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NOTES TOWARDS WG2

John Mason

Questions Posed

- The evolution of theoretical frameworks used in studies of mathematics teacher education
- The influence of other disciplines in this evolution
- ICMI contributions to the dialogue between the mathematics teacher education community and other disciplines
- Outcomes of interdisciplinary dialogue
- The development over time and in different cultures of the societal demands on the profession of the mathematics teacher
- The tensions between the academic foundations of mathematics teacher education and well grounded experience in what constitutes a good teacher
- The changes in structures of mathematics teacher education, division between practice and theory, the demand for research based teacher education
- Different conceptions of the mathematical knowledge needed for teaching and how it can be acquired, and its relation to pedagogical knowledge
- The attention given to preparing teachers of mathematics to address pupil diversity in culture and language, and the education of disadvantaged pupils

Apologia

These are notes prepared to stimulate discussion and for use by the working group organisers. Consequently there are few references and very few examples. To include them would be to write a full-scale paper.

Evolution

One way to interpret the history of mathematics education as a domain of enquiry is as a search for a magic recipe which would resolve inherent tensions in teaching and learning mathematics, and if not enable teachers to teach mathematics to all effectively, at least explain why it is so difficult. Indeed, it is part of the folklore that the continuation of mathematics education as a domain of enquiry requires its continued failure to solve the problems confronting teachers and teacher educators. Attention has been directed at various times to

Instructional design (often aiming for teacher-proof materials)

epistemological obstacles and learner misconceptions;

learner, teacher and institutional myths, assumptions and beliefs about, and attitudes towards teaching and learning;

psychological factors concerning learner and teacher behaviour, learner and teacher affect, and learner and teacher cognition;

social factors to do with the influence of teacher, parental and institutional beliefs and attitudes, perspectives and practices, stances and demands;

methods of enquiry drawn from ethnographic, phenomenological, behaviourist, sociological, linguistic and statistical disciplines.

A charitable reading of the field would be that when each new approach based on outside frameworks, theories and methods of enquiry fails to locate the root problem, another layer accretes on the forest floor that is mathematics education, and the search is on for yet another perspective to import. A less charitable reading would be that rather than ploughing established furrows and

probing more deeply in order to find something fresh which is unencumbered with a burgeoning literature, researchers turn to new approaches, new disciplines, new methods, new discourses which might shed different light, or at least offer a rich seam of new papers to contribute to the literature (and so achieve institutionally imposed publication targets).

Examples of 'approaches' include behaviourism; pragmaticism; genetic epistemology (often translated into simple constructivism); psychological, radical, and social constructivisms; sociocultural-historical; activity theory (in three generations); discourse; enactivism; distributed cognition; and complexity. All of these originate in other disciplines. What seems to be difficult is to weave a fabric out of all of these so as to maintain the complexity of learning and teaching mathematics, while at the same time informing practice while structuring systematic enquiry, but without losing the force and insight of particular approaches.

These various themes and approaches have had knock-on effects on teacher education. Consequently teacher education and professional development have followed parallel paths. The evolution of frameworks has reached a stage in which

> **some people try to stick to a specific 'theory' in order to maintain fidelity with the insight the theory affords, and because they believe that cherry-picking a few constructs misses the core of theories and leads to misapprehension and misapplication**

while

> **some people try to select (cherry-pick) pertinent constructs and integrate them into an idiosyncratic collage of theories.**

The former is academic and necessary in order to get papers published in leading journals which stress the importance of a coherent theoretical 'framework' on which the enquiry is based; the latter is pragmatic and the basis for effective development of and influence on professional practices and pre-service experiences. Papers which are heavily front-loaded with frameworks are often thin on detail, making it impossible for others to reproduce even an analogous situation for themselves. Many papers seem to be reproducing professionally and pragmatically evident findings, which add to the corpus but contribute little fresh insight. The practice when submitting articles to journals of referring particularly to recent articles in those journals may appeal to the editors, but it draws attention away from the rich insights of previous generations. Consequently a significant portion of mathematics education publication does little to enhance the community as a whole, though it may do a great deal for the individuals concerned. This returns me to my theme (Mason 1998a, Mason 1994, PME conference) that mathematics education is more about personal growth and development than it is about communal 'knowledge building'. Each generation has to re-articulate the insights of past generations in their own language and through their own experiences. The result is a reworking of wisdom and insight rather than adding to a body of 'scientific knowledge'.

