Integrable systems, symmetries, and quantization – Exercises

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Recommended Exercises

Exercise 1. Consider a Hamiltonian function $H \in C^\infty(\mathbb{R}^{2n})$, where $\mathbb{R}^{2n}$ is endowed with the canonical symplectic form. Check that H is conserved under the flow, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H(q(t), p(t)\right) = 0,$$

where $(q(t), p(t))$ denotes the flow of the Hamiltonian vector field $\mathcal{X}_H$.

Exercise 2. Suppose that $\dim \Sigma = 2$, that $F : (\Sigma, \omega) \to \mathbb{R}$ is a completely integrable system, and that any singular point is non-degenerate. If $p \in \Sigma$ is singular, prove that the Hessian of F at p is non-degenerate, *i.e.* that its determinant does not vanish. Deduce that F is a Morse function.

Exercise 3 (The mathematical pendulum). Consider the Hamiltonian function $H(x, \theta) = \frac{1}{2}x^2 + \cos\theta$ defined on $T^*S^1 \cong \mathbb{R} \times S^1$ with canonical symplectic form $\mathrm{d}\theta \wedge \mathrm{d}x$.

1. Compute the image of H.
2. Calculate the singular points and values of the system. Prove that all singular points are non-degenerate and determine their Williamson type. *[Hint: Use Exercise 2.]*
3. Describe geometrically the fibres $H^{-1}(t)$ for varying $t \in H(T^*S^1)$.

Exercise 4 (The classical spherical pendulum). Identify $T\mathbb{R}^3 \cong T^*\mathbb{R}^3$ using the standard Euclidean metric, so that $T\mathbb{R}^3$ inherits a symplectic form, which, in standard coordinates (x, y), is given by $\Omega = \sum_{i=1}^{3} \mathrm{d}y_i \wedge \mathrm{d}x_i$. Consider furthermore the unit sphere $S^2 \hookrightarrow \mathbb{R}^3$.

1. Show that $TS^2 \subset T\mathbb{R}^3$ is a symplectic submanifold, *i.e.* the restriction of Ω to TS^2 is non-degenerate. Denote the induced symplectic form by ω.
2. Check that ω can be constructed alternatively by identifying $TS^2 \cong T^*S^2$ using the standard round metric on S^2.

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3. Consider the Hamiltonian function on TS^2

$$H(x,y) = \frac{1}{2}\|y\|^2 + x_3.$$

Prove that the angular momentum $J(x,y) = x_1y_2 - x_2y_1$ Poisson commutes with H.

4. Prove that the Hamiltonian vector field $\mathcal{X}_J$ has a periodic flow. Can you interpret this geometrically?

5. Find the singular points and values of the map $(J,H) : (TS^2,\omega) \to \mathbb{R}^2$ and deduce that it defines a completely integrable system. Furthermore, if you have time, prove that all singular points are non-degenerate and determine their Williamson types (some are easier than others).

Exercise 5 (WKB ansatz). Consider a function on $\mathbb{R}^n$ of the form $\psi(q) = e^{\frac{iS(q)}{\hbar}} a(q)$. Show that ψ is an approximate eigenfunction of the Schrödinger operator $\hat{H}$, with approximate eigenvalue E, in the sense that

$$\hat{H}\psi = E\psi + \mathcal{O}(\hbar)$$

if and only if the real-valued function S satisfies the Hamilton-Jacobi equation

$$H(q, dS(q)) = E.$$

Here H is the classical Hamiltonian $H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$. Deduce that to such a "quantum state" ψ is naturally associated a Lagrangian submanifold of $\mathbb{R}^{2n}$.

Exercise 6 (The quantum spherical pendulum). Let (θ,φ) be the angles defining spherical coordinates on S^2 (θ being the polar angle associated with (x,y)). The quantum spherical pendulum is given by the two self-adjoint operators $\hat{J}_\hbar = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$ and $\hat{H}_\hbar = -\frac{\hbar^2}{2}\Delta + z$ acting on $L^2(S^2)$.

