l_K, l_L) : AF\left(\frac{1}{l_K}, \frac{1}{l_L}\right) = 1\}. \quad (18)$$

For example, the technological menu

$$\Lambda = \{l_K, l_L : l_K^{\alpha} l_L^{1-\alpha} = A\}$$

(where $0 < \alpha < 1$) generates the Cobb-Douglas production function $AF(K, L) = AK^{\alpha} L^{1-\alpha}$.

The following theorem shows that the coefficients of Leontief technologies, which form the technological menu has a simple economic meaning, which allows to associate them to available data.

THEOREM 1. *The elements of the technological menu (18) are all pairs of the average product of capital and the average product of labor, which are feasible in the economy under*

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[6] The principle difference between representations (16) and (17) is that (16) uses the common inner product, $p_K K + p_l L$, while (17) uses the Leontief function, $\min\{l_K K, l_L L\}$, which is the inner product in tropical mathematics with idempotent operation $\oplus = \min$. In particular, tropical mathematics considers analogues of the objects of linear algebra which are defined by use of an idempotent operation. A binary operation $\oplus$ on a set M is called idempotent if $a \oplus a = a$ for each $a \in M$. Basic examples of idempotent operations are $\oplus = \min$ and $\oplus = \max$. See Kondrakov and Shananin, 2011, Matveenko, 2014 and Baldwin and Klemperer, 2015 for examples of application of tropical mathematics in economic analysis.

[7] This point of view is most clearly formulated by Basu and Weil (1998), who write that "each technology is appropriate for one and only one capital-labor ratio". This idea is close to the concept of localized technological progress (Atkinson and Stiglitz, 1969, Nelson and Winter, 1982, ch. 9, Stiglitz, 1989, Antonelli, 1995, 2008). Models of 'appropriate technology' are also constructed in Acemoglu, 2002, 2003, Caselli and Coleman, 2006, Leon-Ledesma and Satchi, 2013.

use of production function $AF(K,L)$ *given technological level A. For any particular bundle of factors* $(\tilde{K},\tilde{L})$ *the maximum in the R.H.S. of (17),*

$$\max_{(l_K,l_L)\in\Lambda} \min\{l_K\tilde{K},\, l_L\tilde{L}\},$$

is achieved in such point of the technological menu, for which Leontief productivities are equal to the average product of capital and the average product of labor, correspondingly:

$$\tilde{l}_K = \frac{AF(\tilde{K},\tilde{L})}{\tilde{K}},\ \tilde{l}_L = \frac{AF(\tilde{K},\tilde{L})}{\tilde{L}}.$$

Moreover, equation (17) is fulfilled with $\arg\max$ *equal to* $(\tilde{l}_K,\, \tilde{l}_L)$.

Proof: See the Appendix.

More on the notion of technological menu

The notion of *technological menu* links two important economic concepts: production function and choice of technology. For any neoclassical production function $F(x)$ there exists a unique set of Leontief technologies, Λ, – a *technological menu* – such that $F(x)$ is the result of optimal choice of "local" Leontief technology from the set Λ.

To formulate this fact more precisely, let us consider production functions $F(x)$ on the space R^n_{++} consisting of all *n*-dimensional vectors *x* with strictly positive coordinates and of the origin[8]. Let M_1 be the unit level set of function $F(.)$, i.e. $M_1 = \{x : F(x) = 1\}$. Now let us define the set $\Lambda = \{l : l^{-1} \in M_1\}$, where $l^{-1} = (l_1^{-1},..., l_n^{-1})$.

As usually, we will use notation $x > y$ if $x_i > y_i$, $i = 1,..., n$ and will assume that $F(0) = 0$, and $F(x) > 0$ for $x > 0$. Function $F(.)$ is called *increasing* if $x > y$ implies $F(x) > F(y)$.

THEOREM 2. *If* $F(.)$ *is increasing, positively homogeneous of degree one (IPH) function, then*

$$F(x) = \max_{l\in\Lambda} \min_i l_i x_i = \min_{l\in\Lambda} \max_i l_i x_i,\ \ x \in R^n_{++}.$$

[8]Thus, except the origin, vectors of costs which have a zero component are excluded from consideration. This does not narrow the class of production functions itself.

Proof: See the Appendix.

The technological menu is closely related to a notion of conjugate function. We should remind that concepts of conjugate space and conjugate function, developed originally in functional analysis, were used in mathematical economics, in particular, in application to sublinear and superlinear functions– see (Makarov and Rubinov, 1975)[9]. From this point of view, Theorem 2 means that the neoclassical production function (which is, of course, an IPH function) is an analogue of both superlinear and sublinear functions simultaneously. Continuing this analogy, we can define conjugate objects for the production function.

