UNIVERSITÉ DE GENÈVE
FACULTÉ DE PSYCHOLOGIE ET DES SCIENCES DE L'ÉDUCATION

International Commission on Mathematical Instruction

DiMaGe
Didactique des Mathématiques à Genève

Teaching and learning Algebra - An international symposium

Geneva Wednesday 7th June 2017

Revoir l'algèbre scolaire par la visualisation et la fabrication des sens

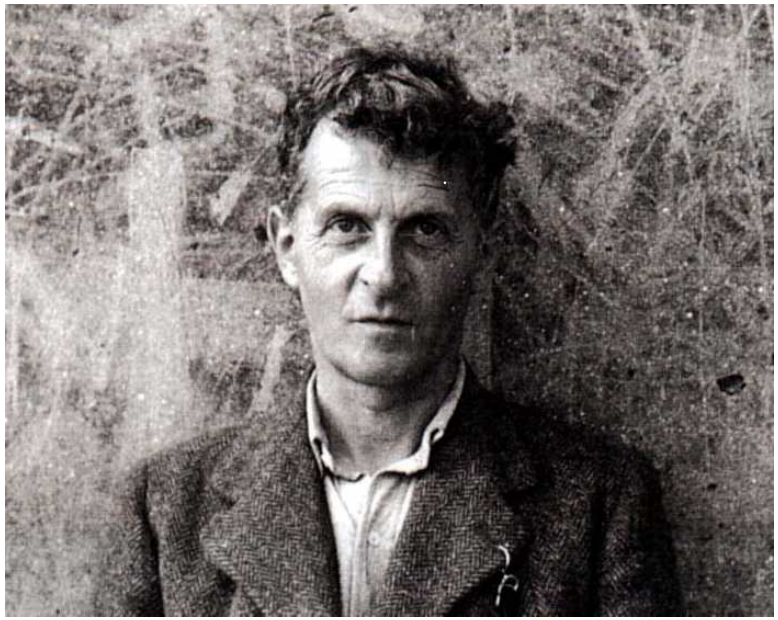
Abraham Arcavi
Department of Science Teaching
Weizmann Institute of Science, Israel

Activité mathématique

↑↓

Sense making

Peut-il être autrement?



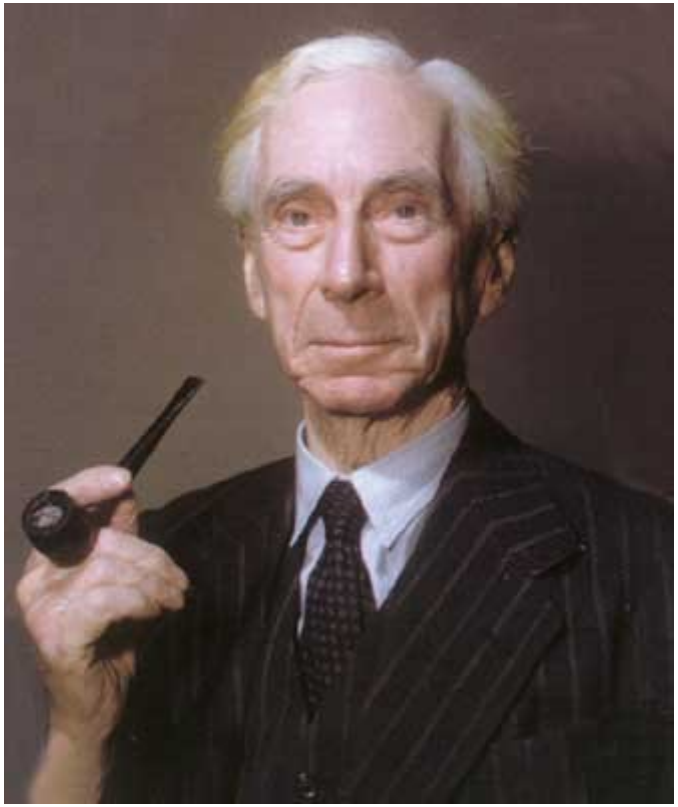
Ludwig J. J. Wittgenstein (1889-1951)

Mathematical propositions are not real propositions and mathematical truth is essentially non-referential and purely syntactical in nature.

Les propositions mathématiques ne sont pas des propositions réelles et la vérité mathématique est essentiellement non-référentielle et purement syntaxique.

Mathematics is the only science where one never knows what one is talking about nor whether what is said is true

Les mathématiques sont la seule science où vous ne savez jamais ce que vous parlez ou si ce qui est dit est vrai



Bertrand Russell (1872-1970)

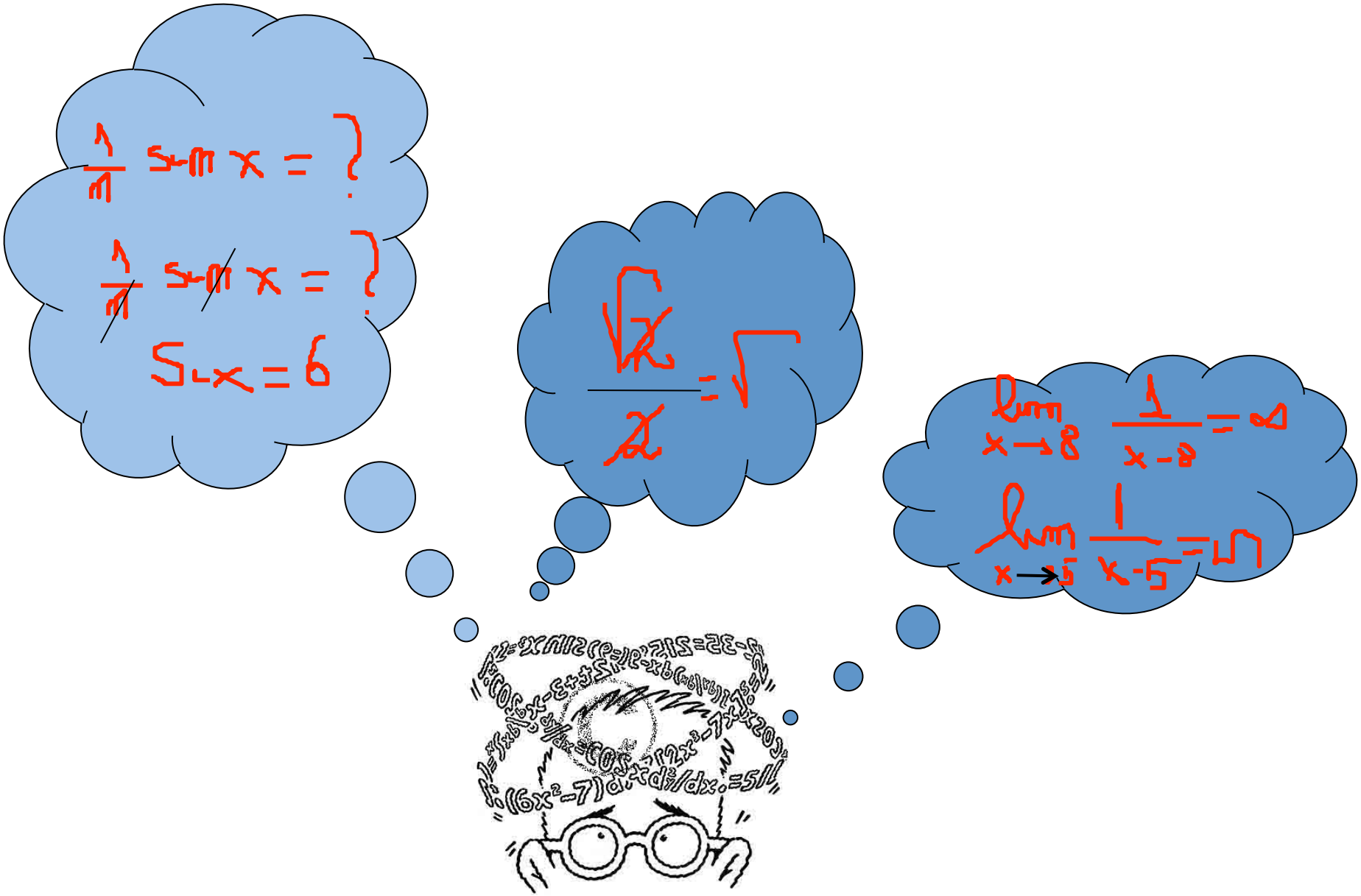
L’ énigme **MU**

- Trois symboles **M**, **I**, et **U** qui peut être combinée, par exemple:
  MIUUU, IUMIMM,

- Quatre règles de transformation (***X***: Toute chaîne)
  - *x*I → *x*IU
  - M*x* → M*xx*
  - *x*III*y* → *x*U*y*
  - *x*UU*y* → *xy*

