Multigrid and Iterative Strategies for Optimal Control Problems

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ABSTRACT

In this minisymposium we focus on optimal control problems, which constitute an important class of PDE-constrained optimization problems. There are many PDEs which can act as the constraints within the problem, such as Stokes-type equations, PDEs with a time-dependent component, and many others – consequently there is considerable potential for applications in applied sciences.

One of the major considerations in the field of optimal control problems is the development of fast and effective methods for their numerical solution. A common approach is to develop efficient strategies for solving the optimality system which characterizes the solution of the problem.

Upon discretization, this generally takes the form of a large and sparse (saddle point) system, the solution of which requires specialized methods tailored to the problem at hand. One technique for solving these systems is to construct preconditioned iterative methods, incorporating techniques such as (algebraic or geometric) multigrid, multilevel and domain decomposition methods to approximate individual blocks of the matrix. Alternatively, one may develop such routines to handle the entire matrix system. A key advantage of both these approaches is that they provide the potential for the exhibition of mesh-independent convergence.

There has been much recent progress in the construction of such methods, which have been applied to many important problems. The aim of this minisymposium is to present state-of-the-art solution strategies and their theoretical underpinning, as well as to highlight possible future directions for this research such as parallelization of the methods and their application to problems arising in industry.

Parallel preconditioning for all-at-once solution of time–dependent PDE constrained optimization problems

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ABSTRACT

We explore domain decomposition strategies for accelerating the convergence of all-at-once numerical schemes for the solution of time-dependent PDE constrained optimization problems on parallel computers. All-at-once schemes aim to solve for all time-steps at the same time, which has the important advantages of preserving physical couplings in the solution, ensuring robustness with respect to the regularization parameter, and accelerating convergence. However, this approach leads to very large linear systems, with resulting costs in computation and memory. We describe a parallel preconditioning strategy for these systems that uses domain decomposition algorithms in the time domain and Schur complement approximations for the resulting local saddle point systems. We describe the motivation behind these algorithms and present numerical results showing their parallel performance. Finally, we discuss possible practical applications of this approach.

A multilevel preconditioner for an optimal Dirichlet boundary control problem

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ABSTRACT

We present a multilevel preconditioner for the discrete optimality system when considering finite element methods for optimal Dirichlet boundary control problems in appropriate energy spaces. While for the interior degrees of freedom a standard multigrid method can be applied, a different approach is required on the boundary. The construction of the preconditioner is based on a BPX type multilevel method. We focus on the robustness with respect to the mesh size and the cost coefficient. Numerical examples illustrate the obtained theoretical results.

An all-at-once multigrid method applied to a Stokes control problem

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ABSTRACT

In this talk we consider the Stokes control problem as model problem:

$$\text{Minimise } j(v,p,f) = \frac{1}{2}\|v - v_D\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|f\|^2_{L^2(\Omega)}$$

subject to the Stokes equations

$$-\Delta v + \nabla p = f \text{ in } \Omega \quad \text{and} \quad \nabla \cdot v = 0 \text{ in } \Omega \quad \text{and} \quad v = 0 \text{ on } \partial\Omega,$$

where v_D is a given desired velocity field and is a given regularization parameter. The discretization of the optimality system (KKT system) characterising the solution of such a PDE-constrained optimisation problem leads to a large-scale sparse linear system. This system is symmetric but not positive definite. Therefore, standard iterative solvers are typically not the best choice. The KKT system is a linear system for two blocks of variables: the primal variables (velocity field v, pressure distribution p and control f) and the Lagrange multipliers introduced to incorporate the partial differential equation. Based on this natural block-structure, we can verify that this system has a saddle point structure where the $(1,1)$-block and the $(2,2)$- block are positive semidefinite. Contrary to the case of elliptic optimal control problems, the $(1,2)$-block is not positive definite but a saddle point problem itself. We are interested in fast iteration schemes with convergence rates bounded away from 1 by a constant which is independent of the discretization parameter (the grid size) and of problem parameters, like in the regularization parameter α in the model problem. To achieve this goal, we propose an all-at-once multigrid approach. In the talk we will discuss the choice of an appropriate smoother and we will give convergence theory.

A robust and optimal AMLI preconditioned MINRES solver for time-periodic parabolic optimal control problems

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ABSTRACT

In this talk, we will consider an optimal control problem with a parabolic time-periodic partial differential equation appearing in its PDE constraints. In order to solve the optimal control problem, we state its optimality system and discretize it by the multiharmonic finite element method leading to a system of linear algebraic equations that decouples into smaller systems, which can be solved totally in parallel. In [1], we construct preconditioners for these systems which yield robust and fast convergence rates for the preconditioned minimal residual method with respect to all parameters. These block diagonal preconditioners are practically implemented by the algebraic multilevel iteration method presented in [2]. The diagonal blocks of the preconditioners consist of a weighted sum of stiffness and mass matrices. In [3], we discuss and prove the robustness and optimality of the AMLI method for solving these reaction-diffusion type problems discretized by the finite element method. Moreover, the multiharmonic finite element analysis of time-periodic parabolic optimal control problems can also be found in [4], where different variational settings are investigated and estimates of the complete discretization error are derived.

References

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