Departing from (Rubinov and Glover, 1998), let us introduce a pair of mutually conjugate sets: $\Omega = \{x : F(x) \le 1\}$ and $\Omega^\circ = \{l : \min_i l_i x_i \le 1 \text{ for all } x \in \Omega\}$. They have a transparent economic meaning: Ω is the set of all bundles of physical resources which allow to produce , by use of production function $F(.)$, not more than one unit of product; while Ω° is the set of Leontief technologies (the technological menu) which is, in some sense, equivalent to production function $F(.)$: the set Ω° allows to produce not more than one unit of product by use of any bundle of physical resources taken from the set Ω [10].

According to Theorem 2, function $F(.)$ satisfies the equation

$$F(x) = \max_{l \in \Omega^\circ} \min_i l_i x_i ,$$

which can serve as a prototype for definition of *conjugate function*:

$$F^\circ(l) = \max_{x \in \Omega} \min_i l_i x_i$$ [11].

The latter has an evident economic sense: for each Leontief technology *l* it shows the maximal output which can be obtained if a bundle of physical resources is taken from the set Ω.

The following theorem provides a formula for calculating conjugate functions.

THEOREM 3. *If* $F(.)$ *is an IPH function then*

$$F^\circ(h) = \frac{1}{F(h^{-1})}, \quad h >> 0 .$$

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[9]Function F on R^n is called sublinear if it is positively homogeneous of degree one and subadditive, i.e. $F(x+y) \le F(x) + F(y)$ for any $x, y \in R^n$. The definition of superlinear function is similar but with $\ge$.

[10]Geometrically, Ω is the set of points which are located not higher than the unit level surface M_1; and Ω° is the set of points located not higher than the surface Λ. It can be checked that $(\Omega^\circ)^\circ = \Omega$.

[11]It can be checked that the conjugate function $F^\circ(.)$ is IPH and $(F^\circ)^\circ(.) = F(.)$.

Proof: See the Appendix.

Results of applications of Theorem 3 to Leontief, Cobb-Douglas and CES production functions are collected in Table 1.

Table 1. Some production functions and their conjugate functions.

	Production function	Conjugate function
Leontief	$F(x) = \min_i l_i x_i$	$F^\circ(h) = \max_i l_i^{-1} h_i$
Cobb-Douglas	$F(x) = Ax_1^{\alpha_1} ... x_n^{\alpha_n}$, where $A > 0,\ \alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1$	$F^\circ(h) = A^{-1} h_1^{\alpha_1} ... h_n^{\alpha_n}$
CES	$F(x) = [\alpha_1 (A_1 x_1)^p + ... + \alpha_n (A_n x_n)^p]^{\frac{1}{p}}$, $A_i > 0,\ \alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1,\ p < 1,\ p \neq 0$	$F(x) = [\alpha_1 (A_1^{-1} h_1)^{-p} + ... \alpha_n (A_n^{-1} h_n)^{-p}]^{-\frac{1}{p}}$

The relations between the sets M_1 and Λ and the functions $F(.)$ and $F^\circ(.)$ introduced above, can be termed *idempotent duality*, as far as the dual relations are based on the functions $\min_i l_i x_i$, $\max_i l_i x_i$, $\max_i h_i x_i$, $\max_i h_i l_i^{-1}$ which are analogues of inner products with the use of idempotent operations $\oplus = \min$ and $\oplus = \max$.

In contrast to the general form production function, an interesting feature of Cobb-Douglas function is that the representation similar to (17) remains if one uses as 'local' production functions not Leontief functions, but CES functions with elasticity of substitution less than 1, while continuing to use the same technological menu.

Let (a) technological menu Λ, consisting of technologies (l_K, l_L) feasible for firm, i.e. factors efficiency vectors, and (b) 'local' CES production function $\varphi(K, L; l_K, l_L)$ be given. The firm possessing quantities (K, L) of production factors chooses technology (l_K, l_L) from the technological menu Λ in order to maximize the 'local' production function**.**

Proposition 5 implies the following statement.

PROPOSITION 6. Let parameters $0 < \alpha < 1,\ p < 0$ be fixed; let 'local' production function $\varphi(K,\ L,\ l_K,\ l_L) = \left(\beta(l_K K)^p + (1-\beta)(l_L L)^p\right)^{\frac{1}{p}}$ and technological menu $\Lambda = \{(l_K, l_L) : l_K^{\alpha} l_L^{1-\alpha} = A\}$ be given. Then

$$\max_{(l_K, l_L)\in\Lambda} \varphi(K, L, l_K,\ l_L) = AK^{\alpha}L^{1-\alpha};$$

i.e. Cobb-Douglas production is the result of solution of the firm's choice problem.

4. Technological menus in economic growth models

Let us remind that Solow (1956) believed that neoclassical production functions must be an alternative to Leontief production function for analysis of economic growth. The representation (16) shows that in fact the presence of 'global' production function with non-zero elasticity of substitution does not deny the presence of Leontief production functions. On the contrary, 'local' Leontief production functions are the "building blocks" that serve as the basis of any neoclassical production function. If a 'global' neoclassical production function is used in an economic growth model, then a choice of 'local' Leontief technology takes place at each step of the trajectory.

