On the cohomology rings of maximal nilpotent Lie subalgebras in split semisimple rational Lie algebras

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Abstract

Let $\mathfrak{g}$ be a split semisimple finite-dimensional Lie algebra over $\mathbb{Q}$ such that the Dynkin diagram for $\mathfrak{g}$ contains no connected subdiagrams of type C_l $(l \geq 3)$, F_4. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the triangular decomposition of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ where $\mathfrak{n}_+$ is a maximal nilpotent Lie subalgebra of $\mathfrak{g}$ corresponding to the system of positive roots Σ. In this work we compute the rational cohomology ring $H^*(\mathfrak{n}_+, \mathbb{Q})$.

Introduction

Let $\mathfrak{g}$ be a split semisimple finite-dimensional Lie algebra over $\mathbb{Q}$, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, Σ be the system of positive roots with respect to $\mathfrak{h}$ and $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_l\}$ be a root basis.

Consider the triangular decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \ .$$

It is known that $\mathfrak{n}_+$ is a maximal nilpotent Lie subalgebra of $\mathfrak{g}$ admitting an integral structure, i.e., one can choose a Chevalley basis $\{E(\alpha) \mid \alpha \in \Sigma\}$ in $\mathfrak{n}_+$ in which all constants of structure are integers; namely,

$$[E(\alpha), E(\beta)] = \begin{cases} N(\alpha,\beta)E(\alpha+\beta) \text{ , if } \alpha+\beta \in \Sigma \ ; \\ 0 \ , \qquad \text{if } \alpha+\beta \notin \Sigma \ , \end{cases} \tag{0.1}$$

where $0 \neq N(\alpha, \beta) \in \mathbb{Z}$. If $\mathfrak{g}$ is simple of type A_l, D_l, E_l then the Chevalley basis is standard. However, the standard Chevalley basis should be slightly modified in the cases B_l, G_2 (some basis elements are multiplied by appropriate integers). Thus for $\mathfrak{n}_+$, when the Lie algebra $\mathfrak{g}$ is simple, we can always have $N(\alpha, \beta) = \pm 1$ in (0.1) except for the cases C_l $(l \geq 3)$, F_4. For semisimple Lie algebras it remains true if the Dynkin diagram for $\mathfrak{g}$ cointains no connected subdiagrams of type C_l $(l \geq 3)$, F_4. More details can be found in Subsection 2.1.

Our main objective is to compute the rational cohomology ring $H^*(\mathfrak{n}_+, \mathbb{Q})$.

In order to state our principal results we introduce some definitions and notation. A subset S of Σ (the case $S = \emptyset$ is admitted) is called a harmonic subset if two following conditions are satisfied:

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H1. If S contains α, β, and $\alpha+\beta$ is a root, then $\alpha+\beta$ belongs to S.
H2. If S contains a root $\alpha+\beta$ then at least one of the positive roots α, β belongs to S.

Let $Hm(\Sigma)$ denote the set of all harmonic subsets of Σ. If $S \subseteq \Sigma$ is a subset of positive roots (not necessarily harmonic) then we denote by $|S|$ the number of its elements.

Let W be the Weyl group of the root system $\Sigma \cup (-\Sigma)$. This group is generated by reflections $w_1, \ldots, w_l$ corresponding to the simple roots $\alpha_i \in \mathfrak{B}$. We denote by $|w|$ the length of an element $w \in W$ with respect to the generating set $\{w_1, \ldots, w_l\}$.

Consider the generating function for the set $Hm(\Sigma)$:

$$H(t) = \sum_{S \in Hm(\Sigma)} t^{|S|} , \tag{0.2}$$

and the growth function for the Weyl group

$$W(t) = \sum_{w \in W} t^{|w|} . \tag{0.3}$$

Evidently, both functions are polynomials. The following theorem is proved in Section 1.

Theorem 1 *For any split semisimple finite-dimensional rational Lie algebra $\mathfrak{g}$ the polynomials $H(t)$ and $W(t)$ coincide. In particular, the number of harmonic subsets of positive roots is equal to the order of the Weyl group W.*

One knows [1, 2] that

$$W(t) = \prod_{i=1}^{l} (1 + t + \ldots + t^{m_i}) ,$$

where $m_1, \ldots, m_l$ are the exponents of the Weyl group W.

Now let

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Sigma} \mathbb{Q} E(\alpha)$$

be the root space decomposition of $\mathfrak{n}_+$. Consider the exterior algebra $\Lambda^* \mathfrak{n}_+$ over $\mathbb{Q}$. Let us give a combinatorial description for an appropriate basis of $\Lambda^* \mathfrak{n}_+$ as a vector space. The set Σ is provided the standard lexicographic ordering with respect to $\mathfrak{B}$. Let $S = \{\beta_1, \ldots, \beta_k\}$ be a subset of positive roots of Σ. Usually we will write the elements of S in the ascending order, i.e., $\beta_1 < \beta_2 < \ldots < \beta_k$. By definition, $E(\emptyset) = 1 \in \Lambda^0 \mathfrak{n}_+$. If $S = \{\beta\}$ then $E(S) = E(\beta)$. For $k \geq 2$

$$E(S) = E(\beta_1) \wedge \ldots \wedge E(\beta_k). \tag{0.4}$$

Note that $E(S) \in \Lambda^{|S|} \mathfrak{n}_+$, i.e., the degree of $E(S)$ is the number of roots in S. Thus all basis elements of $\Lambda^* \mathfrak{n}_+$ are in one-to-one correspondence with all subsets of Σ.

The algebra $\Lambda^* \mathfrak{n}_+$ is given the standard inner product, that is, all elements $E(S)$ form an orthonormal basis. This allows to identify chains and cochains. Consider the standard boundary operator (the differential) d acting on basis elements of kind $E(S)$ as follows:

$$\begin{aligned} d(E(\beta_1) \wedge \ldots \wedge E(\beta_k)) = \\ = \sum_{p<q} (-1)^{p+q-1} [E(\beta_p), E(\beta_q)] \wedge E(\beta_1) \wedge \ldots \widehat{E}(\beta_p) \ldots \widehat{E}(\beta_q) \ldots \wedge E(\beta_k). \end{aligned} \tag{0.5}$$

(as usual, the "hat" means that the corresponding terms are omitted). Consider also the adjoint coboundary operator ∂ and Laplacian $\Delta = d\partial + \partial d$.

It turns out that if $\mathfrak{g}$ is such that the Dynkin diagram for $\mathfrak{g}$ contains no connected subdiagrams of type C_l ($l \geq 3$), F_4 then Laplacian Δ has integer eigenvalues and all basis elements $E(S)$ are eigenvectors for Δ. Let us denote by $\Lambda^*_h\mathfrak{n}_+$ the graded submodule in $\Lambda^*\mathfrak{n}_+$ spanned by all elements of type $E(S)$ where S is a harmonic subset of roots.

Using the above notation we state now our main result proved in Section 2.

