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# Geometry and motion links mathematics and engineering in collections of 19th century kinematic models and their digital representation

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> Research in learning with technological tools has been showing much promise with respect to their use in the generation of learning environments where students have richer opportunities to construct mathematical meanings, to explore and experiment with mathematical ideas and to express these using a wealth of representations.
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> – from Discussion Document of W4

I would like to focus particularly on the use of some digital tools based on actual physical models for exploring geometrical ideas and to give my perspective on two of the questions raised in discussion document:

- What kind of change does the Internet bring to mathematics education?
- What is the relation between human thinking and the tools that have been developed throughout our history?

Let us narrow this last question just about geometrical thinking – how different technologies historically have affected geometrical thinking?

I think that the main aspects of geometry today emerged from four strands of early human activity that seem to have occurred in most cultures: art/pattern, building structures, navigation/stargazing, and motion, machines. These strands developed more or less independently into varying studies and practices that eventually from the 19th century on were woven into what we now call *geometry*. [4] Early human societies used the wheel for transportation, for making pottery, in pulleys, and in pumps. In ancient Greece, Archimedes, Heron, and other geometers used linkages and gears to solve geometrical problems — such as trisecting an angle, duplicating a cube, and squaring a circle. These solutions were not accepted by the followers of Plato's tradition, and this leads to a common misconception that these problems are unsolvable and/or that Greeks did not allow motion in geometry. See, for example, [8] and [5]. It was mentioned in discussion document that: "Euclid's Elements may be considered a modeling theory of geometrical drawing by means of ruler and compasses." It is an interesting paradox here: to do Euclidean constructions one has to use technology - straightedge and compass – and obviously there will be motion involved in this use.

It seems like the motion was first explored in connection with astronomy – geometry of the heavens – planetary motion was translated into geometrical terms while on the plane mostly there were discussed Euclid's constructions. About 365 B.C., Eudoxus visited Egypt where he acquired from the priests of Heliopolis a knowledge of planetary motions and Chaldean astrology, later he completed his book *On Speeds* on motions within our solar system (perhaps his greatest but lost writing). Eudoxus became the first mathematician seriously to attempt to describe the intricate motions of celestial bodies using a mathematical model based on spherical geometry. [2] Geometry and motion came close together for ancient engineers. Problems of trisecting an angle, squaring the circle, and doubling the cube can be easy solved if one involves motion. One of the first mechanical solutions was offered by Menaechmus – he constructed a mechanical device that would trace two appropriate conic curves. As mentioned before, Plato criticized this mechanistic approach and called instead for a purely theoretical solution.

The Romans and Greeks made wide use of gearing in clocks and astronomical devices. Gears were also used to measure distance or speed. One of the most interesting relics from ancient Greece is the "Antikythera machine" which is an astronomical computer. It had many gears in it – some of which were planetary. Problems that originated from the construction of appropriate gears had to be solved geometrically and involved the understanding of motion.

One of the most significant turning points in the development of technology was learning how to transform continuous circular motion into reciprocal or straight-line motion. Rotary motion was available to humans using nature forces – waterwheels, windmills. But this kind of motion was not enough – for example, to saw logs into boards rectilinear motion was needed. This was achieved by the use of gears and linkages. Both later became important subjects of mathematical interest.  To construct the most efficient shape of gear teeth, geometers were studying cycloids (Nicolas of Cusa, 1451, Galileo 1599) and epicycloids (Albrecht Durer, 1525). In 1557 Girolamo Cardano first published a mathematical theory of gears. In 1694 Philip de la Hire published a full mathematical analysis of epicycloids and recommended involute curve for designing gear teeth, but in practice it was not used for another 150 years. In 1733 Charles Camus expanded la Hire's work and developed theories of mechanisms. In 1754 Leonard Euler worked out design principles for involute gearing.

