Mathematical technologies as a vehicle for intuition and experiment: a foundational theme of the ICMI, and a continuing preoccupation

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Foundational theme

The ICMI was formed in 1907, by resolution of the 4th International Congress of Mathematicians, as a means of promoting international exchange of ideas, in the light of the significant movements for the reform of school mathematics teaching then current within educational systems across Western Europe and North America (Howson, 1984), reflecting wider pressure for more practical approaches which would give a more active role to the learner through concrete experience relevant to life (Brock & Price, 1980). At the 5th International Congress of Mathematicians in 1912, the section on Didactics received ICMI reports examining "the current state and modern trends" of mathematics teaching (Fehr, 1913: p. 595). In particular, the report of Sub-Commission A [Mathematics in Secondary Education] focused on "Intuition and experiment in mathematical teaching in secondary schools" (Smith, 1913); it discussed contemporary developments aimed at providing an 'intuitive', 'perceptual', 'experiential' and 'experimental' base for the subject (p. 611), through "applying mathematics seriously to the problems of life, and… visualizing the work" (p. 615). This, then, represented a foundational theme for the ICMI.

Introducing the Subcommission's report, Smith (1913) described a "spirit of unrest" (p. 613) in secondary education, "in particular with respect to this whole question of intuition and experiment in mathematics" (p. 614). His judgement was that "more progress towards the recognition of the rôle of intuition and experiment in secondary mathematics seems to have been made of late in Austria, Germany and Switzerland than in France, England and the United States" (p. 615), and that a key difference of approach lay in "the plan of the Teutonic countries to mix the intuitional and deductive work from the outset, while in France, and now in England, the plan is to let an inductive cycle precede a deductive one [with] the United States [showing the beginnings of a] tendency towards the Anglo-French plan" (p. 615). Indeed, this latter plan seems to have been indicative of a ore qualified view of the place of such work: in England, the report noted, "the prevalent view… is that the proper field for methods of intuition and experiment is in the middle and lower classes rather than the upper" (p. 613), and "many teachers consent to the postponement of abstract methods during the earlier teaching only on the understanding that the abstract character of the higher teaching shall be preserved" (p. 614).

The main areas which the Sub-Commission singled out in relation to this trend to give a more central place to intuition and experiment in the mathematics curriculum were geometrical drawing, graphical methods, practical measuring and numerical computation.

Geometrical drawing

The report described the situation of geometrical drawing as still evolving, and its definition as variable:

> In the matter of geometric drawing and graphic representation of solids, the various countries seemed to be in a transition stage between the period in which this was considered part of the duties of the art teacher and that in which it was to be taken over by the department of mathematics. The tendency is general to consider this work part of mathematics. The nature of the work is not, however, at all settled; even the term "descriptive geometry" has no well-defined meaning. (Smith, 1913: p. 614)

In some countries, there was a very clear linkage to material traditionally taught in art or craft classes or in technical subjects:

> In the curriculum of Bern the course includes work in geometrical ornament, such as parquetry flooring; orthogonal projection of geometric solids; drawing of conics and other plane curves; drawing of machine models; shadow constructions; axonometry; polar perspective; solids of rotation; and plans and elevations. (Smith, 1913: p. 621)

In a paper presented at the previous Congress, Godfrey had portrayed this (re)appropriation of ‘geometrical drawing’ to the mathematics curriculum in broader educational terms:

> Under the old system the instruction in geometrical drawing had been divorced from theoretical geometry, to the detriment of both studies. It was frequently taught as a branch of the fine arts rather than as mathematics. One result of this has been an exaggerated respect for artistic finish and “inking in”; but the worse feature is that “geometrical drawing” came to be identified with a vast collection of special and unrelated rules; the educational value of the subject had sunk to zero” (Godfrey, 1909: pp. 455)

Equally, however, Godfrey linked the new place of geometrical drawing in the mathematics curriculum to the idea that more formal treatment of geometric properties needed to rest on an experiential base:

> The reforming party held that a more vivid realization of the shapes and properties of geometric figures was necessary before these properties could profitably be made the subject of strict logical treatment. (Godfrey, 1909: pp. 453-4)

In particular, this ‘more vivid realization’ depended on students developing a better grasp of the operation of geometrical instruments and their mathematical functionality through direct use of them:

> To experiment in geometry, a child must learn to measure and draw with sufficient accuracy… For another reason, too, these instruments must be provided. Problems of construction are unmeaning unless it is specified what instruments are allowed… Problems of construction, then, cannot be undertaken intelligently unless the learner understands these instrumental restrictions; and he is not likely to understand the restrictions unless he actually handles and uses the legitimate instruments. (Godfrey, 1909: pp. 454-5)

At the same time, Godfrey portrayed such ‘bodily activity’ as essential for pupils, promoting an active attitude on their part, and improving the quality of their thinking:

> Geometrical instruments satisfy the child’s need for bodily activity. He thinks better if he is using his fingers. Ideas are suggested to him by the activity of drawing figures. His attitude becomes active rather than passive. (Godfrey, 1909: pp. 455)

Graphical methods

The widespread use of graph paper in scientific and technical work developed during the 19th century. In the course of an 1833 paper, the astronomer-mathematician Herschel recommended the use of such paper in terms which make it clear that he expected it to be unfamiliar to his readers, since he described both its design and manner of use with care. As interest in this new technology grew amongst scientists and engineers, many manufacturers entered the market. Over the second half of the nineteenth century its price fell by two orders of magnitude and its uptake and use increased enormously (Brock & Price, 1980). By the early twentieth century, the use of squared and graph paper had spread to education, as the Subcommission reported:

> Graphic methods of one form or another are now found in the courses in mathematics… in all countries, having gradually made their way from engineering, through thermodynamics and general physics, to pure mathematics. (Smith, 1913: p. 622)

The report indicated that such uptake, while only recent, was widespread; indeed the report suggested that so enthusiastic was adoption of this new technology that it was in danger of overuse:

> Of the value of squared millimetre paper there is no question anywhere, but it seems equally true that its use has been abused by the over-extensive treatment of equations and by its application to proving the obvious. (Smith, 1913: p.)

Thus, although linked particularly to the treatment of equations and functions, the use of graphical methods extended to many other topics:

> In England about 90% of the schools state that the graphical study of statistics is given… The graphical representation of functions is taught in all… secondary schools. The work is concerned with the plotting of equations and with the approximation of the roots… The use of vectors is found in a large majority of the schools, in connection with mechanics (velocities, acceleration, forces), this latter subject being part of the mathematical course in England… Graphical statics is taught generally. Areas are estimated by squared paper in most schools, but the planimeter is rarely used. (Smith, 1913: p. 622)

Again, in his contribution to the earlier Congress, Godfrey had illuminated a pedagogical motivation for the adoption of such methods, in terms of giving greater emphasis to mathematical rationales as well as rules, and of recruiting the interest of students:

Many teachers and examiners held that the teaching had laid too great stress on manipulative skill, at the expense of intelligent study of the why and wherefore… There was a kind of rebellion against this practice, and teachers sought to lighten the "toil" by introducing graphs, logarithms tables and other interesting matters at a comparatively early stage. All this had a decidedly stimulating effect. It is a revelation to a boy to learn that a function of a variable may be associated with a curve; that he can solve equations, extract roots, etc. by graphical methods. (Godfrey, 1909: pp. 457-8)

Laboratory work

The ICMI Subcommission reported widespread interest in activities such as practical surveying as means of bringing measurement and estimation to life; an example was:

In the Prussian schools the theodolite is usually found, and… along with it are seen simple instruments for angle measure, angle mirrors and prisms, measuring rods, and the like. Simple instruments are often made by the pupils, particularly instruments for the measure of angles. Much is made of out-of-door work in the classes in geometry and trigonometry, in the measuring of heights and distances [by] the pupils making use of the instruments. (Smith, 1913: p. 617)

While the report only refers in passing to laboratory work, Godfrey's account of recent trends in England, delivered at the preceding Congress, suggests that practical measurement, estimation and computation were closely linked to it:

Many schools now arrange that boys of 13-15 shall take, as part of their mathematics, a course of experimental work in the laboratory. During this course they are taught to measure and weigh (incidentally learning to realize the advantages of the decimal notation), to determine the surfaces and volumes of actual objects, to determine densities and specific gravities, to discover the simpler laws of hydrostatics, etc. (Godfrey, 1909: p. 453)

Such laboratory work formed part of a wider movement to develop more 'heuristic' approaches across mathematics and the sciences as a whole:

The development of 'practical' or 'experimental' mathematics as an approach to teaching the subject involving the use of apparatus of various kinds… and perhaps a special room –a mathematical laboratory…– has obvious parallels with the somewhat earlier emergence of practical science teaching.… The growth of both physics and chemistry teaching, as well as demanding the use of various materials and apparatus, was accompanied by the rise of a new approach to teaching, 'heurism', or the method of discovery. (Brock & Price, 1980: p. 370)

In the United States, the most prominent advocate of a 'laboratory method' was Moore, a leading mathematician of the time (Roberts, 2001). Writing in the American *School Review*, Moore's Chicago colleague, Myers elaborates an educational rationale for the mathematical laboratory, which brings out the link to the wider heuristic movement:

Fundamentally, laboratory method means work… on the pupil's part or, better still, method of getting work done *by* the pupil on his own initiative, under the impulse of his natural interests, *and largely under the guidance of his own intelligence.* (Myers, 1903: pp. 730-731),

As with Godfrey, the ideas of illuminating the mathematical rationale of problems and recruiting pupils' interest to them were prominent in Myers' argument:

An advantage that can hardly be overestimated of the laboratory procedure with mathematical classes is that pupils *sense* the difficulties to be overcome *as real* and *natural,* actually needing to be resolved and demanding a knowledge of the mathematical tool as a means of their resolution. In short, it recognizes the educational importance of letting the student know both ***how*** and ***why*** he must use the mathematical tool to get on well in any line of study. (Myers, 1903: p. 732)

Myers also appealed to the ideas of linking school mathematics both to practical needs beyond the school, and to the development of particular intellectual 'faculties':

Laboratory work with real problems, in the formulation and handling of which the pupil habituates himself to the transition from the concrete to the abstract, trains the faculties of analysis and abstraction, teaches him to make his own mathematical problems, to grow his own mathematics, and goes far toward supplementing the too isolated and too abstract teaching of secondary mathematics of today. (Myers, 1903: pp. 735-736)

His listing of "a fairly complete equipment for a mathematical laboratory" was extensive, and indicates the range of applications envisaged, including:

Set of drawing instruments, drawing board, T-square, and 30^{o}- and 45^{o}-triangles for each pupil… Carpenters' tapes, surveyors' tapes, and architects' scales… Three-, five-, and seven-place logarithmic tables… Logarithmic slide rules and computing machines… A surveyor's compass, a transit, and level, and leveling rods and flagpoles… A surveyor's plane table and a sextant… Weighing apparatus, as steelyards, balances, etc.; pendulums, barometers, and thermometers… Force appliances, such as cords, pulleys, etc., and the simple machines… Spherical blackboards, both concave and convex… Three plane blackboards for projective and descriptive work in geometry… Gyroscope taps… (Myers, 1903: pp. 737-738)

Differential adoption

The adoption of such technologies was, of course, affected by considerations of financial and educational cost, as the Subcommission's report notes in relation to slide rules:

> The slide rule has not yet found general acceptance in the secondary schools. The reason for this is thought to be the expense of the instrument, the cheaper ones not being accurate enough to be of value; but the question of time to acquire the necessary facility is also a serious one. In the upper classes the numerical computation is performed almost exclusively by the aid of logarithms. (Smith, 1913: p. 624)

A degree of scepticism about the future of laboratory work (perhaps not wholly innocent of some rivalry between Smith's Teachers College and Moore's University of Chicago) is evident where the report refers to "an extreme laboratory method with a minimum of mathematics" (Smith, 1913: p. 614) as part of the spectrum of innovation under investigation in the United States. However, the laboratory method clearly did pose many challenges of implementation. Writing soon after, in the British *Mathematical Gazette*, Fawdry (1915: p. 36) noted that the form of 'practical mathematics' actually taking root in schools was much more modest than envisaged by its advocates, focusing on topics such as "numerical evaluation of algebraic expressions, accurate construction of geometrical problems, plotting of curves, graphical solutions, [and] use of logarithms in computation", which could "be conducted in a class-room without the use of further apparatus than a box of instruments, some squared paper, and a table of logarithms". In his appraisal of the failure of the laboratory movement in the United States, Roberts points to the impact of a range of broader social factors on educational developments:

> With regard to the school environment, Moore, like many other educators, largely failed to foresee the consequences of changing demographics. In the face of the surge of students into the schools, calls for educational efficiency that had emerged during the last half of the nineteenth century became much more insistent and attractive. The efficiency advocates claimed to offer means to control the flood of students by carefully circumscribing requirements in terms of time and effort. In contrast Moore's "mathematical laboratory", which called for such extravagances as performing all demonstrations in two different ways and for blurring of subject-matter boundaries, could well be seen as a prescription for waste and confusion. Moreover, at the very time that Moore was proposing to justify mathematics education primarily as an aid to science and engineering, the population of high school students was exploding with students, most of whom were not aiming to become scientists or engineers. (Roberts, 2001: p. 694)

It seems that graphical methods, particularly in algebra, were the most conspicuous success of these reform movements in terms of pervasiveness and permanence:

> The use of graphical methods in elementary algebra teachings is universal and entirely a 20th century development. Other aspects of the same movement are the adoption of descriptive geometry by the mathematicians, the use of handy 4-figure tables, and of graphical methods in statics, and, though, in these cases, the victory is less complete than that of the "graph," it is remarkable and equally modern. (Godfrey, 1913: p. 641)

Brock & Price argue that the adoption within school mathematics of the technology of squared (including graph) paper reflected the influence of new educational philosophies and wider pressure to strengthen scientific and technical education. But as well as these drivers external to mathematics itself, the internal motor of Klein's advocacy of 'functional thinking' should not be underestimated. Such functional thinking was portrayed as a core mathematical process, integrating pure and applied mathematics, fusing arithmetic with geometry (Klein, 1908). Even if the Subcommission's report expresses some scepticism as to whether pupils were learning to think functionally through their use of squared paper, this does demonstrate the significance of the association:

> Graphic methods of representing function shave become universal in the last generation. From the idea of a line representing an equation the tendency is at present to that of graphic representation of a function. Just how much the pupil is acquiring the function concept seems often to be questioned, and the whole subject is in the experimental stage at present. (Smith, 1913: p. 614).

Why was it, then, that squared paper prospered whereas many other technologies associated with this reform movement faded away? I suggest that its success depended on the conjunction of the following features:

- Disciplinary congruence with an influential contemporary trend in mathematics.
- External currency in wider mathematical practice within and beyond school.
- Adoptive ease of incorporation in classroom practice and curricular activity.
- Pedagogical benefits across a range of topics considerably outweighing concerns.

