UNIVERSITÉ DE GENÈVE
FACULTÉ DE PSYCHOLOGIE ET DES SCIENCES DE L'ÉDUCATION

International Commission on Mathematical Instruction

DiMaGe
Didactique des Mathématiques à Genève

From Algebra to Functions: promoting mathematical thinking in the classroom

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Overview

1. Sense making in mathematics: Higher Order Thinking and the Method of Variation
2. The Method of Varied Inquiry for developing the sense symbol in the classroom
3. Analysing the MVI
4. Further examples
5. Conclusions

According to Alan Schoenfeld, doing math is a particular form of "sense-making”:

Each discipline has its own version of "sense making": historians have their own way of looking at the world, as well as anthropologists and physicists.

What makes each of these distinct areas are the instruments, norms and habits of mind.

The tools of mathematics are: abstraction, symbolic representation, and symbolic manipulation.

However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman.

Learning to think mathematically means:

(a) developing a mathematical point of view — valuing the processes of mathematization and abstraction and having the predilection to apply them;

(b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure — ***mathematical sense-making****.*

Alan Schoenfeld (2012, 2014)

A consequent main question:

How to trigger and support mathematical sense-making in the classroom? What about Algebra?

A possible answer from many researches:

pursuing High Order Thinking (HOT) in the math class.

Research conducted in the past decade or more in a variety of American classroom contexts has found that greater student learning occurs in classrooms where the high-level cognitive demands of mathematical tasks are consistently maintained throughout the instructional episode.

Silver, E. Cross-national comparisons of mathematics curriculum materials: what might we learn? *ZDM Mathematics Education* (2009) 41:827–832.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, 110(3), 608–645.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform.

Tarr, J. E., Reys, R. E., Reys, B. J., Chavez, O., Shih, J., & Osterlind, S. J. (2008). The impact of middle-grades mathematics curricula and the classroom learning environment on student achievement. *Journal for Research in Mathematics Education*, 39, 247–280.

Higher-order thinking is a general concept of some education reforms based on learning taxonomies (such as Bloom's one, 1956, revised in 2001 by Anderson, Krathwohl, et al.).

The idea is that some types of learning require more cognitive processing than others, but also have more generalized benefits.

In Bloom's taxonomy, for example, skills involving analysis, evaluation and synthesis (creation of new knowledge) are thought to be of a higher order, requiring different learning and teaching methods than the learning of facts and concepts.

A definition of HOT:

Higher-order thinking requires students to **manipulate information and ideas** in ways that transform their meaning and implications. This transformation occurs when students combine facts and ideas in order to **synthesise, generalise, explain, hypothesise or arrive at some conclusion or interpretation**.

Manipulating information and ideas through these processes allows students to **solve problems and discover new (for them) meanings and understandings**.

Lingard et al., (2001); Lincoln (2008)

The main question becomes: How to develop HOT in the classroom?

An answer. NCTM (2012) for helping students to conjecture, invent, and solve problems include:

(i) What would happen if …?
(ii) Do you see a pattern?
(iii) What are some possibilities here?
(iv) Can you predict the next one? How about the last one?
(v) How did you think about the problem?
(vi) What decision do you think he/she should make?
(vii) What is alike and what is different about your method of solution and his/hers?

In Italy we are defining what we call "macro-areas of competences", to be pursued in the classrooms through HOT activities:

KNOWING

PROBLEM SOLVING/POSING

ARGUING

Considering HOT from an international perspective, a problem emerges.

From the one hand in Western culture…

… repetitive learning is often positioned as the opposite of deep learning and understanding (Marton & Saljo, 1976).

Many western educators hold the view that students should be encouraged to understand rather than to memorise what they are learning (Purdie, Hattie & Douglas, 1996) as they believe that understanding is more likely to lead to high quality outcomes than memorizing (Dahlin & Watkins, 2000).

On the other hand…

…in some contries in the East (e.g. in China mainland) learners tend to be oriented towards rote learning and memorization (Marton, Watkins & Tang, 1997).

Educators of those countries have been criticized as not providing a learning environment which is conducive to "good learning" and using a teaching method which is merely "passive transmission" and "rote drilling" (Gu, Huang & Marton, 2004).

According to this criticism, rote learning would lead to poor learning outcomes.

The evidence suggests, however that many students of those Eastern countries consistently outperform their Western counterparts in many international comparative studies of mathematics achievement such as TIMSS and PISA.

This contradictory phenomenon was referred to by Marton, Dall'Alba and Lai (1993) as the "**paradox of the Chinese learner**".

Two related questions arise:

- 1. How can Eastern students do so well in international comparative tests – especially in fields like mathematics and science – if they are "only" rote learners?

  2. Why do they report in both quantitative and qualitative investigations that they are trying to understand what they are learning while their western teachers consider them as mere learners by rote?

(Dahlin and Watkins 2000)

Throughout their work, Marton, Mun Yee Lai and many others have argued that the **theory of variation** may be central to solving the paradox.

According to the Theory of Variation, a key feature of learning involves experiencing a phenomenon in a new light (Marton, 1999).

In other words, "learning amounts to being able to discern certain aspects of the phenomenon that one previously did not focus on or which one took for granted, and simultaneously bring them into one's focal awareness" (Lo, Chik & Pang, 2006, p.3).

Thus, teaching with variation helps students to actively try things out, and then to construct mathematical concepts that meet specified constraints, with related components richly interconnected (Watson & Mason, 2005).

Watson and Chick (2011) highlight the importance of teachers selecting mathematical tasks and examples with adequate variation to ensure that the critical features of the intended concept(s) are exemplified without unintentional irrelevant features.

Thus, a crucial point in the use of variation is that it should be controlled and systematic in every case.

According to Gu et al. (2004), what they call procedural variation is derived from three forms of problem solving:

(1) varying a problem: extending the original problem by varying the conditions, changing the results and generalization;

(2) multiple methods of solving a problem by varying the different processes of solving a problem and associating different methods of solving a problem;

(3) multiple applications of a method by applying the same method to a group of similar problems.

Many studies consistently find that those teachers who pursue the Variation method do not see repetition and understanding as separate but rather as interlocking processes, complementary to each other (Waktins & Biggs, 2001).

Thus, those western educators who reject rote and repetitive learning may have failed to understand the learning strategy in the Eastern context.

Equating repetitive learning with "surface learning without understanding" oversimplifies and misinterprets the intrinsic meaning of that notion of learning.

(Mun Yee Lai & Sara Murray, 2012)

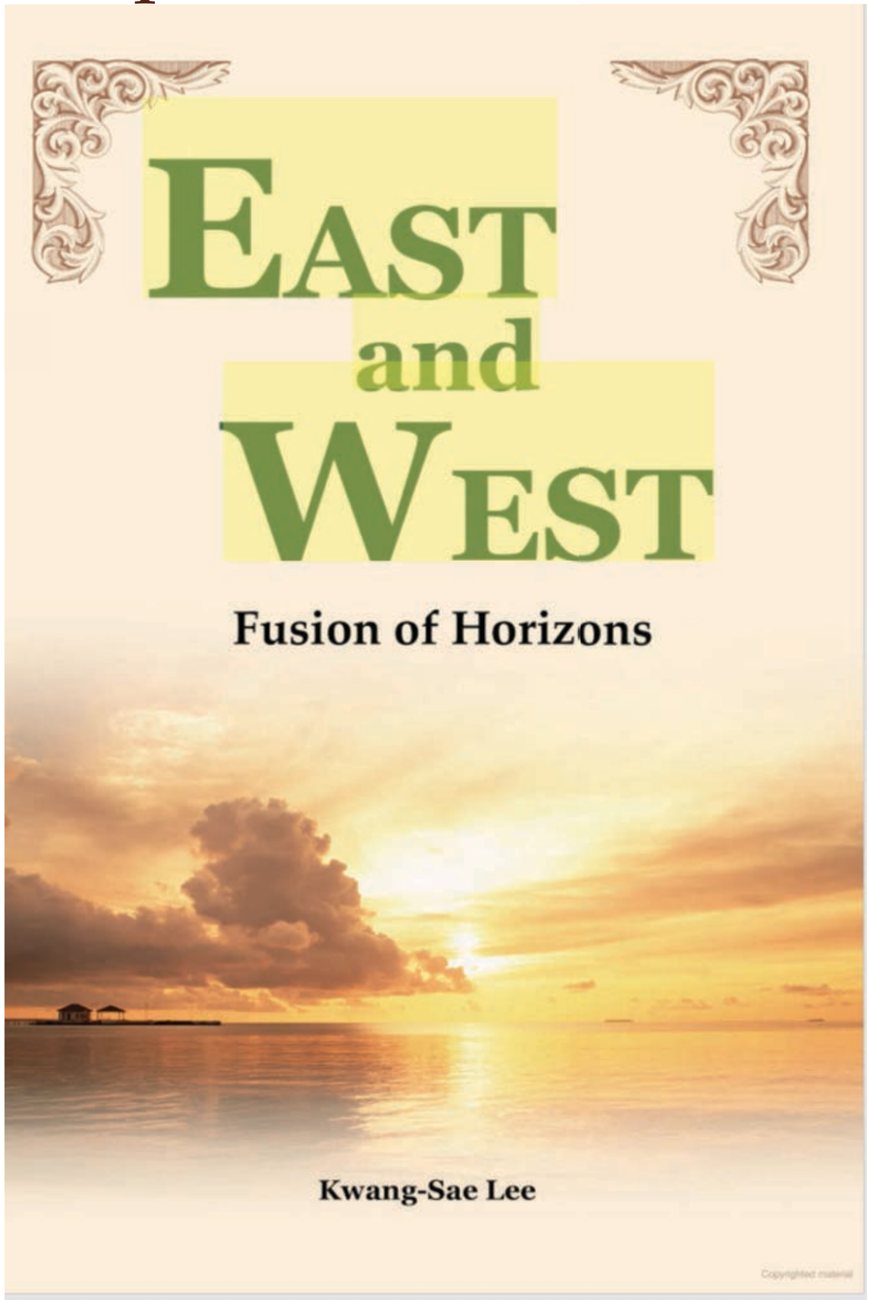
The value of the Method of Variation (MoV) for enhancing student understanding goes some way to unfold the paradox pointed out by Marton, Dall'Alba and Lai.

My claim is that the practices behind HOT and MoV can deeply benefit each other.

Some scholars both from Western and Eastern part (e.g. A. Watson & J. Mason, Fan Lianghuo, Wong Ngai-Ying, Cai Jinfa, Li Shiqi, …) are doing researches and proposing teaching programs towards such a direction.

This interaction MoV+HOT can be improved further suitably using ICTs in the classroom.

I will now discuss some examples, where HOT and MoV can both be present and integrated into what I call the **Method of Varied Inquiry** (**MVI**), where East and West can meet and put together the benefits of HOT and MoV. The examples will consider how to develope what Arcavi calls the **symbol sense** in the classroom.



2. The Method of Varied Inquiry to develop the symbol sense

MVI can be a method for developing the symbol sense in the classroom

—An understanding of and an aesthetic feel for the power of symbols: understanding how and when symbols can and should be used in order to display relationships, generalizations, and proofs which otherwise are hidden and invisible.

—A feeling for when to abandon symbols in favor of other approaches in order to make progress with a problem, or in order to find an easier or more elegant solution or representation.

—An ability to manipulate and to "read" symbolic expressions as two complimentary aspects of solving algebraic problems. On the one hand, the detachment of meaning necessary for manipulation coupled with a global "gestalt" view of symbolic expressions makes symbol-handling relatively quick and efficient. On the other hand, the reading of the symbolic expressions towards meaning can add layers of connections and reasonableness to the results.

A. Arcavi (1994). FLM 14 (3). p. 31

MVI can be a method for developing the symbol sense in the classroom (ct.ed)

—The awareness that one can successfully engineer symbolic relationships which express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions.

—The ability to select a possible symbolic representation of a problem, and, if necessary, to have the courage, first, to recognize and heed one's dissatisfaction with that choice, and second, to be resourceful in searching for a better one as replacement.

—The realization of the constant need to check symbol meanings while solving a problem, and to compare and contrast those meanings with one's own intuitions or with the expected outcome of that problem.

—Sensing the different roles symbols can play in different contexts.

Through the examples I will underline:

- the interweaving between classroom practices, rules (implicit or explicit) and the symbol sense that results for students.
- classroom practices that possibly induce a different sense for mathematics (with the consequent possible rules).

Brown, S.I. & **Walter, M.I.**, (2005). *The Art of Problem Posing*,, Lawrence Erlbaum publ.
Schoenfeld, A. (2012). Problematizing the didactic triangle, *ZDM*.
Arcavi, A. & **Friedlander, A.** (in press), *Tasks and Competencies in the Teaching and Learning of Algebra*, NCTM

Example 1

Step 1: "Looking for patterns"

1	3	3
2	4	8
3	5	15
4	6	24
5	7	35

Question1: what do you observe?

Examples of answers:

O1. There are always two factors, which reproduce the succession of natural numbers, starting from 1 and from 3.

O2. In every line there is a difference of two between the two first numbers.

O3. The third number is the product of the first two.

O4. The products (3, 8, 15, 24, 35) are almost perfect squares: to get them you must add 1.

O4. These perfect squares are the same as the squares of the numbers between the two factors: 3-4-5 → 16; 5-6-7 → 36.

Examples of answers:

O5. Also the differences between the products show an interesting pattern :

1 • 3	3
2 • 4	8
3 • 5	15
4 • 6	24
5 • 7	35

8 - 3	5
15 - 8	7
24 - 15	9
35 - 24	11

O6. …

One can also use an excel spreadsheet:

1	3
2	4
3	5
4	6
5	7
6	8
7	9
8	10
9	11
10	12
11	13
12	14
13	15
14	16
15	17

One can also use an excel spreadsheet:

1	3	3	4
2	4	8	9
3	5	15	16
4	6	24	25
5	7	35	36
6	8	48	49
7	9	63	64
8	10	80	81
9	11	99	100
10	12	120	121
11	13	143	144
12	14	168	169
13	15	195	196
14	16	224	225
15	17	255	256

INSTR.

