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Anantharaman-Delaroche, C. (F-ORLN);
Renault, J. [Renault, Jean N.] (F-ORLN)

★**Amenable groupoids. (English. English summary)**

With a foreword by Georges Skandalis and Appendix B by E. Germain.

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FEATURED REVIEW.

This interesting and important book by two of the leading experts in the field is concerned with studying amenability for locally compact groupoids and measured groupoids. A groupoid is defined as a small category with inverses. In terms of algebra, this means that a groupoid is simply a set with a partially defined multiplication for which the usual properties of a group hold whenever they make sense. Every group is a groupoid, but there are many groupoids that are not groups. For example, an equivalence relation R on a set X is a groupoid with product given by $((x,y),(y,z)) \mapsto (x,z)$ and inverse given by $(x,y)^{-1} = (y,x)$. For any groupoid G, the unit space G^0 is defined as the set $\{xx^{-1}\colon\ x \in G\}$. Groups are groupoids with one unit. There are two natural maps $r, s\colon G \to G^0$ given by $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. These maps are called the range and source maps, respectively. A groupoid G is said to be locally compact (as a groupoid) if G is second countable, locally compact and Hausdorff. The maps $x \mapsto x^{-1}$ and $(x,y) \mapsto xy$ are always assumed to be continuous, and the range and source maps are assumed to be surjective and open. A continuous Haar system λ for a locally compact groupoid G is a family $\{\lambda^x\}$ of measures on G, indexed by $x \in G^{(0)}$, such that λ^x has $G^x = r^{-1}(x)$ as support, is continuous, in the sense that for every $f \in C_c(G)$, the function $\lambda(f)\colon x \mapsto \lambda^x(f)$ is continuous, and is invariant, in the sense that for every $\gamma \in G$, $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$. A Borel groupoid G is a groupoid endowed with a Borel structure such that the range, source, inverse and product maps are Borel. A measured groupoid is a triple (G, λ, μ), where G is a Borel groupoid, λ is a Borel Haar system and μ is a quasi-invariant measure with respect to (G, λ).

The notion of amenability for groups, their actions on spaces, and more generally for semigroups and groupoids, has been studied in various contexts for over fifty years. Historically, the subject began in 1904 with Lebesgue, who asked whether or not a positive, finitely (but not countably) additive, translation-invariant measure μ exists on the real line $\mathbb{R}$ such that $\mu([0,1]) = 1$. Later, a fundamental question of F. Hausdorff in 1914 led to the study of isometry-invariant measures on $\mathbb{R}$ and the well-known Banach-Tarski paradox [see S. Banach and A. Tarski, Fund. Math. **6** (1924), 244–277; JFM 50.0370.02]. In 1929, J. von Neumann [Fund. Math. **13** (1929), 73–116; JFM 55.0151.01] showed that the dichotomy in the Banach-Tarski paradox resides in the different structures of the underlying isometry groups and not in the structure of the spaces themselves. Von Neumann specifically showed that the special rotation group $\mathrm{SO}(3,\mathbb{R})$ contains the free group on two generators, whereas this is not the case for $\mathrm{SO}(2,\mathbb{R})$. As a result it follows that the group itself plays the essential role in the theory and not the spaces.

During the 1940s an important shift in point of view took place. Finitely additive measures μ satisfying $\mu(G) = 1$ (where G is the group under consideration) were replaced with "means", that is, with continuous linear functionals m on $l_\infty(G)$ satisfying $m(1) = 1 = \|m\|$. The correspondence between μ and m is bijective and given by $\mu(E) = m(\chi_E)$. With this shift in point of view the powerful methods of functional analysis and harmonic analysis became available to study amenability of groups. It should be mentioned that the term "amenable" was not used until the penetrating work of M. M. Day in the 1950s [see Trans. Amer. Math. Soc. **69** (1950), 276–291; MR **13**, 357e; Illinois J. Math. **1** (1957), 509–544; MR **19**, 1067c], although the modern notion of amenability was already present in the above-cited work of von Neumann (and others), who studied groups that admitted an invariant mean. It is now well known that the existence of an invariant mean on a locally compact group G is equivalent to many fundamental properties in the harmonic analysis of G, among them the Følner property, the fixed-point property, and weak containment of the trivial representation in the regular representation of G. Thorough treatments of the theory of amenable locally compact groups from different perspectives have been given by A. L. T. Paterson [*Amenability*, Amer. Math. Soc., Providence, RI, 1988; MR 90e:43001] and J.-P. Pier [*Amenable locally compact groups*, Wiley, New York, 1984; MR 86a:43001].