Recently there has been considerable interest in what preparation is required in order to be effective when teaching children mathematics. My position is that the specific content of what is 'taught' is much less important than the scope of what is worked on, the ways of working used, and the relationship built up between teacher and learners. As I shall elaborate shortly, it is not knowledge-of that is important, but knowing-to-act, that is, having something suitable come to mind rather than working on automatic pilot through habit.

Different Metaphors, Different Perceptions

Learner as subject to cause-and-effect

Simple cause-and-effect metaphors pervade both research literature and policy statements. It leads to lists of teacher requirements as if they were packages, harnessed to perceptions of knowledge described by the frozen metaphor as something stored in a person as if in a library. Search for causal mechanisms is supported and amplified by the container metaphor in which skills and knowledge are seen as objects to be passed from one person to another. It enables the teacher or educator to

avoid direct personal involvement with mathematics: as one teacher said, "the text does the teaching; my job is to make it fun". For example, 'delivering' professional development is just one of the many misleading discourses which governs current policy making and implementation. Thus requisites are described in terms of knowledge-of mathematics (definitions, results, proofs), knowledge-of learners, knowledge-of pedagogy (general strategies for teaching) and knowledge-of didactics (specific tactics for teaching specific mathematical topics). This is the basis of Shulman's influential paper (Shulman 1986) which is an elaboration of Dewey's assertion that in order to teach effectively, it is necessary to 'psychologise the subject matter' (Dewey 1933).

Seeking causal connections supports and fosters training of behaviour: teaching comes to be seen as training learner behaviour (procedures rather than concepts) and educating teachers comes to be seen as training teacher behaviour (classroom management, task implementation, modes of interaction) with an emphasis on trying to classify 'which is best under what circumstance'. Unfortunately, 'evidence-based' teaching is just one of the misleading and misconceived by-products of such a perspective.

Learner as sense-making organism

Static knowledge is 'knowing-about', whether as knowing-that, knowing-how or knowing-when. This is what is probed by essays. What actually matters is knowing-to act in a pertinent manner in the moment, whether when planning or when conducting a session, or indeed, when reflecting back upon an incident. Essays written about teaching and learning count for little if the discernments and distinctions which are written about do not come to mind to inform current practice.

Learners seen as agents trying to organise their sense impressions and to make sense of their experience leads to a succession of forms and aspects of a constructivist stance. Skills and knowledge are seen as functionings developed and enriched by learners themselves through engaging in activity. Teaching can be seen as occasioning worlds of experience, and knowledge can be seen *as* action and action *as* knowledge, not just manifestation of knowledge. As 'narrative animals' people construct stories to account for and to organise their experiences, most especially their own actions and those of others which impinge on them. Of course those experiences are themselves the product of the discourses employed by the individual and the social groups to which they belong.

However, the legacy of cause-and-effect means that there is ongoing pressure to convert constructivisms into prescriptions of behaviour, leading to such oxymorons as 'constructivist teaching' and 'constructivist classrooms'. Constructivist perspectives do promote questioning of causal links between 'having been taught' and 'having learned'. As well as challenging education as a causal domain, they justify refusal to be caught up in attempts to deduce or even locate 'best practices'. The quality and effectiveness of practices lies in how the complex functioning of human beings is augmented and informed by the distinctions constituting theories. This highlights the danger of importing theories and perspectives from other disciplines without appreciating fully their purpose and their meaning. Any 'teaching situation' can be described in the discourse of any of the constructivisms, indeed of any theory: what the different discourses provide are foci for attention, not judgements. Informed by such analyses, one might be influenced to make certain choices, but those choices are neither dictated nor necessitated by the discourse. The discourse describes and so reveals sensitivities to notice and to attend. The values which are often interwoven and used to justify choices actually come from other sources.