Step 2: "What if… ?"

Let us change for ex. O2: now suppose that the factors differ by 4 [(~ O2)$_4$]. We get:

1 • 5	5
2 • 6	12
3 • 7	21
4 • 8	32
5 • 9	45

Remembering the observation O4 one can ask (Q1)*:

- what about squares?
- are still they there?
- can we discover them again?

Hunting for squares (Q2):

?

1 • 5	5
2 • 6	12
3 • 7	21
4 • 8	32
5 • 9	45

1 • 5	5	4 + 1	9 – 4
2 • 6	12	9 + 3	16 - 4
3 • 7	21	16 + 5	25 - 4
4 • 8	32	25 + 7	36 - 4
5 • 9	45	36 + 9	49 - 4

What is more "equal" to O4: A or B?

x•y	=	O3
1 • 3	3	4 - 1
2 • 4	8	9 - 1
3 • 5	15	16 - 1
4 • 6	24	25 - 1
5 • 7	35	36 -1

x•y	=	A	B
1 • 5	5	4 + 1	9 – 4
2 • 6	12	9 + 3	16 - 4
3 • 7	21	16 + 5	25 - 4
4 • 8	32	25 + 7	36 - 4
5 • 9	45	36 + 9	49 - 4

(Q1.1)* Why?

1. We have started with observations (looking for patterns):

O1. There are always two factors, which reproduce the succession of natural numbers, starting from 1 and from 3.

O2. In every line there is a difference of two between the two first numbers.

O3. The thiord number is the product of the first two.

O4. The products (3, 8, 15, 24, 35) are almost perfect squares: to get them you must add 1.

O4. Also the differences between the products show an interesting pattern :

2. We have modified O2 going to $(\sim O2)_4$, then we have looked for the situation most equal to O4, let us say $(O4)^*$ and we have given an answer R1 like:

$$(\sim O2)_4 \rightarrow (O4)^*.$$

Can we even go further?

x•y	=	O3
1 • 7	7	□ - d
2 • 8	16	□ - d
3 • 9	27	□ - d
4 • 10	40	□- d
5 • 11	55	□ -d

→

x•y	=	O3
1-4-7	7	16 - 9
2-5-8	16	25 - 9
3-6-9	27	36 - 9
4-7-10	40	49 - 9
5-8-11	55	64 - 9

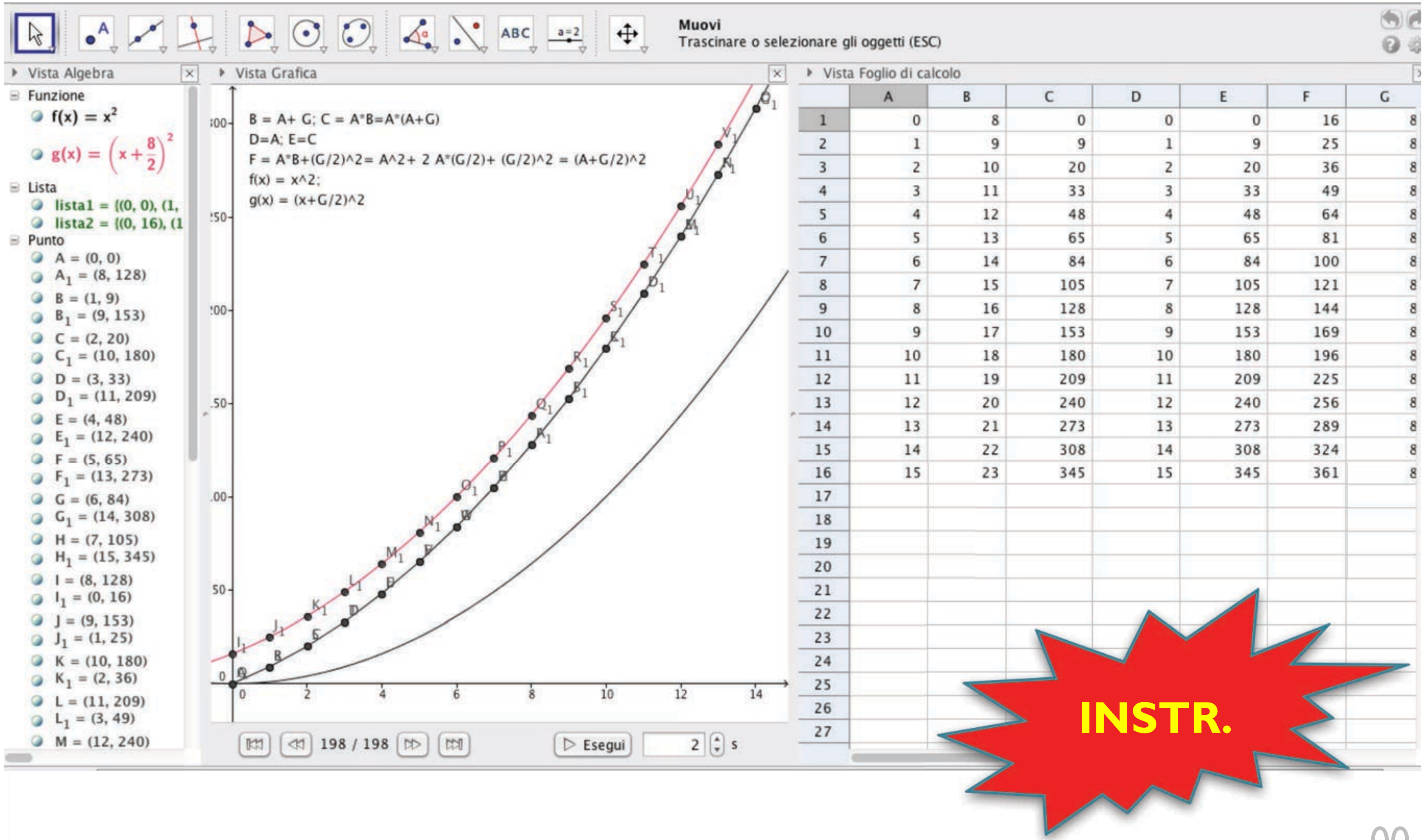
The algebraic language can explain all these regularities, and excel can help students to arrive to that :

n	n+2	n(n+2)	n(n+2)+1	(n+1)	(n+1)^2
1	3	3	4	2	4
2	4	8	9	3	9
3	5	15	16	4	16
4	6	24	25	5	25
5	7	35	36	6	36
6	8	48	49	7	49
7	9	63	64	8	64
8	10	80	81	9	81
9	11	99	100	10	100
10	12	120	121	11	121

INSTR.

$$n(n+2)+1 = n^2 + 2n + 1 = (n+1)^2$$

Using GeGe allows investigations through a "game of frameworks" interacting each other:



It is interesting to discuss the didactical and cognitive difference between a sequence of problems as that generated by our method and problems that directly ask to prove the algebraic formulas behind our tables.

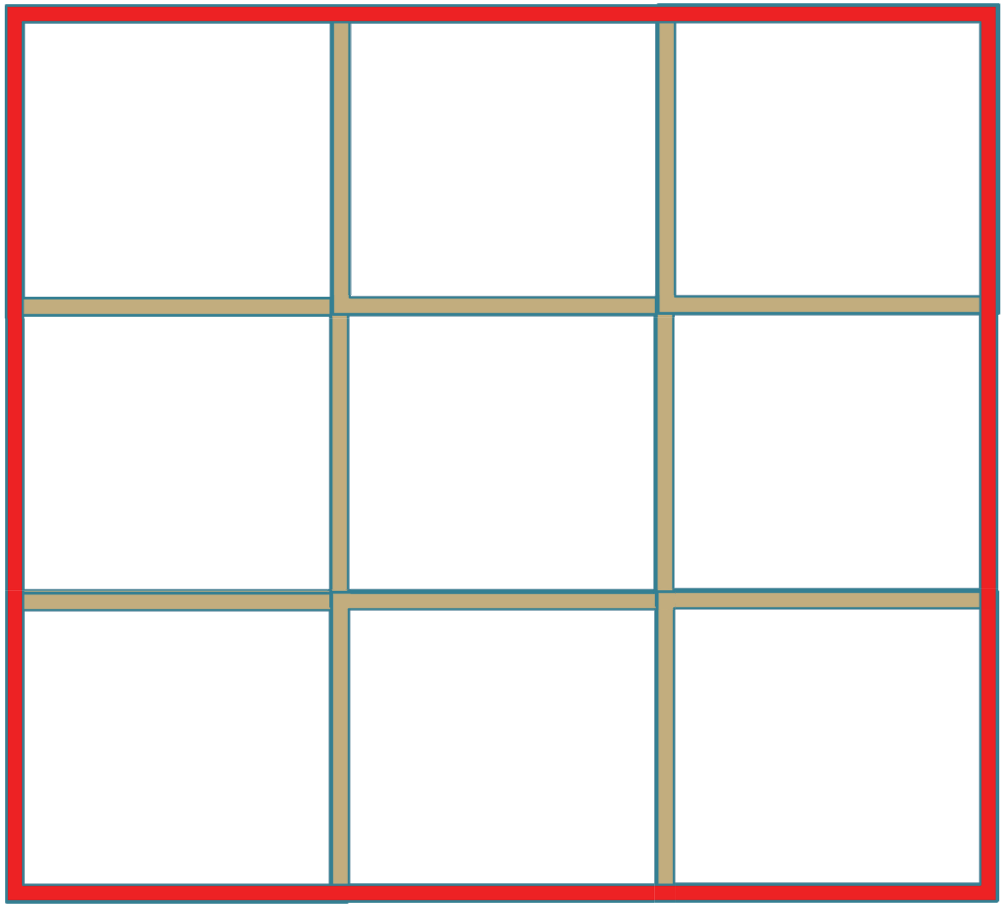
1 • 5	5
2 • 6	12
3 • 7	21
4 • 8	32
5 • 9	45

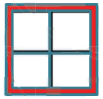
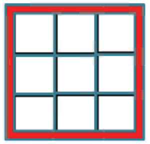
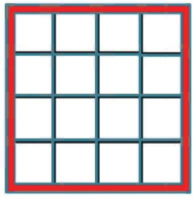
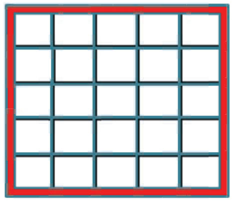
x•y	=	O3
1 • 7	7	$h^2 - k^2$
2 • 8	16	$h^2 - k^2$
3 • 9	27	$h^2 - k^2$
4 • 10	40	$h^2 - k^{22}$
5 • 11	55	$h^2 - k^2$

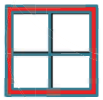
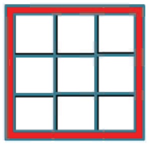
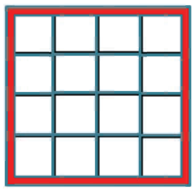
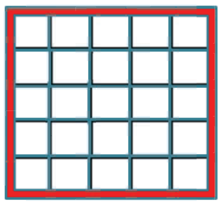
Example 2

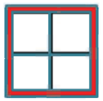
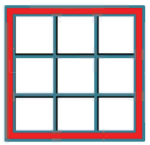
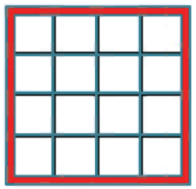
2.1 The geometric pastry chef

A model of the cake with the cuts



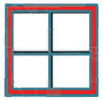
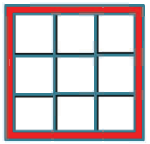
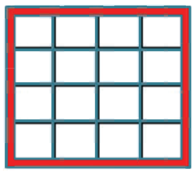
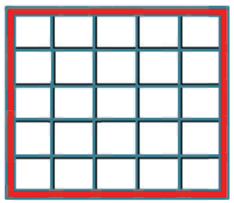
	size	n. 1-squ.s	n. 1-squ.s 0 straw.	n. 1-squ. straw.s on 1 side	n. 1-squ. straw.s on 2 sides	
	2x2					
	3x3					
	4x4					
	5x5					
	axa					

	size	n. 1-squ.s	n. 1-squ.s 0 straw.	n. 1-squ. straw.s on 1 side	n. 1-squ. straw.s on 2 sides	
	2x2	$4 = 2^2$	0	0	4	
	3x3	3^2	1	4=4x1	4	
	4x4	4^2	2^2	8=4x2	4	
	5x5	5^2	3^2	12=4x3	4	
	axa	a^2	$(a-2)^2$	$4(a-2)$	4	

	size	n. 1-squ.s	n. 1-squ.s 0 straw.	n. 1-squ. straw.s on 1 side	n. 1-squ. straw.s on 2 sides	
	1x1	1	0	?	?	
	2x2	$4 = 2^2$	0	0	4	
	3x3	3^2	1	4=4x1	4	
	4x4	4^2	2^2	8=4x2	4	
	axa	a^2	$(a-2)^2$	$4(a-2)$	4	

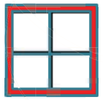
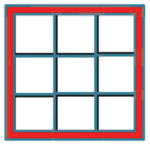
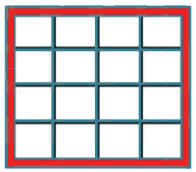
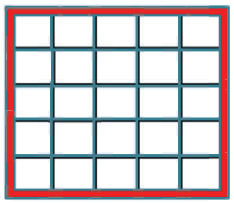
Step 1a