Paths of economic growth models can be described in terms of technologies chosen from the technological menu. As an example let us remind equation of capital dynamics in the discrete time economic growth model in the absence of technological change:

$$k_{t+1}(1+n) = (1-\delta)k_t + f(k_t) - c_t,$$

where k_t is capital, $f(k_t)$ is output, c_t is consumption per capita, n is rate of labor growth, δ is depreciation coefficient. In terms of technological menu an analogous equation describes the path of Leontief technologies $(l_K^t, l_L^t) \in \Lambda$ which are chosen from the technological menu and used:

$$\frac{l_L^{t+1}}{l_K^{t+1}}(1+n) = l_L^t\left(1 + \frac{1-\delta}{l_K^t}\right) - c_t. \quad (19)$$

Equation (19) implies that on each stationary path some definite Leontief technology $l = (l_K, l_L) \in \Lambda$ is used, and per capita consumption is equal to

$$c(l) = l_L \left(1 - \frac{\delta + n}{l_K}\right). \quad (20)$$

The sense of Equation (20) is clear: in steady state each unit of production pays for amortization and creation of new working places, and the rest is consumed. Since expenditures $\delta + n$ are calculated per unit of capital, the coefficient $1/l_K$ recalculates them into expenditures per unit of output. It is clear that consumption is non-negative iff $l_K \geq (\delta + n)$.

Phelps' golden rule corresponds to the solution of the problem of choice of Leontief technology from menu Λ which maximizes per capita consumption (20):

$$\max_{l \in \Lambda} c(l). \quad (21)$$

If the technological menu Λ is graph $G(l_K, l_L) = 0$, then using Lagrange method it is easy to obtain FOC for problem (21):

$$\frac{\dfrac{\partial c}{\partial l_L}}{\dfrac{\partial c}{\partial l_K}} = \frac{\dfrac{\partial G}{\partial l_L}}{\dfrac{\partial G}{\partial l_K}}, \quad (22)$$

i.e.

$$\frac{\partial c}{\partial l_L} dl_L + \frac{\partial c}{\partial l_K} dl_K = 0$$

Condition (22) has simple economic sense: golden rule corresponds to such Leontief technology $(l_K, l_L) \in \Lambda$ deviation from which to any direction in the menu is equally unfruitful from the consumption point of view.

Technological menu which generates Cobb-Douglas production function $AK^{\alpha} L^{1-\alpha}, 0 < \alpha < 1$ is the curve $G(l) = l_K^{\alpha} l_L^{1-\alpha} - A = 0$. Condition (22) characterizing the golden rule steady state is reduced to

$$\frac{l_K - (\delta + n)}{\delta + n} = \frac{1 - \alpha}{\alpha}.$$

Hence, in the golden rule steady state the Leontief technology for which

$$l_K = \frac{\delta + n}{\alpha}$$

is used. Then (20) implies that consumption in the golden rule steady state is $c(l) = l_L(1-\alpha)$, i.e. the saving rate is $s = \alpha$.

For the Solow model with arbitrary constant saving rate s Equation (19) of 'local' technology's non-stationary dynamics transforms into

$$\frac{l_L^{t+1}}{l_K^{t+1}}(1+n) = l_L^t\left(s + \frac{1-\delta}{l_K^t}\right).$$

Convergence to stationary technology for which $l_K = (\delta + n)/s$ takes place.

Then for the case of Cobb-Douglas production function $AK^{\alpha}L^{1-\alpha}, 0 < \alpha < 1$ it is possible to obtain two equations which describe dynamics of 'local' technology separately for coefficients l_L^t and l_K^t:

$$l_L^{t+1} = \left(\frac{A^{\frac{1}{\alpha}}s}{1+n}l_L^t + \frac{1-\delta}{1+n}(l_L^t)^{\frac{1}{\alpha}}\right)^{\alpha},$$

$$l_K^{t+1} = \left[\frac{s}{1+n}(l_K^t)^{\frac{\alpha}{\alpha-1}} + \frac{1-\delta}{1+n}(l_K^t)^{\frac{1}{\alpha-1}}\right]^{\alpha-1}.$$

5. Families of the normalized technological menus and representation of the normalized CES production functions

In this section we introduce notion of family of normalized technological menus. We will show that a family of CES production function is normalized iff their generating family of the technological menus is normalized.

Construction of normalized family of technological menus

Now we define a normalization in the space of technologies. Let us consider technological menus, i.e. sets of 'local' Leontief technologies (l_K, l_L). Let a family of the technological menus have prototype form[12]

$$\beta l_K^{-q} + (1-\beta) l_L^{-q} = H = const, \quad (23)$$

where $0 < \beta < 1$. The technological menus of the family are indexed by parameter q; and coefficient β, generally speaking, depends on q. Let *technological benchmark point* be defined by technology (l_{K0}, l_{L0}), the slope of the technological menu in this point is $-\frac{dl_L}{dl_K} = \nu$, and the value of H is H_0.

