

## School mathematics as a special kind of mathematics

Anne Watson

University of Oxford

### Introduction

In this paper I argue that school mathematics is not, and perhaps never can be, a subset of the recognised discipline of mathematics, because it has different warrants for truth, different forms of reasoning, different core activities, different purposes, and necessarily truncates mathematical activity. In its worst form, it is often a form of cognitive bullying which neither develops students' natural ways of thinking in advantageous ways, nor leads obviously towards competence in pure or applied mathematics as practised by adult experts. The relationship of school mathematics to adult competence is similar to the relationship between doing military drill and military leadership; between being made to eat all your spinach and becoming a chef; between being forced to practise scales and becoming a pianist. There are some connections, but they are about having a focus on a narrow subset of semi-fluent expertise in negative social and emotional contexts, without full purpose, context and meaning. That some people become effective military leaders, beautiful pianists or inspiring cooks is interesting, but what is more interesting is the fact that most people who go through these early experiences do not: instead they merely follow orders, or hate green vegetables, or give up practising their instruments.

The image around which I hang this paper is the cafeteria at the heart of the mathematics faculty at Cambridge. It is a large room with coffee at one end, small tables covered with papers and laptops, each surrounded by four chairs, students (mainly male), some undergraduate, some graduate, pure and applied, some alone, some in casual groups, some in self-organised study groups with their own internal disciplines and plans, taking time, talking, arguing. There are many furrowed brows, people leaning forward with arms and hands working away to express some thought; bodies, words and minds hauling together to communicate how they see relationships and properties, and offering each other handles to work with abstract objects and multi-dimensional extensions. No one is doing exercises or practising techniques; no one is interrupted by bells or instructions to change to the next task; no one is taking their work to teachers to be marked. Every now and then there are whoops of excitement or groans of frustrated realisation.

### Mathematical enquiry

These students are in various stages of transition between school maths and academic maths. With a few exceptions they have rarely worked like this at school and only a few may have been told that this is how they could be working now. Mathematics as a discipline includes these social and cultural characteristics which have contributed to its genesis. Most mathematical advancement does not arise in an isolated, independent way but is a product of its time, within the current paradigms, co-emergent with current technological and economic needs and tools, co-emergent with the *zeitgeist*. Locally and globally, it is a product of social interaction (Davis & Hersh 1981, Burton 2004).

In school things are generally different. I am not assuming any homogeneity about school practices – it is always possible to read such assumptions and argue to exclude your particular patch of the world, or particular teachers, or a particular curriculum. Instead, I focus on the mathematical practices in which, I presume, the students described above are engaged but which cannot be observed in the short-term, nor do they fit neatly into socio-cultural analyses of the learning environment. For me, the starting point is what it means to do mathematics, and to be mathematically engaged.

In the discipline of mathematics, mathematics is the mode of intellectual enquiry, and effective methods of enquiry become part of the discipline – so much so that mathematics theses do not have chapters explaining methodology and methods. This is not just about the availability of coffee, social groupings, the published discourse, or choices between Maple and Mathematica, although these are crucial parts of the picture.

'Doing mathematics' is predominantly about empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalisation, locating isomorphisms, reflecting on answers as raw material for further conjecture, comparing argumentations for accuracy, validity, insight, efficiency and power. It

is also about reworking to find errors in technical accuracy, and errors in argument, and looking actively for counterexamples and refutations. It can also be about creating methods of problem-presentation and solution for particular purposes, and it also involves, after all this, proving theorems.

### **Different kinds of mathematics**

The practices of mathematics just described are very far indeed from the concerns of psychologists who want to construct efficient methods of instruction, seeking for the fastest and most productive ways to teach students how to find answers to broadly isomorphic problems. These areas of research and development do not even begin to offer ways to induct students into the practices listed above. My perspective on the discipline of mathematics is also different from the concerns of the socio-cultural take on classrooms and learners, which tells convincing and robust stories about the existence of differences in practice, and the process of development of identity in different cultural settings, but which cannot put detailed flesh on these stories in terms of the development of specific mathematical ideas.