Theorem 2 *Let $\mathfrak{g}$ be a split semisimple finite-dimensional Lie algebra over $\mathbb{Q}$ such that the Dynkin diagram for $\mathfrak{g}$ contains no connected subdiagrams of type C_l ($l \geq 3$), F_4. Then the graded ring, which is the graded module $\Lambda^*_h\mathfrak{n}_+$ with multiplication:*

$$E(S) \smile E(T) = \begin{cases} 0\,, & \textit{if either } S \cap T \neq \emptyset\,,\ \textit{or } S \cup T \notin Hm(\Sigma), \\ E(S) \wedge E(T) & \textit{otherwise.} \end{cases}$$

(i.e., the multiplication is exterior "modulo non-harmonic products"), is isomorphic to the rational cohomology ring $H^*(\mathfrak{n}_+, \mathbb{Q})$.

The Hilbert-Poincaré polynomial of the graded ring $H^*(\mathfrak{n}_+, \mathbb{Q})$ is equal to the growth polynomial of the corresponding Weyl group, i.e.,

$$\sum_{k \geq 0} \dim\, H^k(\mathfrak{n}_+, \mathbb{Q}) t^k = W(t).$$

It is quite probably that Theorem 2 holds for any split semisimple finite-dimensional Lie algebra over $\mathbb{Q}$. In Subsection 2.8 we also indicate a combinatorial (in some sense) approach which allows to compute in principle the integral cohomology ring for Lie subring $\mathfrak{z}\mathfrak{n}_+$ of $\mathfrak{n}_+$ spanned over $\mathbb{Z}$ by the Chevalley basis.

It is worth to mention some results in this field. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and let $\mathfrak{n}_+$ be a nilpotent subalgebra of $\mathfrak{g}$ from a triangular decomposition. Let V be an irreducible $\mathfrak{g}$-module restricted to $\mathfrak{n}_+$. B. Kostant [4, Theorem 5.14, Corollary 5.14] had given an algebraic description of the complex cohomology groups $H^*(\mathfrak{n}_+, V)$ and a formula to compute their dimensions which follows from Weyl's formula. In particular, from his results one can deduce the form of the Hilbert-Poincaré polynomial pointed out in Theorem 2.

On the other hand, St.Sigg [5] had computed the complex cohomology groups of free nilpotent Lie algebras of class 2. In fact, his results remain valid for rational cohomology. Let us note that in the last case Laplacian also has integer eigenvalues.

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1 Proof of Theorem 1

1.1 Properties of harmonic subsets.

Let $\Sigma \cup (-\Sigma)$ be a root system and $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_l\}$ be a root basis consisting of simple roots. In the sequel we will use the description of (irreducible) root systems and root lattices given in [3, chapter 6].

Let us start indicating some evident properties of harmonic subsets S of Σ. First, the set Σ and the empty set $\emptyset$ are harmonic. Second, a subset $S = \{\alpha\}$ containing a single root is harmonic if and only if $\alpha \in \mathfrak{B}$. Third, a subset S and its complement $\Sigma \setminus S$ are or aren't harmonic simultaneously. In this sense, Conditions H1 and H2 are dual.

Let $S \subseteq \Sigma$ be an arbitrary subset of positive roots. We define the characteristic function $\chi_S : \Sigma \to \{-1, 1\}$ in the following way:

$$\chi_S(\alpha) = \begin{cases} +1 \, , \; \alpha \in S \\ -1 \, , \; \alpha \notin S \, . \end{cases}$$

The odd extension $\tilde{\chi}_S$ of the function χ_S to the system $\Sigma \cup (-\Sigma)$ is defined by the formula $\tilde{\chi}_S(-\alpha) = -\chi_S(\alpha)$. It is clear that the function χ_S is uniquely determined by $\tilde{\chi}_S$ as its restriction to

Let W be the Weyl group of the root system $\Sigma \cup (-\Sigma)$. Recall some facts about this group (see, for instance, [2] for more details). It is well known that W is generated by the fundamental reflections w_i. Each reflection w_i carries a positive root $\alpha \neq \alpha_i$ to a positive root, and $w_i(\alpha_i) = -\alpha_i$.

The action of W on the root system gives rise to the action of W on the root lattice Q and on the set of all functions $\tilde{\chi}_S$ by

$$w\tilde{\chi}_S(\beta) = \tilde{\chi}_S(w^{-1}\beta)$$

where $\beta \in \Sigma \cup (-\Sigma)$. Since there is a one-to-one correspondence between the functions $\tilde{\chi}_S$ and χ_S then there is also the action of the Weyl group on the functions χ_S. In particular, if $\chi_S = w\chi_\emptyset$ for some $w \in W$ then, checking the definitions, one can see that S consists exactly of such positive roots that were transformed by w from negative ones.

Let $|w|$ be the length of $w \in W$ with respect to the generating set $\{w_1, \ldots, w_l\}$. One knows (see, e.g., [2, theorem 2.2.2]) that $|w|$ is just the number of negative (resp. positive) roots transformed by w into positive (resp.negative) roots. Thus the following lemma is almost evident.

Lemma 1.1 *Let $w \in W$ be an arbitrary element and $\chi_S = w\chi_\emptyset$. Then $|S| = |w|$.*

Let us prove two more lemmas.

Lemma 1.2 *Let $\chi_S = w_i\chi_T$ for some w_i and some harmonic subset T. Then S is also harmonic.*

Proof. Condition H1. Let $\alpha, \beta \in S$ and $\alpha + \beta$ be a root.
Case 1. $\alpha, \beta \neq \alpha_i$.
Then $w_i\alpha, w_i\beta > 0$ and

$$+1 = \chi_S(\alpha) = w_i\chi_T(\alpha) = \chi_T(w_i\alpha).$$

Consequently, $w_i\alpha \in T$. Analogously, $w_i\beta \in T$. Then by Condition H1 for T $w_i\alpha + w_i\beta \in T$ (note that $\alpha + \beta \neq \alpha_i$), and it follows that $\chi_T(w_i\alpha + w_i\beta) = +1$. Hence

$$\chi_S(\alpha + \beta) = w_i\chi_T(\alpha + \beta) = \chi_T(w_i\alpha + w_i\beta) = +1.$$

Case 2. $\alpha = \alpha_i$, $\beta \neq \alpha_i$. In this case $w_i\beta > 0$, $w_i\alpha_i = -\alpha_i$, $w_i(\alpha_i + \beta) = \beta - \alpha_i > 0$. By the assumption $\chi_S(\alpha_i) = w_i\chi_T(\alpha_i) = -\chi_T(\alpha_i) = +1$, i.e. $\alpha_i \notin T$. On the other hand, $\chi_S(\beta) = w_i\chi_T(\beta) = \chi_T(w_i\beta) = +1$, i.e. $w_i\beta \in T$.

Using the equality $w_i\beta = \alpha_i + (w_i\beta - \alpha_i)$ and Condition H2 for T we obtain that $w_i\beta - \alpha_i \in T$. Hence

$$\chi_S(\alpha_i + \beta) = w_i\chi_T(\alpha_i + \beta) = \chi_T(w_i\beta - \alpha_i) = +1.$$

Condition H2. It is worth to consider the complements of S and T. It is easy to see that $\chi_S = w_i\chi_T$ if and only if $\chi_{(\Sigma\setminus S)} = w_i\chi_{(\Sigma\setminus T)}$. Hence the verification of Condition H2 for S is just the verification of H1 for $\Sigma\setminus S$. Since S is harmonic if and only if $\Sigma\setminus S$ is harmonic then the lemma is proved. ∎

Lemma 1.3 *A subset S of positive roots is harmonic if and only if there exists such an element $w \in W$ that $\chi_S = w\chi_\emptyset$.*

Proof. Let S be a harmonic subset. If $S = \emptyset$ then $w = 1$. Let $|S| \geq 1$. There always exists a simple root $\alpha_i \in S$. Suppose that it is not true. Consider a root $\beta \in S$ of minimal height ≥ 2. It is well known that β can be represented as a sum $\gamma + \alpha_i$ where the height of γ, α_i is less than the height of β. This contradicts Condition H2.