Look at numbers and formulas with a mathematical eye

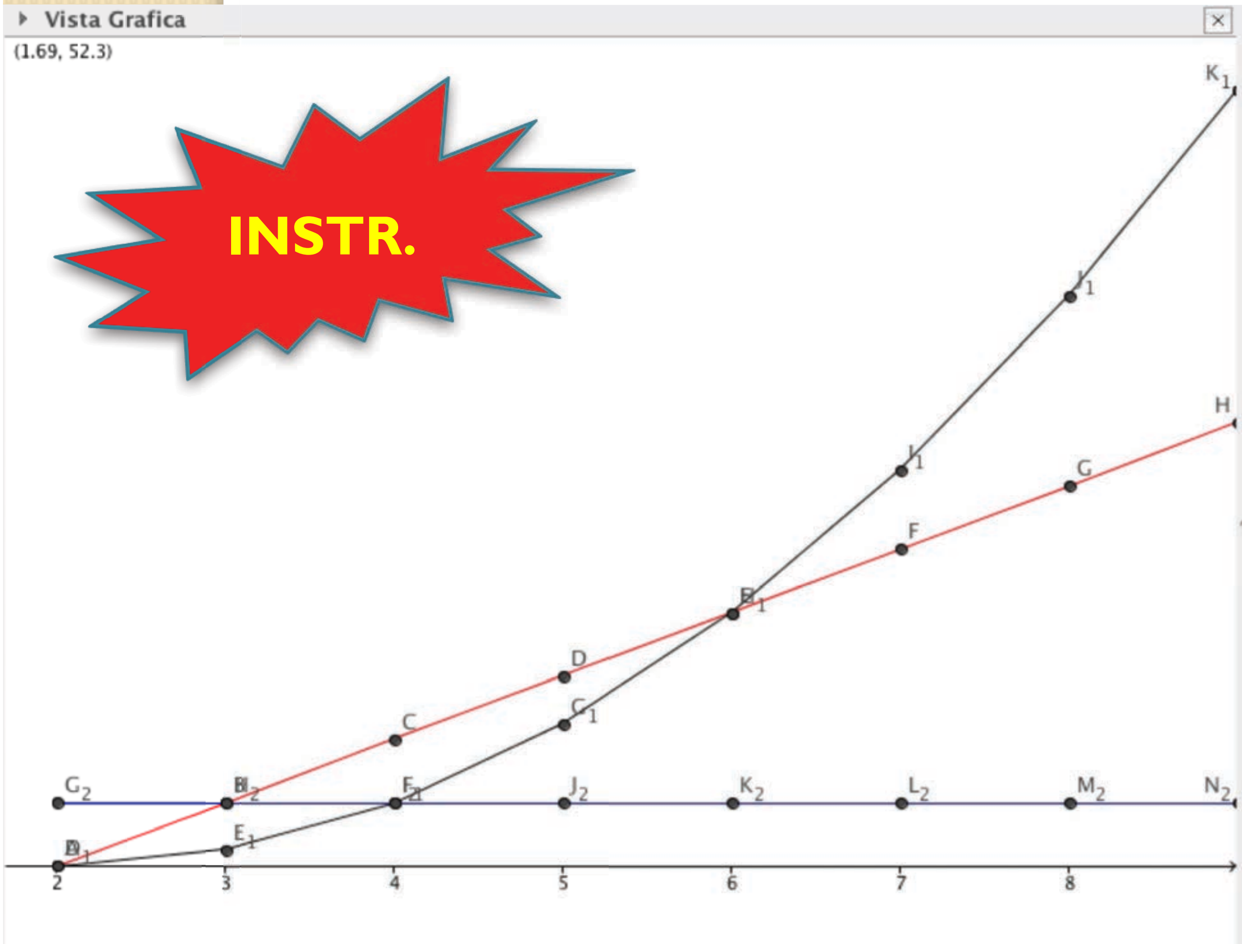
	2x2	$4 = 2^2$	0	0	4	
	3x3	3^2	1	4=4x1	4	
	4x4	4^2	2^2	8=4x2	4	
	5x5	5^2	3^2	12=4x3	4	
	axa	a^2	$(a-2)^2$	$4(a-2)$	4	

1b

How can you represent the data?

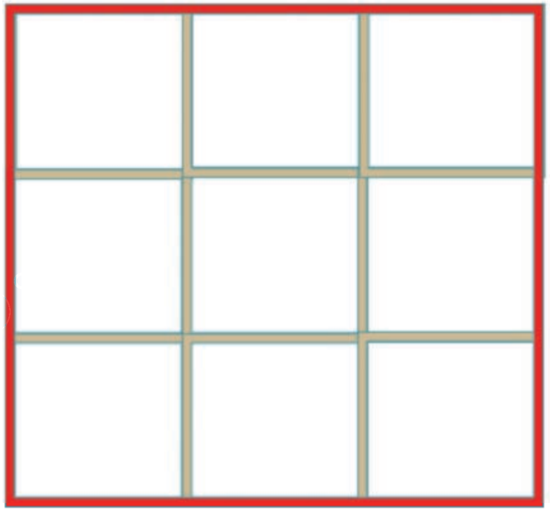
	2x2	$4 = 2^2$	0	0	4	
	3x3	3^2	1	4=4x1	4	
	4x4	4^2	2^2	8=4x2	4	
	5x5	5^2	3^2	12=4x3	4	
	axa	a^2	$(a-2)^2$	$4(a-2)$	4	

1b. How can you represent the data? What can you observe now?



Vista Foglio di calcolo

	B	C	D	E	F
1	0	2	0	2	4
2	4	3	1	3	4
3	8	4	4	4	4
4	12	5	9	5	4
5	16	6	16	6	4
6	20	7	25	7	4
7	24	8	36	8	4
8	28	9	49	9	4
9	32	10	64	10	4
10	36	11	81	11	4
11	40	12	100	12	4
12	44	13	121	13	4
13	48	14	144	14	4
14	52	15	169	15	4
15	56	16	196	16	4
16	60	17	225	17	4
17	64	18	256	18	4
18	68	19	289	19	4
19	72	20	324	20	4
20	76	21	361	21	4
21	80	22	400	22	4
22	84	23	441	23	4
23	88	24	484	24	4
24	92	25	529	25	4
25	96	26	576	26	4
26	100	27	625	27	4

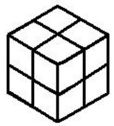
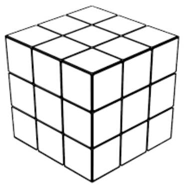
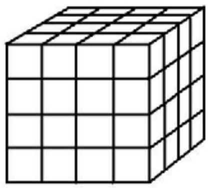
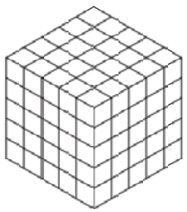


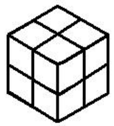
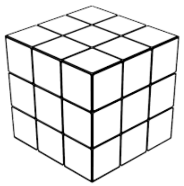
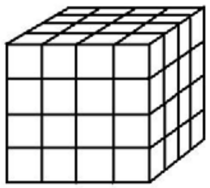
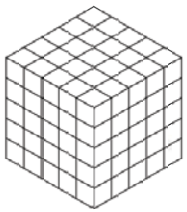
Step 2
What if…?

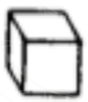
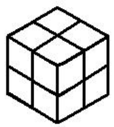
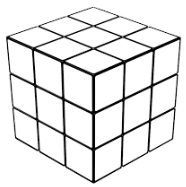
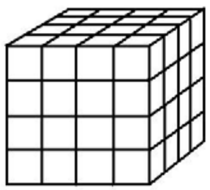
Look for a similar situation (e.g. 3 dim.): what story can you write?

Follow steps 1a, 1b …

The story

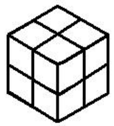
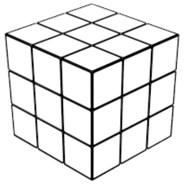
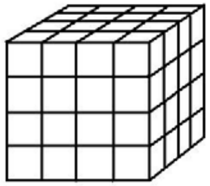
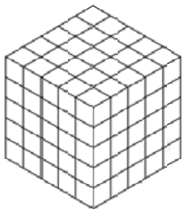
	size	n. 1-cubes	n. 1-cubes no glace	n. 1-cubes 1 face glaced	n. 1-cubes 2 faces glaced	n. 1-cubes 3 faces glaced
	2x2x2					
	3x3x3					
	4x4x4					
	5x5x5					
a 1-cubes	axaxa					

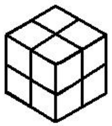
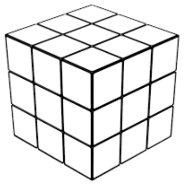
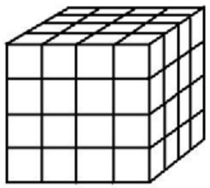
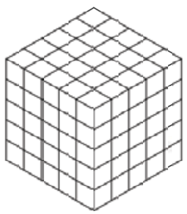
	size	n. 1-cubes	n. 1-cubes no glace	n. 1-cubes 1 face glaced	n. 1-cubes 2 faces glaced	n. 1-cubes 3 faces glaced
	2x2x2	$8 = 2^3$	0	0	0	8
	3x3x3	3^3	1	6=6x1	12=12x1	8
	4x4x4	4^3	2^3	24=6x4	24=12x2	8
	5x5x5	5^3	3^3	54=6x9	36=12x3	8
a 1-cubes	*a*x*a*x*a*	a^3	$(a-2)^3$	$6(a-2)^2$	$12(a-2)$	8

	size	n. 1-cubes	n. 1-cubes no glace	n. 1-cubes 1 face glaced	n. 1-cubes 2 faces glaced	n. 1-cubes 3 faces glaced
	1x1x1	1	1	?	?	?
	2x2x2	$8 = 2^3$	0	0	0	8
	3x3x3	3^3	1	6=6x1	12=12x1	8
	4x4x4	4^3	2^3	24=6x4	24=12x2	8
a 1-cubes	axaxa	a^3	$(a-2)^3$	$6(a-2)^2$	$12(a-2)$	8

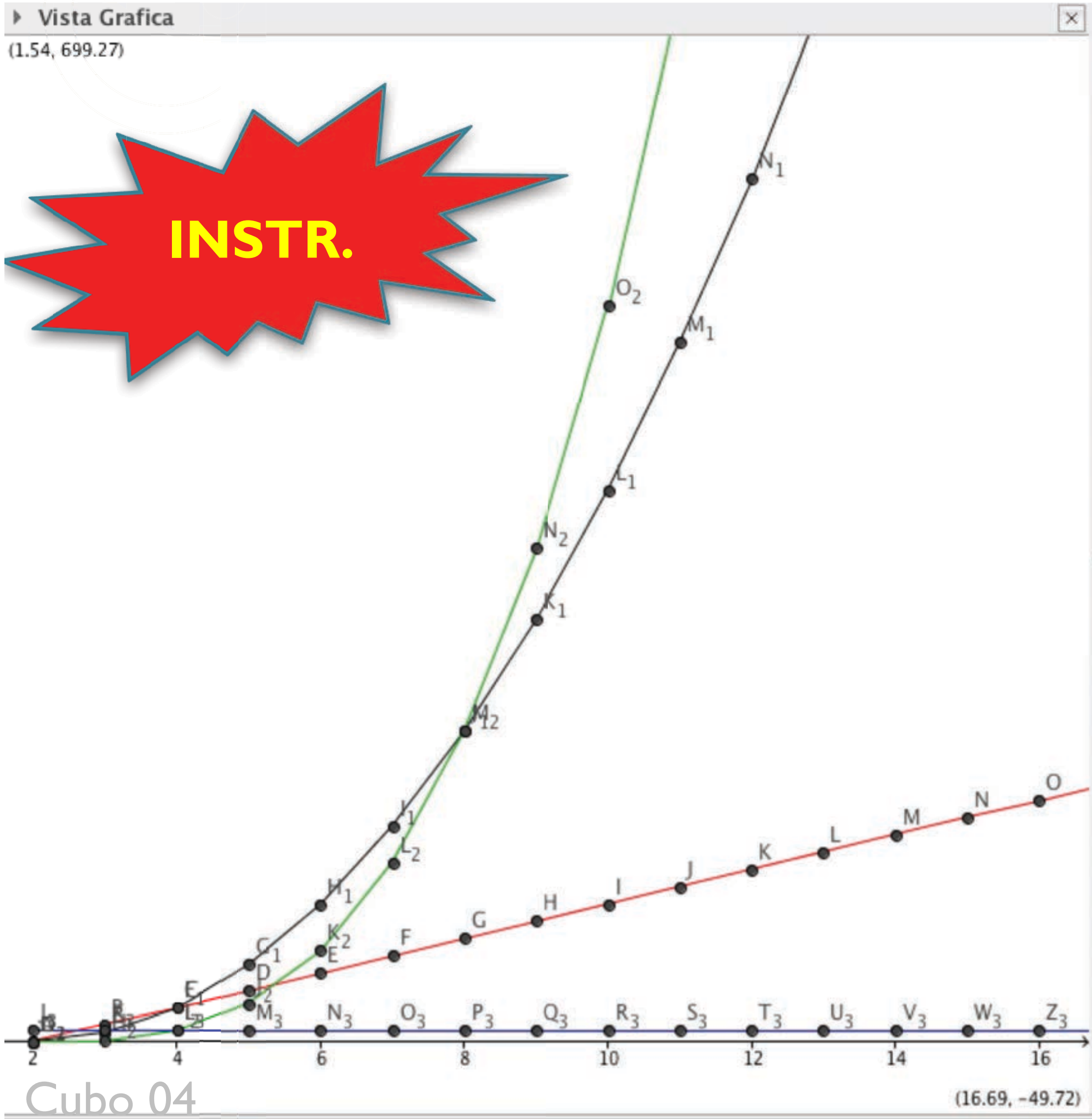
Step 1a

Look at numbers and formulas with a mathematical eye

	2x2x2	$8 = 2^3$	0	0	0	8
	3x3x3	3^3	1	6=6x1	12=12x1	8
	4x4x4	4^3	2^3	24=6x4	24=12x2	8
	5x5x5	5^3	3^3	54=6x9	36=12x3	8
a 1-cubi	*a*x*a*x*a*	a^3	$(a-2)^3$	$6(a-2)^2$	$12(a-2)$	8

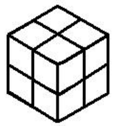
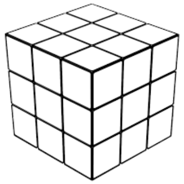
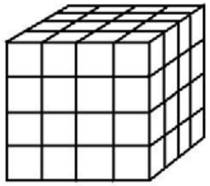
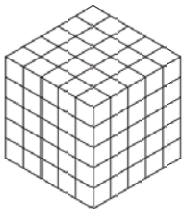
	1b How can you represent the data?					
	2x2x2	$8 = 2^3$	0	0	0	8
	3x3x3	3^3	1	6=6x1	12=12x1	8
	4x4x4	4^3	2^3	24=6x4	24=12x2	8
	5x5x5	5^3	3^3	54=6x9	36=12x3	8
a 1-cubi	axaxa	a^3	$(a-2)^3$	$6(a-2)^2$	$12(a-2)$	8

1b. How can you represent the data? What can you observe now?