Learners, and hence teachers when learning to become teachers and when engaging in professional development, can be seen as agentive wills making use of their natural powers to 'make sense'; teaching is then seen as the construction of tasks and environments which promote, foster and sustain sense making by learners. Striving to do for learners only what they cannot yet do for themselves can become a working principle. Stress can be placed on the individual, on the group as a social entity, or on a combination of these. Sense-making is the most important part of any pedagogically inspired or directed activity, but it often gets the least attention and the least support:

'looking back' and 'reflecting effectively' is the least common of all the actions promoted by teachers and educators.

Stressing the social, to the extent of seeing learning as the adoption of practices in order to fit in socially, leads to emphasising discussion and interaction while downplaying individual construal and activity. Stressing the individual ignores the influence of the social network of competing agencies and forces in which individuals are embedded and the historical-cultural-social roots of the discourse. Stressing the social can extend to seeing mathematics as a discourse, a way of speaking and acting, which is picked up by being in the presence of others and having plenty of opportunity to imitate and rehearse practices. One consequence is the downplaying of the role of mathematical structure, and the way in which mathematics reflects or amplifies the socio-genetic makeup of individuals.

Learner as participant in a dynamically evolving complexity

Seeing learners as agents with wills, desires, propensities, dispositions, habits and awarenesses leads to quite different ways of perceiving teaching and learning. The plethora of forces and influences acting at every moment are so complex that it is highly unlikely that any simplified analysis can lead to significant success when engineered for the many, rather than simply tailored to a few. Rather, teaching and learning are seen as co-emergent phenomena. Will-based agentic human beings interact with and respond to each other. Much of this interaction is socially habituated, but this is precisely where education is required: to bring habits to the surface, question their efficacy, and provide alternative actions. By maintaining complexity rather than trying to reduce it to simpler components, learners, teachers and teacher educators reflect the complexity of organisms. This is not an argument in favour of 'giving up', nor of eschewing discernment of details and choices of action in the moment. Rather this perspective celebrates complexity by seeing teachers, learners, mathematics and the milieu in which they are embedded as more like an organism than a mechanism, as organic rather than mechanistic.

One implication for teaching is that teachers stop looking for 'the best way' to do things independent of their learners and their situation. Rather, they work on themselves (Mason 2002). They work to sensitise themselves to notice opportunities to respond freshly rather than to react habitually. For example, choosing when to intervene and when to hold back; when to speak and when to listen; when to direct attention and when to follow learner attention; when to explain (enter learners' worlds of experience) and when to expound (draw learners into their own world of experience). It is not a matter of developing rigid practices nor even extensive criteria. Rather it is a matter of enjoining sensitivity, richly varied strategies, and developing wisdom. The important feature is to avoid being caught in trying to lay down criteria or principles which govern such choices: as with making robots 'walk', locally principled and informed responses are how the organic world operates, in contrast to the production-line mechanistic world of top-level choices and top-down-prescribed functioning. Organic holistic teacher education would concentrate on individual's being, mathematical as well as social. Trying to write essays 'about' when to listen and when to intervene can actually blunt sensitivity to respond in the moment, through having 'boxed it all up' in academic categories which fail to come to mind when they are actually needed.

From a constructivist-enactivist type of perspective, Dewey's notion of 'psychologising the subject matter' turns into a dynamic and humanistic enterprise because it involves responding rather than reacting to individuals and groups, based on awareness of and sensitivity to learners powers and dispositions, to mathematical themes and heuristics. It fosters an approach to teaching based on prompting the learner to take initiative, make significant mathematical choices, and experience mathematics as a constructive and creative enterprise. Teaching 'about' these things is neither necessary nor sufficient to inform effective teaching. It is very far from sufficient because knowing-about is rarely adequate to inform practice in the moment; it is certainly not necessary either, because people can become aware of these things without being instructed or taught formally, but rather through guided and informed reflection on pertinent experiences.