Thus there is such an $\alpha_i \in S$. By definition, $\chi_S(\alpha_i) = +1$. Let $\chi_T = w_i\chi_S$. By Lemma 1.2 the subset T is harmonic. Moreover,

$$\chi_T(\alpha_i) = w_i\chi_S(\alpha_i) = \chi_S(w_i\alpha_i) = \chi_S(-\alpha_i) = -1,$$

and it is easy to see that $|T| = |S| - 1$. By induction hypothesis we may assume that $\chi_T = u\chi_\emptyset$, and, consequently, $\chi_S = w_i\chi_T = w_iu\chi_\emptyset$.

The converse implication follows from the harmonicity of the empty set and Lemma 1.2 since each $w \in W$ is a product of some reflections w_i. ∎

It remains only to prove Theorem 1. In view of Lemma 1.3 each element $w \in W$ is in the correspondence with the harmonic subset $S = S(w)$ such that $\chi_S = w\chi_\emptyset$, and each harmonic subset can be obtained in such a way. This correspondence is one-to-one because if $u\chi_\emptyset = w\chi_\emptyset$ then $\chi_\emptyset = u^{-1}w\chi_\emptyset$ and by Lemma 1.1 $|u^{-1}w| = 0$, i.e. $u = w$. Moreover, Lemma 1.1 yields $|S(w)| = |w|$. Hence the polynomials $H(t)$ and $W(t)$ defined by (0.2) and (0.3) coincide. This completes the proof of Theorem 1.

Theorem 1 and the harmonicity of the set Σ imply the following well known result.

Corollary 1.4 *Let $w_0 \in W$ be such an element that $\chi_\Sigma = w_0\chi_\emptyset$. Then w_0 has the maximal length $|w_0|$ among all $w \in W$ equal to the number of positive roots $|\Sigma|$. This element w_0 is unique.*

1.2 Conditions O1 and O2. In this part we introduce two technical conditions on irreducible root systems. We will use them below in Subsection 2.3.

Condition O1. Let four different roots $\alpha, \beta, \gamma, \delta \in \Sigma$ be given such that

$$\alpha + \beta = \gamma + \delta$$

is a root. Assume for convenience that $\alpha > \gamma > \beta$ and $\alpha > \delta > \beta$. We will say that Σ satisfies Condition O1 if the following alternative takes place:

1. *either $\alpha - \gamma$ is a root and $\gamma + \beta$ is not a root;*

2. *or $\alpha - \delta$ is a root and $\delta + \beta$ is not a root.*

Condition O2. Suppose that we are given three different roots α, β, γ such that $\alpha > \gamma$, and $\alpha+\beta$, $\beta+\gamma$ are also roots. We say that Σ satisfies Condition O2 if *for all such α, β, γ, if $\alpha+\beta+\gamma$ is not a root then $\alpha - \gamma$ is a root, and if $\alpha+\beta+\gamma$ is a root then $\alpha+\gamma$ is not a root.*

Lemma 1.5 *The root systems of type A_l, D_l, E_l, B_l, G_2 satisfy Conditions O1, O2.*

Proof. Case 1.
Let Σ be a system of positive roots of type A_l, D_l or E_l. We will need the following claim about the root lattices of the corresponding type (claim (7.8.7) from [3]), $(\cdot|\cdot)$ being the standart bilinear form.

Claim A. If α, β are roots then $\alpha \pm \beta$ is a root if and only if $(\alpha|\beta) = \mp 1$.

Let us check Condition O1 first. Since $\alpha+\beta = \gamma+\delta$ is a root then $(\alpha|\beta) = (\gamma|\delta) = -1$. Suppose that $\alpha - \gamma$ and $\alpha - \delta$ are not roots. It follows that

$$(\alpha|\gamma) \leq 0 \ ; \ (\alpha|\delta) \leq 0 \ .$$

Since

$$|\alpha + \beta|^2 = (\alpha + \beta|\gamma + \delta) = (\alpha|\gamma + \delta) + (\beta|\gamma + \delta) > 0$$

and $(\alpha|\gamma + \delta) \leq 0$ by hypothesis then $(\beta|\gamma + \delta) > 0$. Hence, we obtain that either $(\beta|\gamma) > 0$, or $(\beta|\delta) > 0$. For instance, let $(\beta|\gamma) > 0$. Then $\gamma - \beta = \alpha - \delta$ is a root and this contradicts our hypothesis.

We have just seen that either $\alpha - \gamma$, or $\alpha - \delta$ is a root. Let it be, say, $\alpha - \gamma$. Hence $(\alpha|\gamma) = 1$. Since $\alpha + \beta - \gamma = \delta$ is a root then $(\alpha + \beta|\gamma) = 1$. Consequently, $(\beta|\gamma) = 0$, and $\gamma + \beta$ is not a root.

Let us prove that $\alpha - \gamma$ and $\alpha - \delta$ cannot be roots simultaneously. Suppose the contrary. Then $(\alpha|\gamma) = (\alpha|\delta) = 1$, and consequently, $(\alpha|\gamma + \delta) = 2$. It cannot be so for $\alpha \neq \gamma + \delta$. Condition O1 is verified. ■

Let us check now Condition O2. Since $\alpha + \beta$, $\beta + \gamma$ are roots then Suppose that $\alpha + \beta + \gamma$ is not a root. Then $(\alpha + \beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma) \geq 0$ whence $(\alpha|\gamma) \geq 1$, that is, $\alpha - \gamma$ is a root.

Let $\alpha + \beta + \gamma$ be a root. Then $(\alpha + \beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma) = -1$. Thus $(\alpha|\gamma) = 0$, i.e., $\alpha + \gamma$ is not a root.

Case 2. Let Σ be of type B_l. In this case all roots are either long, or short. The following claim easily follows from their description.

Claim B. Let $\alpha > \beta$ be two roots. If at least one of the roots is long then $\alpha \pm \beta$ is a root if and only if $(\alpha|\beta) = \mp 1$. If α, β are short then $\alpha + \beta$, $\alpha - \beta$ are roots if and only if $(\alpha|\beta) = 0$.

Let us verify Condition O1. Just like in the case 1 one can show that either $\alpha - \gamma$, or $\alpha - \delta$ is a root. We will prove that both expressions cannot be roots simultaneously. Suppose the contrary and suppose first that α is a long root. As above, we obtain that $(\alpha|\gamma) = (\alpha|\delta) = 1$, and consequently, $(\alpha|\gamma + \delta) = 2$. This cannot be since $\alpha \neq \gamma + \delta$.