Vista Foglio di calcolo

	A	B	C	D	E	F
1	2	0	2	0	2	0
2	3	12	3	6	3	1
3	4	24	4	24	4	8
4	5	36	5	54	5	27
5	6	48	6	96	6	64
6	7	60	7	150	7	125
7	8	72	8	216	8	216
8	9	84	9	294	9	343
9	10	96	10	384	10	512
10	11	108	11	486	11	729
11	12	120	12	600	12	1000
12	13	132	13	726	13	1331
13	14	144	14	864	14	1728
14	15	156	15	1014	15	2197
15	16	168	16	1176	16	2744
16	17	180	17	1350	17	3375
17	18	192	18	1536	18	4096
18	19	204	19	1734	19	4913
19	20	216	20	1944	20	5832
20	21	228	21	2166	21	6859
21	22	240	22	2400	22	8000
22	23	252	23	2646	23	9261
23	24	264	24	2904	24	10648
24	25	276	25	3174	25	12167
25	26	288	26	3456	26	13824
26	27	300	27	3750	27	15625
27	28	312	28	4056	28	17576
28						

	size	n. 1-cubes	n. 1-cubes no glace	n. 1-cubes 1 face glaced	n. 1-cubes 2 faces glaced	n. 1-cubes 3 faces glaced
	2x2x2	$8 = 2^3$	0	0	0	4
	3x3x3	3^3	2	9	12	4
	4x4x4	4^3	12	28	20	4
	5x5x5	5^3	36	57	28	4
a 1-cubes	axaxa	a^3	$(a-1)(a-2)^2$	$5a^2-16a+12$	$8a-12$	4

3. Analysing the MVI

Let us summarize what we have done from a higher point of view as a **dynamic expansion and a compression** (Sfard, 2008)

I. A situation: observe (Oi), ask questions (Dj), give answers (Rk)

II. Modify one ore more Oi changing the situation → $(\sim Oi)_k$.

III. Fresh observations are generated (Oi)*, further questions (Dj)*, and answers (Rk)*.

Why is it so? (I)

What happens if it is not so? (II–III)

From a starting situation we can get variations of the following patterns:

If (~ Oi) (Dj)*? → (Rk)*	If (Oi) (Dj)? → (Rk)

The dynamics of the patterns: from accepting the situation to its challenge with the consequent investigations.

Thr full process can be so summarized through different levels (*Method of the Varied Investigation*, **MVI**):

- Level 0. Choosing a starting point

Level 1. Listing the observations (Oi); asking questions (Dj)?

Level 2. What if it is not so? (~Oi)

Level 3. Posing consequent fresh problems/questions (Dj)*?

Level 4. Analysing the different (Dj)*

Level 5. Metareflection:

If (~ Oi) then (Dj)*? → (Rk)*

If (Oi) then (Dj)? → (Rk)

MVI is relevant from an epistemological, didactical, and cognitive point of view, with consequences for the teaching practices.

MVI is coherent with the MoV

In fact also MVI promotes the following features:

CONTRAST "... In order to experience something, a person must experience something else to compare it with..."

GENERALISATION. "... In order to fully understand what "three" is, we must also experience varying appearances of three..."

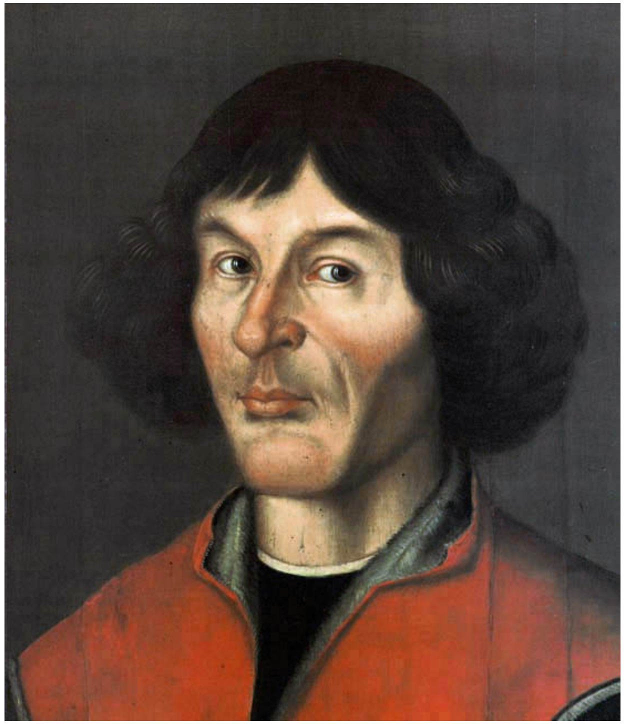
SEPARATION "... In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant."

FUSION "if there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously."

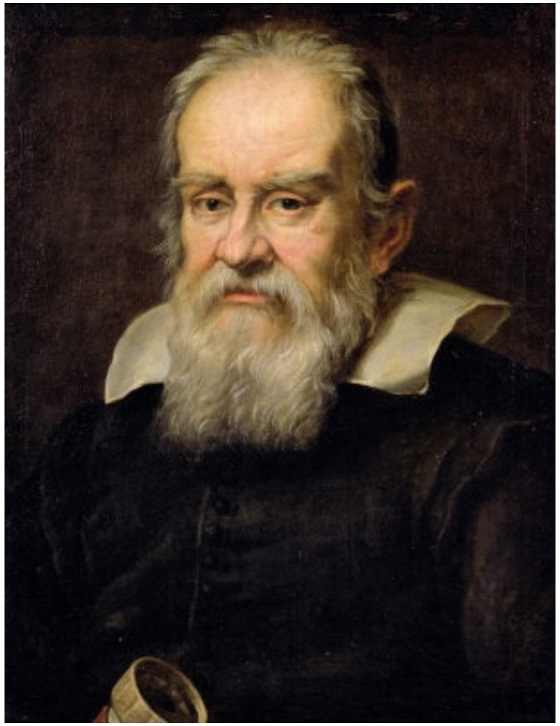
(Marton, F., Runesson, U., & Tsui, A. B. M., 2004, p. 16)

MVI is also coherent with the Method of Scientific Inquiry,

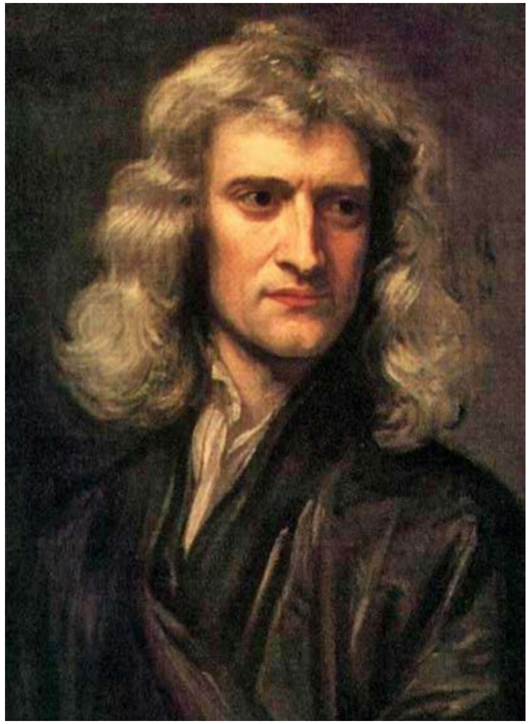
which featured the Scientific Revolution in the Western World.



N. Kopernicus



G. Galilei



I. Newton

"Inquiry-based practices in mathematics involve diverse forms of activity: articulating or elaborating questions, modelling and mathematising; exploring and experimenting; conjecturing; testing, explaining, reasoning, arguing and proving; defining and structuring; connecting, representing and communicating.

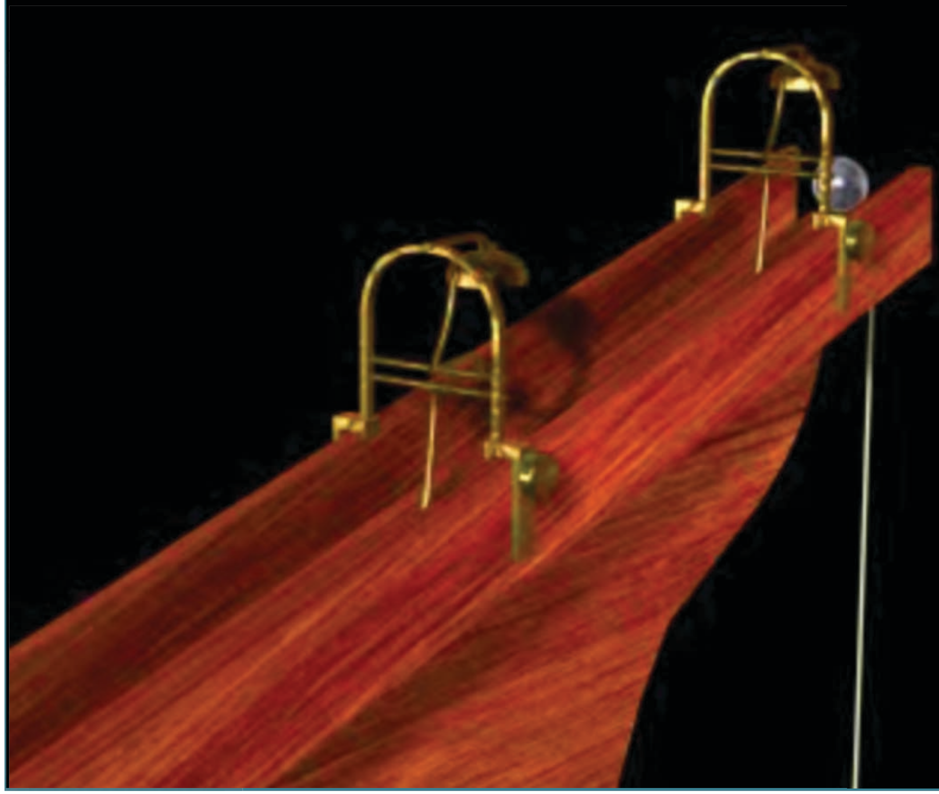
It will also help them develop a more accurate vision of mathematics as a human enterprise, consider mathematics as a fundamental component of our cultural heritage, and appreciate the crucial role it plays in the development of our societies"

(adapted from Fibonacci project, 2012)

An Example:
Galilei as an historical forerunner of MVI

Numbers and Nature

The first modern scientific experiment (1604)



1	1
3	4
5	9
7	16
9	25

Galileo Galilei
(1564-1642)

Discourses and Mathematical Proofs about two New Sciences which concern Mechanics and local Movements by Mr Galileo Galilei Linceo

Leiden 1638

Attenenti alla

MECANICA & I MOVIMENTI LOCALI,

del Signor

GALILEO GALILEI LINCEO,

Filosofo e Matematico primario del Serenissimo Grand Duca di Toscana.

Con vna Appendice del centro di grauità d'alcuni Solidi.