Retaining Complexity

As an example of retaining complexity, consider the question of 'how much mathematics does a teacher need to know?'. The best answer is that this is an inappropriate question, for it is trying to reduce complexity precisely where it is important to retain it. What really matters is what comes to mind when pedagogic and didactic choices which affect learners are being made. Someone can have been exposed to massive amounts of mathematics, but not have much come to mind when appropriate, or not be sufficiently experienced in and aware of their own mathematical thinking to be in a position to guide or stimulate that thinking in others. Someone can also have too much come to mind, either rendering them speechless or inducing them to embark on overly complex exposition. By contrast, someone can have been exposed to comparatively little mathematics but have access to and a disposition for thinking things through, and exposing that thinking to novices. While it is clear that having more experience of mathematical topics and mathematical thinking is an advantage, it is only an advantage if pertinent actions come to mind in the moment (assisted by having come to mind while planning). In other words, to be a 'real teacher' you need not only integrated internalised functionings which give you fluency and confidence, but you need awareness *of* those functionings so as to be aware of choices that you could make. Only then are you in a position to draw learners' attention to those choices (Mason 1998). Your quality as a teacher cannot be judged on the basis of some mixture of what mathematics you 'know', have been exposed to or even can recall or reconstruct; on your grasp of psychology and sociology; on the range of pedagogic strategies and didactic tactics you have deployed or can describe; or on your effectiveness as a mathematical problem solver. Effective teaching involves a complex interweaving of all of these into your 'being', and it is that 'being' which actually matters most.

In the 1980s, when there was a dearth of secondary mathematics teachers in the UK, approaches were made to ex PE teachers who had had some previous encounters with mathematics teaching during their teacher training, with ex physics teachers who were thought to have a smattering of mathematics, and with ex music teachers who were being made redundant due to changes in policy. In many cases it was the music teachers who, despite not knowing all of the mathematics, proved the most effective mathematics teachers. They had ways of finding out when they didn't know, together with ways of working effectively and sesnsitively with learners treating them respectfully as sense-makers. Many of the others who had a less creative more ritualistic perception of mathematics, found it difficult to adapt to ways of teaching beyond fill-and-drill.

Educating Future Generations

The questions that need to be addressed, therefore, are *not* 'what do effective teachers of mathematics need to know' about mathematics; about mathematical pedagogy; about mathematical didactics; about the structure of a topic; about the structure and role of attention, will, desire and habit; about the potential influences of classroom ethos, rubric, socio-mathematical norms; about possible criteria for selecting and presenting tasks, and for choosing ways of fostering and sustaining learner work on those tasks to pedagogic advantage; or about the psychology and sociology of learners in institutions, including the potential influences of assumptions and beliefs about, and attitudes to learning mathematics held and manifested by learners, institutions, and others in their ambit. Rather, the key question is

> **how can teachers and pre-service teachers be stimulated and prompted to probe beneath the surface of their current awareness and sensitivities in order to notice, regarding mathematics; mathematical pedagogy; mathematical didactics; the structure of a mathematical topic; the structure and role of attention, will, desire and habit; possible criteria for selecting and presenting tasks, and for choosing ways of fostering and sustaining learner work on those tasks to pedagogic advantage; and the psychology and sociology of learners in institutions, including the potential influences of assumptions and beliefs about, and attitudes to learning mathematics held and manifested by learners, institutions, and others in their ambit.**

This is no mere semantic shift. This presents a major shift in the approach of policy makers and educators concerning the design and implementation of professional development. There is no 'body of knowledge' to communicate, but there is a vast amount of accumulated wisdom. Wisdom however cannot, by definition, be set down, transferred, or made the object of instruction.

What is needed are ways of working with those who are or who are becoming teachers, so as to stimulate them to question and reflect on their actions; to strengthen their disposition to engage in mathematical thinking themselves, alone, with colleagues, and most especially with and in front of learners; to have suitable possibilities come to mind in the moment when they are needed, whether when planning, when in the classroom, or when reflecting afterwards.

Teachers need access to resources of all kinds: mathematical tasks; experiences which highlight awareness of mathematical principles, powers and themes; pedagogic strategies and constructs; didactical tactics; ways of perceiving and conceiving teaching and learning (Mason 1998). But they also need access to ways of working with these, ways of integrating them into their own functioning, ways of enriching their own professionality (including sense of themselves).