Another interesting application of linkages was pantographs. The idea of a pantograph is based on geometrical proportions that were known since ancient times. It can be called the earliest copying machine, making exact enlarged or reduced copies of written documents. Artists adopted the pantograph for duplicating drawings. It is known that Leonardo da Vinci was using a pantograph to enlarge his sketches and possibly to duplicate them onto canvas. Later pantographs were adapted for duplicating paintings – first the pantograph would be used to trace the outlines and then the shapes would be filled with the paint. Sculptors and carvers adapted the pantograph for tracing master drawings onto blocks of marble or wood. In 18$^{th}$ century the pantograph was used to cut out the typeset letters for printing and engravings. In 19$^{th}$ century pantographs were advanced to even duplicate sculptures. One of the first duplicated sculptures was Michelangelo's sculpture David. Heavy-duty pantographs are still used for engraving and contour milling.

Leonardo da Vinci had ideas about several mechanisms that would trace various mathematical curves. Mechanical devices for drawing curves were used also by Albrecht Dűrer (1471-1528). When French mathematician Rene Descartes (1596-1650) published his *Geometry* (1637) he did not create a curve by plotting points from an equation (as students and computers do now). Descartes always first gave geometrical methods for drawing each curve with some apparatus, and often these apparatus were linkages. This tradition of seeing curves as the result of geometrical actions can be found also in works of Roberval (1602-1675), Pascal (1623-1662) and Leibniz (1646-1716).

A philosophical approach to the description of motion mathematically was developed in the 13$^{th}$ century in the so called Merton School by Thomass Bradwardine (ca.1290-1349) and others, later in 14$^{th}$ century motion was discussed in the writings of Jean Buridan (ca.1300- ca.1358) and Nicole Oresme (ca.1320-1382). Oresme represented motions geometrically by plotting primitive graphs. [2] Many authors talk about motion entering mathematics in the end of the 16$^{th}$ century or 17$^{th}$ century with Tycho Brahe or Galileo's works (see, for example [1],  [5]). It is a widely held opinion that calculus started in 17$^{th}$ century when mathematicians started to investigate motion.

Was there really no motion earlier in mathematics? Was motion in mathematics only connected with calculus?

As we can see already from the previous examples, designing devices that would create some particular type of motion was a geometrical problem already in ancient geometry. I will show by examples of historical kinematic model collections, how we can use motion to elucidate many geometric ideas.

In November 2003, the American Society of Mechanical Engineers (ASME) designated the Cornell Reuleaux Kinematic Model Collection as a (United States) National Historic Landmark Collection. It took another year until the official designation ceremony was possible in the newly opened Duffield Hall that now is the permanent display place for portions of this collection. What is this collection and how it is connected with mathematics?

A distinguished 19$^{th}$ century professor of mechanical engineering, Franz Reuleaux (1829-1905), believed in the use of demonstration models to express mathematical and kinematic ideas. For that purpose he built a large collection of 800 mechanism models in Berlin and marketed 350 of them to universities around the world. Unfortunately much of these collections were destroyed during World War II, but some originals and reproductions of these models can be found in the Deutsche Museum in Munich, the University of Hanover, Kyoto University, and Moscow Bauman Technical School, Karlov University in Prague, and possibly in some other places we do not know yet. But the largest collection of these models is in Cornell University where there are 220 (from the originally acquired 266) Reuleaux models. Since 2002 we are working on developing "Kinematic Models for Design Digital Library" and you can explore these models on our website [7]. There have been added to these Clark models from Museum of Science in Boston, Redtenbacher models from Karlsrue, and models from Illinois Gear. On the website. besides having still images of models, there is historical information, and interactive movies that allow exploring how these models work. The website also has scanned rare books that are important in the history of technology. A significant part of this project is connecting models and mathematical ideas behind those models for the purpose of using them in the classroom. We have developed teaching materials that can be found in the section website called "Tutorials". In this section one can find also a list of biographies for names of people appearing in the historical descriptions of the models and authors of rare books.