IN LEIDA,

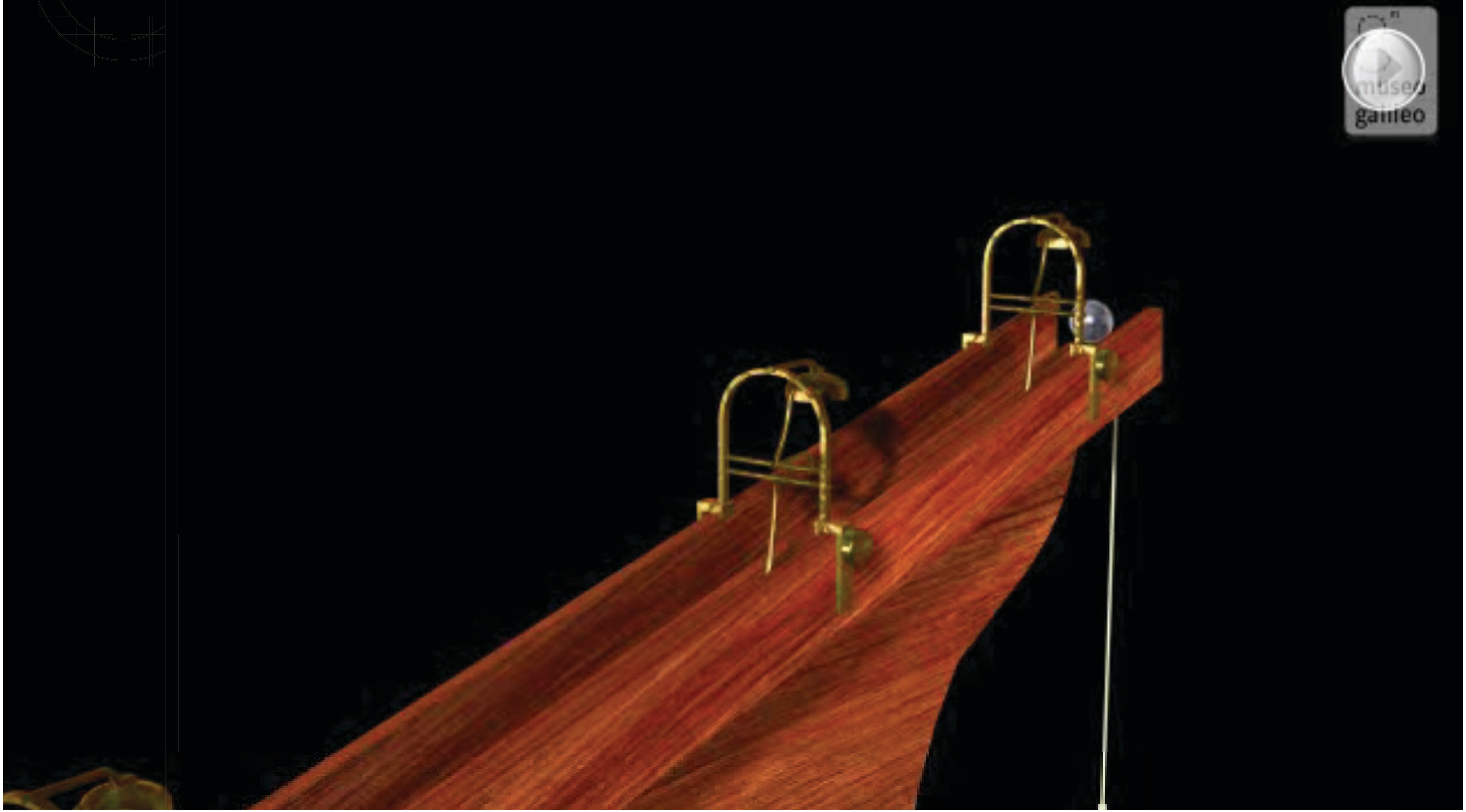
Appresso gli Elsevirii. M. D. C. XXXVIII.

http://www.fmboschetto.it/didattica/DimostrazioniMatematiche.pdf

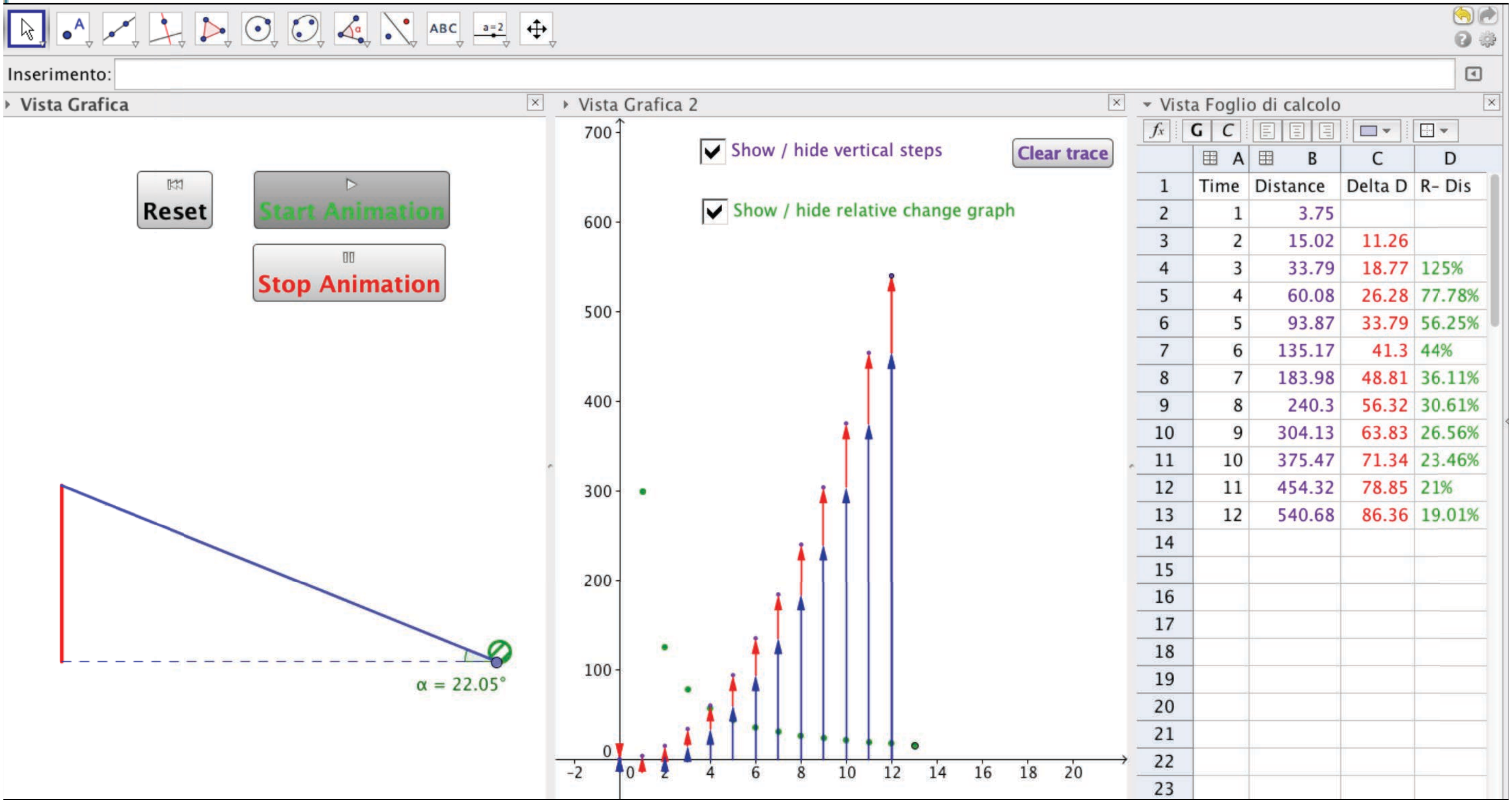
Scientific inquiry in the mathematics classroom

A joint research in Turin with O. Swidan

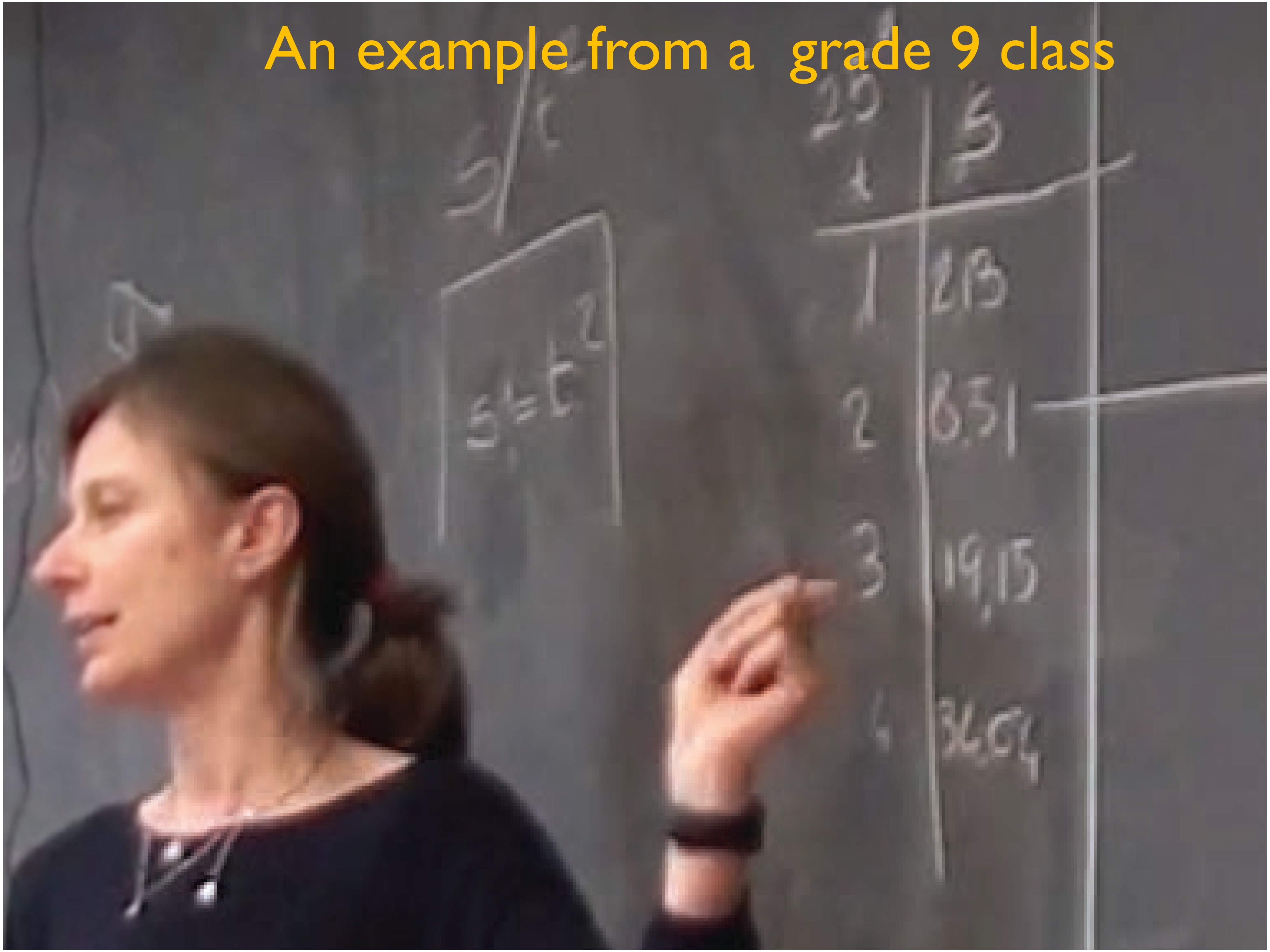
Sense-experiences and mathematical proofs



This experimental apparatus provided a demonstration of the Galilean law of the natural fall of bodies, which stated that the spaces traversed from a position of rest are proportional to the squares of the times of fall.
The pendulum attached to the inclined plane was swung at the same time as the small ball was released. In each successive oscillation of the pendulum the sphere traversed spaces that increased in accordance with the sequence of odd numbers. [about 11 seconds of silence: numbers 1, 3, 5, 7, 9 appear on the screen]
In the first oscillation period the sphere traversed a given interval from its rest position; in the second period it travelled three spaces; in the third period five spaces; in the fourth period seven spaces; and so on.
It follows that the sphere traverses four spaces in two periods from the rest position; nine spaces in three periods; sixteen spaces in the fourth period and so on.


ABC
a=2
Inserimento:
Vista Grafica
Vista Grafica 2
Vista Foglio di calcolo
Reset
Start Animation
Stop Animation
α = 22.05°
Show / hide vertical steps
Show / hide relative change graph
Clear trace
700
600
500
400
300
200
100
0
-2
0
2
4
6
8
10
12
14
16
18
20
G
C
A
B
C
D
Time Distance Delta D R- Dis
1 3.75
2 15.02 11.26
3 33.79 18.77 125%
4 60.08 26.28 77.78%
5 93.87 33.79 56.25%
6 135.17 41.3 44%
7 183.98 48.81 36.11%
8 240.3 56.32 30.61%
9 304.13 63.83 26.56%
10 375.47 71.34 23.46%
11 454.32 78.85 21%
12 540.68 86.36 19.01%

An example from a grade 9 class



Teacher: But it is always 2.36

Student: It could be y = k times x^2, where k is the constant that varies with the inclination

…

Student: As a consequence the space varies changing the coefficients

Teacher: And as a consequence the space varies

Student: Changing all the coefficients

Teacher: Changing all the coefficients

MVI promotes hypothetical thinking

- In fact, it triggers and supports discourses that allow students:
  - to go back on what has been done, seen (etc.), producing interpretations, explanations, answers to questions like "why is it so?”
  - to anticipate events, situations, etc., producing forecasts, talks about hypothetical worlds, answers to questions ”how will it be like?", "how could it be?"

4. Further examples

1. Πάντα ρει

Processes of change :
a **cognitive root** (D. Tall)
for mathematics and science

Change has a double interest for mathematical learning:

- Cognitively: there is a “natural” attention for what is changing, how it changes and what does not change in a given situation.

- Mathematically: changing quantities are analysed not only for their values but above all for their differences in time and for the way of representing and manipulating them.

→FINITE DIFFERENCES :

a) A powerful tool, which allows to approach Calculus very early.

b) A tool, which didactical software can easily implement.