***Research-based* mathematics education could be constructed so as to be consistent with the values and the accumulated wisdom of those who lead it, by stimulating, guiding and directing teachers and would-be-teachers to engage with mathematics and to sensitise themselves to discriminate phenomena emerging from their experience that might otherwise be overlooked, so as to inform their future practice. Where a phenomenon is experienced in the classroom, it is valuable to look for analogous experiences for oneself, and to construct and refine task-exercises for others, so that such experiences can be shared, and alternative actions tried out.**

Efficacy

What can be done to ensure accountability and continuing professional engagement in necessary enquiry? Testing people with mathematical and pedagogical tasks barely illuminates the surface of the important actions which inform future practice. We need to find ways to enable people to display evidence of

- **identifying situations as instances of phenomena (both recognising examples and exemplifying; both seeing the particular in the general and the general through the particular);**
- **seeking fresh ways of acting in those situations whether mathematically, pedagogically or didactically;**
- **collecting evidence on which to reflect (recall the energy levels involved in noticing, marking and recording: see Mason 2002);**
- **trying out task-exercises with colleagues to test whether what has been noticed is accessible to others, and whether their future actions are also informed by being sensitised.**

Efficacy lies in the beholder, as to whether they find their practice informed, and their aims for learners more clearly or effectively achieved. It does not lie in check lists.

Influences of ICMI

ICMI has had its principal effect by bringing together mathematicians, teachers, teacher educators, researchers and scholars from different countries and from different backgrounds. The positive side is that mathematics education as a domain of enquiry is a rich melding of a variety of perspectives; the negative side is that mathematics itself is often overlooked, downplayed, or overlaid. If mathematics educators had managed to develop a collective discourse, it might have been more focused and we might have developed an integrated and more widely shared technical vocabulary and shared discipline. As it is, there is a plethora of parallel or nearly parallel discourses and frequent repetition of enquiry. This may however turn out to be an advantage since, in order to maintain complexity, it is necessary to encourage multiple expressions in slightly differing forms.

Those that truly inform may prevail, although they have to compete with socially reinforced academic-knowledge published in journals.

Mathematics is able to build theorem upon theorem upon theorem, and to provide highways of learning for new entrants to get to the boundaries of the discipline without having to traverse the entire already explored landscape. Mathematics education is as much a personal journey as a structure of results. Each individual has to work at developing their teacher-self (or selves) as an ongoing project. You do not study to become a teacher and then teach; you embark on a career-long process. Each person starts with themselves, and there is no super-highway taking you directly to the frontiers of enquiry, no ladder up to the shoulders of preceding giants. The work, the insights, the articulations of previous generations can inspire and inform, but little can be 'handed-on' without reconstruction by individuals.

Curiously, I am unaware of significant influences of mathematics education on any of the disciplines from which it has drawn inspiration, theories, discourses or ways of working.

ICMI might, as well as continuing to support the cross-fertilisation of mathematics education with other disciplines, also choose to organise forums in which participants work at sharpening the discipline with which and in which they work. For example, developing over time ways of negotiating meaning of technical terms, ways of analysing learner and teacher behaviours, and ways of coordinating different ways of perceiving and conceiving. These would be at least as helpful as the articulation of theoretical frameworks demanded by editors.

ICMI could embark on a programme of novel-like construction so that new generations of teachers and educators can get a vicarious taste of the issues and concerns and the practices invoked to address those issues and concerns at various times in the past. For example, the wheel of history has already made a full revolution between the late 1980s and the present with respect to theme work and investigations: people are again enquiring into how to do it, how to assess it, and how it can be engineered, but without realising that it was well developed in its previous incarnation. Another issue having an airing currently is procedural vs conceptual comprehension. Those who do not learn from the past are doomed to repeat it.

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WHAT TEACHERS REALLY NEED: EDUCATED AWARENESS

Introduction

When I have had a thick cold or a fever, or been seriously ill, it has been as if a veil has been drawn between me and the familiar world s of materiality and of thought. The veil is usually partly transparent, and depending on the seriousness, it may seem as though here are multiple veils. Once I begin to recover, it is as if the veils are drawn back, layer upon layer.

