AN INTRODUCTION TO C*-ALGEBRAS

(Chapters 1 to 9)

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INTRODUCTION

From November 1993 to February 1994, Vaughan Jones gave an introductory Lecture Course on C*-algebras in Lausanne, for the "Troisième Cycle Romand de Mathématiques" (some lectures were prepared by P.H. and by Alain Valette). The audience was very heterogeneous, consisting of a mixture of beginners and of mature mathematicians, most of them working quite outside the field of functional analysis, as well as a few physicists. One of the "leit-motives" was to illustrate the theory with the finite dimensional situation. One of the goals was to get a reasonable understanding of the CAR algebra, as it is used for the representation theory of loop groups of compact Lie groups.

During the academic year 1994/95, Pierre de la Harpe has given a similar set of lectures in Geneva, for the "Diplôme d'Études Supérieures en mathématiques de la région lémanique", with the extra fantasy of writing up notes. More often than not, it has been difficult to obtain an acceptable compromise between the desire to keep some of the lightness of the spoken lectures on one hand, and the heavy need to fill in details as befits a written exposition on the other hand. The result is as follows, so far for the first chapters only. It is possible that these notes will be improved and completed at some future date.

Any comment will be welcome. Thanks are due to Roland Bacher for his help in proofreading the present notes. The first author is responsible for mistakes which could be left in what follows.

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CHAPTER 1. BOUNDED OPERATORS ON HILBERT SPACES

1.A. Recall on Hilbert spaces

In these notes, $\mathcal{H}$ will denote a complex Hilbert space. The *scalar product*

$$\langle \xi | \eta \rangle$$

of two vectors $\xi, \eta \in \mathcal{H}$ is antilinear in ξ and linear in η. The *norm* of $\xi \in \mathcal{H}$ is given by

$$\|\xi\| = \sqrt{\langle \xi | \xi \rangle}$$

and we let

$$\mathcal{H}(1) = \{ \xi \in \mathcal{H} \mid \ \|\xi\| \leq 1 \}$$

denote the closed unit ball in $\mathcal{H}$.

For any subset S of $\mathcal{H}$, the closed linear subspace

$$S^{\perp} = \{ \ \eta \in \mathcal{H} \mid \ \langle \xi \mid \eta \rangle = 0 \ \text{ for all } \ \xi \in S \ \}$$

is the *orthogonal* of S. Observe that $\left(S^{\perp}\right)^{\perp} \supset S$, and that $S \subset T \Rightarrow S^{\perp} \supset T^{\perp}$ for subsets S, T of $\mathcal{H}$. In case S is a linear subspace of $\mathcal{H}$, then $\left(S^{\perp}\right)^{\perp} = \overline{S}$ is the *closure* of S.

Though we assume that the reader has some knowledge about Hilbert spaces, e.g. as in Chapters four and five of [Ru1], we shall recall (without proof) the following five basic facts.

1.1. Cauchy-Schwarz inequality. *One has*

$$| \langle \xi | \eta \rangle | \leq \|\xi\| \ \|\eta\|$$

for all $\xi, \eta \in \mathcal{H}$.

1.2. Projections on convex subsets. *Let C be a non empty closed convex subset of $\mathcal{H}$ and let $\xi \in \mathcal{H}$. There exists a unique vector $\xi_C \in C$ such that*

$$\|\xi - \xi_C\| = \min_{\eta \in C} \|\xi - \eta\| .$$

If C is moreover a closed subspace (meaning linear subspace) of $\mathcal{H}$, then the assignment $\xi \mapsto \xi_C$ is linear and one has

$$\|\xi\|^2 = \|\xi_C\|^2 + \|\xi - \xi_C\|^2 .$$

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1.3. Riesz representation theorem. *Let $\phi : \mathcal{H} \to \mathbb{C}$ be a continuous linear functional. Then there exists a unique vector $\xi \in \mathcal{H}$ such that*

$$\phi(\eta) = \langle \xi | \eta \rangle$$

for all $\eta \in \mathcal{H}$; moreover $\|\xi\| = sup\{|\phi(\eta)| \ : \ \eta \in \mathcal{H}(1)\}$.

This result is due independently to F. Riesz (C.R. Acad. Sc. Paris, **144** (1907) 1409-1411) and M. Fréchet (Ibid., 1414-1416).

1.4 Bounded = continuous for linear mappings. *Let $\mathcal{H}, \mathcal{H}'$ be two Hilbert spaces and let $a : \mathcal{H} \to \mathcal{H}'$ be a linear mapping. Then the three following conditions are equivalent:*

(i) the quantity $\|a\| = \sup\{\|a\xi\| \ | \ \xi \in \mathcal{H}(1)\}$ is bounded,
(ii) a is continuous,
(iii) a is continous at one point of $\mathcal{H}$.

If a fulfills conditions (i) to (iii) above, then a is a *bounded* linear operator from $\mathcal{H}$ to $\mathcal{H}'$ and $\|a\|$ is its *norm.* The set of all such bounded linear operators, furnished with the norm $a \mapsto \|a\|$, is a Banach space denoted by

$$\mathcal{B}(\mathcal{H}, \mathcal{H}').$$

Let $\mathcal{H}''$ be a third Hilbert space, let $a \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and let $b \in \mathcal{B}(\mathcal{H}', \mathcal{H}'')$. It follows straightforwardly from the definitions that $ba \in \mathcal{B}(\mathcal{H}, \mathcal{H}'')$ and that

$$\|ba\| \leq \|b\| \, \|a\|.$$

One writes $\mathcal{B}(\mathcal{H})$ instead of $\mathcal{B}(\mathcal{H}, \mathcal{H})$. The norm $a \mapsto \|a\|$ makes $\mathcal{B}(\mathcal{H})$ a *Banach algebra* which has a unit, namely the identity operator of $\mathcal{H}$ written $id_{\mathcal{H}}$ or simply 1. (A *normed algebra* is a complex algebra A given together with a norm $a \mapsto \|a\|$ such that $\|ab\| \leq \|a\| \, \|b\|$ for all $a, b \in A$. A *Banach algebra* is a normed algebra which is complete.)

1.5 Open mapping Theorem. *Let $\mathcal{H}, \mathcal{H}'$ be two Hilbert spaces and let $a : \mathcal{H} \to \mathcal{H}'$ be a bounded linear operator which is onto. Then a is open, so that in particular there exists a number $\delta > 0$ such that*

$$\{\eta \in \mathcal{H}' \ | \ \|\eta\| < \delta\} \subset a(\{\xi \in \mathcal{H} \ | \ \|\xi\| < 1\}).$$

1.6. Remarks. The three first facts recalled above belong really to Hilbert space theory. On the other hand, 1.4 and 1.5 hold in much more general settings : see e.g. Theorems 1.32 and 2.11 in [Ru2].

On several occasions, we will use other standard results of functional analysis, such as
the analytic form of the Hahn-Banach theorem, on extensions of linear forms
(see e.g. 2.16 and 6.9),
the geometric form of the Hahn-Banach theorem, on separating convex sets (2.16),
the Banach-Steinhaus theorem , on uniform boundedness (2.19),
the Krein-Milman theorem, on extreme points of convex sets (6.14).

1.7. Notations. Among standard examples of Hilbert spaces, there is the space $\mathbb{C}^n$ with its canonical scalar product

$$\langle (\eta_1, ..., \eta_n) \,|\, (\xi_1, ..., \xi_n) \rangle = \sum_{j=1}^{n} \overline{\eta_j} \xi_j ,$$

the space of square-summable sequences indexed by $\mathbb{N}$

$$\ell^2 \;=\; \left\{ \xi = (\xi_n)_{n \geq 0} \;\middle|\; \xi_n \in \mathbb{C} \;\text{ and }\; \sum_{n=0}^{\infty} |\xi_n|^2 \;< \infty \right\}$$

and the space of square-summable measurable functions on a measure space (X, μ)

$$L^2(X, \mu) \;=\; \left\{ \xi : X \to \mathbb{C} \;\middle|\; \int_X |\xi(x)|^2 \, d\mu(x) < \infty \right\}$$

where ξ is (abusively !) identified to its equivalence class modulo the relation of equality μ-almost everywhere; in case the choice of μ is clear (for example the Lebesgue measure on a measurable subset of $\mathbb{R}^n$), one writes simply $L^2(X)$.

1.8. Separability. Most Hilbert spaces arising "naturally" in analysis are separable (i.e. contain countable dense subsets, or equivalently have countable orthonormal bases). But there is for example a "respectable" non separable Hilbert space in the theory of almost periodic functions, of which we recall the following.

Let $\mathcal{C}(\mathbb{R})$ denote the algebra of all continuous functions from $\mathbb{R}$ to $\mathbb{C}$ (for the pointwise product). Let $f \in \mathcal{C}(\mathbb{R})$. For $\epsilon > 0$, a number $t \in \mathbb{R}$ is called an *ϵ-almost period* if $sup_{x \in \mathbb{R}} |f(x+t) - f(x)| < \epsilon$. Say that $f \in \mathcal{C}(\mathbb{R})$ is *almost periodic* if, for any $\epsilon > 0$, there exists $\ell = \ell(f, \epsilon)$ such that any real interval of length ℓ contains an ϵ-almost period of f. One shows that almost periodic functions are bounded, that they constitute a subalgebra $AP(\mathbb{R})$ of $\mathcal{C}(\mathbb{R})$, and that the limit

$$\langle f | g \rangle \;=\; \lim_{T \to \infty} \frac{1}{T} \int_0^T \overline{f(x)} g(x) dx$$

exists for all $f, g \in AP(\mathbb{R})$. The space obtained by completion of $AP(\mathbb{R})$ with respect to this scalar product is a Hilbert space in which

$$\left(t \mapsto e^{i\lambda t} \right)_{\lambda \in \mathbb{R}}$$

is an uncountable orthonormal basis. More on this in [Fav] and in Section VI.5 of [Kat].