Finite differences as a measure of change (squares)

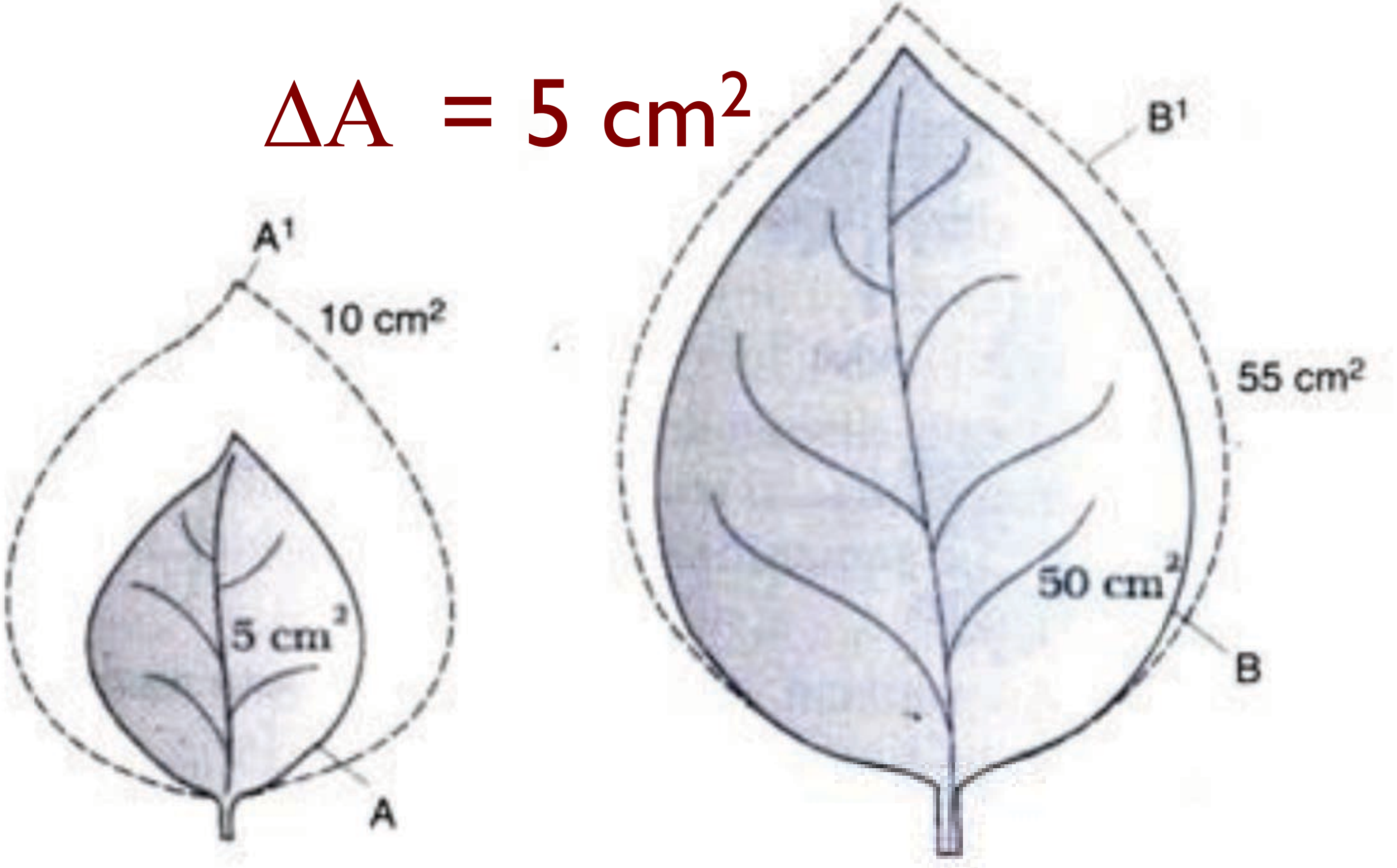
a	$4(a-2)$	a	a^2	a	4	ΔB	ΔD
A	B	C	D	E	F	G	H
2	0	2	0	2	4	4	1
3	4	3	1	3	4	4	3
4	8	4	4	4	4	4	5
5	12	5	9	5	4	4	7
6	16	6	16	6	4	4	9
7	20	7	25	7	4	4	11
8	24	8	36	8	4	4	13
9	28	9	49	9	4	4	15
10	32	10	64	10	4	4	17
11	36	11	81	11	4	4	19
12	40	12	100	12	4	4	21
13	44	13	121	13	4	4	23
14	48	14	144	14	4	4	25
15	52	15	169	15	4	4	27
16	56	16	196	16	4	4	29
17	60	17	225	17	4	4	31

Finite differences as a measure of change (cubes)

a	$12(a\text{-}2)$	a	$6(a\text{-}2)^2$	a	$(a\text{-}2)^3$	a		Δ1B	Δ1D	Δ2D	Δ1F	Δ2F	Δ3F
A	B	C	D	E	F	G	H			K	L	M	N
2	0	2	0	2	0	2	8	12	6	12	1	6	6
3	12	3	6	3	1	3	8	12	18	12	7	12	6
4	24	4	24	4	8	4	8	12	30	12	19	18	6
5	36	5	54	5	27	5	8	12	42	12	37	24	6
6	48	6	96	6	64	6	8	12	54	12	61	30	6
7	60	7	150	7	125	7	8	12	66	12	91	36	6
8	72	8	216	8	216	8	8	12	78	12	127	42	6
9	84	9	294	9	343	9	8	12	90	12	169	48	6
10	96	10	384	10	512	10	8	12	102	12	217	54	6
11	108	11	486	11	729	11	8	12	114	12	271	60	6
12	120	12	600	12	1000	12	8	12	126	12	331	66	6
13	132	13	726	13	1331	13	8	12	138	12	397	72	6
14	144	14	864	14	1728	14	8	12	150	12	469	78	6
15	156	15	1014	15	2197	15	8	12	162	12	547	84	6
16	168	16	1176	16	2744	16	8	12	174	12	631	90	6
17	180	17	1350	17	3375	17	8	12	186	12	721	96	6
18	192	18	1536	18	4096	18	8	12	198	12	817	102	6
19	204	19	1734	19	4913	19	8	12	210	12	919	108	6
20	216	20	1944	20	5832	20	8	12	222	12	1027	114	6
21	228	21	2166	21	6859	21	8	12	234	12	1141	120	6
22	240	22	2400	22	8000	22	8	12	246	12	1261	126	6
23	252	23	2646	23	9261	23	8	12	258	12	1387	132	6
24	264	24	2904	24	10648	24	8	12	270	12	1519	138	6
25	276	25	3174	25	12167	25	8	12	282	12	1657	144	6
26	288	26	3456	26	13824	26	8	12	294	12	1801	150	
27	300	27	3750	27	15625	27	8	12	306		1951		

A finer idea of change

$\Delta A = 5\ cm^2$



The relative change $\Delta_r A = \Delta A / A$

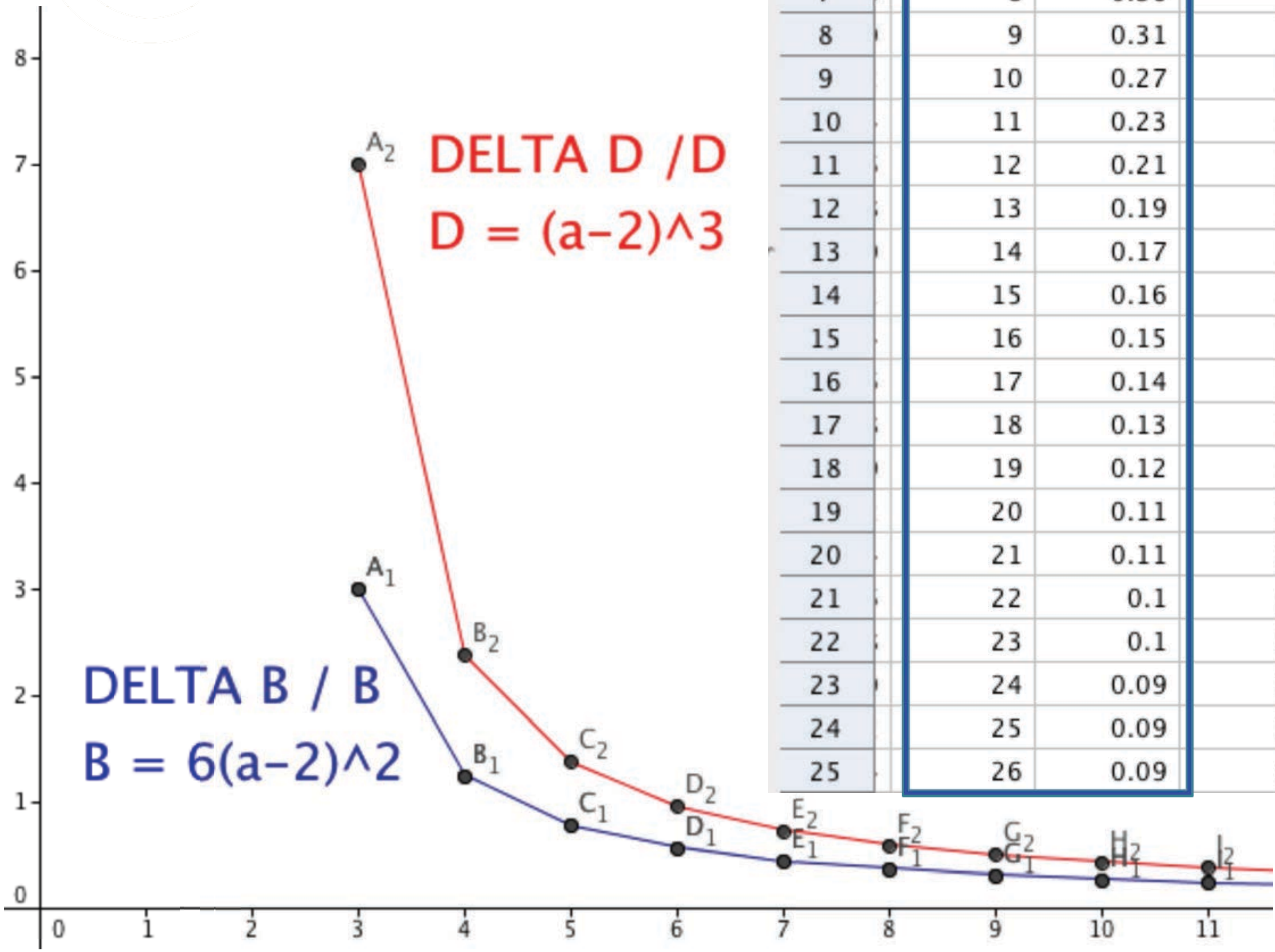
$\Delta_r = 5\ cm^2 / 5\ cm^2$

100%

$\Delta_r = 5\ cm^2 / 50\ cm^2$

10%

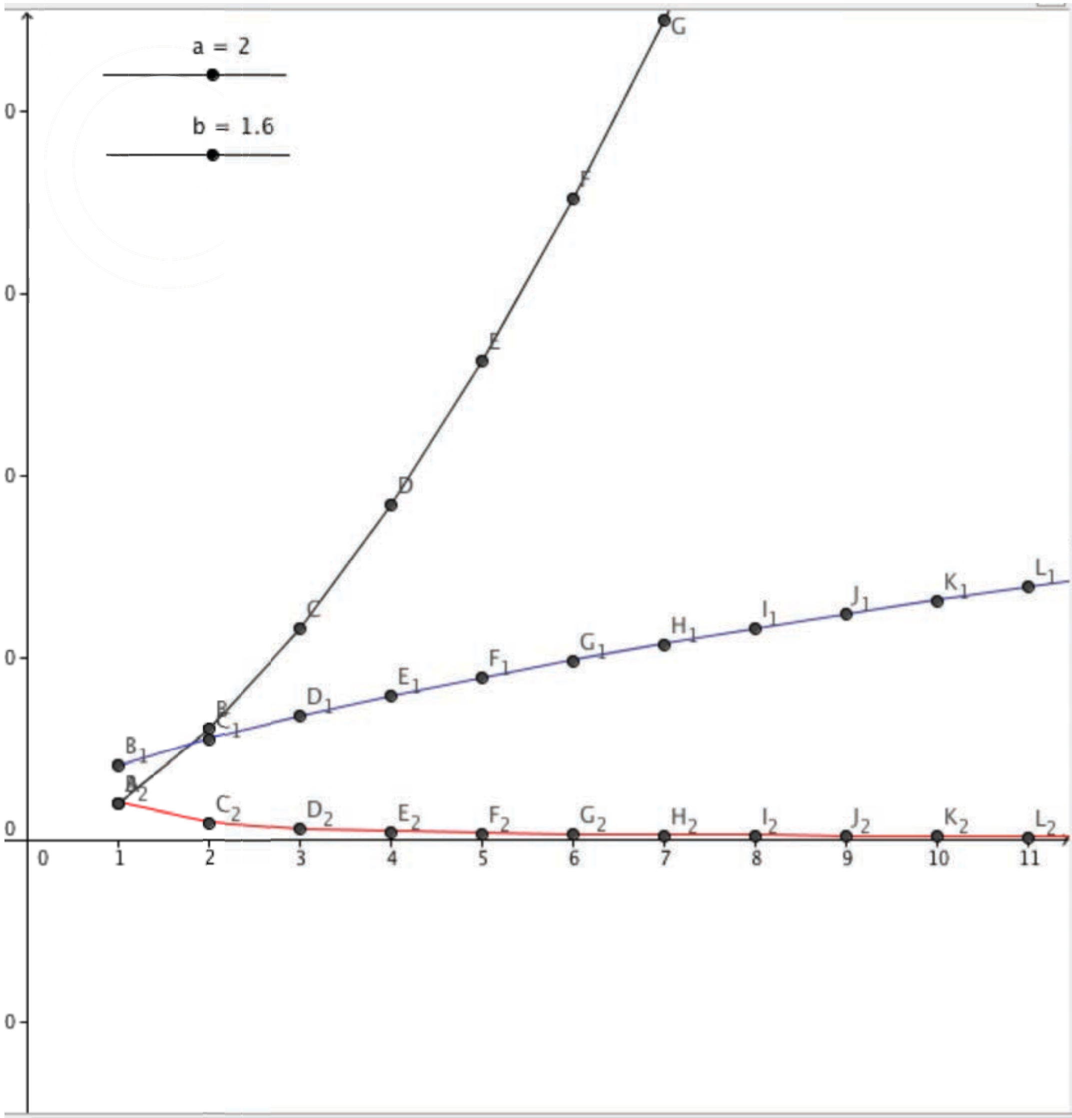
Cubes and relative changes



	C ΔB/B		I	J	ΔD/D	
1	2		2	1	2	
2	3	3	3	7	3	7
3	4	1.25	4	19	4	2.38
4	5	0.78	5	37	5	1.37
5	6	0.56	6	61	6	0.95
6	7	0.44	7	91	7	0.73
7	8	0.36	8	127	8	0.59
8	9	0.31	9	169	9	0.49
9	10	0.27	10	217	10	0.42
10	11	0.23	11	271	11	0.37
11	12	0.21	12	331	12	0.33
12	13	0.19	13	397	13	0.3
13	14	0.17	14	469	14	0.27
14	15	0.16	15	547	15	0.25
15	16	0.15	16	631	16	0.23
16	17	0.14	17	721	17	0.21
17	18	0.13	18	817	18	0.2
18	19	0.12	19	919	19	0.19
19	20	0.11	20	1027	20	0.18
20	21	0.11	21	1141	21	0.17
21	22	0.1	22	1261	22	0.16
22	23	0.1	23	1387	23	0.15
23	24	0.09	24	1519	24	0.14
24	25	0.09	25	1657	25	0.14
25	26	0.09	26	1801		

INSTR.

