Mathematical Knowledge as a Social Construct of Teaching- / Learning Processes –The Epistemology Oriented Mathematical Interaction Research

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1) Mathematics teaching as an autonomous culture

In the course of about the last 30 years, approaches examining the interactive interplay between mathematical knowledge, the teacher and the learning students as a whole have been established more and more. Everyday mathematical instruction and learning processes are taken serious and being analysed under a social interactionist perspective in this kind of research. (e.g. Bauersfeld 1978; 1988; Cobb & Bauersfeld 1995; Krummheuer 1984; 1998; Maier & Voigt 1991; Voigt 1994; Steinbring 2005).

Instruction research about mathematics learning has taken, among others, an ethnographic view towards the research object ›mathematics instruction‹ and has interpreted instruction as a special *culture*. (Cobb & Yackel 1998; Nickson 1994). »Participating in the process of a mathematics classroom is participating in a culture of mathematizing. The many skills, which an observer can identify and will take as the main performance of the culture, form the procedural surface only. These are the bricks of the building, but the design of the house of mathematizing is processed on another level. As it is with culture, the core of what is learned through participation is *when* to do *what* and *how* to do it. … The core part of school mathematics enculturation comes into effect on the meta-level and is ‚learned' indirectly« (Bauersfeld, cited according to Cobb 1994, p. 14).

Essential features constituting a culture are commonly accepted signs and symbols, which constitute an identity of the respective culture. Wilder characterises the meaning of the (specific) symbols for the mathematical culture as follows: »A culture is the collection of customs, rituals, beliefs, tools, mores, etc., which we may call cultural elements, possessed by a group of people …« (Wilder 1986, S. 187). And further: »Without a symbolic apparatus to convey our ideas to one another, and to pass on our results to future generations, there wouldn't be any such thing as mathematics – indeed, there would be essentially no culture at all, since, with the possible exception of a few simple tools, culture is based on the use of symbols. A good case can be made for the thesis that man is to be distinguished from other animals by the way in which he uses symbols….« (Wilder 1986, p. 193). The meaning of the culture concept for scientific mathematics and for school mathematics has been emphasised by different authors. (Wilder 1981; Bishop 1988).

2) Epistemological constraints of mathematical signs in the culture of teaching

Every mathematical knowledge requires *certain sign or symbol systems* in order to gather and code the knowledge. These signs do not initially have a meaning by themselves, but it has to be interactively produced by the students. In order for mathematical sign systems to obtain meaning, they require, generally speaking, appropriate *reference contexts*. Meanings for mathematical concepts are actively constructed by the epistemic agent (e.g. the student or the teacher) as interactions between sign / symbol systems and reference contexts / object domains (Steinbring 1993).

The specific epistemological role of mathematical signs / symbols, which affects interactive construction processes of mathematical knowledge, shall be characterised at the example of the number concept. One can first distinguish two essential functions for mathematical signs / symbols: »(1) A semiotic function: the role of mathematical signs as ›something which stands for something else‹. (2) An epistemological function: the role of the mathematical sign in the context of the epistemological interpretation of mathematical knowledge.« (Steinbring 2005, p. 21)

A comparison between linguistic and mathematical signs reveals the following concerning the first function. The linguistic sign or word ›school‹ first stands for a concrete school – maybe the school, which the students attend. But with ›school‹, one can also designate a big number of different concrete schools – of the same or of a different type. This relation between the word ›school‹ and many concrete schools also covers the ideal construct of the general concept ›school‹ as a place of institutionalised teaching and learning scientific knowledge – and a concrete school is the realisation of this abstract idea. Furthermore the sign ›school‹ can be written in different forms (cursive, block letters, etc.) or languages (école, Schule, scuola, etc.) without there being a change in the illustrated relation between the linguistic sign and the concrete referents or in the abstract idea.

The mathematical sign ›4‹ stands for the conceptual number ›4‹, and that ultimately is an abstract conceptual idea from the beginning. In order to facilitate and to activate child-accordant mathematical learning and understanding processes, there is a multitude of didactical situations and materials to which the sign ›4‹ could relate.

One example for such a referential relation between the sign ›4‹ and an object, which this signs stands for, could be the use of little coloured chips: 

Insofar, the sign ›4‹ relates to the four chips, but does not designate these as the actual objects (as for example the word ›school‹ designates the concrete school of a student), but ultimately ›stands for something else‹ which is meant by the four coloured chips, namely always the abstract concept of the number ›4‹. Comparable to the different writings of the word ›school‹, the mathematical sign ›4‹ can be written in other ways and languages: ›4‹, ›IV‹, ›100‹ (in the binary system), etc. or ›vier‹, ›quatre‹, ›quattro‹, etc. on the one hand, this difference to linguistic signs – namely that mathematical signs/symbols ultimately always relate to a universal mathematical conceptual idea and not to ›concrete mathematical numbers‹ (for example different materials) – illustrates the special epistemological character of mathematical signs.