F. Reuleaux believed that there were scientific principles behind the invention and the creation of new machines – what we call "synthesis" today. This belief in the primacy of scientific principles in the theory and design of machines became the hallmark of his worldwide reputation, especially in the subject of machine kinematics [9]. He also devoted serious attention to education and the role of mathematics:

> …The forces of nature which advance taught us to look to for service are mechanical, physical and chemical; but the prerequisite to their utilization was a full equipment of mathematics and natural sciences. This entire apparatus we now apply, so to say, as a privilege.
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> …The instruction in the polytechnic school has of necessity to adopt as fundamental principles the three natural sciences – mechanics, physics, and chemistry, and the all-measuring master art of mathematics. [12]

Franz Reuleaux incorporated mathematics into design and invention of machines in his work *Kinematics of Machinery* [11]. For mathematicians he is best known for the "Reuleaux triangle", which is one of the curves of constant width. This curved triangle can be seen in some gothic windows, it also appears in some drawings of Leonardo da Vinci and Leonard Euler, but Reuleaux in his *Kinematics* gave the first applications and complete analysis of such triangles, and he also noticed that similar constant-width curves could be generated from any regular polygon with an odd number of sides. We have developed a tutorial about the Reuleaux triangle that can be used in school mathematics classes [17].

Many of the Reuleaux models show change in motion: circular to trigonometric (slider crank), circular to elliptical (double slider crank), circular to straight-line motion (straight-line mechanisms) – analyses of some of these mechanisms involve calculus and inversive geometry (Peaucellier-Lipkin linkage, [14]). Gear mechanisms in the collection use properties of epicycloids and hypocycloids.

Several of Reuleaux models were included in Walter Dyck's *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente* [3]. Walter von Dyck (1856-1934) was one of the creators of the Deutsches Museum of Natural Science and Technology, and he was also appointed as the second Director of the Museum in 1906. The Deutsches Museum was first of its kind and its ideas were soon copied by other science museums around the world. For his catalogue Dyck chose from the Reuleaux models three that demonstrate properties of a cycloid on a sphere. Essentially in that time models served the same purpose as computer animation today. Some of Reuleaux mechanisms demonstrate the effective use of a Möbius band. Another mathematical application, which appears in the Reuleaux model collection, is the use of involutes of circle and the combining of other geometrical figures into the design of pumps [13]. The universal joint used to be a standard mechanism in the drive shaft of all automobiles. A mathematical explanation of the change of the motion in the universal joint can be explained easily using spherical geometry. For more details, see [18, 19].

Reuleaux classified his mechanisms using an alphabet, that is, assigning letters to different groups of his mechanisms. That way he was stressing that each individual mechanism is like a letter in an alphabet and combining them together we are getting words and sentences which denote machines. His style of classification resembles later classification ideas used in topology and theoretical computer science. The largest number (39) of Reuleaux mechanisms is in the so-called S-series – straight line mechanisms. It had been a challenge in technology since ancient times – how to change circular motion into straight-line motion [6]. This problem was crucial for James Watt (1736-1819) when he was working on improving the steam engine. In [16] we describe a short history of this problem and its solution in the 19th century. The mathematical aspects of the Peaucellier-Lipkin linkage involve inversive geometry that is described also in our tutorials [14]. There is renewed interest in the design of different linkages: For example: in mathematics, linkages are discussed in rigidity theory; and, in engineering, linkages are widely used in robotics.

We hope that material from our digital library can be used in schools not only in science or technology courses but also as examples of using mathematical ideas in machine motion design. We have added to the KMODDL library stereo lithographic files that allow 3-d printing of some of the models.

Cornell faculty in mechanical engineering, mathematics, and architecture are using the KMODDL website in the classroom to teach mathematical principles of mechanisms as well as machine design and drawing. Mathematical ideas from this collection have found their place in a geometry textbook [4]. Over the past four years, students and teachers from area schools have visited the collection. We have had visitors to the website seeking ideas for microelectromechanical machines (MEMS), robotic machines, space satellite applications, and biomechanical prostheses. The collection also has attracted scholars from Japan, Italy, Germany and Australia in one year alone. See [10].

This Digital library of Kinematic Models is one of the examples of how the Internet can enrich mathematical education. It is allowing this originally constructed classroom set of models to become widely available – anywhere where there is an access to the Internet one can play with these models interactively. Besides the direct use in the classroom this digital library allows further exploration on your own. It is an interesting example of historic models transformed into digital tools and this way opening new possibilities in geometric design of motion. It is also a rich resource to show our students interdisciplinary connections and practical applications of pure geometric ideas.

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