Let α be a short root. We will get a contradiction with the equality

$$(\alpha|\gamma + \delta) = (\alpha|\gamma) + (\alpha|\delta) \ . \tag{1.1}$$

For the root system of type B_l if both roots γ, δ are either short, or long then $\gamma + \delta$ is a long root. If γ, δ are roots of different length then $\gamma + \delta$ is short. Besides, $\gamma + \delta - \alpha = \beta$ is a root, and one can immediately check that anyway the left-hand the right-hand side of (1.1) differ modulo 2.

For instance, let $\alpha - \gamma$ be a root. Hence $\beta + \gamma$ is not a root. If it is not so then we obtain a contradiction as before considering all cases of root lengths in the equality $(\alpha+\beta|\gamma) = (\alpha|\gamma)+(\beta|\gamma)$.

Let us check Condition O2. Suppose that $\alpha + \beta + \gamma$ is a root and let $\alpha + \gamma$ be a root, too. We obtain a contradiction just when checking Condition O1 if we consider the possible equality $(\alpha + \gamma|\beta) = (\alpha|\beta) + (\gamma|\beta)$.

Suppose that $\alpha+\beta+\gamma$ is not a root as well as $\alpha-\gamma$. Then $(\alpha|\beta+\gamma)\geq 0$, $(\alpha+\beta|\gamma)\geq 0$ and $(\alpha|\gamma)$, $(\beta|\gamma)$, $(\alpha|\beta)\leq 0$. This system of inequalities has a solution if and only if $(\alpha|\beta)=(\beta|\gamma)=(\alpha|\gamma)=0$, i.e., all three roots are short. But then $\alpha-\gamma$ is a root.

Case 3. The case G_2 can be verified directly. This completes the proof of the lemma. ■

1.4 Remark. The root systems of type C_l $(l\geq 3)$, F_4 do not satisfy Condition O2.

2 Proof of Theorem 2

2.1 Chevalley bases. Boundary and coboundary operators. Let a split rational simple Lie algebra $\mathfrak{n}_+$ be constructed by a root system of type A_l, D_l or E_l and let $\varepsilon(\alpha,\beta)$ be a function of asymmetry (see [3, §7.8]). This is a bimultiplicative function on the root lattice $Q\times Q\to\{\pm 1\}$. Some properties of ε will be used later in Subsection 2.3.

The relations (7.8.5) from [3] show that the constants of structure $N(\alpha,\beta)$ in (0.1) can be chosen so that $N(\alpha,\beta)=\varepsilon(\alpha,\beta)$, i.e. the multiplication table of the Lie algebra $\mathfrak{n}_+$ is as follows:

$$[E(\alpha),E(\beta)]=\begin{cases}\varepsilon(\alpha,\beta)E(\alpha+\beta)\ , & \text{if } \alpha+\beta\in\Sigma\\ 0\ , & \text{if } \alpha+\beta\notin\Sigma\ .\end{cases} \tag{2.1}$$

Denote temporarily a Chevalley basis of the algebra $\mathfrak{n}_+$ of type B_l by $\{\tilde{E}(\alpha)\mid\alpha\in\Sigma\}$ (see [3, §7.9] for more details). Denote Σ' the system of positive roots of type D_{l+1}. If $\alpha=v_i\pm v_j\in\Sigma$ is a long root then we set $\alpha'=v_i\pm v_j\in\Sigma'$. If $\alpha=v_i\in\Sigma$ is a short one then we set $\alpha'=v_i+v_{l+1}\in\Sigma'$. Note that the following congruence holds

$$(\alpha+\beta)'\equiv\alpha'+\beta'\pmod{2v_{l+1}}\ . \tag{2.2}$$

Then

$$[\tilde{E}(\alpha),\tilde{E}(\beta)]=\begin{cases}\varepsilon(\alpha',\beta')(p+1)\tilde{E}(\alpha+\beta)\ , & \text{if } \alpha+\beta\in\Sigma\\ 0\ , & \text{if } \alpha+\beta\notin\Sigma\ ,\end{cases} \tag{2.3}$$

where $p\geq 0$ is the maximal integer such that $\alpha-p\beta$ is a root, and ε is the corresponding asymmetry function for the root lattice of type D_{l+1}.

Let us modify the Chevalley basis as follows. If $\alpha=v_i+v_j$ then we set $E(\alpha)=2\tilde{E}(\alpha)$, and if $\alpha\neq v_i+v_j$ (i.e., α is not the sum of two short roots) then $E(\alpha)=\tilde{E}(\alpha)$.

The Lie product in the new basis $\{E(\alpha)\mid\alpha\in\Sigma\}$ (we will call it also the Chevalley basis) is defined by the following brackets

$$[E(\alpha),E(\beta)]=\begin{cases}\varepsilon(\alpha',\beta')E(\alpha+\beta)\ , & \text{if } \alpha+\beta\in\Sigma\\ 0\ , & \text{if } \alpha+\beta\notin\Sigma\ .\end{cases} \tag{2.4}$$

When Lie algebra $\mathfrak{n}_+$ is of type G_2 the system Σ contains six roots: $\Sigma=\{\alpha,\beta,\alpha+\beta,\alpha+2\beta,\alpha+3\beta,2\alpha+3\beta\}$. The elements of the Chevalley basis are temporarily denoted by $\tilde{E}(\alpha),\ldots,\tilde{E}(2\alpha+3\beta)$. Let

$$E(\alpha)=\tilde{E}(\alpha)\ ;\qquad E(\beta)=\tilde{E}(\beta)\ ;\qquad E(\alpha+\beta)=\tilde{E}(\alpha+\beta)\ ;$$
$$E(\alpha+2\beta)=2\tilde{E}(\alpha+2\beta)\ ;\quad E(\alpha+3\beta)=6\tilde{E}(\alpha+3\beta)\ ;\quad E(2\alpha+3\beta)=6\tilde{E}(2\alpha+3\beta)\ .$$

Then all non-trivial Lie brackets are:

$$\begin{gathered}
[E(\alpha), E(\beta)] = E(\alpha+\beta) \ , \\
[E(\beta), E(\alpha+\beta)] = E(\alpha+2\beta) \ , \\
[E(\beta), E(\alpha+2\beta)] = E(\alpha+3\beta) \ , \\
[E(\alpha), E(\alpha+3\beta)] = E(2\alpha+3\beta) \ , \\
[E(\alpha+\beta), E(\alpha+2\beta)] = E(2\alpha+3\beta) \ .
\end{gathered} \tag{2.5}$$

Therefore, in all cases except for C_l, F_4 we can choose such a basis $\{E(\alpha) \mid \alpha \in \Sigma\}$ in which the constants of structure $N(\alpha, \beta) = \pm 1$.

If $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_p$ is the decomposition of a split rational semisimple Lie algebra $\mathfrak{g}$ into a direct sum of simple Lie algebras which are not of type C_l, F_4 then the Chevalley basis for $\mathfrak{n}_+$ is the union of just described Chevalley bases for each simple term $\mathfrak{g}_m$. We will always use this Chevalley basis in the sequel.

Recall that in Introduction we have given a description of the standard complex for Lie algebra $\mathfrak{n}_+$ needed to compute the (co)homology groups $H_*\mathfrak{n}_+$. Besides, we introduced the standard inner product $(\cdot, \cdot)$ on the exterior algebra $\Lambda^*\mathfrak{n}_+$ so that all basis elements of type $E(S)$ form an orthonormal basis.