The generalization of amenability from groups to groupoids in the volume under review is patterned largely after the group case. However, there are substantial and nontrivial hurdles to overcome. As mentioned above, the authors study amenability for both measured groupoids and locally compact groupoids. R. J. Zimmer [Invent. Math. **41** (1977), no. 1, 23–31; MR **57** #10438; Proc. Amer. Math. Soc. **66** (1977), no. 2, 289–293; MR **57** #592; J. Functional Analysis

27 (1978), no. 3, 350–372; MR **57** #12775] initiated the study of amenable measured groupoids in the case of discrete group actions and countable equivalence relations. He introduced amenability through an adaptation of the classical fixed-point property, showed that this fixed-point property is equivalent to the existence of an equivariant conditional expectation, and studied various properties of amenable group actions and their representations.

Following Zimmer's pioneering work, A. Connes, J. Feldman and B. Weiss [Ergodic Theory Dynamical Systems **1** (1981), no. 4, 431–450 (1982); MR 84h:46090] proved the striking result that a countable measured equivalence relation R is amenable if and only if R is hyperfinite, i.e., R is the union of an increasing sequence of finite equivalence relations. Although Zimmer's work has been generalized and applied to a wide variety of problems, e.g., foliations, boundary actions, rigidity, and graphed equivalence relations, its extension to arbitrary measured groupoids has remained an open problem. In this respect, the work of V. A. Kaimanovich [C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 9, 999–1004; MR 98j:28014] and others revealed the need to clarify the various definitions of amenability for groupoids. Earlier, S. Adams, G. A. Elliott and T. Giordano [Trans. Amer. Math. Soc. **344** (1994), no. 2, 803–822; MR 94k:22010] had already taken an important step in this direction by showing, in the case of locally compact group actions, the equivalence of Zimmer's fixed-point property and the existence of an equivariant conditional expectation. Topological amenability was introduced by the second author as the existence of an approximate invariant mean [*A groupoid approach to C^*-algebras*, Lecture Notes in Math., 793, Springer, Berlin, 1980; MR 82h:46075]. This is a strong condition which implies measurewise amenability, i.e., amenability of all quasi-invariant measures. The first-named author proved the surprising result that the converse holds for discrete group actions [Math. Ann. **279** (1987), no. 2, 297–315; MR 89f:46127]. One of the motivations of the present work is to extend this result further.

On the other hand, amenable groupoids have been at the center of recent developments in the theory of operator algebras. For example, if a locally compact group or groupoid admits an amenable action on a compact space, then its reduced C^*-algebra is exact. The question of whether or not every locally compact groupoid admits an amenable action on a compact space has been settled negatively with a counterexample by M. L. Gromov [Geom. Funct. Anal. **2000**, Special Volume, Part I, 118–161; MR 2002e:53056].

A recent classification of purely infinite C^*-algebras, due to E.

Kirchberg, shows that many algebras in this class arise from amenable groupoids [see C. Anantharaman-Delaroche, Bull. Soc. Math. France **125** (1997), no. 2, 199–225; MR 99i:46051]. An amenable groupoid has a nuclear reduced C^*-algebra, from which it follows that this C^*-algebra is simple and purely infinite. On the other hand, a recent theorem of N. Higson and G. G. Kasparov [Invent. Math. **144** (2001), no. 1, 23–74 (Theorem 1.1); MR 2002k:19005] for groups, and its generalization to groupoids by J. L. Tu [K-Theory **17** (1999), no. 3, 215–264; MR 2000g:19004], shows that amenable groups and groupoids satisfy the Baum-Connes conjecture [see P. F. Baum, A. Connes and N. Higson, in *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994; MR 96c:46070]. As a result, their reduced C^*-algebras satisfy the universal coefficient formula of J. Rosenberg and C. Schochet. Thus, remarkably, their KK-theory is determined by their K-theory. In other words, the C^*-algebras of amenable groupoids which are simple and purely infinite are classified up to isomorphism by their K_0 and K_1 groups. Utilizing the above-mentioned result of Tu, Higson [Geom. Funct. Anal. **10** (2000), no. 3, 563–581; MR 2001k:19009] has shown that any locally compact group (or, more generally, any locally compact groupoid) that admits an amenable action on a compact space satisfies the Novikov conjecture.

The book under review has six chapters and several appendices. Chapter 1 provides basic functional-analytic results that are needed to compare the various notions of invariant means or approximate invariant means for a locally compact or a measured groupoid. Because the spaces encountered in the theory of groupoids are fibered over the unit space of the groupoid, it is necessary to extend the theory of L^p-spaces to fibered spaces, and this is done in this chapter.