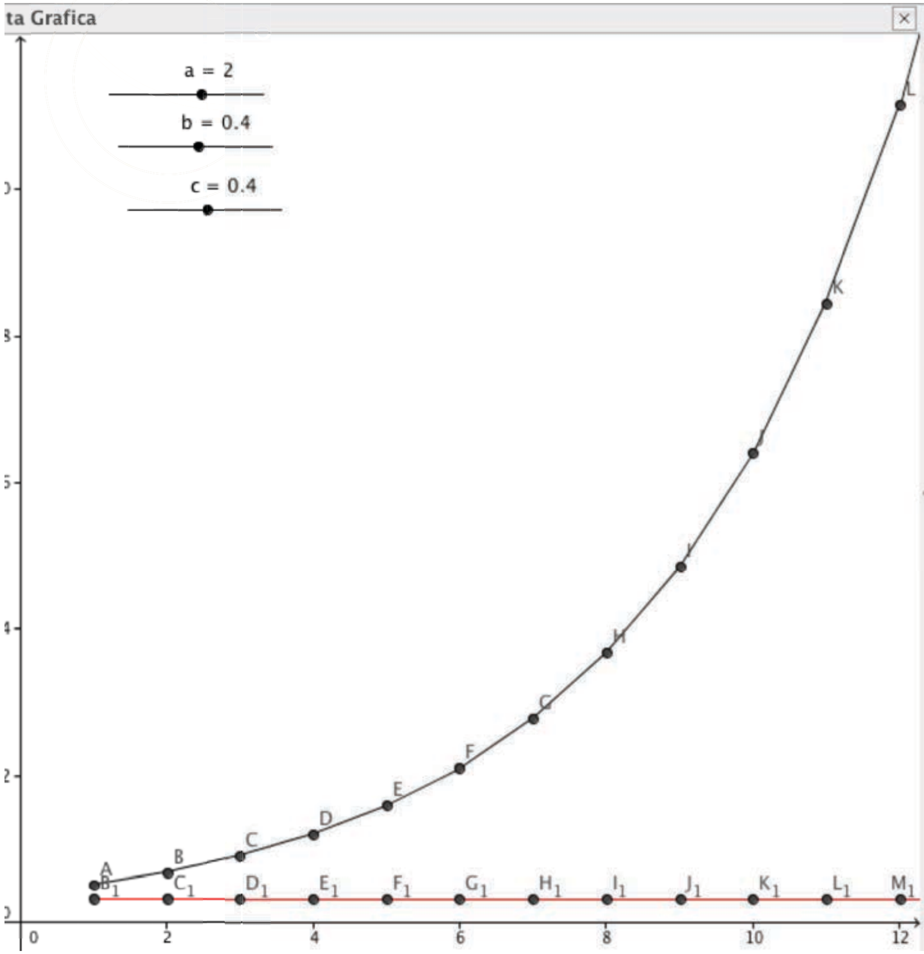
Relative differences: polynomes



	A	B	C	D	E	F
1	n	a*n^b	n	DELTA B	n	DELTA B /B
2	1	2	1	4.06	1	2.03
3	2	6.06	2	5.54	2	0.91
4	3	11.6	3	6.78	3	0.58
5	4	18.38	4	7.89	4	0.43
6	5	26.27	5	8.9	5	0.34
7	6	35.16	6	9.84	6	0.28
8	7	45	7	10.72	7	0.24
9	8	55.72	8	11.55	8	0.21
10	9	67.27	9	12.35	9	0.18
11	10	79.62	10	13.12	10	0.16
12	11	92.74	11	13.85	11	0.15
13	12	106.59	12	14.56	12	0.14
14	13	121.15	13	15.25	13	0.13
15	14	136.41	14	15.92	14	0.12
16	15	152.33	15	16.57	15	0.11
17	16	168.9	16	17.2	16	0.1
18	17	186.1	17	17.82	17	0.1
19	18	203.92	18	18.43	18	0.09
20	19	222.35	19	19.02	19	0.09
21	20	241.37	20	19.6	20	0.08
22	21	260.96	21	20.17	21	0.08
23	22	281.13	22	20.72	22	0.07
24	23	301.85	23	21.27	23	0.07
25	24	323.12	24	21.81	24	0.07
26	25	344.93	25	[illegible]	[illegible]	[illegible]

INSTR.

Relative differences: exponentials

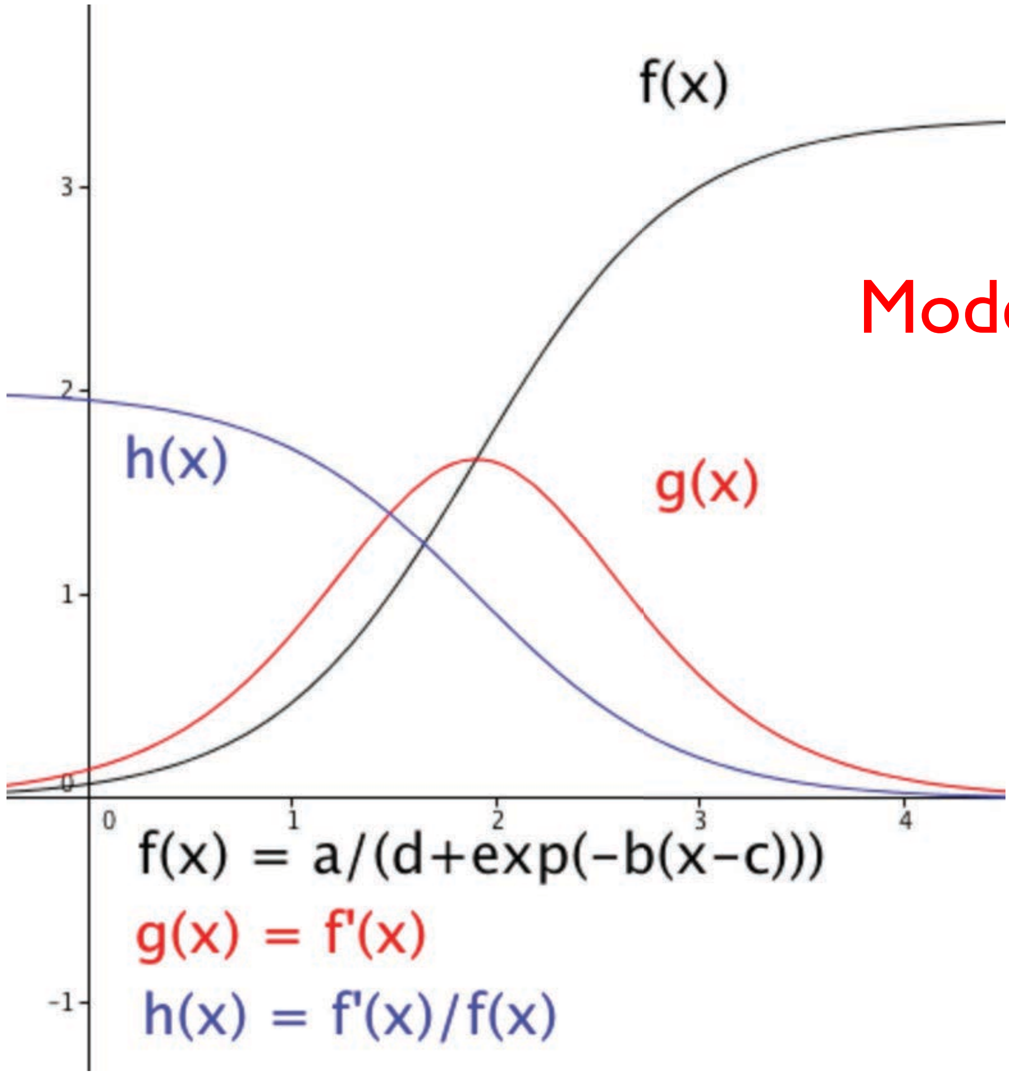


Vista Foglio di calcolo

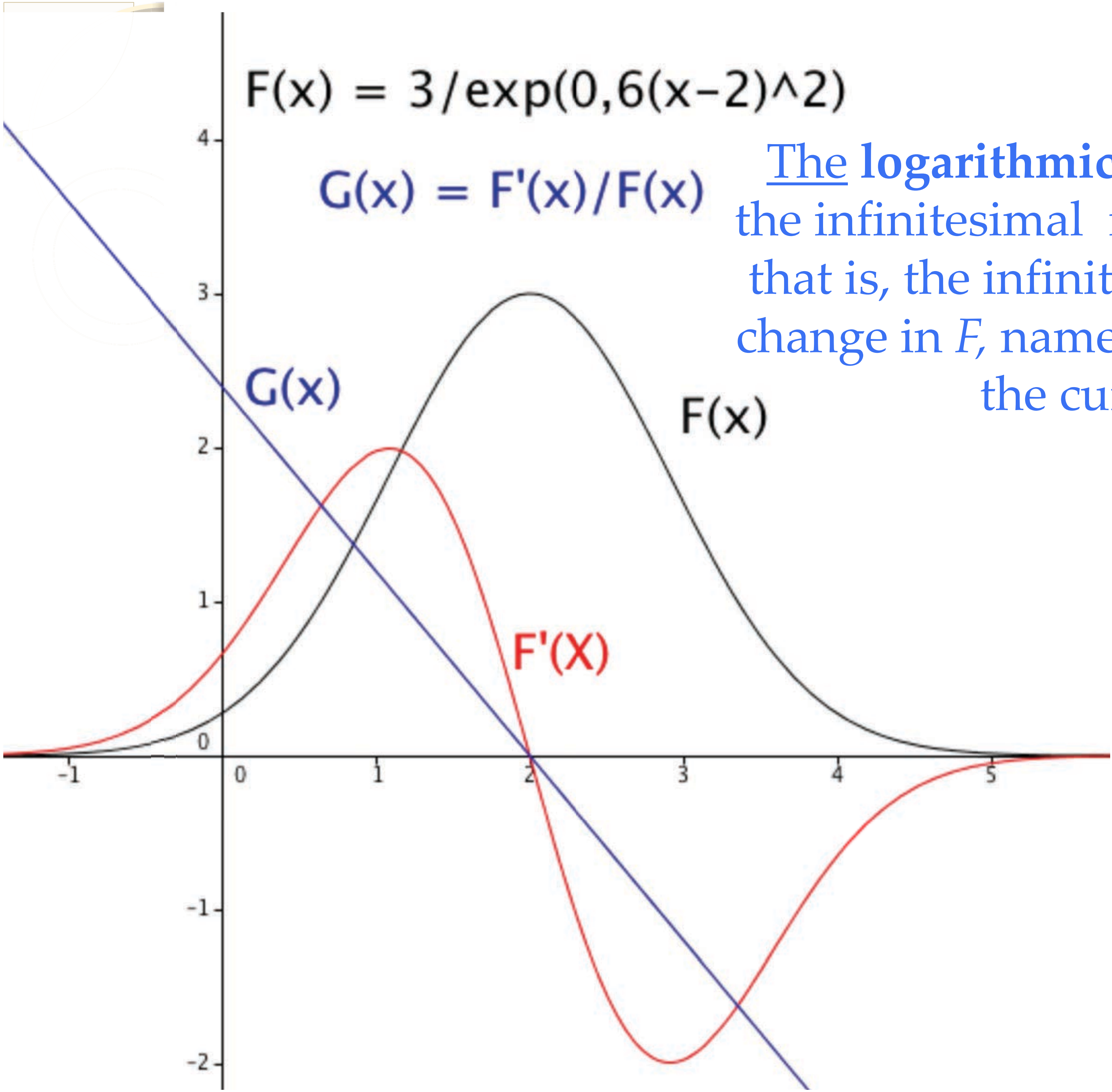
	A	B	C	D	E	F	G	H
1	n	a^n	a^(bn)	n	c*a^(b*n)	DELTA E	n	DELTA E/E
2	1	2	1.32	1	0.53	0.17	1	0.32
3	2	4	1.74	2	0.7	0.22	2	0.32
4	3	8	2.3	3	0.92	0.29	3	0.32
5	4	16	3.03	4	1.21	0.39	4	0.32
6	5	32	4	5	1.6	0.51	5	0.32
7	6	64	5.28	6	2.11	0.67	6	0.32
8	7	128	6.96	7	2.79	0.89	7	0.32
9	8	256	9.19	8	3.68	1.17	8	0.32
10	9	512	12.13	9	4.85	1.55	9	0.32
11	10	1024	16	10	6.4	2.04	10	0.32
12	11	2048	21.11	11	8.44	2.7	11	0.32
13	12	4096	27.86	12	11.14	3.56	12	0.32
14	13	8192	36.76	13	14.7	4.7	13	0.32
15	14	16384	48.5	14	19.4	6.2	14	0.32
16	15	32768	64	15	25.6	8.18	15	0.32
17	16	65536	84.45	16	33.78	10.79	16	0.32
18	17	131072	111.43	17	44.57	14.24	17	0.32
19	18	262144	147.03	18	58.81	18.79	18	0.32
20	19	524288	194.01	19	77.6	24.8	19	0.32
21	20	1048576	256	20	102.4	32.72	20	0.32
22	21	2097152	337.79	21	135.12	43.17	21	0.32
23	22	4194304	445.72	22	178.29	56.96	22	0.32
24	23	8388608	588.13	23	235.25	75.17	23	0.32
25	24	167772...	776.05	24	310.42	99.18	24	0.32
26	25	335544...	1024	25	409.6			

INSTR.