Of course we would expect to see different kinds of learning and different classroom practices where there are different views of mathematics and different curriculum goals, and in particular we would expect to see a political dimension to this as countries connect the outcomes of education with their future workforce. As Luis Radford has said (2003)

While the humanist view of mathematics emphasizes the role this discipline bears in the development of logical thinking, abstraction, rigor and other highly prized faculties that were the clear marks of men of sophisticated spirit since the Enlightenment, the socialist trend stressed the applicability of mathematics. Its importance is seen in terms of the utilitarian dimension of mathematics to master nature in the interest of mankind.

If the curriculum takes a 'humanist' approach, rigorous argument and proof are taken to be important aspects of the subject, yet many teachers treat 'proof' as a topic within mathematics rather than as the way truth is examined and warranted throughout the subject. If this is combined with fallibilist-inspired teaching styles such as investigative work, 'what if...?' questions, and algebra introduced as an expression of generality, then in many classrooms two standards of argument are being offered. One is empirical, arising from specific cases, tables of values, inductive construction of formulae and testing special cases, while the other requires deductive reasoning and a willingness to engage in formal logical argument. The shift learners have to make between inductive to deductive argument is hard. For example, it is well-documented that students who can produce examples, in an inductive mood, easily find the production and understanding of counter-examples, with their deductive role, much harder (Zaslavsky & Ron 1998). To use mathematical argument as the natural and normal method of enquiry in mathematics it needs to be fully embedded in day-to-day lessons as part of classroom discourse. In reality, a curriculum written to in accordance with what Radford calls 'humanist' principles is likely to be presented as a taught curriculum in which topics are presented in a developmental order and few students see the role for logical thinking embedded. If their own logical thinking *does* play a part in mathematics lessons, as it does with some teachers, the curriculum aims might be realized in part.

Learners exposed to utilitarian mathematics have a different experience to those taught with a more abstract view, but solving realistic and everyday problems need not lead them to understand the role of mathematics beyond providing *ad hoc* methods for real problem-solving, or as a service subject which holds tools for moving forward in other domains.

### **Mathematical shifts of understanding in school mathematics**

But the choice between different possible 'mathematics' is, to my mind and in my experience, not the core problem about mismatch between school mathematics and the discipline of mathematics, as I intend to show. I am going to describe several shifts of understanding which have to be made for learners to be successful within and beyond school mathematics, whatever the dominant view of the discipline (this is not an exhaustive list!):

*Additive to multiplicative reasoning:* A shift from seeing additively to seeing multiplicatively is expected to take place during late primary or early secondary school. Not everyone makes this shift successfully, and multiplication seen as 'repeated addition' lingers as a dominant image for many students. This is unhelpful for learners who need to work with ratio, to express algebraic relationships, to understand polynomials, to recognise and use transformations and similarity, and in many other mathematical and other contexts.

*Probabilistic reasoning:* The concept of probability, understood mathematically, offers a different warrant for truth than is associated with either deductive logic or induction from empirical evidence. To understand probability as a tool, or to see the world probabilistically, requires abstraction and imagination well beyond observable phenomena. Moreover, one cannot merely follow algorithms to get answers except in very simple contexts; often learners have to decide for themselves whether events are independent, exclusive or not. The shift here is from being told everything about a situation to having to identify characteristics and properties for oneself, before conceptually-based action.

*Integration:* In the UK, integration used to be the first context in the school curriculum where learners could not merely apply methods and be sure they would get some kind of answer. However students now are told not only what method to use, but also what substitutions to make if substitution is the method examiners wish to see. The shift from being told what methods and tools to use, to developing sensible selection criteria for what is possible, has to be made to become a mathematician.

*Geometrical reasoning:* Questions involving application of theorems can be avoided in UK national tests at 16+ and students still be awarded the highest grades. Theorems and proof of any kind, let alone geometrical contexts, do not play a part in higher school examinations. In countries where reasoning from axioms still has a place in the school curriculum these may be taught mechanically, as a kind of memory or question-spotting activity, rather than as a demonstration of deductive reasoning to explore phenomena and to establish a particular kind of truth. A shift from knowing what to look for, to selecting what to look at, then to deciding what to use and constructing multi-stage arguments for oneself, has to be made to become a mathematician.