The *normalized family of technological menus* corresponding the technological benchmark point, can be defined as the set of technological menus with different values of parameter q which have prototype form (23) and for the technology (l_{K0}, l_{L0}) achieve the same $H = H_0$ and ν for all q:

$$\beta l_{K0}^{-q} + (1-\beta) l_{L0}^{-q} = H_0,$$

$$\frac{\beta}{1-\beta}\left(\frac{l_{K0}}{l_{L0}}\right)^{-q-1} = \nu.$$

Figure 1 illustrates the normalized family of technological menus.

[12] The prototype form of technological menu can be justified by the ideas models of technological progress. In Jones' (2005) model random productivities of capital and labor corresponding to new technological ideas are drawn from independent Pareto distributions; this model leads to Cobb-Douglas production function. Versions of this model which lead to normalized CES functions are based on one or another joint distribution based on Pareto distribution (Growiec, 2008a, Matveenko, 2010) or on independent Weibull distributions (Growiec, 2008b). There are also versions leading to non-normalizable families of CES production functions of the shape $A(K^p + L^p)^{1/p}$; they are based on independent exponential distributions (Matveenko, 2011) or on independent Weibull distributions (Hrendash and Matveenko, 2015).

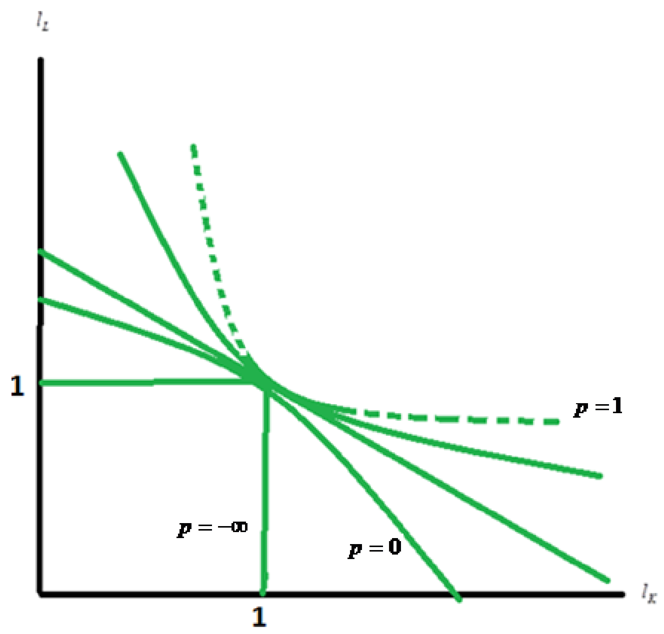

Figure 1. Normalized family of technological menus. Each of the alternative technological menus includes the benchmark technology (l_{K0}, l_{L0}), is characterized by the same $H = H_0$, and has the same slope (MRTS) in the technological benchmark point.

The technological benchmark point can be interpreted as the best existing (or frontier) technology. The prototype form can be interpreted as a forecast for the future technological development which would be able to create such technological menus which (i) include the frontier benchmark technology and (ii) possess a spectrum of various elasticities of substitution.

A different interpretation of the family of prototype technologies is a set of possible (expected or alternative) technologies which could exist under different conditions, e.g. in different countries.

In the technological benchmark point:

$$\nu = -\frac{dl_L}{dl_K}(l_{K0}, l_{L0}) = \frac{\beta(-q)l_K^{-q-1}}{(1-\beta)(-q)l_L^{-q-1}} = \frac{\beta}{1-\beta}\left(\frac{l_L}{l_K}\right)^{q+1}.$$

Hence,

$$\beta = \frac{\nu}{\nu + \left(\frac{l_{L0}}{l_{K0}}\right)^{q+1}}.$$

We see that, generally, β depends on q. Substituting β into (23) we have

$$\frac{\nu}{\nu+\left(\frac{l_{L0}}{l_{K0}}\right)^{q+1}} l_K^{-q} + \left(1-\frac{\nu}{\nu+\left(\frac{l_{L0}}{l_{K0}}\right)^{q+1}}\right) l_L^{-q} = H_0$$

and after transformation,

$$\frac{\nu l_{K0}}{\nu l_{K0}^{q+1}+l_{L0}^{q+1}}\left(\frac{l_K}{l_{K0}}\right)^{-q} + \frac{l_{L0}}{\nu l_{K0}^{q+1}+l_{L0}^{q+1}}\left(\frac{l_L}{l_{L0}}\right)^{-q} = H_0 . \quad (24)$$

In the benchmark point, Equation (24) turns into

$$\frac{\nu l_{K0}+l_{L0}}{\nu l_{K0}^{q+1}+l_{L0}^{q+1}} = H_0 .$$

Hence, Equation (24), describing the family of technological menus, has the form

$$\eta\left(\frac{l_K}{l_{K0}}\right)^{-q} + (1-\eta)\left(\frac{l_L}{l_{L0}}\right)^{-q} = 1, \quad (25)$$

where $\eta = \frac{\nu l_{K0}}{\nu l_{K0}+l_{L0}} \in (0,1)$ does not depend on q.