Exercice:
En utilisant les règles, essayez de transformer MI à MU

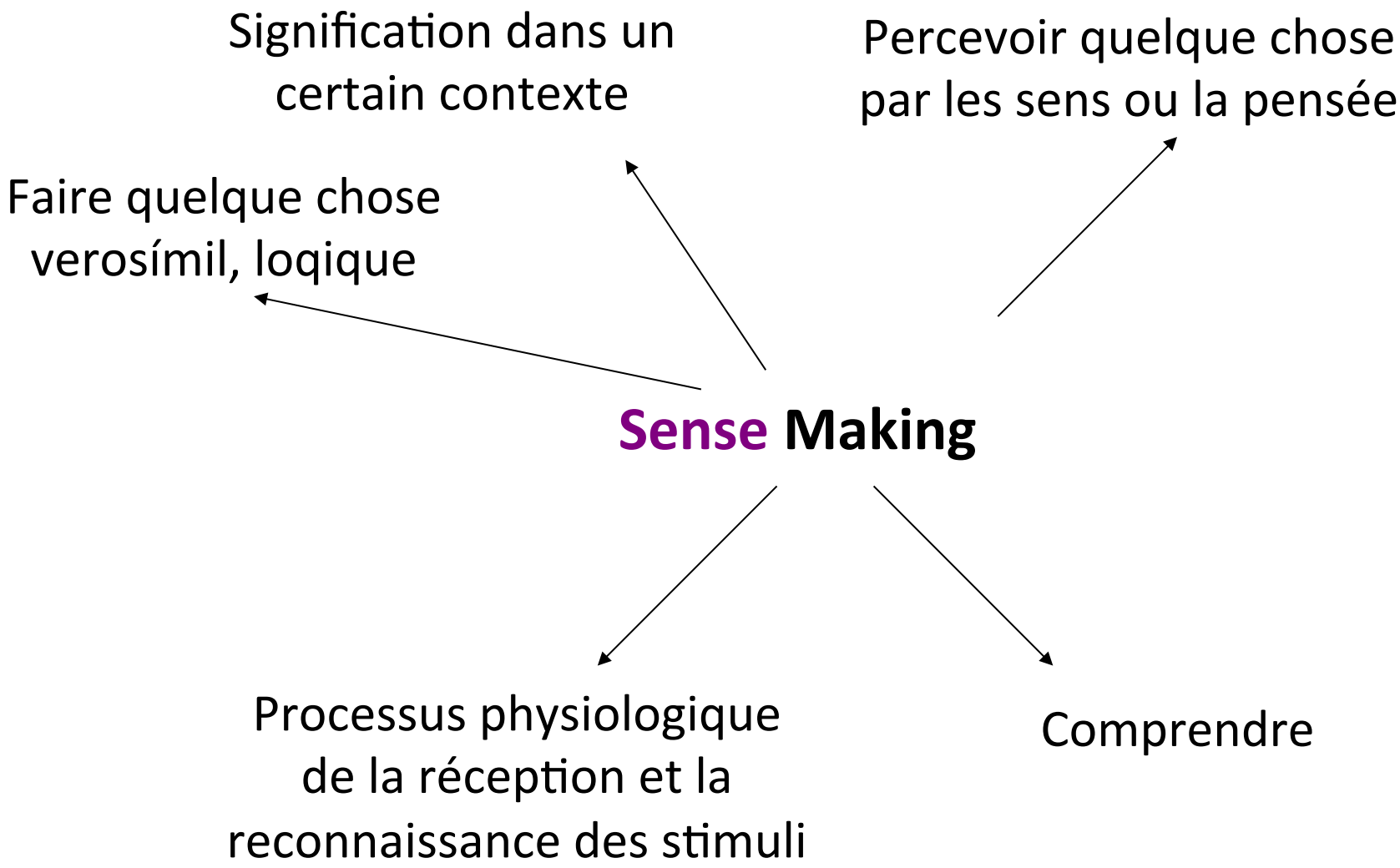
Hofstadter, D. (1979). *Gödel, Escher, Bach: An Eternal Golden Braid*. Basic books.


$\frac{1}{x}\sin x = ?$
$\frac{1}{x}\sin x = ?$
$5-x=6$
$\frac{\sqrt{x}}{x}=\sqrt{\ }$
$\lim_{x\to 8}\frac{1}{x-8}=\infty$
$\lim_{x\to 5}\frac{1}{x-5}=\infty$

Activité mathématique

↑↓

Sense making


Signification dans un certain contexte
Percevoir quelque chose par les sens ou la pensée
Faire quelque chose verosímil, loqique
Sense Making
Processus physiologique de la réception et la reconnaissance des stimuli
Comprendre

Fabrication des sens – Sense making

Le processus par lequel une personne peut attribuer le sens au monde perçoit.

Sense making nous permet de relier de nouvelles informations aux croyances antérieures, de sorte que améliore le fonctionnement de l'individu dans le monde réel.

Processus d'acquisition de connaissance de la situation dans un environnement d'incertitude.

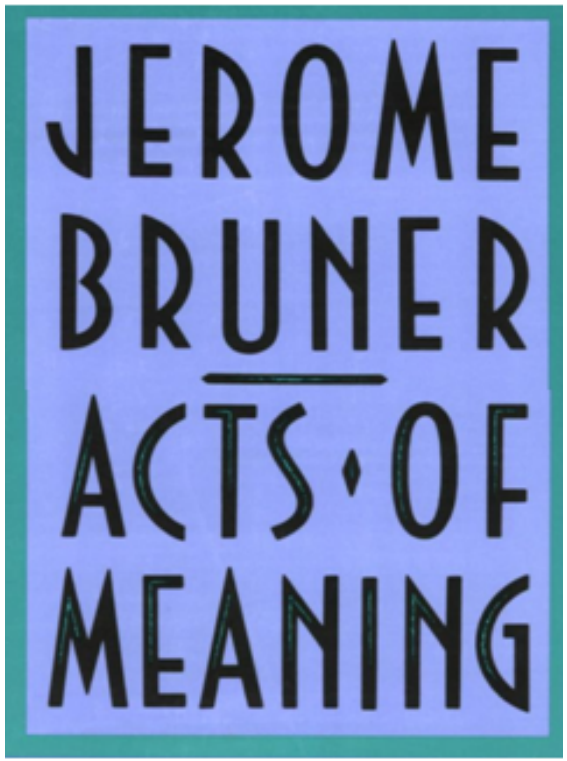
Il consiste à identifier les qualités et les attributs aux personnes et aux choses, le cadrage, la reconnaissance des formes, la reconnaissance des émotions, et avec tout cet apprentissage.

Fabrication des sens – Sense making
d'où provient-il?

“… the tremendous amount of experience we gain in early life that tells us such an enormous amount about the world we live in and the objects that surround us.”

“... l'énorme quantité d'expérience que nous acquérons au début de la vie qui nous dit une quantité énorme sur le monde dans lequel nous vivons et les objets qui nous entourent.”

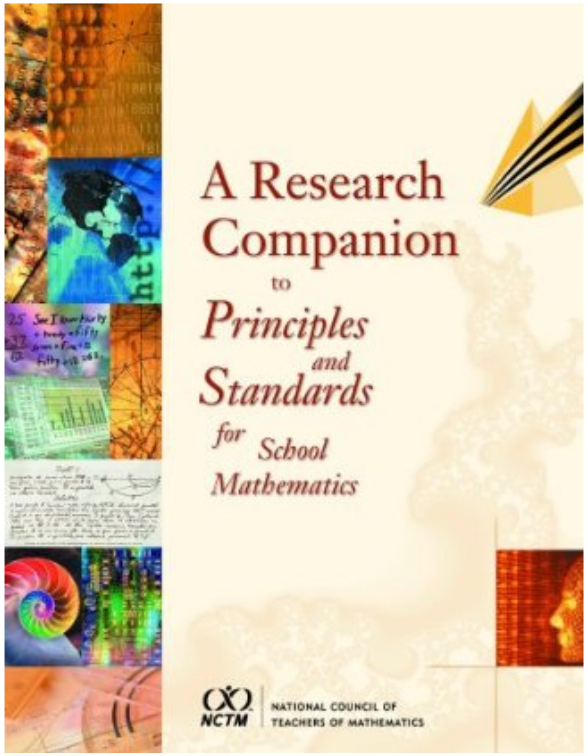
Bondi, H. (1964). *Relativity and Common Sense. A New Approach to Einstein*. Dover Publications.



« … culture et la quête du sens dans la culture sont la cause appropriée de l'action humaine»

***"…culture and the quest for meaning within culture are the proper cause for human action."* (p. 20)**

«Le ... essentiellement universel, besoin de sens et le besoin de comprendre nous-mêmes et le monde qui nous entoure ... [est] la force motrice de toutes nos activités intellectuelles»



"…the…essentially universal, need for meaning, and the need to understand ourselves and the world around us, …[is] the driving force behind all our intellectual activities" (Anna Sfard, p. 356)

Balancing the Unbalanceable: The NCTM Standards in Light of Theories of Learning Mathematics

En algèbre…

- Tentatives qui ont échoué
- Tentatives “en bonne santé”
- “Répression”
- Stratégies d'enseignement

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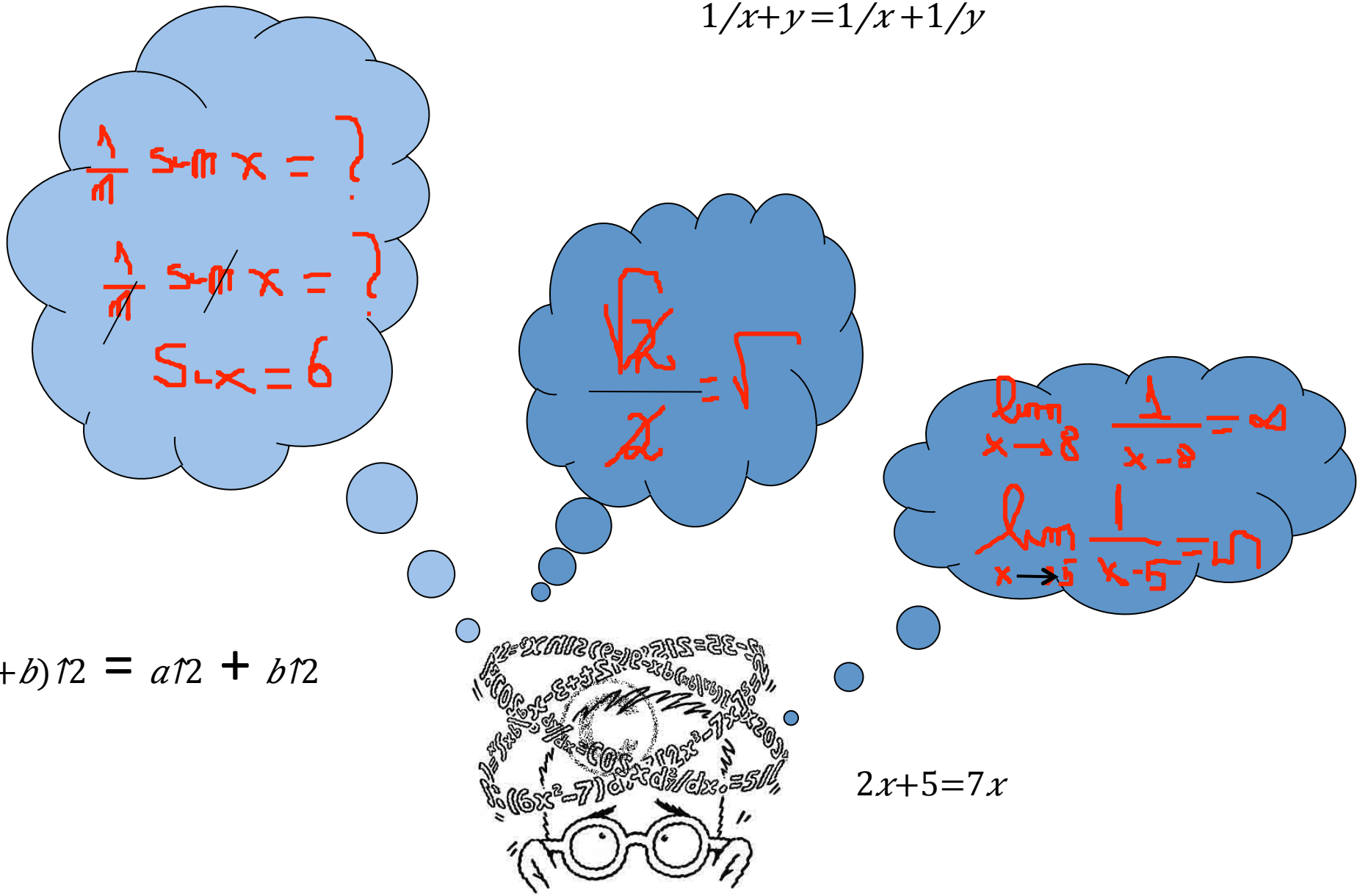
$1/x+y=1/x+1/y$