Consider the boundary operator d defined by the formula (0.5). Let $S \subseteq \Sigma$ be a fixed subset of roots. Let us define $\{T \leftarrow S\}$ (the notation $\{S \rightarrow T\}$ also will be used) as the set of such subsets $T \subseteq \Sigma$ that can be obtained from S by removing a pair of roots α, β and adding the root $\alpha + \beta$ whenever it is possible. This notation $T \leftarrow S$ agrees with the notion of a differential graph introduced below in Subsection 2.2. Note that $|T| = |S| - 1$ for such a subset T.

In view of this notation and the rule (0.1) we can rewrite the differentiation formula (0.5) as follows

$$dE(S) = \sum_{T \leftarrow S} L(S,T) E(T) \tag{2.6}$$

where $L(S,T) = \pm N(\alpha, \beta)$. We do not precise the sign ± 1 which is determined by the position of the roots α, β in the subset S and the position of $\alpha + \beta$ in T (recall that all roots in S and T are written in the ascending order).

The inner product allows to identify chains and cochains and to define the coboundary operator ∂ of degree $+1$. The cohomology groups of the complex $(\Lambda^*\mathfrak{n}_+, \partial)$ are just the cohomology groups $H^*(\mathfrak{n}_+, \mathbb{Q})$ that we want to compute.

It is not hard to check that ∂ acts as follows

$$\partial E(T) = \sum_{S \rightarrow T} L(S,T) E(S) \tag{2.7}$$

and the Leibnitz formula is satisfied

$$\partial(E(S) \wedge E(T)) = \partial E(S) \wedge E(T) + (-1)^{|S|} E(S) \wedge \partial E(T) \ .$$

Finally, consider Laplacian $\Delta = d\partial + \partial d$ which is a self-adjoint nonnegative definite operator of degree 0. The following lemma explains the role of harmonic subsets.

Lemma 2.1 *A subset $S \subseteq \Sigma$ is harmonic if and only if $dE(S) = \partial E(S) = 0$ (hence $\Delta E(S) = 0$).*

Proof. In view of (2.6) and (2.7) $dE(S) \neq 0$ if and only if S contains at least one pair of such roots α, β that $\alpha + \beta \notin S$, i.e., Condition H1 is not satisfied for the subset S. Analogously, $\partial E(S) \neq 0$ if and only if S contains at least one root $\alpha + \beta$ such that S includes neither α, nor β, i.e., Condition H2 is not satisfied. ■

Hence the complex $\Lambda^* \mathfrak{n}_+$ with respect to d (or ∂) is a direct sum of complexes

$$\Lambda^* \mathfrak{n}_+ = \Lambda^*_h \mathfrak{n}_+ \oplus \Lambda^*_{nh} \mathfrak{n}_+$$

where $\Lambda^*_h \mathfrak{n}_+$ is spanned as a vector space by those $E(S)$ for which S is a harmonic subset, and $\Lambda^*_{nh} \mathfrak{n}_+$ is spanned by remaining part of basis elements $E(S)$. We call $\Lambda^*_h \mathfrak{n}_+$ the harmonic part, and $\Lambda^*_{nh} \mathfrak{n}_+$ the non-harmonic part of the standard (co-)chain complex $\Lambda^* \mathfrak{n}_+$. It is clear that the restriction of d (or ∂) to the harmonic part is identically zero.

2.2 Differential graph Γ. In this subsection we attach to the standard complex $(\Lambda^* \mathfrak{n}_+, d)$ (and also $(\Lambda^* \mathfrak{n}_+, \partial)$) a combinatorial object called differential graph. We will need it to compute the eigenvalues of Laplacian.

Let $\Gamma = \Gamma(\mathfrak{n}_+)$ (in fact, Γ depends on the choice of basis) be the oriented graph whose vertices are in one-to-one correspondence with the subsets $S \subseteq \Sigma$. By definition, the weight of a vertex S is the number $|S|$. Each vertex S is the initial vertex of edges joining S with all vertices of the set $\{T \,|\, T \leftarrow S\}$. The edge with initial vertex S and terminal vertex T is labeled by the number $L(S,T)$ according to the formula (2.6). Note that the change of direction of all arrows corresponds to the change of d on ∂.

The degree of the vertex S in the graph Γ is denoted $\deg(S)$. The distance $\rho(S,T)$ between two vertices of Γ is defined as usual, i.e., it is the length (number of edges) of the shortest path joining two given vertices. Note that Γ is not usually a connected graph. In particular, in view of Lemma 2.1 a vertex S is isolated if and only if the subset S is harmonic. Finally, the sum of label's squares for all edges incident to the vertex S is called the height $\lambda(S)$ of S.

Let S_1, S_2 be two vertices of the same weight and within the distance 2 from each other. All possibilities for a path of length 2 joining these two vertices are presented on Figure 1 (labels are omitted).

variant A

variant B

Figure 1

Below the variant A will be schematically denoted as $S_1 \leftarrow S_3 \rightarrow S_2$, and the variant B as $S_1 \rightarrow S_4 \leftarrow S_2$.

2.3 Conjugate paths in the case of the root lattice with conditions O1 and O2. It turns out that each path of length 2, joining two different vertices S_1, S_2 of the same weight, defines a unique second path (called conjugate) with the same properties, provided Conditions O1, O2 are satisfied for the root lattice. The relation to be conjugate is symmetric. Thus, all paths of length 2 joining two vertices S_1, S_2 are pairwise conjugate in the sense explained below.

Note that we do not require in this subsection that the roots in subsets S must be written in the ascending order.

Variant A. Consider two paths $S_3 \to S_1$ and $S_3 \to S_2$. The arrow from S_3 to S_1 means that for some roots α, β

$$S_3 = \{\alpha, \beta, \widehat{\alpha+\beta}, \ldots\}\ ; \quad S_1 = \{\widehat{\alpha}, \widehat{\beta}, \alpha+\beta, \ldots\}$$

(the "hat" means the absence of a root, and "..." mean irrelevant roots).

In a similar way,

$$S_3 = \{\gamma, \delta, \widehat{\gamma+\delta}, \ldots\}\ ; \quad S_2 = \{\widehat{\gamma}, \widehat{\delta}, \gamma+\delta, \ldots\}.$$

Note that $\{\alpha, \beta\} \neq \{\gamma, \delta\}$, since $S_1 \neq S_2$. First, let the sets $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ be disjoint.

Case A11. $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset,\ \ \alpha+\beta \neq \gamma+\delta$. In this case

$$S_1 = \{\alpha+\beta, \gamma, \delta, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma+\delta}, \ldots\}\ ; \quad S_2 = \{\gamma+\delta, \alpha, \beta, \widehat{\gamma}, \widehat{\delta}, \widehat{\alpha+\beta}, \ldots\}.$$

Then the conjugate path is $S_1 \to S_4 \leftarrow S_2$, where $S_4 = \{\alpha+\beta, \gamma+\delta, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \ldots\}$. $\blacksquare$

Now let us consider the cases when $\alpha+\beta = \gamma+\delta$. Up to a change of notation we may suppose that $\alpha > \gamma > \beta$, $\alpha > \delta > \beta$. Lemma 1.5 implies (up to a notation again) that $\alpha - \gamma$ is a root, and $\gamma + \beta$ isn't.