Another motivation for introducing non separable Hilbert spaces comes from the study of the Calkin algebra and is alluded to in Remark 6.9.

1.B. Adjoints and norms of operators.

Let $\mathcal{H}, \mathcal{H}'$ be two Hilbert spaces and let $a : \mathcal{H} \to \mathcal{H}'$ be a bounded linear operator.

1.9. Proposition. *There exists a unique bounded linear operator $a^* : \mathcal{H}' \to \mathcal{H}$ such that*

$$\langle a^* \eta | \xi \rangle = \langle \eta | a \xi \rangle$$

for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}'$.

Proof. For each $\eta \in \mathcal{H}'$, one has a continuous linear form

$$\begin{cases} \mathcal{H} \to & \mathbb{C} \\ \xi \mapsto & \langle \eta | a\xi \rangle \end{cases}$$

and thus by Riesz theorem a unique vector $\zeta \in \mathcal{H}$ such that

$$\langle \zeta | \xi \rangle = \langle \eta | a \xi \rangle$$

for all $\xi \in \mathcal{H}$. If a^* is defined to be the assignment $\eta \mapsto \zeta$, it is easy to check that a^* is linear and bounded. □

1.10. Definition. The operator $a^* \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ is the *adjoint* of a.

One has obviously $(a^*)^* = a$ as well as $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ for all $a, b \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and $\lambda, \mu \in \mathbb{C}$.

1.11. Proposition. *One has*

$$\|a\| = sup \{ \, | \langle \eta | a \xi \rangle | \, : \, \xi \in \mathcal{H}(1) \, , \eta \in \mathcal{H}'(1) \, \} = \|a^*\|$$

for all $a \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$.

Proof. By the Cauchy-Schwarz inequality and by the definition of $\|a\|$, one has

$$| \langle \eta | a \xi \rangle | \leq \|\eta\| \; \|a\xi\| \leq \|\eta\| \; \|a\| \; \|\xi\|$$

for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}'$, so that

$$\|a\| \geq sup \{ \, | \langle \eta | a \xi \rangle | \, : \, \xi \in \mathcal{H}(1) \, , \eta \in \mathcal{H}'(1) \, \} \, .$$

For the opposite inequality, we may assume $a \neq 0$ and we choose $\epsilon > 0$ such that $\epsilon < \|a\| / 2$. Choose then $\xi \in \mathcal{H}(1)$ such that $\|a\xi\| \geq \|a\| - \epsilon$ and set $\eta = a\xi / \|a\xi\| \in \mathcal{H}'(1)$. Then $| \langle \eta | a \xi \rangle | = \|a\xi\| \geq \|a\| - \epsilon$. Hence

$$sup \{ \, | \langle \eta | a \xi \rangle | \, : \, \xi \in \mathcal{H}(1) \, , \eta \in \mathcal{H}'(1) \, \} \geq \|a\| \, .$$

As $| \langle \eta | a \xi \rangle | = | \langle \xi | a^* \eta \rangle |$ for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}'$, the last equality follows. □

1.12. Corollary. *One has*

$$\|a^*a\| = \|a\|^2$$

for all $a \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$.

Proof. One has

$$\|a^*a\| \leq \|a^*\| \ \|a\| = \|a\|^2$$

and

$$\|a\|^2 = \sup_{\xi \in \mathcal{H}(1)} \langle a\xi | a\xi \rangle \ = \sup_{\xi \in \mathcal{H}(1)} \langle \xi | a^*a\xi \rangle \ \leq \ \|a^*a\|$$

so that $\|a^*a\| = \|a\|^2$. □

1.13. Remark. Let A denote an *involutive algebra*, namely a complex algebra A given together with an involution $\begin{cases} A \to A \\ a \mapsto a^* \end{cases}$ such that

$$\begin{aligned} (a+b)^* &= a^* + b^* \\ (\lambda a)^* &= \overline{\lambda} a^* \\ (ab)^* &= b^* a^* \\ (a^*)^* &= a \end{aligned}$$

for all $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$.

Let $a \mapsto \|a\|$ be a norm on A such that $\|ab\| \leq \|a\| \, \|b\|$ for all $a \in A$. Then the equality

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$ implies the equality $\|a^*\| = \|a\|$ for all $a \in A$. Indeed, assuming the first of these, one has

$$\|a^*\|^4 = \|aa^*\|^2 = \|aa^*aa^*\| \leq \|a\| \ \|a^*a\| \ \|a^*\| \leq \|a\|^3 \ \|a^*\|$$

so that $\|a^*\|^3 \leq \|a\|^3$ for all $a \in A$. Similarly $\|a\|^3 \leq \|a^*\|^3$.

An involutive algebra A with a norm satisfying $\|a^*\| = \|a\|$ for all $a \in A$ is called a *normed involutive algebra,* and a *Banach involutive algebra* if it is moreover complete. In Chapter 4, we will define abstract C^**-algebras:* they are Banach involutive algebras satisfying $\|a^*a\| = \|a\|^2$. Group algebras such as $\ell^1(\mathbb{Z})$ and $L^1(\mathbb{R})$ provide examples of Banach involutive algebras which are not C^*-algebras (see n^o 4.9).

A $*$*-representation* of an involutive algebra A on a Hilbert space $\mathcal{H}$ is a linear map $\Phi : A \to \mathcal{B}(\mathcal{H})$ such that $\Phi(ab) = \Phi(a)\Phi(b)$ and $\Phi(a^*) = \Phi(a)^*$ for all $a, b \in A$.

1.14. Norm of finite dimensional operators. Consider an operator $a \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and assume that the space $\mathcal{H}$ is finite dimensional. Let $\mu_1, ..., \mu_n$ denote the eigenvalues of $a^*a \in \mathcal{B}(\mathcal{H})$. Then

$$\|a\| \ = \ \sqrt{\max_{1 \leq j \leq n} \mu_j}.$$

The proof is left as an exercise for the reader. (For an arbitrary operator a, see n^{os} 2.8 and 4.22 below.)

1.15. Multiplication operators. Set $\mathcal{H} = L^2(\mathbb{R})$, choose $f \in L^\infty(\mathbb{R})$ and define $M_f \in \mathcal{B}(\mathcal{H})$ by

$$(M_f\xi)(x) = f(x)\xi(x)$$

for all $\xi \in \mathcal{H}$ and for almost all $x \in \mathbb{R}$. Then it is easy to check that

$$\|M_f\| = \|f\|_\infty$$

where $\|f\|_\infty$ denotes the essential supremum of f.

This can be generalized to $\mathcal{H} = L^2(X,\mu)$ and $f \in L^\infty(X,\mu)$ for any measure space (X,μ). These *multiplication operators* are basic examples of the theory; the reader is encouraged to study Chapter 6 of [Hal].

1.c. Classes of bounded operators.

In this section, we consider a Hilbert space $\mathcal{H}$ and bounded operators in $\mathcal{B}(\mathcal{H})$.

1.16. Self-adjoint and positive operators. The operator $a \in \mathcal{B}(\mathcal{H})$ is called *self-adjoint* if $a^* = a$; an operator a is self-adjoint if and only if $\langle \xi \mid a\xi\rangle \in \mathbb{R}$ for all $\xi \in \mathcal{H}$, as it follows easily from the so-called *polarization identity*

$$\begin{aligned}\langle \xi \mid a\eta\rangle \;=\; & \frac{1}{4}\langle \xi+\eta|a(\xi+\eta)\rangle \;-\; \frac{1}{4}\langle \xi-\eta|a(\xi-\eta)\rangle \\ -\; & \frac{i}{4}\langle \xi+i\eta|a(\xi+i\eta)\rangle \;+\; \frac{i}{4}\langle \xi-i\eta|a(\xi-i\eta)\rangle\end{aligned}$$

which holds for any operator a and any pair (ξ,η) of vectors in $\mathcal{H}$.

For example a multiplication operator M_f on $L^2(X,\mu)$ is self-adjoint if and only if the function $f \in L^\infty(X,\mu)$ is real-valued.

The operator $a \in \mathcal{B}(\mathcal{H})$ is called *positive* if $\langle \xi \mid a\xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$. Such an operator is necessarily self-adjoint, as we have just seen. (Aside : on a *real* Hilbert space, there exist operators a such that $a^* \neq a$ and $\langle\xi|a\xi\rangle \geq 0$ for all vectors ξ.)

For example, any operator of the form $a = b^*b$ for some $b \in \mathcal{B}(\mathcal{H})$ is positive, and conversely (if a is positive, then $a = b^*b$ with $b = \sqrt{a}$; see Problem 95 in [Hal] and § 4.E below). A multiplication operator M_f on $L^2(X,\mu)$ is positive if and only if the function $f \in L^\infty(X,\mu)$ satisfies $f(x) \in \mathbb{R}_+$ for μ-almost all $x \in X$.