$(a+b)^2 = a^2 + b^2$

$2x+5=7x$

En algèbre…

- Tentatives qui ont échoué
- Tentatives “en bonne santé”
- “Répression”
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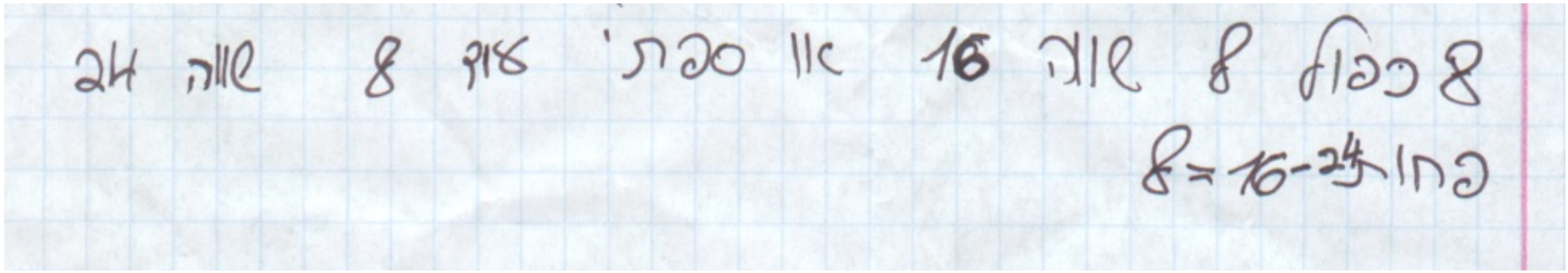
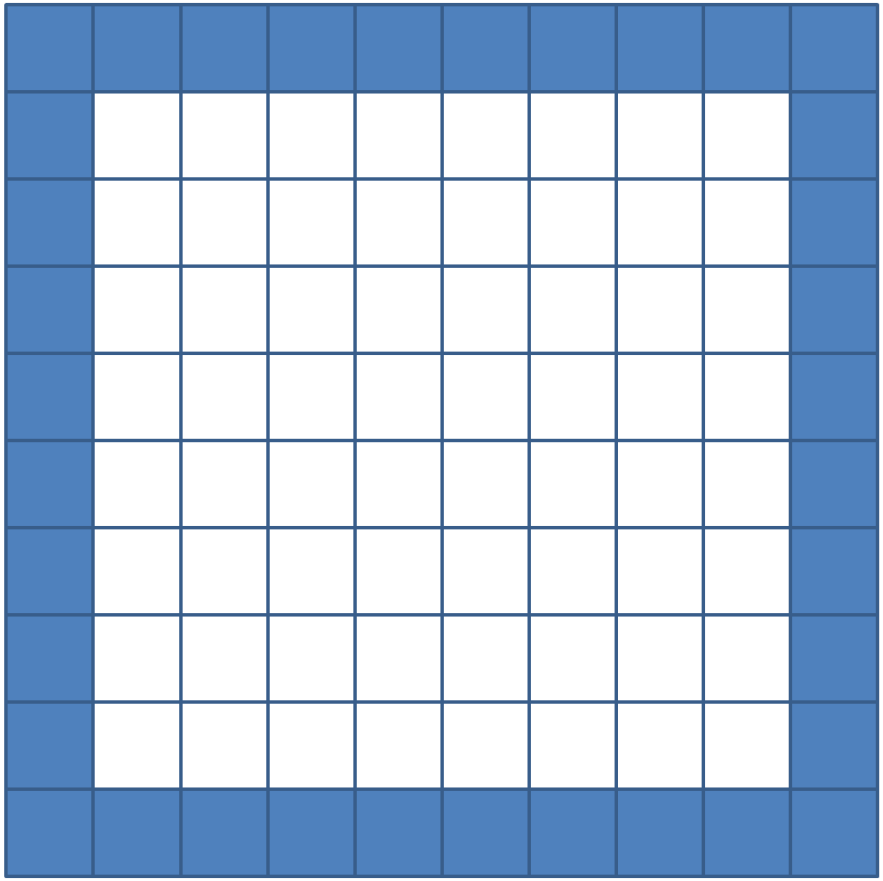
Je pensais un certain nombre, je l'ai multiplié par 3, j'ai soustrait 10, et le résultat que j'ai obtenu était 8. Quel est le numéro?

Je pensais un certain nombre, je l'ai multiplié par 3, j'ai soustrait 16 y et le résultat que j'ai obtenu était le nombre que je pensais





$4 \times 10 - 4$

$4 \times 8 + 4$

$2 \times 10 + 2 \times 8$

4×9

$10 \times 10 - 8 \times 8$

6×6

“Situation Definition” - “Définition de la situation”

Boaler, J. & Humphreys C. (2005). *Connecting Mathematical Ideas: Middle School Video Cases to Support Teaching and Learning. Heinemann.*

1- Choose a two digit number such that the tens digit is greater by two than the units digit

97

1- Choose a two digit number such that the tens digit is greater by two than the units digit

97

2- Create a new number by reversing the digits

79

1- Choose a two digit number such that the tens digit is greater by two than the units digit

2- Create a new number by reversing the digits

3- Subtract the second number from the first

$$\begin{array}{r} 97 \\ 79 \\ \hline 18 \end{array}$$

1- Choose a two digit number such that the tens digit is greater by two than the units digit

2- Create a new number by reversing the digits

3- Subtract the second number from the first

4- Repeat the same with another number

$$\begin{array}{r} 97 \\ 79 \\ \hline 18 \end{array}$$

$$\begin{array}{r} 42 \\ 24 \\ \hline 18 \end{array}$$

1- Choose a two digit number such that the tens digit is greater by two than the units digit

2- Create a new number by reversing the digits

3- Subtract the second number from the first

4- Repeat the same with another number

5- What do you notice? Can you explain?

$$\begin{array}{r} 97 \\ 79 \\ \hline 18 \end{array}$$

$$\begin{array}{r} 42 \\ 24 \\ \hline 18 \end{array}$$

1- Choose a two digit number such that the tens digit is greater by two than the units digit

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3- Subtract the second number from the first

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5- What do you notice? Can you explain?

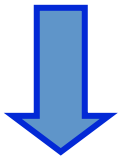
$$\begin{array}{r} 97 \\ 79 \\ \hline 18 \end{array}$$

$$\begin{array}{r} 42 \\ 24 \\ \hline 18 \end{array}$$

$$\begin{array}{rcl} - & x+2 & x \\ & x & x+2 \end{array}$$

$$- \quad x + 2 \qquad x$$

$$x \qquad x + 2$$

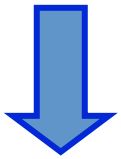


1

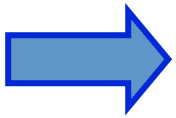
$$- \quad x + 2 \qquad x$$

$$x \qquad x + 2$$

$$\begin{array}{rcc} - & x+2 & x \\ & x & x+2 \\ \hline \end{array}$$



$$\begin{array}{rcc} & \overset{1}{\longrightarrow} & \\ - & x+2 & x \\ & x & x+2 \\ \hline \end{array}$$



$$\begin{array}{rcc} - & x+1 & x+10 \\ & x & x+2 \\ \hline & 1 & 8 \end{array}$$

Given number $\overline{AB}$ $A=B+2$

$$\overline{AB} + 2 = \overline{AA}$$

$$\overline{BA} - 2 = \overline{BB}$$

$$\overline{AA} - \overline{BB} = 22$$

$$\overline{AB} - \overline{BA} = 18$$

$$\frac{2x+3}{4x+6} = 2 \longrightarrow 2x + 3 = 2(4x + 6)$$

$$2x + 3 = 8x + 12 \longrightarrow -9 = 6x$$

$$x = -1.5$$

Arcavi, A. (1994) Symbol sense: Informal sense making in formal mathematics. *For the learning of mathematics* 14(3), 24-35

En algèbre…

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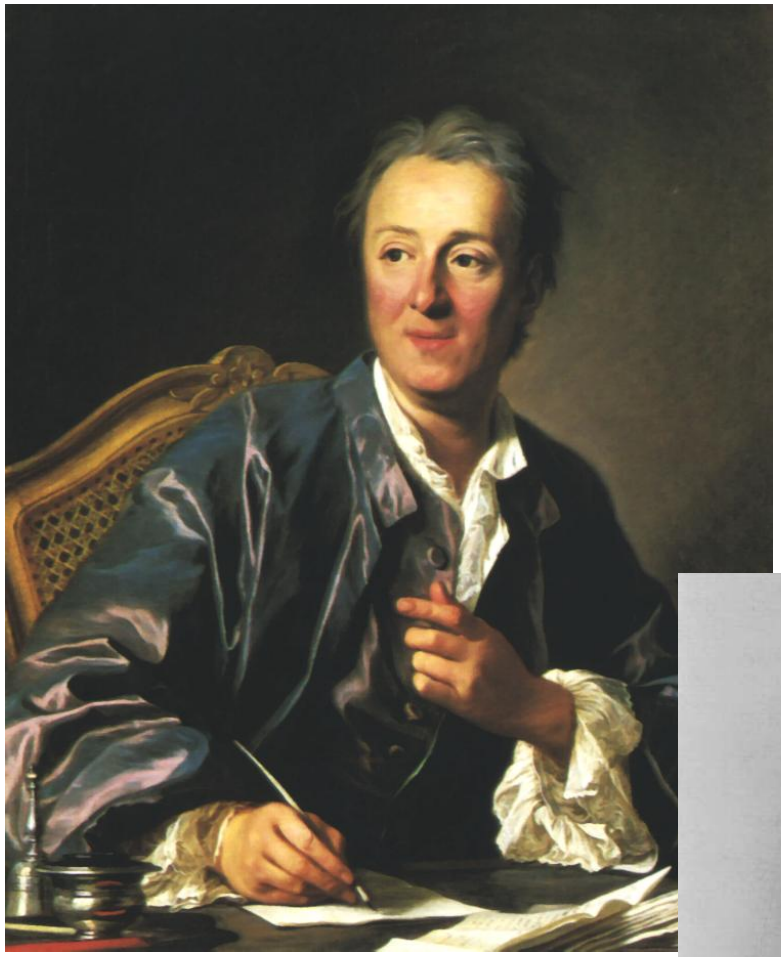
Augustus De Morgan



Denis Diderot (1713-1784)

Catherine the Great
(1729-1796)

Leonhard Euler (1707-1783)

“Monsieur!

$$\frac{a+b^n}{n}=x$$

Donc Dieu existe,

Repondez!”