*Problem-solving:* I use 'problem-solving' in Polya's sense rather than to mean answering word problems designed to rehearse application of algorithms. Even so, a problem can be solved and nothing new be learnt, even about problem-solving. A mathematician will usually have a purpose in mind when solving a problem, so that the outcome is used to reflect on the context in which the problem arose, to decide if something unexpected has arisen, to raise further questions, or in some other way to enrich or extend knowledge. The answer, if there is one, is not the end of the process. A shift from getting answers to gaining insight or constructing arguments has to be made. In non-mathematical contexts, once the problem is solved there is no motive for extending the work hypothetically.

*Modelling:* Mathematical modelling of realistic or artificial situations is a feature of many mathematics curricula. While generalisation might take place in order to create the model, and this might be explored further to look for extreme cases or further variation, the modelling process itself does not require more abstraction or structural understanding than the situation being modelled.

### **The discipline of school mathematics**

It is not mere coincidence that these are all inherently 'hard-to-teach' topics. They all offer epistemological obstacles which require shifts of reasoning, or new ways to act with imagery, or encapsulation of previous experience to be overcome. Yet in all these curriculum contexts, and many others, there are features which are peculiar to school mathematics, and the way it is generally taught, which are not part of the discipline as practised by adult experts:

- There is a strong focus on answers and generalisations rather than structural insight and abstraction
- There is avoidance by teachers, tests, and curricula, of the need for uncertain choices
- Curricula seem to cling to topics and approaches which can be represented experientially, diagrammatically or in concrete ways, rather than in abstract and imaginary ways
- Inductive, empirical, and ad hoc reasoning are privileged over deductive or probabilistic reasoning

Importantly, it is becoming more and more the case that, in an effort to ensure that more students can gain school mathematics qualifications, the difficult shifts which would have to be made for school mathematics to be a subset of the discipline of mathematics are being edited *out* of mathematics as a school subject, rather than edited *in* as the discipline itself becomes more complex, more post-modern and less certain. I am talking here of all kinds of classrooms: reform and traditional; classrooms in which students are expected to behave like little mathematicians and those where they are expected to behave like acquiescent cognitive machines; classrooms in the developing world and those in high-achieving countries. I am not arguing for or against particular kinds of school curriculum.

These features, peculiar to school mathematics, can hinder the progress of students at university on pure and applied courses, alongside inappropriate work habits, challenges to identity, and unrealistic expectations of the subject (sometimes promulgated by popularisers). Mathematics as a discipline, by contrast to school mathematics, is concerned with thought, structure, alternatives, abstract ideas, deductive reasoning and an internal sense of validity and authority. It is also concerned with uncertainties about ways forward in its own realms of enquiry. To do maths includes holding nagging questions in the mind while carrying on with life, and not expecting answers to be found, problems to be solved, within the confines of a particular room or timescale.

The problem is that, whichever approach is taken by the curriculum, most students are taught and examined on mathematics of a kind which is done, both in the academy and in other workplaces, by machines. They are taught not as an apprenticeship to adult mathematics users, but as bottom-up preparation for future mathematical activity which might one day be meaningful, either as an intellectual or an economic activity. Furthermore, they are taught this in regimented settings, with short time-scales, by teachers who themselves have limited experience of the mathematical practices described above. Those students who continue beyond school might be motivated by a so-far satisfied need to have right answers, or by getting a kick from the resolution of puzzles, or by discovering special features which intrigue them, or by anticipating the delayed gratification of getting a further qualification, or of finally being able to study at the cutting edge. They might also have what Krutetskii identified as common features of mathematically gifted students: propensities to see the world and organize their mental activity in certain mathematical ways (1976). The list he identified has little in common with the contents of formal taught lessons and assessment regimes, and much to do with the list of practices above, and what Cuoco and his colleagues have called mathematical 'habits of mind' (1996). Neither a curriculum based on rigour, historical genesis and conceptual development, nor a curriculum based on modelling and authentic contextual questions, can achieve the education of the mind required to engage consistently, habitually, with the normal intellectual practices of mathematicians.