Case A12. $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset,\ \ \alpha+\beta = \gamma+\delta,\ \ \alpha-\gamma \in S_3$. Then

$$S_1 = \{\alpha+\beta, \gamma, \delta, \alpha-\gamma, \widehat{\alpha}, \widehat{\beta}, \ldots\}\ ; \quad S_2 = \{\gamma+\delta, \alpha, \beta, \alpha-\gamma, \widehat{\gamma}, \widehat{\delta}, \ldots\}.$$

The conjugate path is $S_1 \to S_5 \leftarrow S_2$, where $S_5 = \{\alpha+\beta, \alpha, \delta, \widehat{\gamma}, \widehat{\beta}, \widehat{\alpha-\gamma}, \ldots\}$. In fact, the arrow $S_1 \to S_5$ exists in view of the equality $\alpha = \gamma + (\alpha-\gamma)$, and the arrow $S_2 \to S_5$ in view of $\delta = (\alpha-\gamma)+\beta$. $\blacksquare$

Case A13. $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset,\ \ \alpha+\beta = \gamma+\delta,\ \ \alpha-\gamma \notin S_3$.

$$S_1 = \{\alpha+\beta, \gamma, \delta, \widehat{\alpha}, \widehat{\beta}, \widehat{\alpha-\gamma}, \ldots\}\ ; \quad S_2 = \{\gamma+\delta, \alpha, \beta, \widehat{\gamma}, \widehat{\delta}, \widehat{\alpha-\gamma}, \ldots\}.$$

Let $S_6 = \{\alpha+\beta, \gamma, \beta, \alpha-\gamma, \widehat{\delta}, \ldots\}$. In view of O1 one can check that

The arrow $S_6 \to S_1$ exists by virtue of $\alpha = \gamma + (\alpha-\gamma)$, and arrow $S_6 \to S_2$ by virtue of $\delta = (\alpha-\gamma)+\beta$.

Note that in contrast with A11, A12 $|S_3| = |S_6|$, i.e. the picture looks as follows:

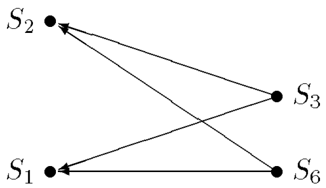

Figure 2

Let us suppose that the sets $\{\alpha, \beta\}$, $\{\gamma, \delta\}$ have exactly one common root. We may assume that this root is $\beta = \delta$, and, besides, $\alpha > \gamma$. Then $S_3 = \{\alpha, \beta, \gamma, \widehat{\alpha+\beta}, \widehat{\beta+\gamma}, \ldots\}$.

Case A21. $\alpha+\beta+\gamma$ is a root, $\alpha+\beta+\gamma \notin S_3$.

$$S_1 = \{\alpha+\beta, \gamma, \widehat{\alpha}, \widehat{\beta}, \widehat{\alpha+\beta+\gamma}, \ldots\}\ ; \quad S_2 = \{\beta+\gamma, \alpha, \widehat{\gamma}, \widehat{\beta}, \widehat{\alpha+\beta+\gamma}, \ldots\}.$$

Let $S_7=\{\alpha+\beta+\gamma,\widehat{\alpha},\widehat{\beta},\widehat{\gamma},\widehat{\alpha+\beta},\widehat{\beta+\gamma},\ldots\}$. Then the conjugate path is $S_1\to S_7\leftarrow S_2$. ■

Case A22. $\alpha+\beta+\gamma$ is a root, $\alpha+\beta+\gamma\in S_3$.

$$S_1=\{\alpha+\beta,\gamma,\alpha+\beta+\gamma,\widehat{\alpha},\widehat{\beta},\ldots\}\ ;\quad S_2=\{\beta+\gamma,\alpha,\alpha+\beta+\gamma,\widehat{\gamma},\widehat{\beta},\ldots\}.$$

Let $S_8=\{\alpha+\beta,\beta+\gamma,\alpha,\gamma,\widehat{\beta},\widehat{\alpha+\beta+\gamma},\ldots\}$. There is the arrow $S_1\leftarrow S_8$ since $\alpha+\beta+\gamma=(\alpha+\beta)+\gamma$, and the arrow $S_8\to S_2$ since $\alpha+\beta+\gamma=\alpha+(\beta+\gamma)$. The situation is the same as on the Figure 2. ■

Let $\alpha+\beta+\gamma$ not to be a root. In view of Lemma 1.5 $\alpha-\gamma$ is a root.

Case A23. $\alpha+\beta+\gamma$ is not a root, $\alpha-\gamma\in S_3$.

$$S_1=\{\alpha+\beta,\gamma,\alpha-\gamma,\widehat{\alpha},\widehat{\beta},\widehat{\beta+\gamma},\ldots\}\ ;\quad S_2=\{\beta+\gamma,\alpha,\alpha-\gamma,\widehat{\gamma},\widehat{\beta},\widehat{\alpha+\beta},\ldots\}.$$

Let $S_9=\{\alpha,\alpha+\beta,\widehat{\beta+\gamma},\widehat{\beta},\widehat{\alpha-\gamma},\widehat{\gamma}\ldots\}$. There is the arrow $S_1\to S_9$ due to $\alpha=(\alpha-\gamma)+\gamma$, and the arrow $S_2\to S_9$ due to $\alpha+\beta=(\alpha-\gamma)+(\beta+\gamma)$. ■

Case A24. $\alpha+\beta+\gamma$ is not a root, $\alpha-\gamma\notin S_3$.

$$S_1=\{\alpha+\beta,\gamma,\widehat{\alpha},\widehat{\beta},\widehat{\alpha-\gamma},\widehat{\beta+\gamma},\ldots\}\ ;\quad S_2=\{\beta+\gamma,\alpha,\widehat{\gamma},\widehat{\beta},\widehat{\alpha-\gamma},\widehat{\alpha+\beta},\ldots\}.$$

Let $S_{10}=\{\gamma,\beta+\gamma,\alpha-\gamma,\widehat{\alpha+\beta},\widehat{\alpha},\widehat{\beta},\ldots\}$. There is the arrow $S_{10}\to S_1$ since $\alpha+\beta=(\alpha-\gamma)+(\beta+\gamma)$, and the arrow $S_{10}\to S_2$ since $\alpha=(\alpha-\gamma)+\gamma$. ■

Variant B. One might consider several cases as above but it is much easier to proceed as follows. Let I be a transformation of the set of vertices of Γ such that $I(S)=\Sigma\setminus S$. It is not hard to verify that if we have a path $S_1\to S_4\leftarrow S_2$ (variant B) then we have the path $I(S_1)\leftarrow I(S_4)\to I(S_2)$ (variant A). We can find the conjugate for the last path and then apply to it the transformation I. Such a procedure provides the conjugate for the original path. Note that I is responsable for the Poincaré duality. One can also check that the relation to be conjugate is symmetric.