“Diderot, to whom algebra was Hebrew, was embarrassed and disconcerted; ...
He asked permission to return to France at once, which was granted”

De Morgan, A Budget of Paradoxes
The Open Court Publishing Co. 1915, Vol II, p.4

"Diderot, à qui l'algèbre était l'hébreu, était embarrassé et déconcerté; ... Il a demandé la permission de retourner en France à la fois, qui a été accordé "

Exercices répétitifs/Inutilement complexes

6. Simplify $\dfrac{2340136986301369863013698630136962 9}{3450072992700729927007299270072958 2}$.

7. Simplify $\dfrac{\frac{1}{198}-\frac{4}{199}+\frac{6}{200}-\frac{4}{201}+\frac{1}{202}}{\frac{1}{197}-\frac{4}{198}+\frac{6}{199}-\frac{4}{200}+\frac{1}{201}}$.

8. Simplify $\dfrac{\frac{1}{198}-\frac{2}{99}+\frac{3}{100}-\frac{2}{101}+\frac{1}{204}}{\frac{1}{194}-\frac{2}{98}+\frac{3}{99}-\frac{2}{100}+\frac{1}{202}}$.

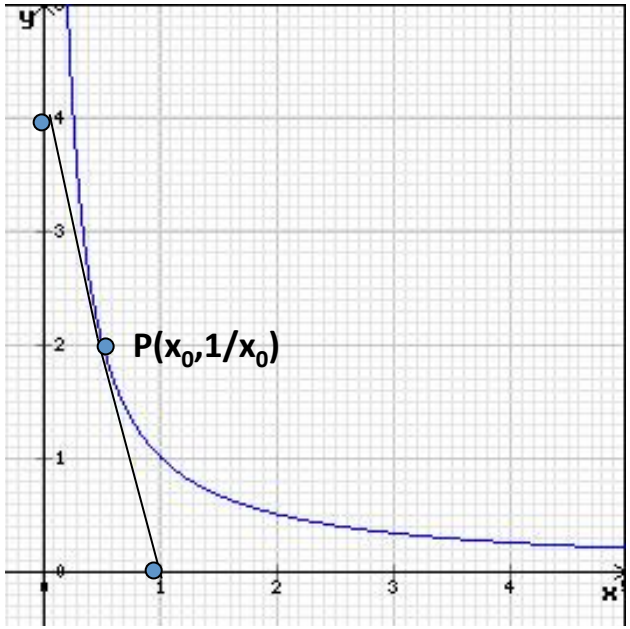
9. Simplify $\dfrac{\frac{1}{2}-\frac{1}{3}-\frac{1}{7}-\frac{1}{43}-\frac{1}{1807}-\frac{1}{3263443}}{\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{7}\cdot\frac{1}{43}\cdot\frac{1}{1807}\cdot\frac{1}{3263443}}$.

10. Simplify $\dfrac{3+1\frac{1}{2}+1\frac{2}{7}+1\frac{14}{67}+1\frac{938}{5623}+1\frac{5274374}{37772347}}{3\,.\,1\frac{1}{2}\,.\,1\frac{2}{7}\,.\,1\frac{14}{67}\,.\,1\frac{938}{5623}\,.\,1\frac{5274374}{37772347}}$.

11. Simplify $\dfrac{10+1\frac{1}{9}+1\frac{9}{91}+1\frac{819}{9181}+1\frac{7519239}{92480761}}{10\times1\frac{1}{9}\times1\frac{9}{91}\times1\frac{819}{9181}\times1\frac{7519239}{92480761}}$.

Given the function $f(x)=1/x$. P is a point on the graph of the function (in the first quadrant). A tangent line to the graph through P creates (with the axes) a right-angled triangle. What should the coordinates of P be, in order for the hypotenuse of that triangle to be maximum/ minimum?

IA's solution process:



- Sketched the graph
- Found derivative of f(x), $f'(x)= -1/x^2$
- Wrote equation of tangent line at P,

$$y-y_0=(-1/x_0^2)(x-x_0)$$

- Found coordinates of intersection with axes

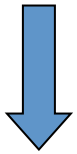
$(2x_0, 0)$ and $(0, 2/x_0)$

- Wrote the function for the segment length

$$g(x) = \sqrt{4x_0^2 + \frac{4}{x_0^2}}$$

- Asked himself: max or min?

“I have a friend who always does that [plays with the problem and makes sense of it], after such an effort, he usually has neither time nor energy to do the symbols, he does not get credit for what he may have done and fails the exams. If I don’t have to, I do only the symbols, which is what the teacher and the exam want.”



Effects of deep rooted classroom practices

En algèbre…

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$$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{8}$$

$$\frac{1}{9}+\frac{1}{8\cdot 9}=\frac{1}{9}+\frac{1}{72}=\frac{8+1}{72}=\frac{9}{72}=\frac{1}{8}$$

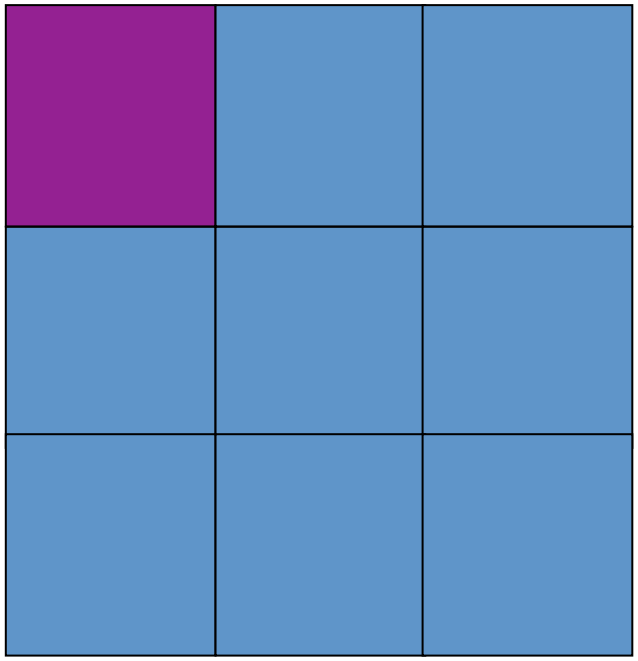
$$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{8}$$

$$\frac{1}{9}\cdot\left(1+\frac{1}{8}\right)=\frac{1}{9}\cdot\frac{9}{8}=\frac{1}{8}$$

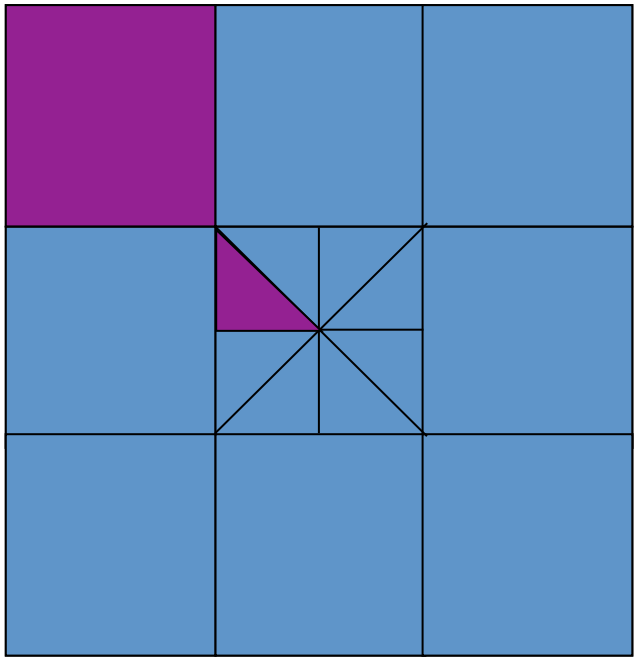
$$\frac{1}{9}+\frac{1}{8}\cdot\frac{1}{9}=\frac{1}{8}$$

$$\frac{1}{9}+\overbrace{\frac{1}{8}-\frac{1}{9}}=$$

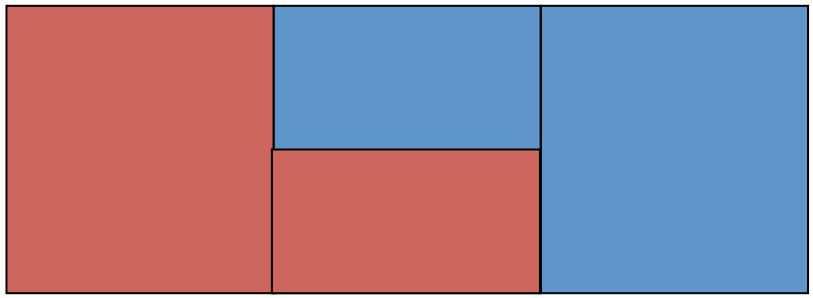
$$\frac{1}{9} + \frac{1}{8} \cdot \frac{1}{9} = \frac{1}{8}$$