1.17. Proposition. *If $a \in \mathcal{B}(\mathcal{H})$ is self-adjoint then*

$$\|a\| = sup\{ \ |\langle \xi | a\xi \rangle| \ : \ \xi \in \mathcal{H}(1) \ \}$$

(compare with Proposition 1.11.)

Proof. Set $K = sup\{ \ |\langle \xi | a\xi \rangle| \ : \ \xi \in \mathcal{H}(1) \ \}$. For all $\xi, \eta \in \mathcal{H}(1)$, one has

$$\begin{aligned} \mathcal{R}e\, \langle \eta | a\xi \rangle &= \frac{1}{2} \{ \langle \eta | a\xi \rangle + \langle \xi | a\eta \rangle \} \\ &= \frac{1}{4} \{ \langle \xi + \eta | a(\xi + \eta) \rangle + \langle \xi - \eta | a(\xi - \eta) \rangle \} \end{aligned}$$

and then also

$$\begin{aligned} &|\mathcal{R}e\, \langle \eta | a\xi \rangle \,| \\ &\qquad = \frac{K}{4} \left\{ \ \|\xi + \eta\|^2 + \|\xi - \eta\|^2 \right\} \\ &\qquad = \frac{K}{2} \left\{ \ \|\xi\|^2 + \|\eta\|^2 \ \right\} \\ &\qquad \leq K \end{aligned}$$

It follows that

$$|\langle \eta | a\xi \rangle| = \max_{\theta \in \mathbb{R}} |\mathcal{R}e \left\langle e^{i\theta}\eta | a\xi \right\rangle| \leq K$$

namely that $\|a\| \leq K$ by Proposition 1.11. The opposite inequality $K \leq \|a\|$ is an immediate consequence of the Cauchy-Schwarz inequality. □

1.18. Projections. An operator $p \in \mathcal{B}(\mathcal{H})$ is called a *projection* if $p^2 = p$ and $p^* = p$. Using 1.2, one shows easily that there is a canonical bijection between projections in $\mathcal{B}(\mathcal{H})$ and closed subspaces of $\mathcal{H}$ given by $p \leftrightsquigarrow p(\mathcal{H})$.

Projections in $\mathcal{B}(\mathcal{H})$ are consequently *ordered:* $p_1 \leq p_2$ if $p_1(\mathcal{H})$ is a subspace of $p_2(\mathcal{H})$, equivalently if $p_1 p_2 = p_1$. Similarly, two projections p_1, p_2 in $\mathcal{B}(\mathcal{H})$ are said to be *orthogonal* if the spaces $p_1(\mathcal{H})$ and $p_2(\mathcal{H})$ are orthogonal, equivalently if $p_1 p_2 = 0$.

A multiplication operator M_f on $L^2(X, \mu)$ is a projection if and only if the function $f \in L^\infty(X, \mu)$ is the caracteristic function of a measurable subset of X.

1.19. Isometries. An operator $w \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is an *isometry* if it satisfes $w^* w = 1$, or equivalently $\|w(\xi)\| = \|\xi\|$ for all $\xi \in \mathcal{H}$.

The basic observation to record is that, when $\mathcal{H}$ is infinite dimensional, an isometry $w \in \mathcal{B}(\mathcal{H})$ needs not be onto. The most famous example is the *unilateral shift* $s \in \mathcal{B}(\ell^2)$ defined by

$$s\,(\xi_0, \xi_1, \xi_2, ...) = (0, \xi_0, \xi_1, , ...)$$

for all $\xi = (\xi_n)_{n \geq 0} \in \ell^2$. Using Fourier analysis, one may also view ℓ^2 as the space of continuous functions on the closed unit disc of $\mathbb{C}$ which are holomorphic in the open unit disc, and s as the operator of multiplication by z. More on this operator in Chapters 9 and 14 of [Hal].

1.20. Unitaries. An operator $u \in \mathcal{B}(\mathcal{H})$ is *unitary* if it is isometric and onto, or equivalently (as a consequence of the open mapping Theorem) if $u^*u = uu^* = 1$. The *unitary group* of $\mathcal{H}$ is

$$\mathcal{U}(\mathcal{H}) = \{ \ u \in \mathcal{B}(\mathcal{H}) \mid uu^* = u^*u = 1 \ \} .$$

In case $dim_{\mathbb{C}}\mathcal{H} = n < \infty$, it is a compact Lie group usually denoted by $U(n)$.

A multiplication operator M_f on $L^2(X, \mu)$ is unitary if and only if $|f(x)| = 1$ for μ-almost all $x \in X$.

Let $(X, \mathcal{B}, \mu)$ be a probability space. For a measure preserving transformation $T : X \to X$, one defines a unitary operator u_T on $L^2(X)$ by $(u_T\xi)(x) = \xi(T^{-1}x)$ for all $\xi \in L^2(X)$ and $x \in X$. The study of this operator u_T is important in ergodic theory; see e.g. §§ 25 and 26 in [Wal] and Chapter 2 in [Zim].

1.21. Partial isometries. For an operator $w \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$, the five following conditions are equivalent

(i) $(w^*w)^2 = w^*w$,
(ii) $(ww^*)^2 = ww^*$,
(iii) $ww^*w = w$,
(iv) $w^*ww^* = w^*$,
(v) there are two closed subspaces $E \subset \mathcal{H}$, $E' \subset \mathcal{H}'$ such that w is the composition
of the projection of $\mathcal{H}$ onto E,
of an isometry of E onto E',
and of the inclusion of E' into $\mathcal{H}'$.

If these conditions hold, then

w^*w is the projection of $\mathcal{H}$ onto E,
ww^* is the projection of $\mathcal{H}'$ onto E',

and w is called a *partial isometry* with *initial space* E and *initial projection* w^*w, with *final space* F and *final projection* w^*w. For example, the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ defines a partial isometry on the Hilbert space $\mathbb{C}^2$ with initial space the first axis and with final space the second axis.

For more on partial isometries, see Chapter 13 of [Hal].

1.22. Normal operators and eigenvalues. An operator a on $\mathcal{H}$ is *normal* if $a^*a = aa^*$, or equivalently if $\|a^*\xi\| = \|a\xi\|$ for all $\xi \in \mathcal{H}$.

For example, self-adjoint operators and unitary operators are obviously normal, but the unilateral shift is not.

If a normal operator a has two eigenvectors corresponding to different eigenvalues, these vectors are orthogonal (the argument is the same as in finite dimensions). There are however two important facts to note. The first one is that a normal operator may have no eigenvector at all, as it is the case for the self-adjoint multiplication operator M defined on $L^2([0,1])$ by

$$\big(Mf\big)(t) = tf(t)$$

for all $f \in L^2([0,1])$ and $t \in [0,1]$. The second one is that an operator on a separable Hilbert space which is not normal may have uncountably many eigenvalues. For example, if s is the unilateral shift introduced in n^o 1.19, one has

$$s^*(1, z, z^2, z^3, ...) = z(1, z, z^2, z^3, ...)$$

for all $z \in \mathbb{C}$ such that $|z| < 1$ (for more on the spectrum of s^*, see Solution 67 in [Hal]).

P. de la Harpe and V. Jones, July 1995.

CHAPTER 2. ALGEBRAS OF OPERATORS : DEFINITIONS AND FINITE DIMENSIONAL EXAMPLES

In many parts of mathematics, an "algebra" is understood to have a unit. This is *not so* in functional analysis, where examples of algebras include spaces of continous functions vanishing at infinity such as $\mathcal{C}_o(\mathbb{R})$, group algebras such as $L^1(\mathbb{R})$ or $C^*(\mathbb{R})$, and various other C*-algebras *without* units.

However, in this chapter, we concentrate on C*-algebras with units and we postpone to 4.5 a general discussion on "adding units" (but see Remark 2.24.i).

2.A. C*-ALGEBRAS OF OPERATORS

Let $\mathcal{H}$ be a complex Hilbert space.

2.1. Definition. A *C*-algebra of operators* on $\mathcal{H}$ is an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed for the norm topology. (A subalgebra A of $\mathcal{B}(\mathcal{H})$ is *involutive* if $a^* \in A$ whenever $a \in A$; see 1.13. For the definition of the norm on $\mathcal{B}(\mathcal{H})$, see 1.4; for the equality $\|a^*a\| = \|a\|^2$, see 1.12.)

Given a C*-algebra A on $\mathcal{H}$, a *sub-C*-algebra* B of A is an involutive subalgebra of A which is closed for the norm topology.

2.2. Trivial examples. The algebra $\mathcal{B}(\mathcal{H})$ itself is a C*-algebra of operators on $\mathcal{H}$. So is the algebra of complex multiples of the identity, which is isomorphic to $\mathbb{C}$.

2.3. Algebras of continuous functions. Let X be a compact space, let μ be a positive measure on X such that $\mu(U) > 0$ for any nonempty open subset U of X, and let $L^2(X, \mu)$ denote the resulting Hilbert space.

Let $\mathcal{C}(X)$ be the algebra of continuous functions on X. Recall from 1.15 that each $f \in \mathcal{C}(X)$ defines a multiplication operator $M_f \in \mathcal{B}\left(L^2(X;\mu)\right)$. Then

$$A = \left\{ \ M_f \in \mathcal{B}\left(L^2(X;\mu)\right) \ \mid f \in \mathcal{C}(X) \ \right\}$$

is a C*-algebra of operators on $\mathcal{H}$. The condition on μ implies that the mapping $f \mapsto M_f$ is injective, so that $\mathcal{C}(X)$ and A are isomorphic algebras.