I was reminded of this by Antony Gormley's 'sculpture' *Blind Sight* exhibited at the Hayward gallery in 2007. It consists of a large room-sized cuboid filled with some sort of steam. On entering the room, it is like being in a very dense fog. You come across other people, and the boundaries, very suddenly. Leaving the room, everything is clear again, as if the fog has lifted, the veils pulled back.

I have several times, when ill, contemplated the analogy that even when I am fully recovered and functioning reasonably in the worlds of materiality and of thought, there may be veils or fog of which I am unaware. Gormley's room works partly because I am aware of what has been veiled from me by the fog. But in ordinary life I miss all sorts of things. It is as if there are veils or fog which prevent me from being aware of objects, details, relationships, and properties.

A confirming instance for me was a visit made to the valley of Ollantaytambo in Peru many years ago. I saw what our guide pointed out, of course, but only relatively recently people have discovered that if you stand in certain places in the valley, the entire escarpments on both sides have been carved or shaped so as to represent figures of importance to the Incas. But I was completely oblivious of them when I was there.

Teaching Mathematics

For me there is a strong analogy between the veil-fog metaphor and the awareness, the sensitivities to notice of pre-service and in-service teachers. Pre-service teachers especially are rarely attuned to the many dimensions of experience which make teaching mathematics possible. I am aware of four domains:

> Learner as a collection of agentive selves held together by habits and desires, awarenesses and intention, attention and will, interwoven with emotions and intellect and manifested as behaviour.
>
> Milieu as the micro environment (on several levels) within which the individual acts, participates and is enculturated.
>
> Self as a similar collection of agentive selves, with the added necessity of shifting so as to dwell in the (worlds of) others not just in their own.
>
> Mathematics as a collection of behaviours, dispositions, habits of mind and particularly manifestation of natural human powers, built around core Awarenesses which have been developed into the sub-discipline which comprise mathematics at any time.

One implication of this line of thinking is that teaching is seen as a caring profession. The major implication is that what is needed in order to be effective as a teacher is an awakening to the potential awarenesses peculiar to these domains, and a commitment to extending and enriching those Awarenesses, developing refined and honed sensitivities to notice mathematical and pedagogic opportunities with suitable actions coming readily to mind.

An important feature of this line of thinking is that it is not a matter of a threshold of knowledge which enables someone to teach well, but rather a flexibility and commitment to full participation, which includes ongoing personal and professional development.

Mathematical Awarenesses

Taking the perspective described here means that learners need to experience and integrate into their functioning various actions which lie at the core of mathematical topics and themes. Setting tasks, concomitant activity, accumulated experience and working on that experience is the surface structure of mathematical classrooms. Below the surface lies the essential purpose of the tasks, activity and experience, for at the heart of mathematics lie core awarenesses. Thus coordinating a counting poem with pointing, together with discerning a 'unit' are the actions which underpin the awareness which constitutes counting; the action of discerning a repeated unit lies at the core of measuring as well as counting; the action of combining and breaking up underpins addition and subtraction; the actions of scaling and of repeating underpin multiplication; the projection of motion along a path into motion in two (usually orthogonal) directions and the coordination of action in two directions to form a path underpins graphing; the action of selection from a range of possibilities underpins probability, and so on.

Each mathematical topic is based on core actions which learners can carry out under instruction (otherwise they cannot be expected to make much sense of the topic!). These actions constitute the core awarenesses around which the topic is built. A learner needs to encounter the use of previously familiar actions, possibly in a new form or with extensions or variations; an effective teacher needs to be aware of these awarenesses; an effective teacher educator needs to be aware of how to locate and bring to the surface these core awarenesses, derived from the literature and from experience.

If a teacher is not aware of relevant core awarenesses, then they are not in an position to choose appropriate tasks, nor to choose appropriate pedagogical strategies and didactic tactics. For example, partitioning a rectangle into equal pieces with vertical divisions, or with horizontal divisions is a core action which underpins the awareness of fractions as operators on objects. An associated pedagogical awareness is the fact that many learners have a predisposition for vertical partitions, and so require continued exposure to horizontal ones as well so that they develop the flexibility to choose for themselves, which is essential to use the rectangular area model for fraction operations (Kyriakides 2006).