$$\frac{1}{9} + \frac{1}{8} \cdot \frac{1}{9} = \frac{1}{8}$$

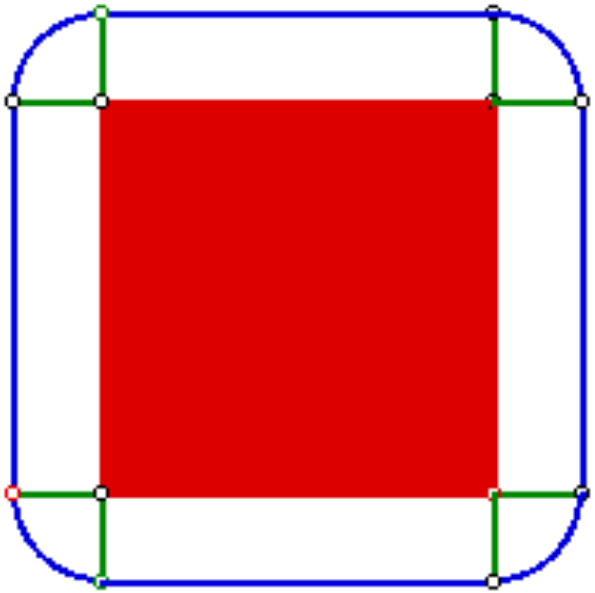


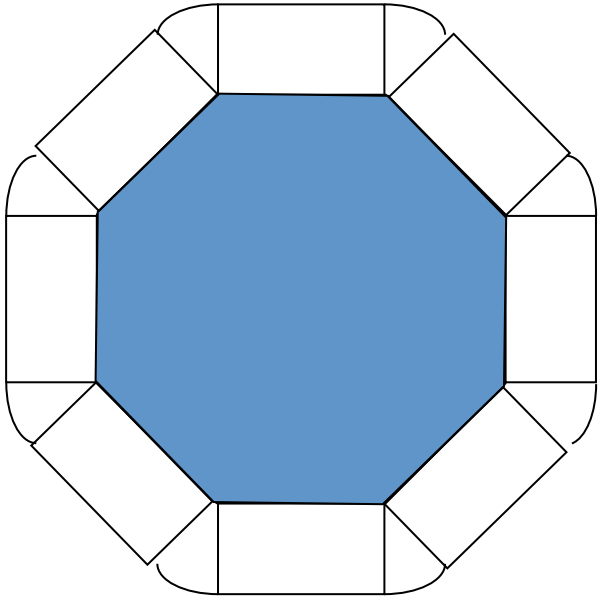
$$\frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} =$$



$$2\pi(r+2)-2\pi r$$

$$4\pi$$





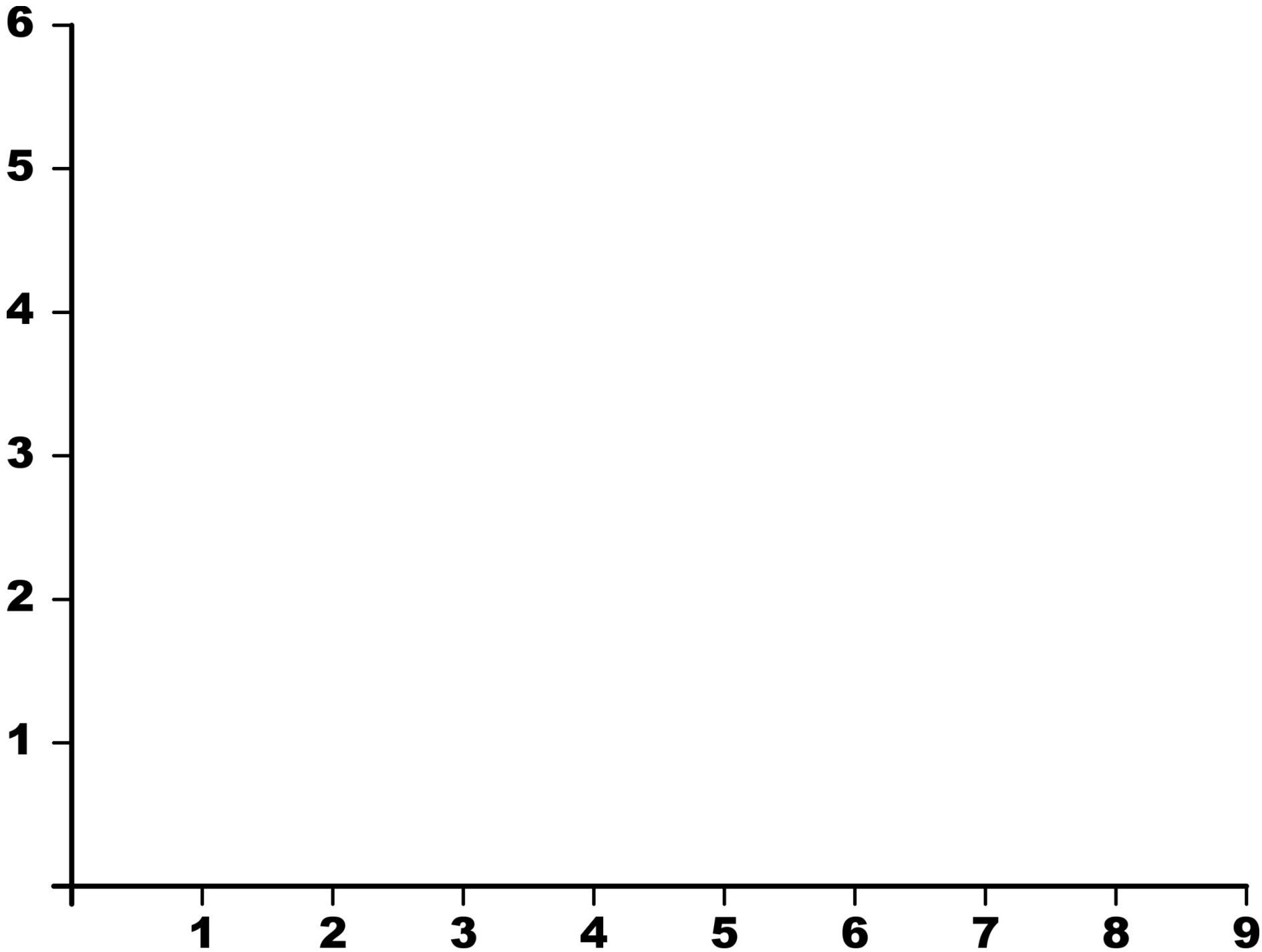
Papert, S. (1980) *Mindstorms: Children, computers, and powerful ideas*. Basic Books, NY., p.147