2.4 Remark on reducible root systems. Suppose that $\Sigma\cup(-\Sigma)$ is a reducible root system such that each its irreducible component is not of type C_l, F_4. Note that the previous considerations concerning the construction of conjugate paths in Γ remain valid. Indeed, Conditions O1, O2 were used in all cases except A11. It it easy to see that in these cases we worked within one part of the root lattice spanned by an appropriate irreducible component of the root system.

2.5 Eigenvectors and eigenvalues of Laplacian.

Using the above notation let us state the following result.

Proposition 2.2 *Let Σ be the system of positive roots such that each irreducible component of Σ is not of type C_l $(l\geq 3)$, F_4. Then any basis element $E(S)\in\Lambda^*\mathfrak{n}_+$ is an eigenvector of Laplacian Δ associated with the eigenvalue* $\deg(S)=\lambda(S)$, *i.e.,* $\Delta E(S)=\deg(S)E(S)$.

Proof. The relations (2.1), (2.4) and (2.5) imply that all labels of edges in Γ are equal to ±1, and consequently $\lambda(S)=\deg(S)$ for any vertex S.

Let $E(S)$ be any basis element of $\Lambda^*\mathfrak{n}_+$. Apply (2.6) and (2.7) in order to compute $\Delta E(S)=(d\partial+\partial d)E(S)$. It follows that

$$\begin{aligned}\Delta E(S)=(\sum_{S\leftarrow S}L^2(S,T)+\sum_{S\to P}L^2(P,S))\,E(S)+\\ +\sum_{\substack{S\to W\leftarrow R\\ R\neq S}}L(S,W)L(W,R)\,E(R)+\sum_{\substack{S\leftarrow U\to V\\ V\neq S}}L(S,U)L(U,V)\,E(V)=\\ =\deg(S)E(S)+e(S)\ ,\end{aligned}\tag{2.8}$$

where

$$e(S) = \sum_{\substack{S \to W \leftarrow R \\ R \neq S}} L(S,W)L(W,R)\,E(R) + \sum_{\substack{S \leftarrow U \to V \\ V \neq S}} L(S,U)L(U,V)\,E(V)\ . \tag{2.9}$$

Let us focus the attention on the sum $e(S)$. This is exactly the sum of labels' products for all paths of length 2 that join the vertex S with another vertex of the same weight $|S|$.

Let us show that $e(S) = 0$. Collecting similar terms we rewrite (2.9) as follows:

$$e(S) = \sum_{|F|=|S|,\ \rho(S,F)=2} l(S,F)\,E(F), \tag{2.10}$$

where

$$l(S,F) = \sum_{\substack{S \leftarrow U \to F \\ S \to U \leftarrow F}} L(S,U)L(U,F)\ . \tag{2.11}$$

In other words, $l(S,F)$ is exactly the sum of labels' products for all paths of length 2 that join the vertex S with a fixed vertex F of the same weight $|S|$

As it was explained in Subsection 2.3, all such paths are pairwise conjugate. Hence we have only to prove that in each pair the summands of kind $L(S,U)L(U,F)$ are of opposite signs.

Let us make an important remark on this subject. Let π be a permutation of roots in a subset $S = \{\alpha_1, \dots, \alpha_n\}$ that determines the vertex S of Γ, i.e., the roots are written in some not necessarily ascending order. Then the element

$$E(S,\pi) = E(\alpha_{\pi(1)}) \wedge \ldots \wedge E(\alpha_{\pi(n)}) \in \Lambda^n \mathfrak{n}_+\ , \tag{2.12}$$

differs from $E(S) = E(S, \mathrm{id})$ by the sign $(-1)^{\operatorname{sgn}\pi}$.

Suppose that we want to deal with $E(S,\pi) = (-1)^{\operatorname{sgn}\pi} E(S,\mathrm{id})$ of type (2.12) from the very beginning. Then in the differential graph Γ we have to multiply the labels of all edges incident to S by $(-1)^{\operatorname{sgn}\pi}$. Let the vertex S be the common vertex for two conjugate paths (on Figure 2 it is either S_1, or S_2). It is not hard to see that after a permutation of the roots in S the sum of labels' products for two conjugate paths will either change the sign if the permutation is odd, or will remain the same if the permutation is even.

If the vertex S is one of the intermediate vertices of a pair of conjugate paths (on the Figure 2 it is either S_3, or S_6) then the sum of label's products of these two paths will not be changed under the transformation π.

Thus, if the sum of labels' products for two conjugate paths equals zero for a given ordering of roots in the subset S, then it equals zero for any ordering. Clearly, we should take into account the changes of labels' signs in Γ if we permute roots in some subsets S.

Let us return to the proof of the proposition. Using standard properties of a chosen asymmetry function $\varepsilon(\cdot,\cdot)$ (see Subsection 2.1 and [3, §7.8]) we get in the case A11 that $\varepsilon(\alpha,\beta)\varepsilon(\gamma,\delta) - \varepsilon(\gamma,\delta)\varepsilon(\alpha,\beta) = 0$. In fact, this is a direct consequence of the relation $d^2 = 0$ and holds without any assumption on the root system.

Next, let us first consider irreduceble root systems of types A_l, D_l, E_l. We will use the bimultiplicativity and the following properties of ε. If α is a root then $\varepsilon(\alpha, \pm\alpha) = -1$. If α, β and $\alpha \pm \beta$ are roots then $\varepsilon(\alpha,\beta) = -\varepsilon(\beta,\alpha)$. We obtain in the cases A12, A13

$$\begin{gathered}
\varepsilon(\alpha,\beta)\varepsilon(\gamma,\delta) + \varepsilon(\gamma,\alpha-\gamma)\varepsilon(\beta,\alpha-\gamma) = \\
= \varepsilon(\alpha,\beta)\varepsilon(\gamma,\alpha+\beta-\gamma) - \varepsilon(\alpha,\gamma)\varepsilon(\alpha-\gamma,\beta) = \\
= \varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma)\varepsilon(\gamma,\beta) - \varepsilon(\alpha,\gamma)\varepsilon(\alpha,\beta)\varepsilon(\gamma,\beta) = 0\ .
\end{gathered}$$

In the cases A21, A22

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\gamma)+\varepsilon(\alpha+\beta,\gamma)\varepsilon(\beta+\gamma,\alpha)=\varepsilon(\alpha,\beta)\varepsilon(\beta,\gamma)-\varepsilon(\alpha+\beta,\gamma)\varepsilon(\alpha,\beta+\gamma)=0\ .$$

In the cases A23, A24

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\gamma)+\varepsilon(\gamma,\alpha-\gamma)\varepsilon(\beta+\gamma,\alpha-\gamma)=\varepsilon(\alpha,\beta)\varepsilon(\beta,\gamma)+\varepsilon(\beta,\alpha)\varepsilon(\beta,\gamma)=0\ .$$

If the root lattice is of type B_l then, according to (2.4), we should substitute all roots $\alpha,\dots$ in the just verified relations by roots $\alpha',\dots$ and use the congruence (2.2). Since the relations hold for D_{l+1} then the proposition is also proved for B_l.

Finally, the case G_2 can be treated directly. The corresponding graph Γ has 12 isolated vertices, 24 vertices of degree 1 belonging to 12 connected components and 28 vertices of degree 2 belonging to 7 connected components. The eigenvalues of Laplacian are $0,1,2$ in the last case.

The proof for a pair of conjugate paths in the case of variant B is analogous.