### **The roles of unifying concepts**

A further striking difference is in the role of unifying concepts in mathematics as a discipline and in school mathematics. In mathematics as a discipline these orientate much of what is studied and researched. Such theories provide opportunities for links and connections within the discipline, new languages for discussing ideas, and new questions to be explored. In school mathematics, however, the shaping of a curriculum with unifying concepts could be unhelpful. A unifying concept makes sense to mathematicians precisely because it offers unification of previously disparate ideas of which they have a range of experience. For school students, to be told about a unifying concept before experiencing several examples of it reduces the concept to 'something else to be learnt' or 'something else to be taught' rather than an enabling encapsulation. Linearity, for example, crops up throughout school and could have been included in the 'hard to teach' list above, because it requires a shift from looking at functions as representations of relationships to comparing properties of functions. However, this only makes sense when a learner has experienced several different linear and non-linear situations so that there is a need for a word for those which behave in certain ways. Older mathematicians are supposed to have had a range of experiences to draw on when they meet, for example, vector spaces so that they have examples which might be activated by hearing definitions, theorems and questions (Watson & Mason 2002).

### **Constraints on teaching which alter the discipline**

It certainly is the role of school mathematics to provide a range of experiences of various kinds so that students understand the usefulness of mathematics, and can do various necessary calculations and estimations. It would also be helpful for students to understand the purpose and value of practice, and of basic standards of accuracy. School could introduce students to ways of working on mathematics, to the kinds of questions mathematicians ask and the subject matter worked on, but would have to include in that some of the practice of number systems, and algebraic manipulation, ways of using diagrams, which are assumed within the discipline. At the most extreme, schools use entirely different kinds of questions, enquiry, warrants and work habits than those of the discipline. At best, they introduce what is to come, while moulding it to fit the institutional constraints, rather than to fit the development of mathematical ideas.

Many teachers and projects try to make their classrooms more and more like mathematical workshops, but the school context overlays these with purposes which are not about the development of mathematics, but are more about learning what to do and how to be. In the best of these classrooms, teachers are explicit about appropriate forms of statement and discussion in ways which are not made explicit in mathematics faculties, but even the best teachers cannot be present at every student's side to be explicit about the many practices of professional

mathematics listed above. Mathematics as a discipline includes discussion and critique of its own modes of enquiry, but does not include 'teaching' these modes of enquiry, or practising core techniques, as school mathematics has to do.

Another aspect of school mathematics which differs from the discipline of mathematics is the nature of authority. Vergnaud (1997), Freudenthal (1973) and others have emphasised that mathematics includes its own methods of validation, and a good teacher can enable students to use the structures of mathematics to verify their own work and ideas. In the education system as a whole, however, validation and authority are highly structured through textbooks, examinations, and curriculum systems rather than by mathematicians working as a community.

## Conclusion

In conclusion, therefore, I find myself claiming that school mathematics is necessarily not a subset of the discipline of mathematics, whatever the nature of mathematics being taught, whatever the way it is taught. Essentially, this is for the following reasons:

- Schools have to prepare students for future study and/or employment and this requires them to be explicit about ways to develop recall, fluency, accuracy, and ways of working; these constitute the major part of the goals of school mathematics
- Many 'hard-to-teach' topics require shifts away from everyday thinking in mathematics lessons to special forms of mathematical thinking which require knowledgeable teachers. We have to recognise that many teachers of mathematics do not have personal experience of what it means to do mathematics over time, exploring questions which have intellectual purpose, not pedagogic purpose
- Limited time slots, curricular pressures, and assessment regimes constrain or prevent the development of the kinds of questions and ways of working which characterise the discipline
- Authority lies outside mathematical argument.

It is not only ways of working and goals that are different between school maths; it is the way that these shape the forms of mathematical enquiry available that makes school maths a different discipline, with its own rules, purposes, authorities and warrants.

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