$$\frac{a}{b}, \frac{c}{d}$$

$$\frac{a+c}{b+d}$$

$$\frac{a}{b} < \frac{c}{d}$$

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$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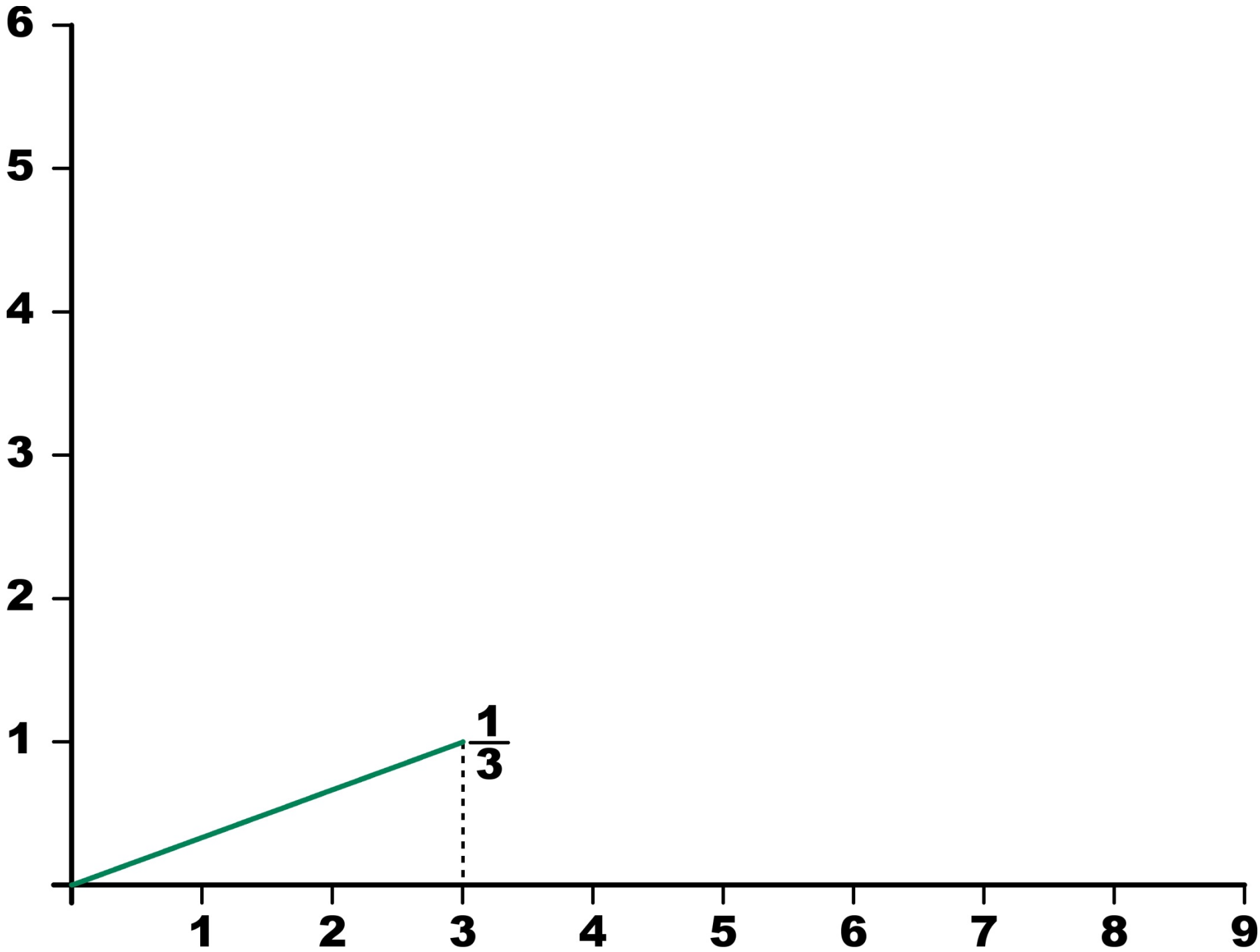
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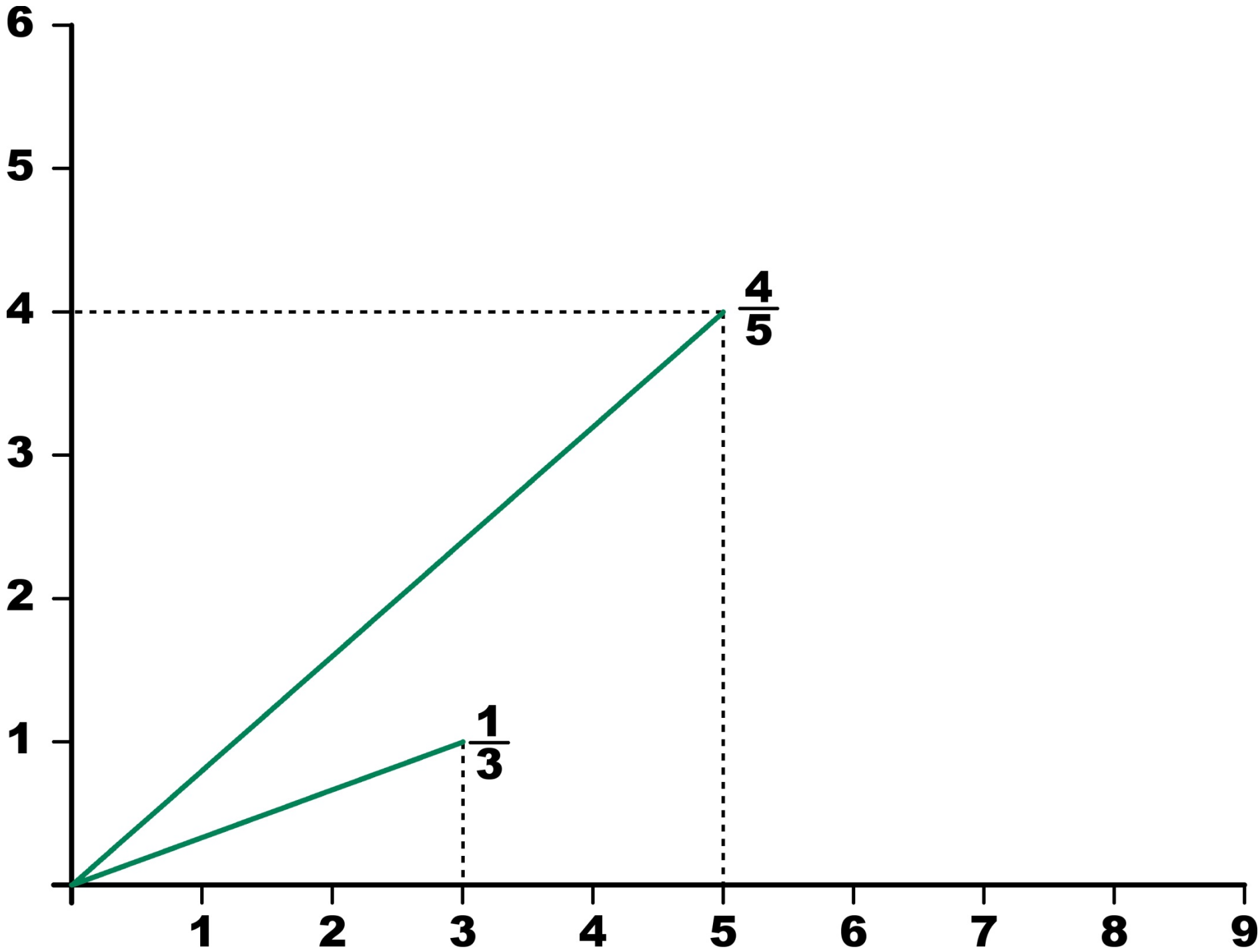
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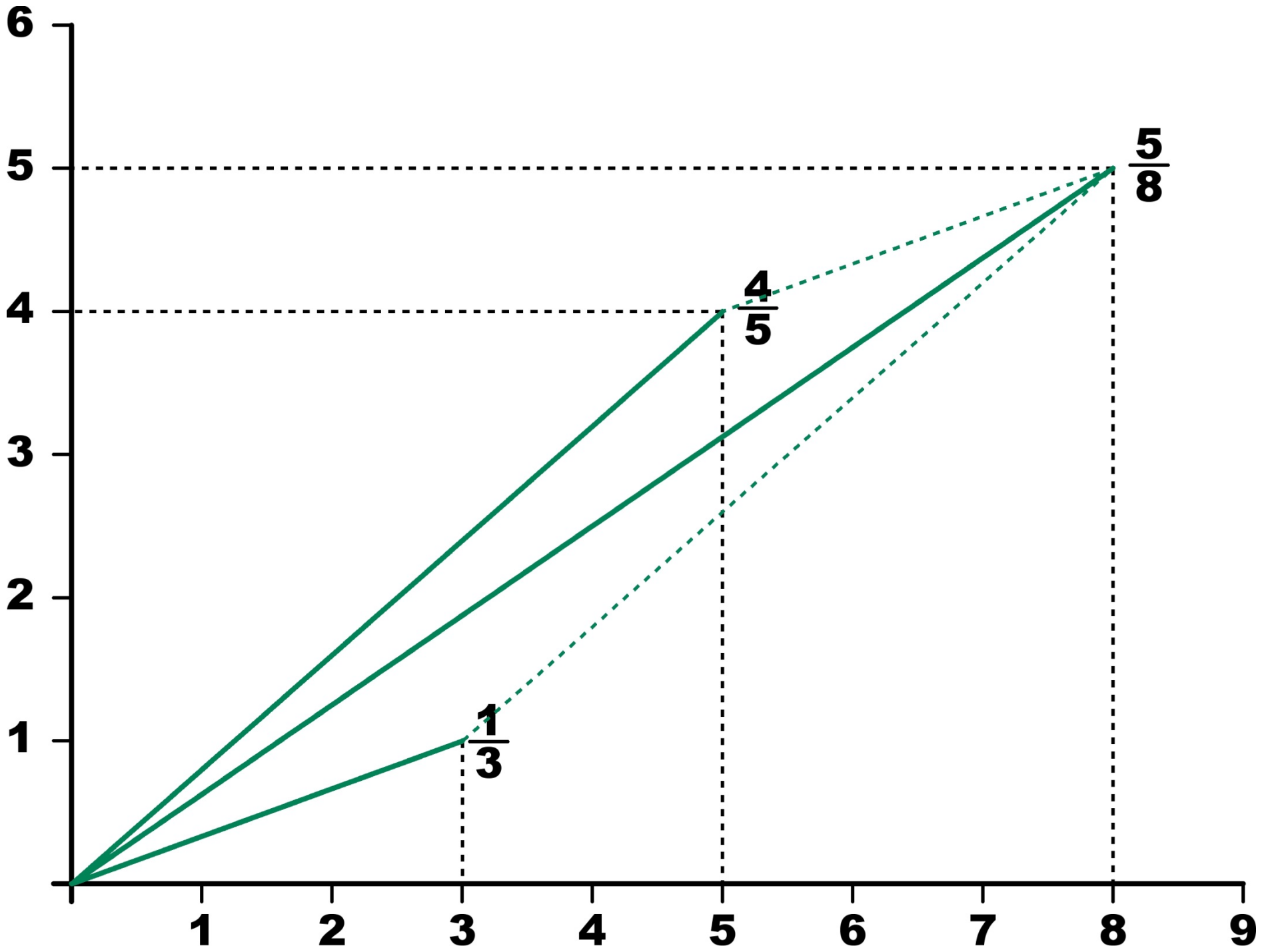
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Suite arithmétique

$$S_{10} = 65$$

$$a_{10} = 20$$

$$a_1 = ?$$

$$d = ?$$

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\left(\frac{1}{4}\right)^6+\ldots=$$

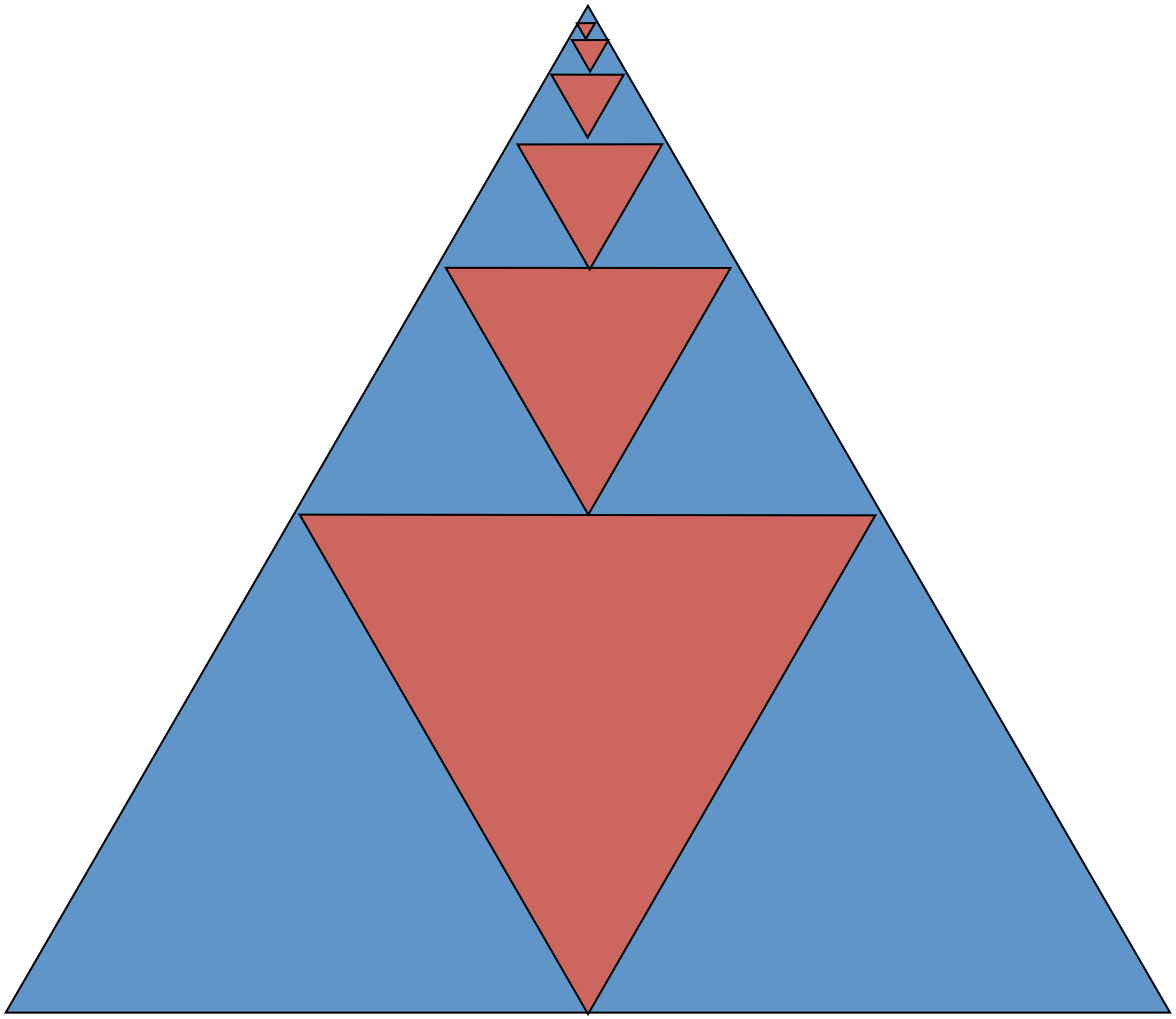
0<q<1,

$$\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}$$

$$a = q = \frac{1}{4}$$

$$\frac{1/4}{3/4} = \frac{1}{3}$$

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\left(\frac{1}{4}\right)^6+\ldots=\frac{1}{3}$$



Mabry, R.: 1999, '$\frac{1}{4}+(\frac{1}{4})^2+(\frac{1}{4})^3+\cdots=\frac{1}{3}$', *Mathematics Magazine* 72(1), 63.