If the root system is reducible then we may apply the previous results in view of Remark 2.4. This completes the proof of the proposition. ■

Corollary 2.3 *In the above notation let $\Pi\in\pi_0\Gamma$ be a connected component of Γ. Then all vertices S of Π have the same degree* $\deg(S)=\deg(\Pi)$.

Proof. We will use the evident property of d and ∂ to be commuting with Δ. On the one hand, the formula (2.6) yields

$$\Delta dE(S)=\sum_{T\leftarrow S}L(S,T)\Delta E(T)=\sum_{T\leftarrow S}L(S,T)\deg(T)E(T).$$

On the other hand,

$$d\Delta E(S)=\deg(S)\sum_{T\leftarrow S}L(S,T)E(T)=\sum_{T\leftarrow S}L(S,T)\deg(S)E(T).$$

One can also obtain similar relations for the coboundary operator ∂.

Compare the above two formulas. We see that if the vertices S and T are connected by the edge in Γ then the degrees of S and T are equal. Thus, the degree is a constant function on connected components of the graph Γ. ■

2.6 Proof of Theorem 2. It follows from Proposition 2.2 that the eigenvalues λ of Laplacian Δ are non-negative integers. The complex $\Lambda^*\mathfrak{n}_+$ can be decomposed into the direct sum of subcomplexes

$$\Lambda^*\mathfrak{n}_+=\bigoplus_{\lambda\geq 0}K_\lambda(\mathfrak{n}_+)\ ,$$

where $K_\lambda(\mathfrak{n}_+)$ is the eigenspace associated with the eigenvalue λ. Since both operators ∂ and d commute with Δ then $K_\lambda(\mathfrak{n}_+)$ are in fact subcomplexes (with respect to either ∂, or d).

Next, the subcomplex $(K_\lambda(\mathfrak{n}_+),\partial)$ for $\lambda\neq 0$ is contractible with trivial homology (a contracting homotopy is $H=d/\lambda$). The kernel of Laplacian

$$K_0(\mathfrak{n}_+)=\Lambda^*_h\mathfrak{n}_+$$

is the subcomplex with trivial differential which provides the desired cohomology groups. The exterior product coincides with cochain product on $\Lambda^*\mathfrak{n}_+$ whence we obtain the multiplicative structure.

The second statement of the theorem follows from the observation that any element $E(S)$ belongs to the corresponding exterior power $\Lambda^{|S|}\mathfrak{n}_+$ and from Theorem 1 concerning the generating function $H(t)$ for harmonic subsets. This completes the proof of Theorem 2. ∎

2.7 Remark. It seems to be true that Theorem 2 holds for any Lie algebras $\mathfrak{n}_+$, in particular, for those of type C_l, F_4. We suppose that Laplacian restricted to the non-harmonic part $\Lambda^*_{nh}\mathfrak{n}_+$ has strictly positive integer eigenvalues (or it is, at least, non-singular). However, Proposition 2.2 is not true for the above two types.

2.8 Remarks on connected components of Γ and integral cohomology. Let $\{E(\alpha) \mid \alpha \in \Sigma\}$ be the Chevalley basis of the rational Lie algebra $\mathfrak{n}_+$. Since all constants of structure are integers we can consider the Lie ring

$$\mathfrak{z}\mathfrak{n}_+ = \bigoplus_{\alpha\in\Sigma} \mathbb{Z}E(\alpha) ,$$

and construct the differential graph Γ as before. Note that the exterior algebra $\Lambda^*\mathfrak{z}\mathfrak{n}_+$ should be regarded over $\mathbb{Z}$.

Let $\pi_0\Gamma$ be the set of connected components of Γ. It follows from the definition of this graph that for each connected component $\Pi \in \pi_0\Gamma$ the graded $\mathbb{Z}$-submodule

$$\Lambda^*(\Pi) =< E(S) \mid S \in \Pi > \subset \Lambda^*\mathfrak{z}\mathfrak{n}_+$$

is a subcomplex with respect to either ∂, or d, whence

$$\Lambda^*\mathfrak{z}\mathfrak{n}_+ = \bigoplus_{\Pi\in\pi_0\Gamma} \Lambda^*(\Pi)$$

We obtain from (2.13) the following decomposition

$$H^*(\mathfrak{z}\mathfrak{n}_+) = \bigoplus_{\Pi\in\pi_0\Gamma} H^*(\Pi) ,$$

where $H^*(\Pi)$ is the integral (co)homology of the subcomplex $\Lambda^*(\Pi)$.

If a connected component Π consists of a unique vertex S corresponding to the harmonic subset S then $H^*(\Pi) \cong \mathbb{Z}$. If Π has more than one vertex then $H^*(\Pi)$ obviously are torsion groups, but not necessarily trivial. For instance, for the Lie ring $\mathfrak{z}\mathfrak{n}_+$ of type G_2 we can compute the following cohomological groups:

$$H^0(\mathfrak{z}\mathfrak{n}_+) = \mathbb{Z};\ H^1(\mathfrak{z}\mathfrak{n}_+) = \mathbb{Z}^2;\ H^2(\mathfrak{z}\mathfrak{n}_+) = \mathbb{Z}^2;\ H^3(\mathfrak{z}\mathfrak{n}_+) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z};$$
$$H^4(\mathfrak{z}\mathfrak{n}_+) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z};\ H^5(\mathfrak{z}\mathfrak{n}_+) = \mathbb{Z}^2;\ H^6(\mathfrak{z}\mathfrak{n}_+) = \mathbb{Z}.$$

Note that 2-torsion cocycles appear in the situation when a connected component looks like that on Figure 2. One can also verify that the integral cohomology groups for Lie rings $\mathfrak{z}\mathfrak{n}_+$ of type A_l have 2-torsion when $l \geq 3$, and 3-torsion when $l \geq 4$.

2.9 Some applications. In conclusion let us mention one algebraic and one geometric application of the above results.

Corollary 2.4 *Let $G = UT_n(\mathbb{Z})$ be the group of upper triangular matrices of order n with integer entries, $\mathfrak{g}$ be the Lie algebra $\mathfrak{n}_+$ of type A_{n-1}. Then $\forall k \geq 0$ $H^k(G,\mathbb{Q}) \cong H^k(\mathfrak{g},\mathbb{Q})$.*

The proof follows at once from [6].

Let $G = UT_n(\mathbb{R})$ be the Lie group of upper triangular real matrices of order n. Then $UT_n(\mathbb{Z})$ is a lattice of G. The group $UT_n(\mathbb{R})$ is contractible, hence the compact manifold

$$\mathcal{M}_n = UT_n(\mathbb{R})/UT_n(\mathbb{Z})$$

is an Eilenberg-MacLane space of type $K(UT_n(\mathbb{Z}),1)$. It follows (see, for instance, [7, Chap.2, Prop. 4.1]) that there is an isomorphism $H^*(\mathcal{M}_n) \cong H^*(UT_n(\mathbb{Z}))$. We also have the following result in view of Corollary 2.4.

Corollary 2.5 *The rational cohomology groups of the manifold $\mathcal{M}_n$ are isomorphic to the corresponding cohomology groups of the Lie algebra $\mathfrak{n}_+$ of type A_{n-1}.*

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