EXERCISES ON HOLOMORPHIC POISSON BRACKETS
POISSON 2016

1. Poisson surfaces

1.1. Given any constant $\lambda \in \mathbb{C}$, define a Poisson bracket on $\mathbb{C}^2$ by the formula

$$\{x, y\}_\lambda = \lambda\, xy,$$

where x, y are the standard coordinates on $\mathbb{C}^2$. Show that the brackets $\{\cdot,\cdot\}_\lambda$ and $\{\cdot,\cdot\}_{\lambda'}$ are isomorphic if and only if $\lambda = \pm\lambda'$. Conclude that the isomorphism class of a Poisson structure on $\mathbb{C}^2$ depends on more information than just the divisor on which it vanishes.

1.2. Amongst all the possible singularities of a curve in $\mathbb{C}^2$, there are three special classes called A, D and E—the simple singularities [1]. They are the zero sets of the polynomials in the following table:

A_k, $k \geq 1$	D_k, $k \geq 4$	E_6	E_7	E_8
$x^2 + y^{k+1}$	$x^2 y + y^{k-1}$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$

Let f be one of these polynomials, and define a Poisson structure π on $\mathbb{C}^2$ by

$$\pi = f \partial_x \wedge \partial_y.$$

Let $\widetilde{\pi}$ be the Poisson structure obtained by blowing up π at the origin in $\mathbb{C}^2$. Describe the divisor $\mathsf{D} \subset \widetilde{\mathbb{C}}^2$ on which $\widetilde{\pi}$ vanishes.

1.3. Let π be a holomorphic Poisson structure on $\mathbb{C} \times \mathbb{P}^1$, and let $\mathsf{D} \subset \mathbb{C} \times \mathbb{P}^1$ be its divisor of zeros. Describe the possible intersections of D with the locus $\{0\} \times \mathbb{P}^1 \subset \mathbb{C} \times \mathbb{P}^1$.

1.4. Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a ruled surface, i.e. a holomorphic locally trivial fibration, where Y is a smooth compact curve and the fibres are isomorphic to the projective line $\mathbb{P}^1$. Let π be a Poisson structure on X, and let $\mathsf{D} \subset \mathsf{X}$ be its divisor of zeros.

(a) Using the results of the previous exercise, show that D has the form

$$\mathsf{D} = \widetilde{\mathsf{Y}} + \sum_{p \in \mathsf{Y}} m_p \mathsf{F}_p$$

where $\mathsf{F}_p = \phi^{-1}(p) \subset \mathsf{X}$ is the fibre over p, the multiplicities $m_p \in \mathbb{Z}_{\geq 0}$ are zero for all but finitely many $p \in \mathsf{Y}$, and the divisor $\widetilde{\mathsf{Y}}$ falls into one of the following two classes:

(i) $\widetilde{\mathsf{Y}} \subset \mathsf{X}$ is a reduced curve and the restriction

$$\phi|_{\widetilde{\mathsf{Y}}} : \widetilde{\mathsf{Y}} \to \mathsf{Y}$$

is generically two-to-one.

(ii) $\widetilde{\mathsf{Y}} = 2\mathsf{S}$, where $\mathsf{S} \subset \mathsf{X}$ is a section of the fibration.

(b) Show that case (i) above is impossible if the genus of Y is at least two. *Hint:* use adjunction, and the fact that a compact curve admits no nonconstant maps to curves of higher genus. If you wish, you may assume that $\widetilde{\mathsf{Y}}$ is smooth, but this hypothesis is not actually necessary.

2. Poisson threefolds

2.1. Construct a Poisson structure on $\mathbb{C}^3$ that has no nonconstant Casimir functions. *Hint:* use the vector fields $x\partial_x, y\partial_y$ and $z\partial_z$ in coordinates x, y, z.

2.2. Let $\mathsf{X} \subset \mathbb{P}^4$ be the smooth quadric threefold, given by

$$\mathsf{X} = \left\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \,\middle|\, x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \right\}.$$

Then the anticanonical divisors $\mathsf{D} \subset \mathsf{X}$ are precisely the divisors of the form

$$\mathsf{D} = \{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathsf{X} \,|\, F(x_0, \ldots, x_4) = 0\}$$

where F is a homogeneous cubic polynomial. Taking this fact as given, construct a Poisson structure on X using pencils. Describe the base locus of your pencil.

3. Poisson subspaces and degeneracy loci

3.1. Let w, x, y, z denote the standard coordinates on $\mathbb{C}^4$. Find equations for the degeneracy loci $\mathsf{D}_0(\pi)$ and $\mathsf{D}_2(\pi)$ of the following Poisson structures, and describe the degeneracy loci geometrically:

(a) The Poisson bivector

$$\pi = wx\,\partial_w \wedge \partial_x + uv\,\partial_u \wedge \partial_v$$

(b) The Poisson structure π with elementary brackets

$$\begin{aligned} \{w, x\} &= x & \qquad \{x, y\} &= z^2 + \lambda\, xy \\ \{w, y\} &= y & \qquad \{y, z\} &= x^2 + \lambda\, yz \\ \{w, z\} &= z & \qquad \{z, x\} &= y^2 + \lambda\, zx, \end{aligned}$$

where $\lambda \in \mathbb{C}$ is a constant.

(c) The bivector

$$\pi = Z \wedge \partial_w$$

where Z is a vector field satisfying $[Z, \partial_w] = 0$.

3.2. Let X be a complex manifold, and let Y be an analytic subspace, defined by an ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$. The ***reduced subspace*** $\mathsf{Y}_{\mathrm{red}} \subset \mathsf{Y} \subset \mathsf{X}$ is the unique analytic subspace of X that has the same underlying points as Y, but has no nilpotent elements in its algebra of functions. It is defined by the ***radical ideal***

$$\sqrt{\mathcal{I}} = \left\{ f \in \mathcal{O}_{\mathsf{X}} \,\middle|\, f^k \in \mathcal{I} \text{ for some } k \in \mathbb{Z}_{>0} \right\}$$

(a) Suppose that $Z \in \mathcal{T}_{\mathsf{X}}$ is a vector field on X such that Z is tangent to Y, in the sense that

$$Z(\mathcal{I}) \subset \mathcal{I}.$$

(Hence the flow of Z preserves the ideal defining Y.) Show that Z is also tangent to $\mathsf{Y}_{\mathrm{red}}$.

(b) Suppose that π is a Poisson structure on X, and Y is a Poisson subspace. Conclude that $\mathsf{Y}_{\mathrm{red}}$ is also a Poisson subspace.

(c) Equip $\mathsf{X} = \mathbb{C}^3$ with the linear Poisson bivector

$$\pi = x\,\partial_y \wedge \partial_z + y\,\partial_z \wedge \partial_x + z\,\partial_x \wedge \partial_y,$$

corresponding to the Lie algebra $\mathfrak{so}\,(3,\mathbb{C})$. Find an analytic subspace $\mathsf{Y} \subset \mathsf{X}$ such that $\mathsf{Y}_{\mathrm{red}}$ is a Poisson subspace, but Y is not. Thus the converse of (b) fails in general.

3.3. Let π be the linear Poisson structure associated with the Lie algebra $\mathfrak{sl}\,(3,\mathbb{C})$. Show that the origin is the only symplectic leaf of dimension ≤ 2.

References

[1] V. I. Arnol′d, *Normal forms of functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities*, Funkcional. Anal. i Priložen. **6** (1972), no. 4, 3–25. MR0356124 (50 #8595)