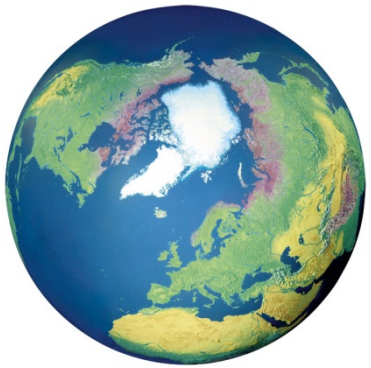
$$\frac{1}{9}+\left(\frac{1}{9}\right)^2+\left(\frac{1}{9}\right)^3+\left(\frac{1}{9}\right)^4+\left(\frac{1}{9}\right)^5+\left(\frac{1}{9}\right)^6+\ldots=$$

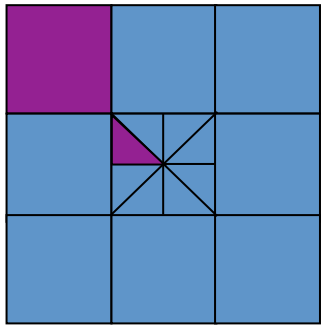
$0<q<1,$ $$\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}$$

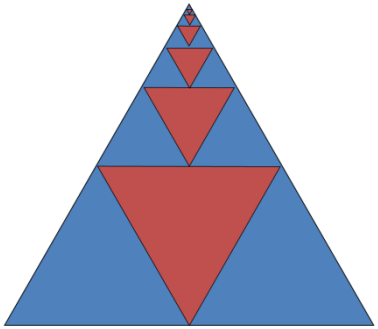
$$a = q = \frac{1}{9}$$

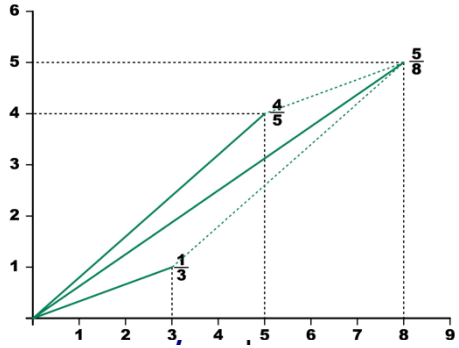
$$\frac{1/9}{8/9} = \frac{1}{8}$$

Ce que ces exemples ont en commun?









Visualization est :

- la compétence,
- le processus,
- le produit

de

- créer,
- interpréter,
- utiliser,
- reflechir

- mentalement,
- sur papier, o
- avec des outils technologiques,

- icônes
- images,
- diagrammes,
- graphiques

avec l'objectif de

- représenter et communiquer des informations,
- résoudre des problèmes, développer nouvelles idées,
- consolider notre compréhension.

Ressources pour la solution

Formules

$$S_n = na_1 + \frac{n(n-1)}{2}d$$

$$A_n = a_1 + (n-1)d$$

Substitution

$$\downarrow$$

$$65 = 10a_1 + 45d$$

$$20 = a_1 + 9d$$

Résoudre des équations

Quel genre de mathématiques avons-nous ici?

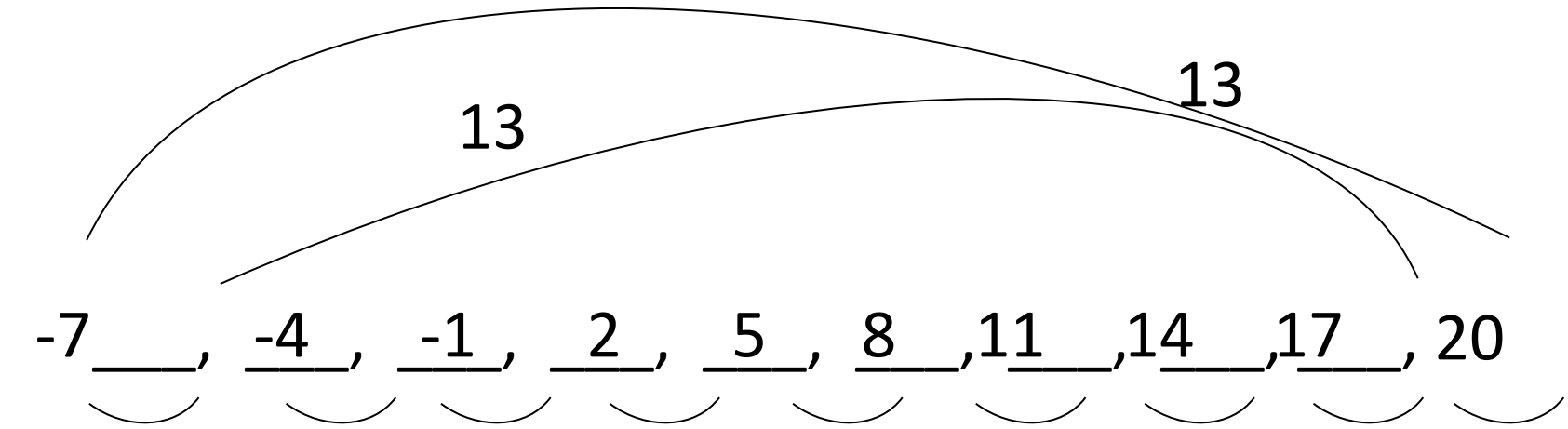
Suite arithmétique

$$S_{10} = 65$$

$$a_{10} = 20$$

$$a_1 = ?$$

$$d = ?$$



5 arcs are 65, each arc is $\frac{65}{5} = 13$

Since each arc is 13 and the tenth term is 20, the first will be -7

There are nine jumps from -7 to 20,

The distance is 27

We divide $\frac{27}{9} = 3$

The difference is b=3

9. א. $S_{10}=65$

$-7, -4, -1, 2, 5, 8, 11, 14, 17, 20$

5 קשתות. בין 65, נתן הסכום אחת שווה: $\frac{65}{5}=13$

מכאן שכל הסכום שווה 13 והאיבר ה-10 הוא 20 והאיבר

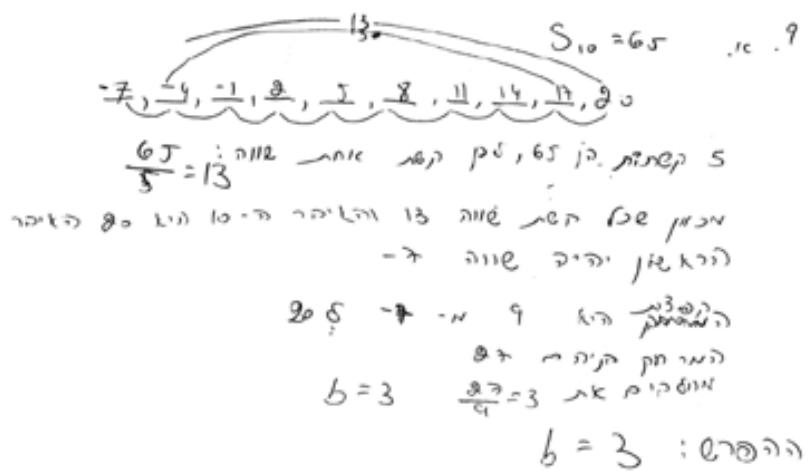
הראשון יהיה שווה -7

ההפרש הוא 9 מ- -7 עד 20

השאר חוק הניה ה- 20

מוסיקים את $\frac{27}{9}=3$ $b=3$

ההפרש: $b=3$



$S_{10} = 65$

$a_{10} = 20$

$a_1 = ?$

$d = ?$

$$S_n = na_1 + \frac{n(n-1)}{2}d$$

$$A_n = a_1 + (n-1)d$$

$$\downarrow$$

$$65 = 10a_1 + 45d$$

$$20 = a_1 + 9d$$

- Numérique
- Solution particulière
- Outils visuels
- Justifications verbales
- Transparence
- Common sense

- Symbolique
- Justifications syntaxiques

Coût vs bénéfice

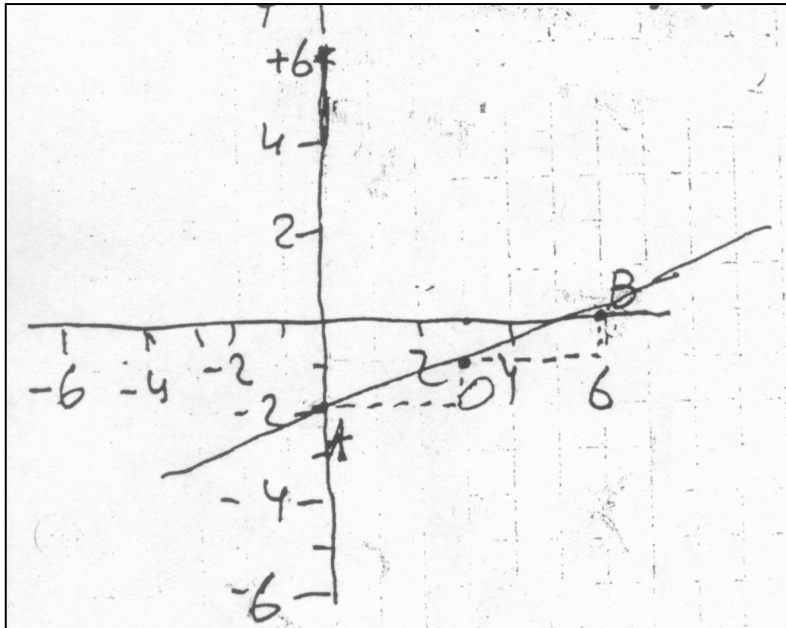
O (3, -1) is the midpoint of AB. A and B on the axes. Find their coordinates.