1. The Laplacian in spherical coordinates satisfies:

$$\Delta f = \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial f}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 f}{\partial\theta^2}$$

for every $f \in C^\infty(S^2)$. Prove formally that $\hat{J}_\hbar$ and $\hat{H}_\hbar$ commute.

The joint spectrum of $\hat{J}_\hbar$ and $\hat{H}_\hbar$ consists of the pairs (λ_1,λ_2) such that there exists a nonzero $v \in L^2(S^2)$ with $\hat{J}_\hbar v = \lambda_1 v$ and $\hat{H}_\hbar v = \lambda_2 v$. In order to compute it, we will use the fact that there exists an orthonormal basis $(Y_{\ell,m})_{\ell\geq 0, -\ell\leq m\leq \ell}$ of $L^2(S^2)$ such that

$$-\Delta Y_{\ell,m} = \ell(\ell+1)Y_{\ell,m}, \quad -i\frac{\partial Y_{\ell,m}}{\partial\theta} = mY_{\ell,m}.$$

2. Compute the eigenvalues of $\hat{J}_\hbar$.

More precisely, $Y_{\ell,m} = C_{\ell,m}P_\ell^m(\cos\varphi)\exp(im\theta)$ where

$$C_{\ell,m} = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}$$

and the P_ℓ^m are Legendre polynomials satisfying the recursion relation

$$(\ell-m+1)P_{\ell+1}^m = (2\ell+1)XP_\ell^m - (\ell+m)P_{\ell-1}^m.$$

3. Compute $\hat{H}_\hbar Y_{\ell,m}$.

4. Check on this formula that $\hat{H}_\hbar$ indeed preserves the eigenspaces of $\hat{J}_\hbar$.

5. Use a computer to obtain numerically the joint spectrum of $\hat{J}_\hbar$ and $\hat{H}_\hbar$ *via* the spectrum of a matrix obtained by looking at the action of the restriction of $H_\hbar$ to each eigenspace of $\hat{J}_\hbar$ on a finite number of $Y_{\ell,m}$.

Exercise 7. Given a local model of Williamson type (k, k_e, k_h, k_{ff}), prove that all its singular points are non-degenerate and that the origin is a singular point of Williamson type (k, k_e, k_h, k_{ff}).

Optional Exercises

Exercise 8. Check that the motion of a particle in $\mathbb{R}^3$ under the influence of a magnetic field $\mathbf{B}$ is Hamiltonian with function $H(q,p) = \frac{1}{2}\|p - A(q)\|^2$, where A is a solution to the equation $\operatorname{curl} A = \mathbf{B}$.

Exercise 9. A *Lagrangian fibration* is a smooth map $\phi : (M,\omega) \to B$ whose regular fibres are Lagrangian submanifolds.

1. Prove that a completely integrable system gives rise to a Lagrangian fibration.

2. Conversely show that, locally, any Lagrangian fibration gives rise to a completely integrable system.

3. Suppose that ϕ is a surjective submersion with connected fibres. Prove that there exists a unique Poisson structure $\{\cdot,\cdot\}_B$ on B satisfying, for any $f, g \in C^\infty(B)$,
$$\{\phi^* f, \phi^* g\} = \phi^*\{f,g\}_B,$$
where $\{\cdot,\cdot\}$ is the Poisson bracket on M induced by ω. Can you describe $\{\cdot,\cdot\}_B$ explicitly?

Exercise 10 (Non-commutative integrable Hamiltonian systems). A *non-commutative integrable system* is a smooth map
$$F = (f_1, \dots, f_j, f_{k+1}, \dots, f_{2n-k}) : (M,\omega) \to \mathbb{R}^{2n-k}$$
satisfying

- for all $i = 1, \dots, k$, $j = 1, \dots, 2n-k$, $\{f_i, f_j\} = 0$;
- for almost all $p \in M$, $\operatorname{rk} D_p F = 2n - k$.

For the purposes of this exercise, assume that, in fact, F is a surjective submersion onto a submanifold $B \subset \mathbb{R}^{2n-k}$ with connected fibres.