Observe that the measure μ does not play an important role for A. This is a first motivation for the space-free definition of Section 4.A.

2.4. Separability. A C*-algebra is *separable* if it contains a countable dense subset.

It is easy to see that the algebra $\mathcal{B}(\mathcal{H})$ is separable if and only if $\mathcal{H}$ (or equivalently $\mathcal{B}(\mathcal{H})$) is finite dimensional.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Indeed, suppose $\mathcal{H}$ is infinite dimensional. If $\mathcal{H}$ is separable, one may identify $\mathcal{H}$ with ℓ^2. Then, for any subset S of $\mathbb{N}$, the characteristic function of S provides a multiplication operator $a_S \in \mathcal{B}(\ell^2)$; moreover $\|a_S - a_T\| \geq 1$ if S, T are distinct subsets of $\mathbb{N}$. As there are uncountably many subsets of $\mathbb{N}$, it follows that $\mathcal{B}(\mathcal{H})$ is not separable. (A variant of this argument, using the multiplication operators in $L^2([0,1])$ defined by the characteristic functions of subintervals, appears in Solution 83 of [Hal].) If $\mathcal{H}$ is not separable, $\mathcal{H}$ is the direct sum of ℓ^2 and of another space, and the argument above can be adapted easily.

We leave it as an exercise for the reader to check that the algebra $A \approx \mathcal{C}(X)$ of Example 2.3 is separable if and only if the compact space X is separable.

In the same sense that "most" compact spaces "of interest" are separable, "most" C*-algebras "of interest" are separable. But there are important counterexamples to this statement, and $\mathcal{B}(\mathcal{H})$ is of course the first of them. Note however that, viewed as a von Neumann algebra with the strong topology, $\mathcal{B}(\mathcal{H})$ is separable if and only if $\mathcal{H}$ is separable (see 2.19 below).

2.5. Matrix algebras. Given an integer $n \geq 1$ and the Hilbert space $\mathbb{C}^n$, we identify the algebra $\mathcal{B}(\mathbb{C}^n)$ with the algebra $M_n(\mathbb{C})$ of n-by-n complex matrices. Thus $M_n(\mathbb{C})$ is a C*-algebra of operators on $\mathbb{C}^n$, the involution is given by

$$(a^*)_{j,k} = \overline{a_{k,j}}$$

for all $a \in M_n(\mathbb{C})$ and $j, k \in \{1, ..., n\}$, and the norm is given by

$$a \longmapsto \|a\| = \sup_{\substack{\xi \in \mathbb{C}^n \\ \|\xi\| \leq 1}} \|a\xi\| = \sqrt{\max_{1 \leq j \leq n} \mu_j}$$

where $\mu_1, ..., \mu_n$ denote the eigenvalues of a^*a, as in 1.14. Recall the basic fact

$$\|a^*a\| = \|a\|^2$$

for all $a \in M_n(\mathbb{C})$.

2.6. Lemma. *On the involutive algebra $M_n(\mathbb{C})$, the only norm ν such that $\nu(a^*a) = \nu(a)^2$ for all $a \in M_n(\mathbb{C})$ is the operator norm $a \mapsto \|a\|$.*

Proof. Consider some matrix $a \in M_n(\mathbb{C}^n)$ and the matrix $d = a^*a$. It is enough to show that $\nu(d) = \|d\|$.

Let $\mu_1, ..., \mu_n$ denote the eigenvalues of d, arranged in such a way that $\mu_1 \geq ... \geq \mu_n \geq 0$. For $r \in \mathbb{R}_+$, the limit $\lim_{k\to\infty} \left(\frac{d}{r}\right)^k$ is 0 if $r > \mu_1$ and does not exist if $r < \mu_1$. It follows that

$$\begin{aligned} \mu_1 &= inf \left\{ r \in \mathbb{R}_+ \;\middle|\; \lim_{k\to\infty} \left(\frac{d}{r}\right)^k = 0 \right\} \\ &= inf \left\{ r \in \mathbb{R}_+ \;\middle|\; \lim_{l\to\infty} \left(\frac{d}{r}\right)^{2^l} = 0 \right\}. \end{aligned}$$

Now saying that a sequence of vectors in a finite dimensional complex vector space (here the space of matrices of order n) converges to 0 means by definition that the sequence of the norms of these vectors converges to 0 (for *some* norm, because all norms are equivalent in finite dimensions). In particular one has

$$\mu_1 = inf \left\{ r \in \mathbb{R}_+ \;\middle|\; \lim_{l\to\infty} \nu\left(\left(\frac{d}{r}\right)^{2^l}\right) = 0 \right\}.$$

By hypothesis on ν, one has $\nu(d^2) = \nu(d^*d) = \nu(d)^2$, and thus also $\nu\left(d^{2^l}\right) = \nu(d)^{2^l}$ for each $l \geq 0$. Hence

$$\nu\left(\left(\frac{d}{r}\right)^{2^l}\right) = \left(\frac{\nu(d)}{r}\right)^{2^l}$$

and

$$\mu_1 = \nu(d).$$

As

$$\mu_1 = \|a^*a\| = \|d\|$$

by 1.14, one has $\nu(d) = \|d\|$ and the proof is complete.

Observe that one has in particular $\nu(ab) \leq \nu(a)\nu(b)$ for all $a, b \in M_n(\mathbb{C}^n)$. □

2.7. Proposition. *Let A be a subalgebra of $M_n(\mathbb{C})$ such that $a^* \in A$ as soon as $a \in A$. The only norm $\nu : A \to \mathbb{R}_1$ such that $\nu(a^*a) = \nu(a)^2$ for all $a \in M_n(\mathbb{C})$ is the operator norm $a \mapsto \|a\|$.*

Proof. The proof of Lemma 2.6 applies verbatim. □

2.8. Remark. (i) The previous proposition shows that

involutive subalgebras of $M_n(\mathbb{C})$

are the same as

sub-C*-algebras of $M_n(\mathbb{C})$ with a norm satisfying $\|a^*a\| = \|a\|^2$.

For algebras containing 1, we will see in Proposition 2.13 below that they are also the same as von Neumann algebras on $\mathbb{C}^n$.

(ii) Note that one has $\nu(ab) \leq \nu(a)\nu(b)$ for all a and b in A as a consequence of the proof of the proposition, not as an a priori hypothesis.

(iii) Proposition 2.7 holds in infinite dimensions. More precisely, let A be a complex involutive algebra which is also a Banach space for a norm ν such that $\nu(a^*a) = \nu(a)^2$ for all $a \in A$. Then one has necessarily $\nu(ab) \leq \nu(a)\nu(b)$ for all $a, b \in A$ by an argument due to Araki and Elliott [ArE]. It is then classical that $\nu(a)^2$ is the spectral radius of a^*a, so that ν is the *unique* norm for which A is a C*-algebra: see Corollary 4.22 below.

2.9. Matrix units. In the matrix algebra $M_n(\mathbb{C})$, for each j and k in $\{1, ..., n\}$, let $e_{j,k}$ denote the matrix which has a 1 at the intersection of the j^{th} row and of the k^{th} column, and 0 's elsewhere. Then one has

$$\begin{aligned}(e_{j,k})^* &= e_{k,j} \\ e_{j,k}e_{l,m} &= \delta_{k,l}e_{j,m} \\ \sum_{i=1}^{n} e_{i,i} &= 1\end{aligned}$$

for all $j, k, l, m \in \{1, ..., n\}$.

Given any C*-algebra, a system $(w_{j,k})_{1\leq j,k\leq n}$ of n^2 elements in A is called a *system of matrix units of order* n if one has

$$\begin{aligned}(w_{j,k})^* &= w_{k,j} \\ w_{j,k}w_{l,m} &= \delta_{k,l}w_{j,m}\end{aligned}$$

for all $j, k, l, m \in \{1, ..., n\}$ and if the $w_{j,k}$'s are not all zero (note that $w_{j,k} \neq 0$ for some j, k implies $w_{j,k} \neq 0$ for all $j, k \in \{1, ..., n\}$). Observe that the $w_{j,j}$'s of such a system are pairwise orthogonal projections, and that the $w_{j,k}$'s $(j \neq k)$ are partial isometries.

To any such system corresponds an injective homomorphism ϕ given by $\phi(e_{j,k}) = w_{j,k}$ from the C*-algebra $M_n(\mathbb{C})$ onto the subalgebra of A linearly generated by the $w_{j,k}$'s; when A has a unit, $\phi(1) = 1$ if and only if

$$\sum_{i=1}^{n} w_{i,i} = 1.$$

Consider a C*-algebra A which is $*$-isomorphic to $M_n(\mathbb{C})$ for some $n \geq 1$ and a sequence $p_1, ..., p_m$ of pairwise orthogonal and minimal projections in A (with of course $m \leq n$). We leave it to the reader to check that one may find a system of matrix units $(w_{j,k})_{1\leq j,k\leq n}$ in A such that $w_{j,j} = p_j$ for all $j \in \{1, ..., m\}$.