The mediation between signs and structured reference contexts requires a conceptual mediation (Steinbring 2005, p. 22). The epistemological triangle (for an extensive description see Steinbring 2005, 2006) is a theoretical instrument used to analyse the coherence of – yet unfamiliar – mathematical signs / symbols, of – partly familiar –

reference contexts for the signs / symbols and of fundamental mathematical concept principles, with regulate the intermediation between signs and reference contexts.

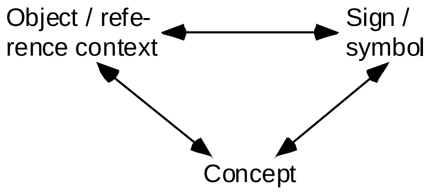

Fig. 1 The Epistemological Triangle

This epistemological triangle should be understood as a theoretical schema, in which the corners *reciprocally* ›define‹ each other, thus none of the three elements can be explicitly or unequivocally given in order to then deductively determine the other elements. A fundamental conceptual idea is necessary in order to regulate the mediation between sign and reference context and in the further development of mathematical knowledge the fundamental conceptual knowledge is enhanced and differentiated (see Steinbring 2005).

In contrast to an empirical understanding of numbers as representing concrete objects or as names of sets, such a conception is fundamentally criticized from philosophical and epistemological perspectives. »I … argue, … that numbers could not be objects at all; for there is no reason to identify any individual number with any one particular object than with any other (not already known to be a number)« (Benacerraf 1984, pp. 290/1).

But if numbers are not objects, what else are they? »To be the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5, …… Any object can play the role of 3; that is any object can be the third element in some progression. What is peculiar to 3 is that it defines that role - not being a paradigm of any object which plays it, but by representing the relation that any third member of a progression bears to the rest of the progression« (Benacerraf 1984, p. 291).

Mathematical knowledge does not relate directly to concrete or real objects. Mathematical concepts and mathematical knowledge, coded in signs and symbols, represent *abstract relations*, *structures* and *patterns*.

3) The interactive constructions of mathematical knowledge – social and epistemological conditions

The interpretation that mathematical knowledge as a *theoretical* – and not empirically fixed – knowledge *develops* and changes in such development processes regarding its epistemological status, by means of becoming more abstract, general and universal, makes it possible in social processes of teaching and learning, in comparison to the historical development of mathematics, which was bound into social and cultural contexts, to understand the interactive generation of mathematical knowledge as a relatively independent procedure within the frame of the teaching culture.

The interaction-research approach of the *social epistemology of mathematical knowledge* (Steinbring 2005) understands itself as an important, independent complete model inasmuch as the particularity of the social existence of mathematical knowledge is an essential component of this theoretical approach of interaction analysis. In this theoretical conception of the social epistemology of mathematical knowledge, the epistemological particularity of the subject matter ›mathematical knowledge‹ dealt with in the interaction constitutes a basis for its theoretical examination. In this theoretical investigation mathematical knowledge is seen from a different perspective: the subject matter of ›mathematics‹ is, according to the considerations in the previous considerations, not understood as a pre-given, finished product, but interpreted according to the epistemological conditions of its dynamic, interactive development.

Every qualitative analysis of mathematical communication always has to start – explicitly or implicitly – from assumptions about the status of mathematical knowledge. There are different ways of coping with this requirement. Epistemology-based interaction research in mathematics education proceeds on the assumption that a specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities and the manner of how to analyze and interpret mathematical communication. This assumption includes the following view of mathematics: Mathematical knowledge is not conceived as a ready made product, characterized by correct notations, clear cut definitions and proven theorems. If mathematical knowledge in learning processes could be reduced to this description, the interpretation of mathematical communication would become a direct and simple concern. When observing and analyzing mathematical interaction one would only have to diagnose whether a participant in the discussion has used the ›correct‹ mathematical word, whether he or she has applied a learned rule in the appropriate way, and then has gained the correct result of calculation, etc.

Mathematical concepts are *constructed* in interaction processes as symbolic relational structures and are coded by means of *signs and symbols* that can be combined logically in mathematical operations. This interpretation of mathematical knowledge as »symbolic relational structures that can be consistently combined« represents an assumption which does not require a fixed, pre-given description for the mathematical knowledge (the symbolic relations have to be actively constructed and controlled by the subject in interactions). Further, certain epistemological characteristics of this knowledge are required and explicitly used in the analysis process; i.e. mathematical knowledge is characterized in a consistent way as a structure of relations between (new) symbols and reference contexts.

The intended construction of meaning for the unfamiliar, new mathematical signs, by trying to build up reasonable relations between signs and possible contexts of reference and of interpretation, is a fundamental feature of an epistemological perspective on mathematical classroom interaction. This intended process of constructing meaning for mathematical signs is an essential element of every mathematical activity whether this construction process is performed by the mathematician in a very advanced research problem, or whether it is undertaken by a young child when trying to understand elementary arithmetical symbols with the help of the position table. The focus on this construction process allows for viewing mathematics teaching and learning at different school levels as an authentic mathematical endeavour.

(Remark: Essential parts of this paper are based on Steinbring, 2005).

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