$$\sqrt{(x-3)^2+(0+1)^2}=\sqrt{(0-3)^2+(y+1)^2}$$

$$\sqrt{(x-3)^2+(0+1)^2}=\frac{1}{2}\sqrt{(x-0)^2+(y-0)^2}$$

$$\frac{x+0}{2}=3$$

$$\frac{y+0}{2}=-1$$



תרגיל 2

A על ציר y

B על ציר x

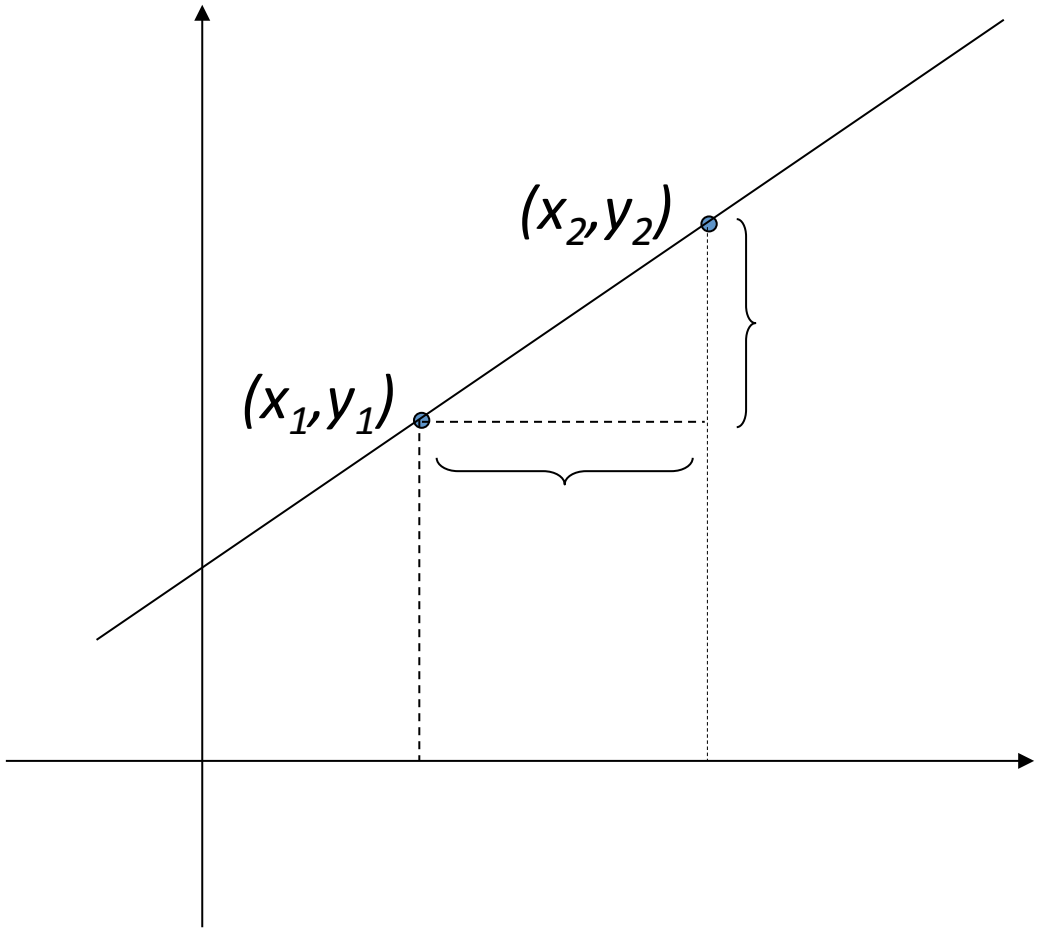
תשובה –

0 , -2 = A

6 , 0 = B

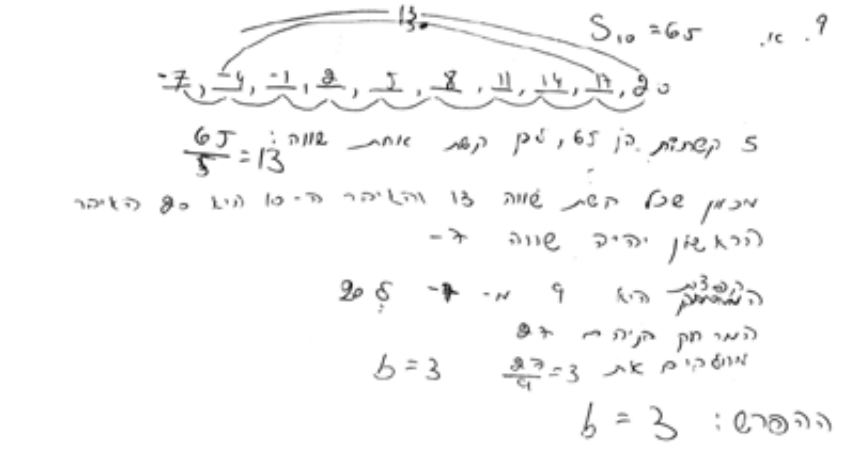
Pente d'une ligne

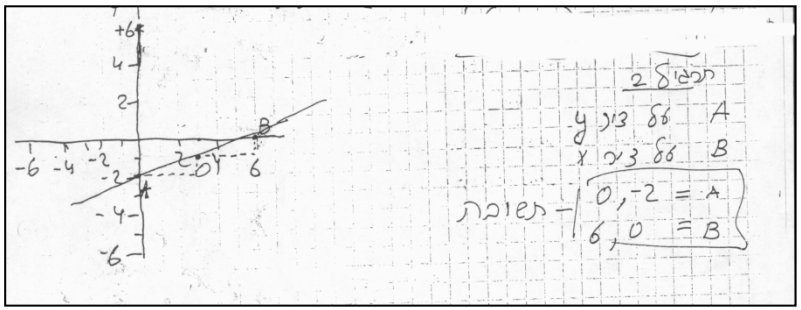
$$\frac{y_2 - y_1}{x_2 - x_1}$$

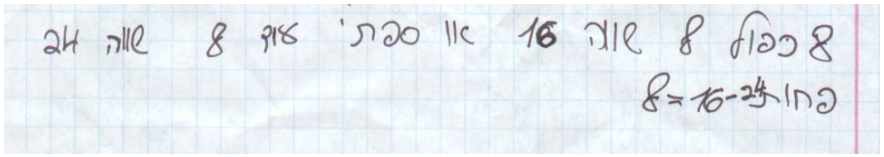


Dans une ferme de poulets et des vaches
il y a 70 têtes et 186 pieds

Ce que ces exemples ont en commun?







De-formalisation

Développer un regard mathématique


PROVINCIA
Serie J6
04369
Serie 5
34221
929


ロープウエーへ
ゆっくり歩いて10分
ときどき走って 7分
10 MIN. WALK (7 IF RUN A LITTLE!)
TO ROPEWAY STN.

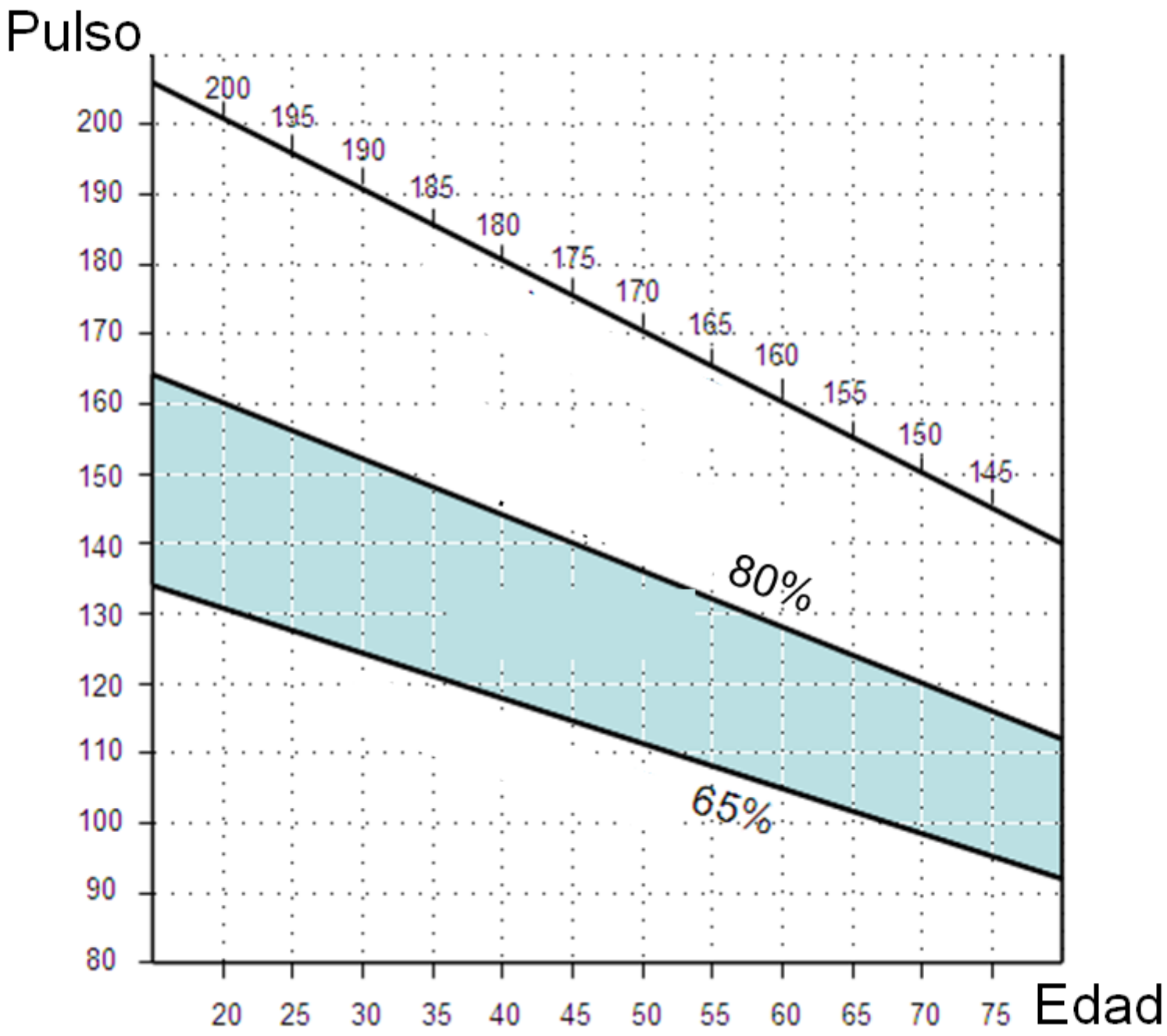

MAN
19·740·15


מעלית מס' 12
9 נוסעים PASS.
800 ק"ג KG.


0
450 kg 6 PERS.
CE 0099 R.A.E.: 5939


乗用
FOR PASSENGER
定員
CAPACITY
15 名
15 PRS
積載
LOADING
1000 kg
1000 kg
禁 煙
NO SMOKING


Capacidad máxima:
5 personas ó 600 Kg.
Maximum capacity:
5 people or 600 Kg.


Pulso
200
195
190
185
180
175
170
165
160
155
150
145
80%
65%
200
190
180
170
160
150
140
130
120
110
100
90
80
20
25
30
35
40
45
50
55
60
65
70
75
Edad