THEOREM 4. *For each normalized family,* M *, of technological menus there is a normalized family,* Φ *, of CES production functions, such that each menu* $m \in M$ *generates a function* $f \in \Phi$ *, and vice versa, each function* $f \in \Phi$ *is generated by a menu* $m \in M$.

Proof: See the Appendix.

The equations $\pi = \frac{k_0}{\mu_0 + k_0}$, $\eta = \frac{\nu}{\nu + k_0}$ and $\pi = \eta$ imply $\mu_0 \nu = k_0^2$.

Figure 2 illustrates unit level lines for the normalized family of CES production functions.

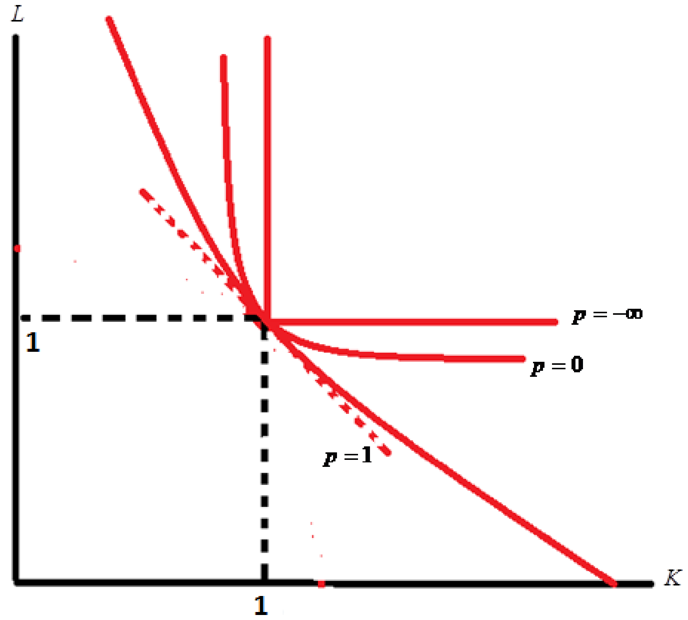

Figure 2. Unit level lines for the normalized family of CES production functions.

Induced family of normalized technological menus

It is interesting to consider the case when the technological benchmark point is not arbitrary but is generated by some particular technological menu. Let a function $G(l_K, l_L)$ be given which possesses constant returns to scale and other standard neoclassical properties, and let the unit level line of this function determine technological menu

$$\Lambda(G) = \{(l_K,\, l_L) : G(l_K, l_L) = 1\}.$$

Let us take as technological benchmark values some 'local' Leontief technology $(l_{K0}, l_{L0}) \in \Lambda(G)$, the corresponding slope $\nu = \frac{\partial G}{\partial l_K} / \frac{\partial G}{\partial l_L}(l_{K0}, l_{L0})$ and the value $H_0 = G(l_{K0}, l_{L0}) = 1$. As we have shown, the normalized family of technological menus with prototype form (23) is (25), where

$$\eta = \frac{\nu l_{K0}}{\nu l_{K0} + l_{L0}} = \frac{\frac{\partial G}{\partial l_K} l_{K0}}{\frac{\partial G}{\partial l_K} l_{K0} + \frac{\partial G}{\partial l_L} l_{L0}}.$$

By Euler theorem,

$$\eta = \frac{\partial G}{\partial l_K} l_{K0},,$$

and correspondingly,

$$1 - \eta = \frac{\partial G}{\partial l_L} l_{L0}.$$

Thus, the normalized family of technological menus has form

$$\left\{ (l_K, l_L) : \frac{\partial G(l_{K0}, l_{L0})}{\partial l_K} l_{K0}^{1+q} l_K^{-q} + \frac{\partial G(l_{K0}, l_{L0})}{\partial l_L} l_{L0}^{1+q} l_L^{-q} = 1 \right\}. \quad (26)$$

We will say that this family is *induced by technological menu* $\Lambda(G)$ *at point* (l_{K0}, l_{L0}).

Each of technological menus from family (26) generates corresponding CES production function

$$\left(\frac{1}{\eta} l_{K0}^q K^q + \frac{1}{1-\eta} l_{L0}^q L^q \right)^{\frac{1}{q}}. \quad (27)$$

Functions (27) form a family of normalized CES production functions.

On the other hand, technological menu $\Lambda(G)$ generates production function $F(K, L) = G^{\circ}(K, L)$. The latter, in its turn, induces normalized family of CES production functions.