In Chapter 2 the authors define and examine amenability for locally compact groupoids. A particular case of amenability is proper amenability, studied in the first section, which is defined in terms of the existence of an equivariant system of probability measures. A key result in the chapter is Theorem 2.2.17, which gives the invariance of amenability under equivalence of groupoids.

Chapter 3 treats the theory of measured groupoids. A mean is defined as a conditional expectation from $L^\infty(G)$ onto $L^\infty(G^{(0)})$, where $G^{(0)}$ is the unit space G. This is a well-defined notion; however, there are several candidates for invariance of such means. Although it is an open question as to whether these various notions of invariance coincide, the authors show that the existence of an invariant mean of one kind implies the existence of invariant means of the other kinds. A measured groupoid G is said to be amenable if there exists an invariant mean on G. Among many other things, the authors give equivalent definitions for amenable measured groupoids that are analogous to conditions of Day, H. Reiter, and R. Godement for groups, respectively. They study a Følner condition enjoyed by an arbitrary amenable measured groupoid as well as additional growth conditions that arise in graph theory and foliation theory. One of the principal results in the chapter (Theorem 3.3.7) states that a locally compact groupoid G with a Haar system and countable orbits is measurewise amenable if and only if G is amenable in the sense that it has an approximate invariant continuous mean (Definition 2.2.8). This result provides a partial answer to the question of the equivalence of various definitions of amenability for groupoids.

Chapter 4 is concerned primarily with showing the equivalence, for a measured groupoid, of the authors' definition of amenability in terms of an invariant mean and Zimmer's fixed-point property (Theorem 4.2.7). Measurable equivariant bundles of Banach spaces on G-spaces arise naturally in this work and are used to construct invariant measurable sections of G-fields of compact convex sets. These results are used to give a characterization of amenable measured groupoids in terms of vanishing bounded 1-cohomology. Results in this chapter also enable the authors to construct conditional expectations in Chapter 6 that are needed in their study of injectivity of von Neumann algebras associated with measured groupoids.

In Chapter 5 the authors extend basic properties of amenable locally compact groups to topological and measured groupoids. The topological setting is briefly studied in Section 1, and a more detailed study of measured groupoids is provided in Section 3. With suitable interpretations, it is shown that closed subgroupoids, quotients, extensions, and inductive limits of amenable groupoids are again amenable. Groupoid extensions play a particularly important role (see Theorems 5.3.14 and 5.3.31). The second section is concerned with the class of groups that admit an amenable action on a compact space. The important hyperfiniteness result of Connes, Feldman and Weiss mentioned above in the case of a measured equivalence relation is established following the original arguments. The proof depends on the Følner property defined in (3.2.20), which holds for all amenable measured groupoids.

In the last chapter, titled "Operator algebras", the authors show how the amenability of a locally compact groupoid or a measured groupoid is reflected in its convolution algebra or in its representations. The argument that an amenable measured groupoid has only injective representations (Corollary 6.2.2) follows Zimmer's original proof. Conversely, Zimmer showed that, for a discrete group action, the injectivity of the von Neumann algebra of its regular representation implies the amenability of the action. The authors show in the general case of a measured groupoid G that an additional condition is required. Namely, they show that the injectivity of the von Neumann algebra $W^*(G)$ of the regular representation together with the existence of an invariant mean for the adjoint action of the von Neumann algebra $W^*(G')$, where G' is the isotropy, implies the amenability of the measured groupoid. They further show that a measured groupoid is amenable if and only if its trivial representation is weakly contained in its regular representation. A recent result of J.-M. Vallen characterizing amenability by the existence of an approximate unit in the Fourier algebra $A(G)$ is also treated. In the case of a locally compact groupoid G with a Haar system, it is shown that amenability of G implies that its full and reduced C^*-algebras coincide. However, the converse remains open even for discrete group actions. Finally, in the last section of the chapter, it is shown that groups which admit an amenable action on a compact space are C^*-exact.

The authors' appendices provide necessary background on several topics, among them Connes' transverse measure theory for groupoids, the structure of the von Neumann algebra of a measured groupoid, and basic facts on measurable bundles of Banach spaces. A final appendix by E. Germain gives an explicit construction of an approximate invariant mean for the action of a hyperbolic group on its boundary, thus providing a simple proof of a result due to S. Adams.

In summary, this excellent book provides a comprehensive overview of amenable groupoids in both the measure-theoretic and topological settings. The authors have given careful definitions, provided numerous examples, and treated the main properties of amenable groupoids. They are to be highly commended for writing an up-to-date, elegant volume that will be of interest to all serious researchers in the field.

Robert S. Doran (1-TXC)