2.9$^{\text{bis}}$. Exercice. *Let A be a C*-algebra with unit and let M be a sub-C*-algebra of A containing the unit. Consider the relative commutant*

$$M' \cap A \;=\; \{a \in A \mid am = ma \;\; \textit{for all} \;\; m \in M\}.$$

If $M \approx M_n(\mathbb{C})$ is a full matrix-algebra, the map

$$\phi \;:\; \begin{cases} M \otimes (M' \cap A) & \longrightarrow \quad A \\ \qquad m \otimes a & \longmapsto \quad ma \end{cases}$$

is an isomorphism.

[*Indication.* Let $(e_{i,j})_{1\leq i,j\leq n}$ be matrix units in M. For each $a \in A$ and $i, j \in \{1, ..., n\}$, set

$$a_{i,j} \;=\; \sum_{k=1}^{n} e_{k,i} \; a \; e_{j,k}.$$

Check that $a_{i,j}e_{p,q} = e_{p,q}a_{i,j}$ for all $p, q \in \{1, ..., n\}$, so that $a_{i,j} \in M' \cap A$, and that

$$\sum_{1 \leq i,j \leq n} e_{i,j} \; a_{i,j} \;=\; \sum_{1 \leq i,j \leq n} e_{i,i} \; a \; e_{j,j} \;=\; a$$

so that ϕ is onto. If $\sum_{1 \leq i,j \leq n} e_{i,j} \otimes a_{i,j} \in Ker\phi$, then

$$\sum_{k=1}^{n} e_{k,p} \left(\sum_{1 \leq i,j \leq n} e_{i,j} \; a_{i,j} \right) e_{q,k} \;=\; \sum_{k=1}^{n} e_{k,k} a_{p,q} \;=\; a_{p,q} \;=\; 0$$

for all $p, q \in \{1, ..., n\}$, so that ϕ is injective.]

2.B. The von Neumann Density Theorem

2.10. Commutants. Let $\mathcal{H}$ be a Hilbert space and let S be a subset of $\mathcal{B}(\mathcal{H})$. The *commutant* of S is the subalgebra

$$S' = \left\{ a \in \mathcal{B}(\mathcal{H}) \;\middle|\; as = sa \;\text{ for all }\; s \in S \right\}$$

of $\mathcal{B}(\mathcal{H})$. One writes S'' for $(S')'$, and S''' for $(S'')'$, etc. Observe that

$$(*) \qquad \begin{aligned} &S'' \supset S \qquad \text{and} \\ &S \subset T \implies S' \supset T' \end{aligned}$$

for all $S, T \subset \mathcal{B}(\mathcal{H})$.

Let S be a subset of $\mathcal{B}(\mathcal{H})$ which is self-adjoint (i.e. $s \in S \iff s^* \in S$). Then S' is an involutive sub-algebra of $\mathcal{B}(\mathcal{H})$. It is of course true that S' is closed with respect to the norm, so that S' is a C*-algebra of operators on $\mathcal{H}$; but the norm topology is often not the most interesting on S' (it is rarely separable).

2.11. Lemma. *Let S be a self-adjoint subset of $\mathcal{B}(\mathcal{H})$, let E be a closed subspace of $\mathcal{H}$ and let p be the orthogonal projection of $\mathcal{H}$ onto E. Then*

$$E \;\text{ is } S\text{-invariant} \quad \iff \quad p \in S'.$$

Proof. Let us show $\implies$. Suppose E is S-invariant. Then $sp = psp$ for all $s \in S$. As S is self-adjoint on has also $ps = (s^*p)^* = (ps^*p)^* = psp$ for all $s \in S$. This shows that $p \in S'$. □

2.12. Definition. A *von Neumann algebra* on $\mathcal{H}$ is an involutive subalgebra A of $\mathcal{B}(\mathcal{H})$ such that $A'' = A$. A *factor* on $\mathcal{H}$ is a von Neumann algebra A such that the center $A' \cap A$ is reduced to $\mathbb{C}id_{\mathcal{H}}$.

For each self-adjoint subset S of $\mathcal{B}(\mathcal{H})$, the commutant S' is a von Neumann algebra on $\mathcal{H}$. Indeed, by (*), one has on one hand $S' \subset S'''$, and on the other hand $S \subset S'' \implies S' \supset S'''$. This shows one may define von Neumann algebras on $\mathcal{H}$ as *commutants of self-adjoints subsets* in $\mathcal{B}(\mathcal{H})$.

For example, let G be a group and let $\pi : G \to U(\mathcal{H})$ be a unitary representation. Then $\pi(G)'$ is a von Neumann algebra on $\mathcal{H}$. Lemma 2.11 shows that projections in $\pi(G)'$ are in natural bijection with closed G-invariant subspaces of $\mathcal{H}$. (Usually, in practice, G is locally compact and π is continuous, but this is not necessary for the observation above.)

2.13. Proposition (von Neumann's Density Theorem in finite dimensions). *Let A be a involutive subalgebra of $M_n(\mathbb{C})$ which contains the unit matrix. Then $A^{''} = A$, i.e. A is a von Neumann algebra on $\mathbb{C}^n$.*

Proof. Let $\mathcal{K}$ denote the orthogonal sum of n copies of $\mathbb{C}^n$. Let

$$\pi : M_n(\mathbb{C}) \longrightarrow \mathcal{B}(\mathcal{K})$$

denote the representation of $M_n(\mathbb{C})$ on $\mathcal{K}$ given by the diagonal action. Consider also an orthonormal basis $(v_1, ..., v_n)$ of $\mathbb{C}^n$, the vector

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathcal{K}$$

and the orthogonal projection p of $\mathcal{K}$ onto the subspace $\pi(A)v$. Lemma 2.11 shows that $p \in \pi(A)'$.

We may write any element in $\mathcal{B}(\mathcal{K})$ as a n-by-n matrix with entries in $M_n(\mathbb{C})$; in other words we may identify $\mathcal{B}(\mathcal{K})$ with $M_n\left(M_n(\mathbb{C})\right)$. Then, we claim that

$$\pi(A)' = M_n(A').$$

(Caution: the first prime denotes a commutant in $\mathcal{B}(\mathcal{K})$ and the second a commutant in $\mathcal{B}(\mathbb{C}^n)$.) Indeed, for

$$x = (x_{j,k})_{1 \le j,k \le n} \in M_n\left(M_n(\mathbb{C})\right) = \mathcal{B}(\mathcal{K}),$$

the equations

$$\begin{aligned}
\pi(a)x &= \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{pmatrix} \\
&= \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{pmatrix} \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a \end{pmatrix} = x\pi(a)
\end{aligned}$$

for all $a \in A$ are equivalent to

$$x_{j,k} \in A' \quad \text{for all} \quad j, k \in \{1, ..., n\}.$$

Similarly one has

$$\pi(A^{''}) \subset M_n(A')'$$

(it is easy to see, but not useful just here, that the equality holds).

Consider now a self-adjoint element $b \in A^{''}$. By the previous step one has

$$\pi(b) \in \pi(A^{''}) \subset M_n(A')' = \pi(A)^{''}$$

so that $\pi(b)$ commutes with $p \in \pi(A')$; the same holds for $\pi(b^*)$. Hence the subspace $\pi(A)v$ is invariant by $\pi(b)$, by Lemma 2.11 again. In particular

$$\pi(b)\Big(\pi(1)v\Big) = \begin{pmatrix} bv_1 \\ \vdots \\ bv_n \end{pmatrix} \in \pi(A)v$$

so that b acts on each of $v_1 \dots v_n$ as an operator of A. This means precisely that $b \in A$.

We have shown that any element $b \in A^{''}$ is also in A. □

2.14. Comment. This proposition will be an essential tool in Section 2.C for the study of involutive algebras of operators in finite dimensions. The end of the present Section 2.B will not be used in these notes before Section 6.C; Theorem 2.17 however is one of the most fundamental result in the theory of operator algebras. It first appeared in 1929 [vNe]. There is a bicommutant theorem in pure algebra of which the ultimate form seems to be due to Jacobson [Jac]. But results like Proposition 2.13 are much older, probably going back to E. Noether [Di2, page 11].

2.15. Topologies on $\mathcal{B}(\mathcal{H})$. We know already the norm topology on $\mathcal{B}(\mathcal{H})$. There are many other useful topologies on $\mathcal{B}(\mathcal{H})$, of which we define two here. For a good introduction to some of the others, see [StZ] (Chapter I, Comments to Chapter 5, and Chapter 8).

Strong topology. For any $a \in \mathcal{B}(\mathcal{H})$ and $\xi \in \mathcal{H}$, set

$$\mathcal{V}_\xi(a) = \left\{ b \in \mathcal{B}(\mathcal{H}) \;\middle|\; \|(b-a)\xi\| < 1 \right\}.$$

Finite intersections of the $\mathcal{V}_\xi(a)$'s constitute a basis of neighbourhoods of a in $\mathcal{B}(\mathcal{H})$ for a locally convex Hausdorff topology on $\mathcal{B}(\mathcal{H})$ which is called the *strong topology.*

Weak topology. For any $a \in \mathcal{B}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, set

$$\mathcal{V}_{\xi,\eta}(a) = \left\{ b \in \mathcal{B}(\mathcal{H}) \;\middle|\; |\langle \eta | (b-a)\xi \rangle| < 1 \right\}.$$

Finite intersections of the $\mathcal{V}_{\xi,\eta}(a)$'s constitute a basis of neighbourhoods of a in $\mathcal{B}(\mathcal{H})$ for a locally convex Hausdorff topology on $\mathcal{B}(\mathcal{H})$ which is called the *weak topology.*

It is easy to check that the one-sided multiplication $L_a : b \mapsto ab$ and $R_a : b \mapsto ba$ are continuous on $\mathcal{B}(\mathcal{H})$ for the strong topology, and also for the weak topology; however $(a,b) \mapsto ab$ is neither strongly nor weakly continuous. The adjoint $a \mapsto a^*$ is continuous for the weak topology, but not for the strong topology.