The following theorem tells that the families of CES production functions which can be obtained in two ways from technological menu $\Lambda(G)$ do coincide.

THEOREM 5. *Production function* $F(K, L) = G^{\circ}(K, L)$ *at point* (K_0, L_0) *such that* $k_0 = l_{L0} / l_{K0}$, *induces the same normalized family of CES production functions which is generated by the family of technological menus (26), induced by technological menu* $\Lambda(G)$ *at point* (l_{K0}, l_{L0}).

Proof: See the Appendix.

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APPENDIX: Proofs

Proof of Proposition 1. In arbitrary point (K, L) factor shares ratio is equal to

$$\lambda = \frac{\frac{\partial F}{\partial K}\frac{K}{F}}{\frac{\partial F}{\partial L}\frac{L}{F}} = \frac{\frac{\partial F}{\partial K}}{\frac{\partial F}{\partial L}}\frac{K}{L} = \frac{\frac{\pi}{K_0^p}K^{p-1}}{\frac{1-\pi}{L_0^p}L^{p-1}}\frac{K}{L} = \frac{\pi}{1-\pi}\left(\frac{k}{k_o}\right)^p .$$

Capital share is

$$\frac{\pi(K/K_0)^p}{\pi(K/K_0)^p + (1-\pi)(L/L_0)^p};$$

in the benchmark point it is π, and labor share is, correspondingly, $1-\pi$. If $k = k_0$ then $\lambda = \pi/(1-\pi)$. If $k > (<) k_0$ then λ increases (decreases) in p as well as in the elasticity of substitution $\sigma = 1/(1-p)$. □

Proof of Proposition 2. The well-known expressions for the marginal products, $\frac{\partial F}{\partial K} = f'$, $\frac{\partial F}{\partial L} = f - kf'$, imply

$$\mu_0 = -\frac{dK}{dL} = \frac{\partial F/\partial L}{\partial F/\partial K} = \frac{f(k_0)}{f'(k_0)} - k_0 = \left(k_0^{\frac{\sigma-1}{\sigma}} + \gamma_2\right)k_0^{\frac{1}{\sigma}} - k_0 = \gamma_2 k_0^{\frac{1}{\sigma}} .$$

Hence, $\gamma_2 = \mu_0 k_0^{-\frac{1}{\sigma}}$.

Evidently, $y_0 = \gamma_1\left(k_0^{\frac{\sigma-1}{\sigma}} + \gamma_2\right)^{\frac{\sigma}{\sigma-1}}$. Hence, $\gamma_1 = y_0\left(k_0 k_0^{\frac{-1}{\sigma}} + \mu_0 k_0^{-\frac{1}{\sigma}}\right)^{-\frac{\sigma}{\sigma-1}} = y_0 k_0^{\frac{1}{\sigma-1}}\left(k_0 + \mu_0\right)^{-\frac{\sigma}{\sigma-1}}$.

We come to

$$f(k) = y_0\left(\frac{k_0}{k_0+\mu_0}\left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}} + \frac{\mu_0}{k_0+\mu_0}\right)^{\frac{\sigma}{\sigma-1}}$$

which is the normalized CES production function in the intensive form (5). □

Proof of Lemma 1. Capital share for function F is equal to

$$\frac{\partial F}{\partial K}(K_0, L_0)\frac{K_0}{F(K_0, L_0)} = \frac{\partial F}{\partial K}(K_0, L_0)\cdot\frac{K_0}{\frac{\partial F}{\partial K}(K_0, L_0)K_0 + \frac{\partial F}{\partial L}(K_0, L_0)L_0} = \frac{k_0}{k_0+\mu_0} = \pi .$$

Correspondingly, labor share is $1-\pi$. □

Proof of Proposition 3. According to the theorem about a generalized mean (Hardy et al., 1952, Theorem 16, pp. 26-27) while comparing cases $p<0$ and $p\to 0$, the induced production function satisfies

$$Y = AK_0^{\alpha}L_0^{1-\alpha}\left(\alpha\left(\frac{K}{K_0}\right)^p + (1-\alpha)\left(\frac{L}{L_0}\right)^p\right)^{\frac{1}{p}} \le$$

$$\le AK_0^{\alpha}L_0^{1-\alpha}\left(\frac{K}{K_0}\right)^{\alpha}\left(\frac{L}{L_0}\right)^{1-\alpha} = AK^{\alpha}L^{1-\alpha}.$$

But at $k_0 = K/L$, i.e. at $(K_0,\, L_0) = \gamma(K,L),\ \gamma>0,$ we have $Y = AK^{\alpha}L^{1-\alpha}$. Hence, $k_0 = K/L$ is the maximum point. □