Continuing preoccupation

The parallels of this historical episode to today's situation are intriguing. The role of new technologies in mathematics education remains distinctive in having been the focus of two ICMI Studies within the last 25 years. Much of the educational advocacy of new technologies continues to be associated with reforming aspirations to engage students in more authentic mathematical activity and enquiry, involving greater emphasis on intuition and experiment. Consequently, we live at a time when there is an ever richer diversity of materials and tools available for use in the mathematics classroom. However, the main issue for curriculum developer and classroom teacher alike remains one of developing coherent use of a relatively small selection of them to form an effective resource system. This depends, in turn, on the more fundamental issue of coordinating working environment, resource system, activity format and curriculum script to underpin classroom practice which is viable within the time economy (Ruthven, forthcoming).

What will our successors discern as the eventual influence of computer-based technologies on mathematics education in the early twenty-first century? The evidence gathered in 2003 by the most recent TIMSS study (Mullis et al., 2004) provides a simple indicator of the degree to which use of such technologies has become a regular part of mainstream practice in today's schools – operationalised as use by a class in about half of the lessons or more. Focusing on the 24 educational systems for which information is available at both Grade 4 and Grade 8 levels provides a good basis for comparison between primary and secondary levels of schooling, as well as between calculator and computer technologies (Ruthven, 2007). Across these educational systems, regular calculator use is extremely rare at primary level – reported in only 8% of classes in the system at the upper quartile of the distribution– and more variable at secondary level – reported in 12% of classes in the system at the lower quartile against 67% of classes in the system at the upper quartile. Regular computer use in lessons is rare at both levels, reported in 7% of classes in the system at the upper quartile of the primary distribution, and in only 3% of classes in the system at the upper quartile of the secondary distribution. An important contributory factor is that computers may simply not be available. Yet, focusing on those systems where half or more of classes were reported as having access to computers for lessons, the incidence of regular use is still low: in the systems at the upper quartile of the distributions, only 10% at primary level and 4% at secondary.

Such evidence suggests that computer-based technologies in contemporary mathematics education are, in the large, following the more general educational trend towards limited uptake and influence of new information and communication technologies over the last century. Reviewing the educational reception of wave upon wave of such technologies, Cuban (1986, 2001) suggests that a recurrent pattern of response can be found; a cycle in which initial exhilaration, then scientific credibility, give way to practical disappointment, and consequent recrimination. He reports that while new technologies have broadened teachers' instructional repertoires to a degree, they remain relatively marginal to classroom practice, and are rarely used for more than a fraction of the school week. For scholars of school reform, this reception of new technologies forms part of a much wider pattern of largely unsuccessful attempts to change the structures of curriculum, pedagogy and assessment at the heart of schooling.

In the historical precedent on which this paper has focused, squared paper emerged as one conspicuous success, bucking this generally disappointing trend. Likewise, in our era, the TIMSS findings identify the use of calculators as a relative success in terms of uptake, though a more qualified one – prevalent only at secondary level, and only in some educational systems. In terms of the factors suggested as contributing to the success of squared paper, the situation of calculators is correspondingly less clear-cut. While, with the development of computational technologies, numerical methods have become more significant within mathematics, they hardly represent a central disciplinary idea on the lines of Klein's 'functional thinking', although in the earlier days of computer-based technologies, 'algorithmic thinking' was proposed as the modern equivalent. Nevertheless, computation by machine clearly has an established external currency in wider mathematical practice within and beyond school, and use of a calculator – designed as a cheap, portable, personal technology – represents the most ready educational realisation of this, due to its corresponding adoptive ease of incorporation in classroom practice and curricular activity (Ruthven, 1996). However, at primary level, where the curriculum has traditionally been organised around highly valorised methods of written and mental calculation, the pedagogical benefits of calculator use remain controversial, and appropriate forms of curricular (re)organisation underdeveloped (Ruthven, 1999). At secondary level, a relatively immediate and wide-ranging pedagogical benefit of calculator use is more readily apparent, simply as a pragmatic means of giving pupils access to efficient and effective means of computation to support work on other mathematical topics.

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