Growth phenomena in Biology and Economy: reasoning about changes as education to rational decisions

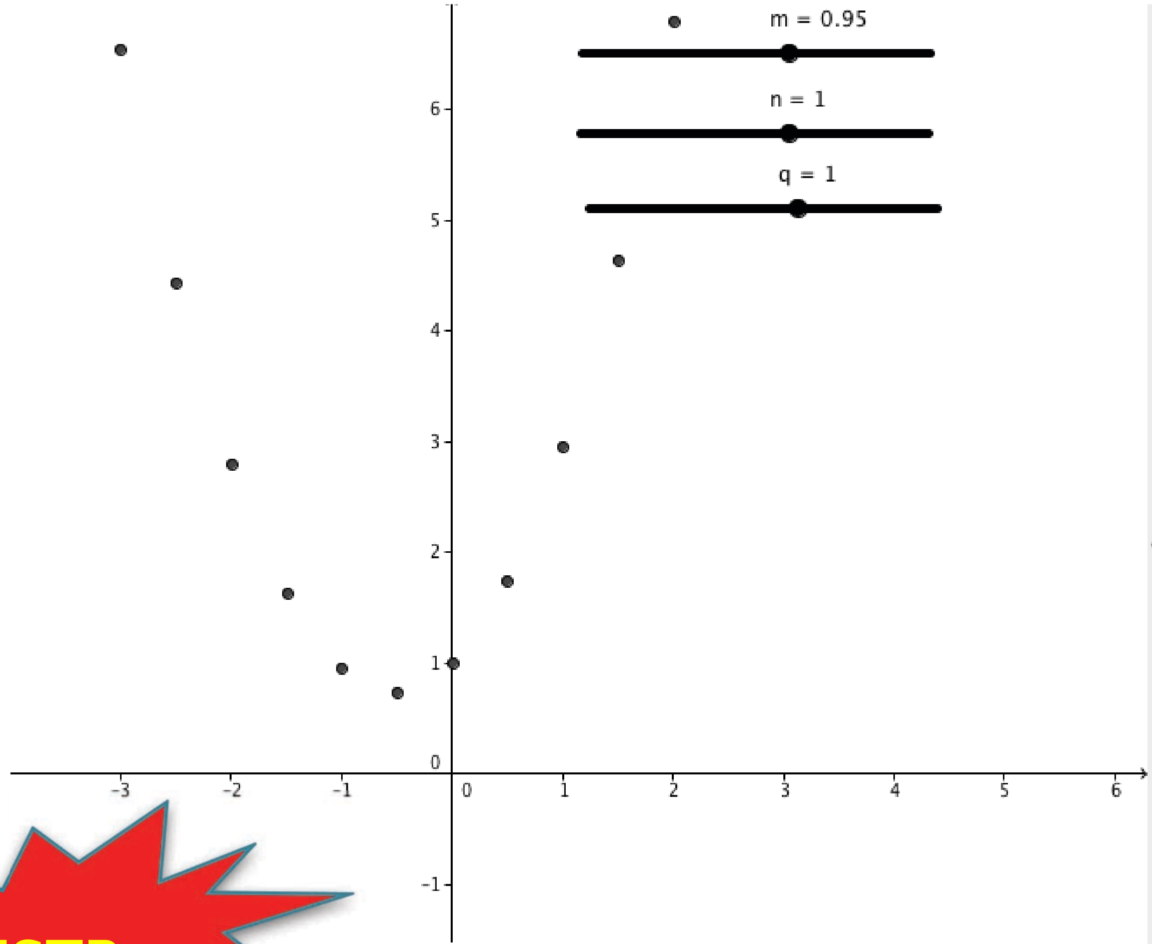


Model of Verhulst



<u>The</u> **logarithmic derivative** of *F* the infinitesimal relative change, that is, the infinitesimal absolute change in *F*, namely F′, scaled by the current value of *F*.

Finite differences: another example



$$y = ax^2 + bx + c$$

	A	B	C	D
1	-10	86	diff1	diff2
2	-9.5	77.24	-8.76	
3	-9	68.95	-8.29	0.47
4	-8.5	61.14	-7.81	0.47
5	-8	53.8	-7.34	0.48
6	-7.5	46.94	-6.86	0.47
7	-7	40.55	-6.39	0.48
8	-6.5	34.64	-5.91	0.47
9	-6	29.2	-5.44	0.48
10	-5.5	24.24	-4.96	0.48
11	-5	19.75	-4.49	0.48
12	-4.5	15.74	-4.01	0.48
13	-4	12.2	-3.54	0.48
14	-3.5	9.14	-3.06	0.48
15	-3	6.55	-2.59	0.48
16	-2.5	4.44	-2.11	0.48
17	-2	2.8	-1.64	0.48
18	-1.5	1.64	-1.16	0.48
19	-1	0.95	-0.69	0.48
20	-0.5	0.74	-0.21	0.48
21	0	1	0.26	0.48
22	0.5	1.74	0.74	0.48
23	1	2.95	1.21	0.48
24	1.5	4.64	1.69	0.48
25	2	6.8	2.16	0.48
26	2.5	9.44	2.64	0.48
27	3	12.55	3.11	0.48
28	3.5	16.14	3.59	0.48

Students observe that: $y = ax^2 + bx + c$

- Modifying the value of *c*, only the column of *y* changes, not the others: hence the way the function increases/decreases does not depend on *c*;

- Modifying the value of *b*, columns *y*, Δ_1y change, Δ_2y does not: hence the way the function increases/decreases does depend on *b*, *but it is not so for the concavity;*

- Modifying the value of *a*, y, D_1y, D_2y change: hence concavity depends only on *a*.

- A difficult question: why is it so?

The number sense as pre-algebra: dice in 2° grade

5. Conclusions

MVI has important didacical consequences

MVI promotes everyday life ways of reasoning: the teacher supports their use in her/his teaching practices, promoting the transition to the mathematical context.

Doing so, she/he allows the construction of mathematical **skills**, in which **knowledge** is intertwined with *argumentative* skills of the students in situations where they **pose and solve problems**.

MVI induces an open attitude to the inquiry, in which the students:

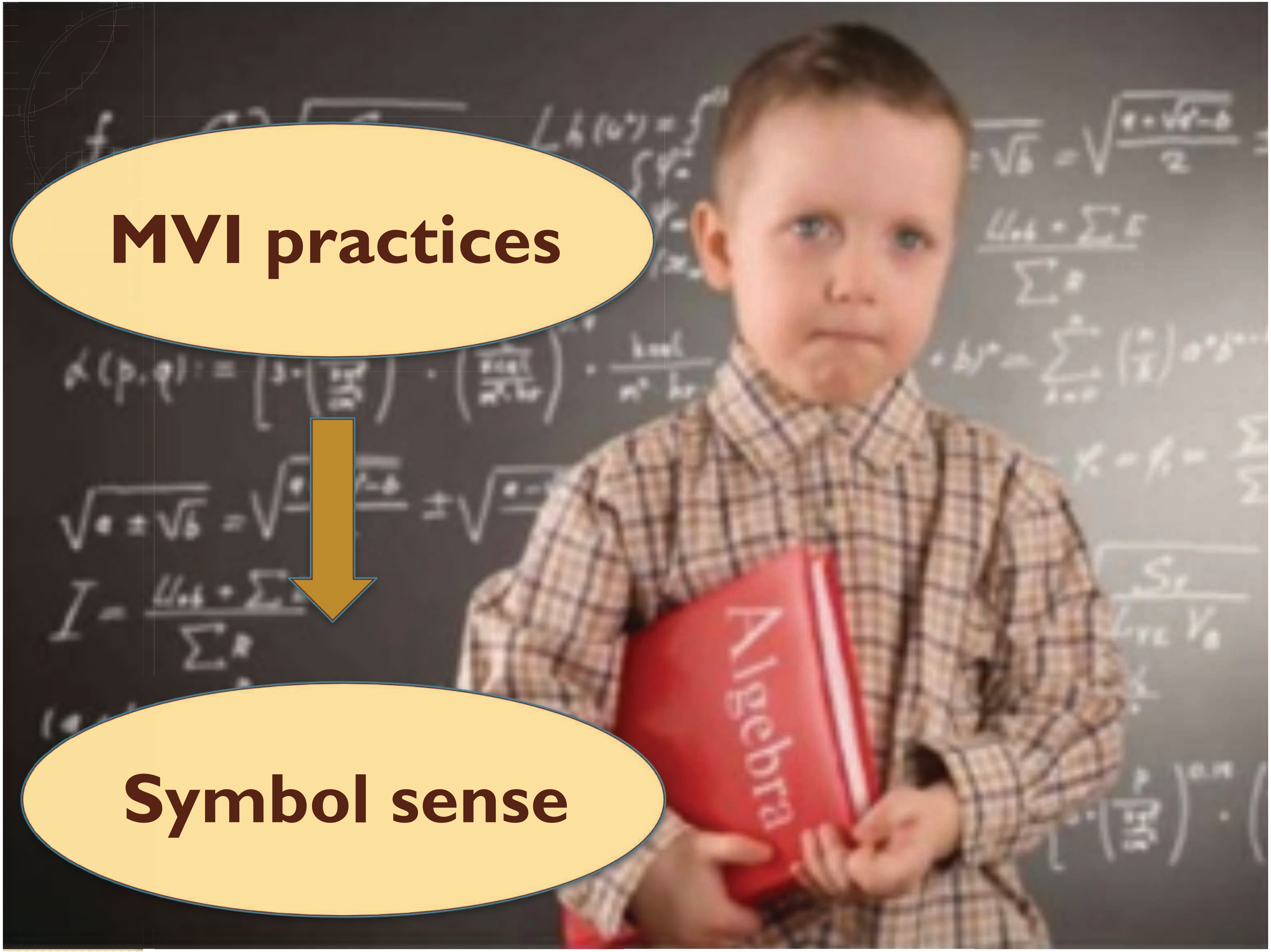
- pose and solve problems;
- are not embalmed in the typical closed scheme: given situation → solve / show;
- produce hypotheses, definitions, arguments: MVI supports the transition from forms of "natural" arguments (abductions) to forms of mathematical reasoning (deductions).

MVI can be implemented within rich technological enivronments, which can concretely "instrument" the dynamic interactions between different frames (numerical, algebraic, graphic).

Educational and cognitive advantages of MVI

- The variations are generated by the students themselves (with the support of the teacher, stronger at the beginning, more and more attenuated when the method is progressing in the class): THEN the control in posing the problem passes from the "others" to themselves: it so generated a broader conception of what is a problem and a greater emotional sharing.
- Variations deal with the same subject under different points of view: THEN they generate a deeper and wider understanding.

The result can be a shared and positive sense for mathematics.

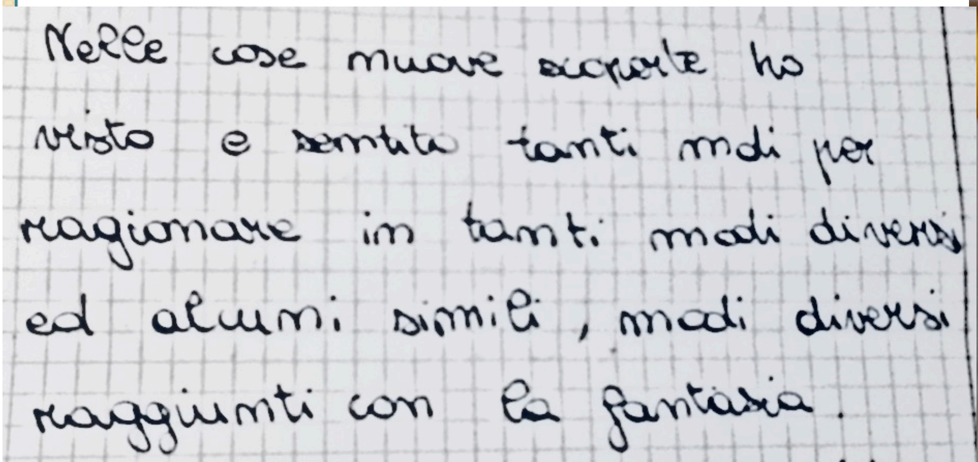


MVI practices

Symbol sense

In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi.

In mathematics the art of asking questions is more valuable than solving problems. (G. Cantor, 1867)



Nelle cose nuove scoperte ho visto e sentito tanti modi per ragionare in tanti modi diversi ed alcuni simili, modi diversi raggiunti con la fantasia.

In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi.

In mathematics the art of asking questions is more valuable than solving problems. (G. Cantor, 1867)

Thank you!

In the new discovered things
I have seen and heard many
different ways of reasoning,
reached with the phantasy.
(a 2° grade student)