Proof of Proposition 4. The induced production function is

$$Y = A(\beta K_0^q + (1-\beta)L_0^q)^{\frac{1}{q}}\left[\frac{\beta K_0^q}{\beta K_0^q + (1-\beta)L_0^q}\left(\frac{K}{K_0}\right)^p + \frac{(1-\beta)L_0^q}{\beta K_0^q + (1-\beta)L_0^q}\left(\frac{L}{L_0}\right)^p\right]^{\frac{1}{p}} =$$

$$= A\left(\beta K_0^q + (1-\beta)L_0^q\right)^{\frac{1}{q}-\frac{1}{p}}(\beta K_0^{q-p}K^p + (1-\beta)L_0^{q-p}L^p)^{\frac{1}{p}} =$$

$$= A\left(\beta k_0^q + (1-\beta)\right)^{\frac{1}{q}-\frac{1}{p}}(\beta k_0^{q-p}K^p + (1-\beta)L^p)^{\frac{1}{p}}.$$

Its derivative with respect to k_0 is

$$A\beta\frac{(q-p)}{p}\left(\beta k_0^q + (1-\beta)\right)^{\frac{1}{q}-\frac{1}{p}-1}k_0^{q-p-1}\left(\beta k_0^{q-p}K^p + (1-\beta)L^p\right)^{\frac{1}{p}-1}\left(k_0^pL^p - K^p\right) = 0,$$

which can be represented as $H(k_0)\left(k_0^pL^p - K^p\right),$ where $H(k_0)>0,\ H'(k_0)>0$. Thus, FOC for maximization with respect to k_0 is $k_0 = K/L$. The second derivative in this point is $H(k_0)pk_0^{p-1}L^p,$ which is negative under $p<0$; i.e. it is point of maximum. □

Proof of Proposition 5. The following inequality takes place:

$$\left(\frac{K}{\overline{K}}\right)^{\chi}\left(\frac{L}{\overline{L}}\right)^{1-\chi} \ge \tilde{g}_{p,(\overline{K},\overline{L})}(K,\, L). \quad (28)$$

Applying again the theorem about a generalized mean (Hardy et al., 1952, Theorem 16, pp. 26-27), we see that for any positive $K, L, \overline{K}, \overline{L}$, and $p<0,$ the equality in (28) is achieved iff $K/\overline{K} = L/\overline{L}$. If $(K, L) \in M$, then $(K, L) = (\overline{K}, \overline{L})$ and both sides of (28) are equal to 1, i.e. the value of function $F(K, L) = AK^{\alpha}L^{1-\alpha}$. Hence, for any $(K, L) \in M$ and for any $p<0$ (15) is held. Because of homogeneity, Equation (28) holds not only for $(K, L) \in M$, but for all $(K, L) \in R_{+}^{2}$. □

Proof of Theorem 1. Equation $AF(K,L)=Y$, where Y is output, is equivalent to equation $AF(K/Y, L/Y)=1$. Hence, any admissible pair of average products of capital and labor satisfies equation $AF\left(\frac{1}{Y/K}, \frac{1}{Y/L}\right)=1$. Thus, the set of such pairs of average products is a subset of the technological menu (18). Conversely, let (l_K, l_L) be a technology from the technological menu. For an arbitrary Y let us consider the bundle $K=Y/l_K$, $L=Y/l_L$. Equation (18) implies that $AF\left(\frac{1}{Y/K}, \frac{1}{Y/L}\right) = AF(1/l_K, 1/l_L)=1$, hence $AF(K,L)=Y$. The first part of the theorem is proved.

To prove the second part of the theorem one needs to show that for any pair $(l_K, l_L) \in \Lambda$ the following inequality is fulfilled:

$$\min\{l_K\tilde{K}, l_L\tilde{L}\} \le \min\{\tilde{l}_K\tilde{K}, \tilde{l}_L\tilde{L}\} = AF(\tilde{K},\tilde{L}). \quad (29)$$

Assume the opposite: there exists such $(l_K, l_L) \in \Lambda$ that

$$l_K\tilde{K} > AF(\tilde{K},\tilde{L}),\ l_L\tilde{L} > AF(\tilde{K},\tilde{L}) \quad (30)$$

Then, according to the first part of the theorem, there exists such point (K, L) that

$$l_K = \frac{AF(K,L)}{K},\ l_L = \frac{AF(K,L)}{L}.$$

Inequalities (30) take form

$$\frac{AF(K,L)}{K}\tilde{K} > AF(\tilde{K},\tilde{L}),\ \frac{AF(K,L)}{L}\tilde{L} > AF(\tilde{K},\tilde{L}),$$

which is equivalent to

$$AF\left(\tilde{K}, \frac{L\tilde{K}}{K}\right) > AF(\tilde{K}, \tilde{L}), \ AF\left(\frac{K\tilde{L}}{L}, \tilde{L}\right) > AF(\tilde{K}, \tilde{L}),$$

which, in its turn, is equivalent to

$$\frac{L\tilde{K}}{K} > \tilde{L}, \ \frac{K\tilde{L}}{L} > \tilde{K}.$$

The latter system is incompatible, and this contradiction proves validity of (29). □

LEMMA 2. *If* $F(.)$ *is increasing function, homogeneous of degree* $q > 0$*, then*

$$\min_i l_i x_i \le F(x) = 1 \le \max_i l_i x_i$$

for each $x \in M_1$ *and* $l \in \Lambda$.