The most important topology for physics is the weak topology, because the quantities $\langle \eta | (b-a)\xi \rangle$ are related to the "transition probabilities".

2.16. Proposition. *(i) The weak topology on $\mathcal{B}(\mathcal{H})$ is weaker than the strong topology, which is itself weaker than the norm topology.*

(ii) A linear form $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is strongly continuous if and only if it is weakly continuous.

(iii) A convex subset of $\mathcal{B}(\mathcal{H})$ is strongly closed if and only if it is weakly closed.

(iv) On the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$, the weak topology coincides with the strong topology; they make $\mathcal{U}(\mathcal{H})$ a topological group.

(v) The comparisons of (i) are strict as soon as $\mathcal{H}$ is infinite dimensional.

Proof. (i) We leave this to the reader.

(ii) We follow [StZ]. Assume that ϕ is strongly continuous. There exist $\xi_1, ..., \xi_n \in \mathcal{H} - \{0\}$ such that

$$|\phi(x)| \ \le \ \sum_{k=1}^{n} \|x\xi_k\|$$

for all $x \in \mathcal{B}(\mathcal{H})$. On $\mathcal{B}(\mathcal{H})^n$, define a semi-norm p by

$$p(x_1, ..., x_n) \ = \ \sum_{k=1}^{n} \|x_k\xi_k\| \, ;$$

on the diagonal $\mathcal{D}$ of $\mathcal{B}(\mathcal{H})^n$, define a linear form $\tilde{\phi}$ by

$$\tilde{\phi}(x, x, ..., x) \ = \ \phi(x).$$

As $|\tilde{\phi}(x, ..., x)| \le p(x, ..., x)$ for all $(x, ..., x) \in \mathcal{D}$, the Hahn-Banach theorem shows that there exists a linear form Φ on $\mathcal{B}(\mathcal{H})^n$ which extends $\tilde{\phi}$ and which is such that

$$|\Phi(x_1, ..., x_n)| \ \le \ p(x_1, ..., x_n)$$

for all $x_1, ..., x_n \in \mathcal{B}(\mathcal{H})$.

For each $k \in \{1, .., n\}$, let ϕ_k be the linear form on $\mathcal{B}(\mathcal{H})$ defined by

$$\phi_k \ = \ \Phi(0, ..., 0, x, 0, ..., 0) \qquad (x \text{ in } k\text{th place}).$$

Then $|\phi_k(x)| \le \|x\xi_k\|$ for all $x \in \mathcal{B}(\mathcal{H})$. As any vector in $\mathcal{H}$ is of the form $x\xi_k$ for some $x \in \mathcal{B}(\mathcal{H})$, it follows that the linear form

$$\begin{cases} \mathcal{H} & \longrightarrow \ \mathbb{C} \\ x\xi_k & \longmapsto \ \phi_k(x) \end{cases}$$

is well defined and bounded. By Riesz' theorem, there exists $\eta_k \in \mathcal{H}$ such that $\phi_k(x) = \langle \eta_k | x\xi_k \rangle$, and this holds for all $x \in \mathcal{B}(\mathcal{H})$. As $\phi(x) = \Phi(x, ..., x) = \sum_{k=1}^{n} \phi_k(x)$, one has finally

$$\phi(x) \ = \ \sum_{k=1}^{n} \langle \eta_k | x\xi_k \rangle$$

so that ϕ is clearly weakly continuous.

The converse implication follows from (i).

(iii) This is a straight consequence of (ii) and of the geometric form of the Hahn-Banach Theorem.

(iv) Let us check that the mapping $\mathcal{U}(\mathcal{H})_{weak} \to \mathcal{U}(\mathcal{H})_{strong}$ is continuous. For this, given a strong neighbourhood of 1 in $\mathcal{U}(\mathcal{H})$ of the form

$$\mathcal{O}_{strong} = \left\{ \, u \in \mathcal{U}(\mathcal{H}) \mid \|(u-1)\xi\| < 1 \, \right\}$$

for some $\xi \in \mathcal{H}$ and an element $u_0 \in \mathcal{O}_{strong}$, it is enough to find a weak neighbourhhod $\mathcal{O}_{weak}$ of u_0 in $\mathcal{U}(\mathcal{H})$ such that $\mathcal{O}_{weak} \subset \mathcal{O}_{strong}$. If $\epsilon = 1 - \|(u_0-1)\xi\|$, one may set

$$\mathcal{O}_{weak} = \left\{ \, v \in \mathcal{U}(\mathcal{H}) \;\middle|\; \begin{array}{c} |\langle u_0\xi|(v-u_0)\xi\rangle| < \frac{1}{2}\epsilon^2 \\ |\langle \xi|(u_0^* - v^*)u_0\xi\rangle| < \frac{1}{2}\epsilon^2 \end{array} \right\}$$

(recall that the map $a \mapsto a^*$ is weakly continuous). For $v \in \mathcal{O}_{weak}$, one has then

$$\|(v-u_0)\xi\|^2 \;=\; \langle \xi|(1-u_0^*v)\xi\rangle + \langle \xi|(1-v^*u_0)\xi\rangle \;<\; \epsilon^2$$

and consequently

$$\|(v-1)\xi\| \;\le\; \|(v-u_0)\xi\| + \|(u_0-1)\xi\| < 1,$$

namely $v \in \mathcal{O}_{strong}$.

(v) For simplicity of notations, we assume $\mathcal{H}$ to be separable. Let $(e_n)_{n\ge 1}$ be an orthonormal basis of $\mathcal{H}$.

For each $n \ge 1$, let p_n denote the projection of $\mathcal{H}$ onto the span of $\{e_1, ..., e_n\}$. Then p_n converges to $id_{\mathcal{H}}$ strongly, but not in the norm topology. Hence the strong topology is strictly weaker than the norm topology.

Let s be the unilateral shift defined by $se_n = e_{n+1}$ for all $n \ge 1$ (see 1.19). Then, for any $p, q \ge 1$, the scalar product $\langle e_p|s^k e_q\rangle$ is zero for k large enough. It follows that $\lim_{k\to\infty}\langle \eta|s^k\xi\rangle = 0$ for all $\xi, \eta \in \mathcal{H}$, namely that the powers s^k converge weakly to 0 when $k \to \infty$. As $\left\|s^k\xi\right\| = \|\xi\|$ for all $k \ge 1$ and $\xi \in \mathcal{H}$, the powers s^k do not converge strongly to 0. Hence the weak topology is strictly weaker than the strong topology. □

2.17. Von Neumann's Density Theorem. *Let A be an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ which contains $id_{\mathcal{H}}$. Then A is strongly dense in A''.*

Proof. Let $b \in A''$ be a self-adjoint element, let $\xi_1, ..., \xi_n$ be vectors in $\mathcal{H}$, let $\epsilon > 0$, and let

$$\mathcal{V} = \left\{ \, x \in \mathcal{B}(\mathcal{H}) \;\middle|\; \|(x-b)\xi_j\| < \epsilon \;\text{ for }\; j \in \{1, ..., n\} \, \right\}$$

be a basic strong neighbourhood of b in $\mathcal{B}(\mathcal{H})$. We have to show that there exists an element in $A \cap \mathcal{V}$, and we follow the proof of Proposition 2.13.

Let $\mathcal{K}$ denote the orthogonal sum of n copies of $\mathcal{H}$ and let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ denote the diagonal representation. Consider the vector

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathcal{K}$$

and the orthogonal projection p of $\mathcal{K}$ onto the *closure* in $\mathcal{K}$ of the supspace $\pi(A)\xi$. Then $p \in \pi(A)'$ by Lemma 2.11.

Using precisely the same argument as in the proof of Proposition 2.13, one sees that

$$\pi(b)(\xi) = \begin{pmatrix} b\xi_1 \\ \vdots \\ b\xi_n \end{pmatrix} \in \overline{\pi(A)\xi}.$$

In particular, there exists $a \in A$ such that $\|(b-a)\xi_j\| < \epsilon$ for all $j \in \{1, ..., n\}$, namely there exists $a \in A \cap \mathcal{V}$. □

2.18. Corollary. *Let A be an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ which contains $id_{\mathcal{H}}$. The following are equivalent:*

(i) A is a von Neumann algebra, i.e. $A'' = A$,
(ii) A is strongly closed in $\mathcal{B}(\mathcal{H})$,
(iii) A is weakly closed in $\mathcal{B}(\mathcal{H})$.