1. Prove that there exists a unique Poisson structure $\{\cdot,\cdot\}_B$ on B satisfying , for any $f, g \in C^\infty(B)$,
$$\{\phi^* f, \phi^* g\} = \phi^*\{f,g\}_B,$$
where $\{\cdot,\cdot\}$ is the Poisson bracket on M induced by ω. Moreover, show that it is regular, *i.e.* the rank of $\pi_B^\sharp : T^*B \to TB$ is constant, where π_B is the bivector associated to $\{\cdot,\cdot\}_B$.

2. Show that the fibres of F are isotropic.

Exercise 11 (**The Darboux-Caratheodory theorem).**
Prove the Darboux-Caratheodory theorem.

Hints: Let $F = (f_1, \ldots, f_n)$. Let $c = F(m) \in \mathbb{R}^n$. First show that the common level set $\Lambda_c := \{(q,p) \in \mathbb{R}^{2n}; F(q,p) = c\}$ is lagrangian *[See exercise 9]*. Take any lagrangian submanifold through m that is locally transversal to Λ_c (this exists!). Now for any point m' near m, define the coordinates functions $(\sigma_1(m'), \ldots, \sigma_n(m'))$ as the (locally) unique numbers such that the image of m under the joint flow $\varphi_{f_1}^{-\sigma_1} \circ \cdots \circ \varphi_{f_n}^{-\sigma_n}$ belongs to Λ_c. Prove that $\{f_i, \sigma_j\} = \delta_{i,j}$. Prove that the submanifolds $(\sigma_1, \ldots, \sigma_n) = \text{const}$ are lagrangian, and hence $\{\sigma_i, \sigma_j\} = 0$. Conclude that the local coordinates $(\sigma_1, \ldots, \sigma_n, f_1, \ldots, f_n)$ on $\mathbb{R}^{2n}$ (near m) are indeed canonical.

Exercise 12. Given a completely integrable system $F : (M, \omega) \to \mathbb{R}^n$ with compact fibres, suppose that $p \in M$ has rank $0 \le k \le n$. Prove that its orbit $\mathcal{O}_p$ under the Hamiltonian $\mathbb{R}^n$-action is an immersed submanifold of dimension k.

Exercise 13. Let $\mathrm{Sym}(2n; \mathbb{R})$ denote the vector space of symmetric, bilinear forms on $\mathbb{R}^{2n}$.

1. If $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(\mathbb{R}^{2n})$ induced by the canonical symplectic form, prove that $\mathrm{Sym}(2n; \mathbb{R}) \subset \big(C^\infty(\mathbb{R}^{2n}), \{\cdot, \cdot\}\big)$ is a Lie subalgebra.

2. By abuse of notation, let $\{\cdot, \cdot\}$ denote the Lie bracket on $\mathrm{Sym}(2n; \mathbb{R})$ given by part 1. Show that there is an isomorphism of Lie algebras $(\mathrm{Sym}(2n; \mathbb{R}), \{\cdot, \cdot\}) \cong \mathfrak{sp}(2n; \mathbb{R})$.

Exercise 14. Given a completely integrable system $F = (f_1, \ldots, f_n) : (M, \omega) \to \mathbb{R}^n$ with compact fibres, suppose that $p \in M$ has rank $0 \le k < n$. Suppose that, for all $i = 1, \ldots, n-k$, $\mathrm{d}_p f_i = 0$ and recall the existence of a Lie algebra homomorphism $\mathbb{R}^{n-k} \to \mathfrak{sp}(2n; \mathbb{R})$ from the lectures. If A is in the image of this homomorphism, show that $T_p\mathcal{O}_p \subset \ker A$ and that $\operatorname{im} A \subset (T_p\mathcal{O}_p)^{\Omega}$.

Exercise 15. Suppose that $p \in M$ is a non-degenerate singular point of a completely integrable system $F : (M, \omega) \to \mathbb{R}^n$. Prove that all points in the orbit $\mathcal{O}_p$ through p are non-degenerate and have the same Williamson type as p.

Exercise 16. If you have time and the will, try to fill in as many of the gaps in the proofs of the results presented in the lectures (or ask for references!).