Proof. Let us prove that $\min_i l_i x_i \le 1$. Assume the opposite: $l_i x_i > 1$ for all $i = 1,...,n$. Then $x > l^{-1}$. We can pick up such number $\lambda > 1$ that $x > \lambda l^{-1}$. Then

$$F(x) > F(\lambda l^{-1}) = \lambda^q F(l^{-1}) > F(l^{-1})$$

which is impossible, since $x, \ l^{-1} \in M_1$. The inequality $\max_i l_i x_i \ge 1$ is proved in a similar way.

□

Proof of Theorem 2. Each vector $x > 0$ can be represented as $x = F(x)\bar{x}$, where $\bar{x} \in M_1$. By Lemma 2, for each $l \in \Lambda$,

$$\min_i l_i \bar{x}_i \le F(\bar{x}) = 1.$$

This implies

$$\min_i l_i x_i = F(x) \min_i l_i \bar{x}_i \le F(x).$$

Here the inequality is fulfilled as equality if $l = \bar{x}^{-1}$, thus the maximum is achieved:

$$\max_{l \in \Lambda} \min_i l_i x_i = F(x).$$

The equation

$$\min_{l \in \Lambda} \max_i l_i x_i = F(x)$$

is proved similarly. □

Proof of Theorem 3.

$$F^{\circ}(h) = \min_{l \in \Lambda} \max_i h_i l_i^{-1} = \frac{1}{\max_{l \in \Lambda} \min_i h_i^{-1} l_i} = \frac{1}{F(h^{-1})}. \ \square$$

Proof of Theorem 4. According to Theorem 2 for each q technological menu (25) generates production function which possesses constant returns to scale and is characterized by the following unit level set:

$$M = \left\{(\bar{K},\, \bar{L}) : \eta\left(l_{K0}\bar{K}\right)^q + (1-\eta)\left(l_{L0}\bar{L}\right)^q = 1\right\}$$

Because of homogenuity, this production function is determined by equation

$$F(K,\, L) = \left(\eta(l_{K0}K)^q + (1-\eta)(l_{L0}L)^q\right)^{\frac{1}{q}} \qquad (31)$$

For all $K > 0, L > 0$. Since $K_0 = 1/l_{K0},\ L_0 = 1/l_{L0}$, for each q equations $F(K_0,\, L_0) = 1$ and $-\frac{dL}{dK}(K_0,\, L_0) = \frac{\eta}{1-\eta} \cdot \frac{l_{K0}}{l_{L0}}$ take place, and equation (31) with different values of parameter q describes the normalized family of CES production functions with benchmark point $K_0 = 1/l_{K0},\ L_0 = 1/l_{L0}, Y_0 = 1,\ \mu_0 = \frac{(1-\eta)l_{L0}}{\eta l_{K0}}$. $\square$

Proof of Theorem 5. Let technological menu $\Lambda(G)$ generate production function *F*. The ratio of factor shares and its analogue calculated by use of function G are equal to

$$\frac{\dfrac{\partial F(K_0, L_0)}{\partial L}\dfrac{L_0}{F(K_0, L_0)}}{\dfrac{\partial F(K_0, L_0)}{\partial K}\dfrac{K_0}{F(K_0, L_0)}} = \frac{\mu}{k_0}, \quad \frac{\dfrac{\partial G(l_{K0}, l_{l0})}{\partial l_K} l_{K0}}{\dfrac{\partial G(k_0, l_0)}{\partial l_L} l_{L0}} = \frac{\nu}{k_0}.$$

Since each of the pairs of shares is equal to 1 in sum, and since $\mu / k_0 = k_0 / \nu$, the following equalities take place:

$$\frac{\partial F(K_0, L_0)}{\partial L}\frac{L_0}{F(K_0, L_0)} = \frac{\partial G(l_{K0}, l_{L0})}{\partial l_L} l_{L0},$$

$$\frac{\partial F(K_0, L_0)}{\partial K}\frac{K_0}{F(K_0, L_0)} = \frac{\partial G(l_{K0}, l_{L0})}{\partial l_K} l_{K0}.$$

Substituting into Equation (9), we obtain that the family of normalized CES functions which is induced by function F have the form

$$Y = F(K_0, L_0)\left[\frac{\partial G}{\partial l_K}(l_{K0}, l_{L0})k_{K0}\left(\frac{K}{K_0}\right)^p + \frac{\partial G}{\partial l_L}(l_{K0}, l_{L0})l_{L0}\left(\frac{L}{L_0}\right)^p\right]^{\frac{1}{p}},$$

which can be rewritten as (27). □