Proof. One has (i) $\Longrightarrow$ (iii) because commutants are weakly closed, (iii) $\Longrightarrow$ (ii) because the strong topology is stronger than the weak one, and (ii) $\Longrightarrow$ (i) by von Neumann's Density Theorem. □

2.19. Separability. Let $\mathcal{H}$ be a separable Hilbert space and let $(\xi_n)_{n\in\mathbb{N}}$ be a countable dense sequence in the unit ball of $\mathcal{H}$. On the unit ball of $\mathcal{B}(\mathcal{H})$, the strong topology can be defined by the distance

$$d_s(a,b) \;=\; \sum_{n\in\mathbb{N}} \frac{1}{2^n} \|(a-b)(\xi_n)\|$$

and the weak topology by

$$d_w(a,b) \;=\; \sum_{m,n\in\mathbb{N}} \frac{1}{2^{m+n}} \big|\langle \xi_m \mid (a-b)(\xi_n)\rangle\big|.$$

Each of these makes the unit ball of $\mathcal{B}(\mathcal{H})$ a second countable complete metric space. If A is a von Neumann algebra on $\mathcal{H}$, the same facts hold for the unit ball of A. It follows that a von Neumann algebra on a separable Hilbert space can always be generated by a countable set of elements. More on this (together with the canonical references to Bourbaki) in [DvN, § I.3].

Recall that *sequences* are appropriate to study the topology of *metrizable* spaces (see Bourbaki, Topologie générale, chap. 9, § 2, n^o 6). To study general topological spaces, one needs more efficient tools such as *filters* or *nets* (the choice depends on cultural background: see e.g. Problem L in Chapter 2 of [Kel]). This is demonstrated by the following fact; we copy the argument from [DvN, § I.3, exercice 3].

If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then $\mathcal{B}(\mathcal{H})$ with the strong topology is not metrizable. Indeed, consider an orthonormal basis $(e_n)_{n\geq 1}$ of $\mathcal{H}$ and, for each $n \geq 1$, the orthogonal projection P_n of $\mathcal{H}$ onto $\mathbb{C}e_n$. Set $T_{m,n} = P_m + mP_n$ and consider the set $\mathcal{T} = \{T_{m,n}\}_{m\geq 1, n\geq 1}$. We claim firstly that 0 is in the strong closure of $\mathcal{T}$, and secondly that no sequence from $\mathcal{T}$ converges strongly to 0 in $\mathcal{B}(\mathcal{H})$. The two claims imply clearly that the strong topology on $\mathcal{B}(\mathcal{H})$ is not metrizable.

For the first claim, consider $\epsilon > 0$ and $\xi = \sum_{n\geq 1} \xi_n e_n \in \mathcal{H}$. For m large enough, one has $|\xi_m| < \epsilon$; then, for n large enough, one has $|\xi_n| < \epsilon/m$; thus

$$\|T_{m,n}(\xi)\| \;\leq\; |\xi_m| + m|\xi_n| \;\leq\; 2\epsilon.$$

It follows that 0 is in the strong closure of $\mathcal{T}$. For the second claim, assume ab absurdo that some sequence $\left(T_{m_j,n_j}\right)_{j\geq 1}$ of elements of $\mathcal{T}$ converges strongly to 0. The Banach-Steinhaus theorem implies that $\sup_{j\geq 1} \left\|T_{m_j,n_j}\right\| \leq \infty$. It follows firstly that $m_j \leq m$ for some integer m, and one may assume that $m_j = m$ is independent on j. But then $\left(P_m + mP_{n_j}\right)_{j\geq 1}$ does not converge strongly to zero, in contradiction with the hypothesis. This proves the second claim.

2.20. Exercise. *(i) Let $\mathcal{H}$ be a Hilbert space and let p, q be two projections in $\mathcal{B}(\mathcal{H})$. Then the sequence of n^{th} term $(pqp)^n$ converges strongly, and its limit is the projection onto the space $p(\mathcal{H}) \cap q(\mathcal{H})$.*

(ii) Let A be a von Neumann algebra on $\mathcal{H}$ and let $p, q \in A$ be two projections. Then the projection $p \wedge q$ of $\mathcal{H}$ onto the closed linear span of $p(\mathcal{H})$ and $q(\mathcal{H})$ belongs to A.

Hints. Prove first (i) when $\mathcal{H}$ is finite dimensional. For the general case, use Proposition 4.44 below. The claim in (ii) follows from that in (i) because $p \wedge q = 1 - \Big((1-p)\vee(1-q)\Big)$. If this hint is not enough, see e.g. Problem 95 in [Hal].

2.c. Operator algebras with units on finite dimensional spaces

2.21. Preview. Let A be an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ for some finite dimensional Hilbert space $\mathcal{H}$. For the time being, assume that A contains the identity operator of $\mathcal{H}$, written $id_{\mathcal{H}}$ or simply 1. We are going to show that one has an isomorphism of the form

$$A \approx \bigoplus_{j=1}^{r} M_{n_j}(\mathbb{C})$$

or in other words that A is a *multi-matrix algebra.* We are also going to make precise the embedding of $\bigoplus_{j=1}^{r} M_{n_j}(\mathbb{C})$ in $\mathcal{B}(\mathcal{H})$.

One could prove this using classical results of algebra: a finite dimensional involutive algebra is semi-simple, hence the direct sum of its simple two-sided ideals, and each of these is isomorphic to some full matrix algebra because the underlying field is $\mathbb{C}$. (Some further points on this are sketched in Appendix II.a of [GHJ].) But we are using below another method, dealing firstly with abelian C*-algebras of operators, secondly with factors, and finally with the general case.

Recall from 2.13 that one has

$$\begin{aligned} &\{ \text{ involutive subalgebras of } M_n(\mathbb{C}) \text{ with } 1 \ \} \\ = &\{ \text{ von Neumann algebras on } \mathbb{C}^n \} \end{aligned}$$

(see also 2.8.i).

2.22. Lemma. *Let A denote an involutive subalgebra of $M_n(\mathbb{C})$, let $a \in A$ be a self-adjoint element and let $\mu_1, ..., \mu_s$ denote the distinct non zero eigenvalues of a. Then there exist orthogonal projections $q_1, ..., q_s \in A$ such that*

$$a = \sum_{k=1}^{s} \mu_k q_k.$$

Proof. By the spectral theorem for self-adjoint matrices, one may write $a = \sum_{k=1}^{s} \mu_k q_k$ with the q_k 's being orthogonal projections in $M_n(\mathbb{C})$. For each $k \in \{1, ..., s\}$, there exists a polynomial f_k such that $f_k(\mu_l) = \delta_{k,l}$ (Kronecker delta), and $f_k(0) = 0$, for example the Lagrange polynomial

$$f_k(T) \ = \frac{T}{\mu_k} \frac{\prod_{1 \leq l \leq s, l \neq k} \ (T - \mu_l)}{\prod_{1 \leq l \leq s, l \neq k} \ (\mu_k - \mu_l)} \ .$$

One has consequently $q_k = f_k(a) \in A$ for each $k \in \{1, ..., s\}$. □

2.23. Proposition. *Let A be an abelian involutive subalgebra of $M_n(\mathbb{C})$. Let $\mathcal{P} = \{p_1, ..., p_r\}$ be the set of minimal projections in A. Then*

$$A = \bigoplus_{j=1}^{r} \mathbb{C} p_j .$$

Proof. Let $a \in A$ be self-adjoint. By Lemma 2.22, one may write $a = \sum_{k=1}^{s} \mu_k q_k$ with the μ_k 's in $\mathbb{R}^*$ and the q_k 's in A.

Let $k \in \{1, ..., s\}$. For each $j \in \{1, ..., r\}$, the projections p_j and q_k commute, so that $p_j q_k$ is also a projection; one has $p_j q_k \in \{0, p_j\}$ and $q_k = \sum_{j=1}^{r} p_j q_k$ by definition of $\mathcal{P}$. Hence $a = \sum_{j=1}^{r} \lambda_j p_j$, each λ_j being the non zero eigenvalue μ_k indexed by that k for which $p_j q_k \neq 0$.

As each $a \in A$ is the sum of two self-adjoint elements, this ends the proof. □

2.24. Remarks. (i) The abelian C*-algebra A of Proposition 2.23 contains the unit $1 = id_{\mathcal{H}}$ if and only if $\sum_{j=1}^{r} p_j = 1$. Observe however that, in all cases, $\sum_{j=1}^{r} p_j$ is a multiplicative unit in A which is consequently an algebra *with unit.*

(ii) The algebra A of 2.23 is isomorphic to the "algebra of continuous functions" of 2.3 for a space X with r points and for the counting measure on X.

2.25. Lemma. *Let A be a sub-C*-algebra of $M_n(\mathbb{C})$ containing 1. Then any element a in A is a linear combination of four unitaries in A.*

Proof. As

$$a = \|a\| \frac{a+a^*}{2\|a\|} - i\|a\| \frac{ia - ia^*}{2\|a\|},$$

it is enough to check that any self-adjoint element b of norm 1 in A is a linear combination of two unitaries. By Lemma 2.22 there exist real numbers $\mu_1, ..., \mu_s$ in $[-1,1]$ (zero allowed this time !) and orthogonal projections $q_1, ..., q_s$ in A such that $\sum_{k=1}^s q_k = 1$ and $b = \sum_{k=1}^s \mu_k q_k$. Then

$$u = \sum_{k=1}^{s} \left(\mu_k + i\sqrt{1-\mu_k^2} \right) q_k$$

is unitary and $b = \frac{1}{2}(u + u^*)$, which ends the proof.

Functional calculus (Theorem 4.24) shows that this proof carries over to any C*-algebra with unit, with $u = b + i\sqrt{1-b^2}$. □

2.26. Proposition. *Let A be a factor on $\mathcal{H} \approx \mathbb{C}^n$ and let $p, q \in A$ be two projections distinct from 0. Then there exists $a \in A$ such that $paq \neq 0$.*

Proof. For each unitary u in A, let $p_u = upu^* \in A$ be the projection of $\mathcal{H}$ onto $u(p(\mathcal{H}))$. Let e denote the projection of $\mathcal{H}$ onto the subspace E of $\mathcal{H}$ spanned by the $u(p(\mathcal{H}))$'s; one has $e \in A$ by 2.20.i. The space E is obviously invariant by any unitary in A, hence also by any element in A (see Lemma 2.25). Hence $e \in A'$ (Lemma 2.11). As A is a factor, $e \in \mathbb{C}id_{\mathcal{H}}$. As $e \neq 0$ (because $p \neq 0$), one has $e = 1$.

Suppose now ab absurdo that $paq = 0$ for all $a \in A$. Then $upu^*q = 0$ for all unitary element $u \in A$, hence $eq = q = 0$, which is preposterous. □

The previous argument carries over to infinite dimensions, so that Proposition 2.26 holds for an arbitrary factor (see also [DvN], chap. I, § 1, Corollary 3 of Proposition 7). But the following argument works in finite dimensions only.

Another proof. Let du denote the Haar measure of mass 1 on the compact group

$$U(A) = \{ u \in A \mid u^*u = uu^* = 1 \}$$

and set $z = \int_{U(A)} uqu^* du$. By invariance of the Haar measure, one has $vz = zv$ for all $v \in U(A)$, hence $z \in Z(A)$ by the previous lemma. Moreover, if $tr : M_n(\mathbb{C}) \to \mathbb{C}$ denotes the usual trace, one has

$$tr(z) = \int_{U(A)} tr(uqu^*)du = \int_{U(A)} tr(q)du = tr(q) \neq 0$$

and consequently $z \neq 0$. It follows that z is a nonzero multiple of the identity.

Suppose ab absurdo that $paq = 0$ for all $a \in A$. Then $puqu^* = 0$ for all $u \in U(A)$, so that $pz = 0$ by integration on $U(A)$. But this is absurd because $p \neq 0$ and z is a nonzero multiple of 1. □

2.27. Lemma. *Let A be an involutive subalgebra of $M_n(\mathbb{C})$ and let $p \in A$ be a projection distinct from 0. Then p is minimal (among projections of A distinct from 0, for the ordering recalled in 1.18) if and only if $pAp \approx \mathbb{C}$.*

Proof. Assume firstly that p is minimal. Let $a \in pAp$ be a self-adjoint element distinct from 0. By Lemma 2.22, one may write $a = \sum_{k=1}^{s} \mu_k q_k$ with the μ_k 's in $\mathbb{R} \setminus 0$ and the q_k 's in pAp. For each $k \in \{1, ..., s\}$, one has therefore $pq_k = q_k$, so that $q_k = p$ by minimality of p. In other words, one may write $a = \mu p$. It follows that $dim_{\mathbb{C}}(pAp) = 1$.

Assume secondly that p is not minimal, and let $p_1, p_2 \in A$ be two nonzero orthogonal projections in A such that $p = p_1 + p_2$. It is then obvious that $dim_{\mathbb{C}}(pAp) \geq 2$. □

2.28. Remark. The previous lemma does *not* hold for infinite dimensional C*-algebras. Indeed, it is easy to check that 1 is a minimal projection in $\mathcal{C}([0,1])$, or more generally in $\mathcal{C}(X)$ for any *connected* compact space X. But an appropriate phrasing holds for any von Neumann algebra (see e.g. Proposition 4.10 in [StZ]).

2.29. Proposition. *Let A be a factor on $\mathcal{H} \approx \mathbb{C}^n$. Then there exists a divisor m of n such that $A \approx M_m(\mathbb{C})$.*

Moreover, for each minimal projection p of A, the space $p(\mathcal{H})$ is a (n/m)-dimensional subspace of $\mathcal{H}$.

Proof. Let $e \in A$ be a projection which is maximal for the property "eAe is a full matrix algebra" (this makes sense because of Lemma 2.27). For the first claim, we have to show that $e = 1$. We assume ab absurdo that $e \neq 1$ and we shall reach a contradiction.

Lef $f \in A, f \neq 0$ be a projection which is minimal for the property "$fe = 0$". Then f is minimal in A, so that $fAf \approx \mathbb{C}$ by Lemma 2.27. By Proposition 2.26, one may choose $a \in A$ such that $eaf \neq 0$. One has also $(eaf)^*eaf = fa^*eaf \neq 0$. Hence there exists $\lambda \in \mathbb{C} \setminus 0$ so that $fa^*eaf = \lambda f$; indeed

$$\lambda \|f\xi\|^2 \;=\; \lambda \langle \xi \mid f\xi \rangle \;=\; \langle eaf\xi \mid eaf\xi \rangle \;\geq\; 0$$

for all $\xi \in \mathbb{C}^n$, and $\lambda > 0$. Set $w = \lambda^{-1/2} eaf$. As $w^*w = \frac{1}{\lambda} fa^*eaf = f$, the matrix w is a partial isometry with initial projection f. As $eww^* = ww^*$, the final projection of w is a sub-projection of e; moreover, as f is minimal in A, it follows that ww^* is also a minimal projection in A, and a fortiori in eAe.

Let $m \geq 1$ be the integer such that $eAe \approx M_m(\mathbb{C})$ and let $(w_{j,k})_{1 \leq j,k \leq m}$ be a system of matrix units in eAe. One may assume that $w_{m,m} = ww^*$ (see the end of 2.9). We may now extend $(w_{j,k})_{1 \leq j,k \leq m}$ to a system of matrix units of order $m+1$ according to the scheme

$$\begin{array}{cccc} w_{1,1} & \cdots & w_{1,m} & w_{1,m}w \\ \vdots & \vdots & \vdots & \vdots \\ w_{m,1} & \cdots & w_{m,m} & w_{m,m}w \\ w^*w_{m,1} & \cdots & w^*w_{m,m} & f \end{array}$$

so that $(e+f)A(e+f)$ is a full matrix algebra of order $m+1$. (More precisely we set $w_{k,m+1} = w_{k,m}w$ and $w_{m+1,k} = w^*w_{m,k}$ for all $k \in \{1, ..., m\}$, as well as $w_{m+1,m+1} = f$.)

This is in contradiction with the definition of e. It follows that $e = 1$, namely that A is a full matrix algebra.

Consider now the restriction to A of the usual trace $tr : M_n(\mathbb{C}) \to \mathbb{C}$. As $w_{1,k}^* w_{1,k} = w_{k,k}$ and $w_{1,k} w_{1,k}^* = w_{1,1}$, the projections $w_{1,1}$ and $w_{k,k}$ have the same trace, for all $k \in \{1, ..., m\}$; this value of the trace is precisely n/m, and the proof is complete. □

2.30. Definition. The integer n/m in 2.29 is called the *multiplicity* of the representation of $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$.

For example

$$\begin{cases} M_2(\mathbb{C}) & \longrightarrow & M_6(\mathbb{C}) \\ x & \longmapsto & \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \end{cases}$$

is a representation of multiplicity 3. Proposition 2.29 shows that *any* involutive subalgebra of $M_6(\mathbb{C})$ which contains 1 and which is isomorphic to $M_2(\mathbb{C})$ is conjugated to the image of the above inclusion.

2.31. Theorem. *Let $\mathcal{H} \approx \mathbb{C}^n$ be a finite dimensional space and let A be an involutive subalgebra of $\mathcal{B}(\mathcal{H}) \approx M_n(\mathbb{C})$ containing 1. Let $p_1, ..., p_r$ be the minimal projections in the center $Z(A)$ of A. Then there are strictly positive integers $n_1, ..., n_r$ such that $p_j A p_j \approx M_{n_j}(\mathbb{C})$ for all $j \in \{1, ..., r\}$, and*

$$A \approx \bigoplus_{j=1}^{r} M_{n_j}(\mathbb{C}).$$

Moreover, if μ_j denotes the multiplicity of the representation of $p_j A p_j$ in $\mathcal{B}(p_j \mathcal{H})$, then $\sum_{i=1}^{r} \mu_j n_j = 1$.

Proof. One has

$$A = \bigoplus_{j=1}^{r} p_j A p_j$$

by definition of the p_j 's and $Z(A) = \bigoplus_{j=1}^{r} \mathbb{C} p_j$ by Proposition 2.23. The theorem follows from the previous proposition. □

2.32. Remark. Some studies have been devoted to "real C*-algebras". In finite dimensions, they are direct sums of matrix algebras over the reals, the complex or the quaternions.

2.D. Examples of operator algebras on finite dimensional spaces

2.33. Example. *Let V be a Hilbert space of dimension $n \geq 2$. Let $(v_j)_{j \in \mathbb{Z}/n\mathbb{Z}}$ be an orthonormal basis indexed by the cyclic group of order n. Let ω be a primitive n^{th} root of*

1. *Define two unitary operators* a, b *on* V *by*

$$a(v_j) = v_{j+1}$$
$$b(v_j) = \omega^j v_j$$

for all $j \in \mathbb{Z}/n\mathbb{Z}$, *so that* $ba = \omega ab$. *Then the von Neumann algebra generated by* a *and* b *is* $\mathcal{B}(V) \approx M_n(\mathbb{C})$.

Proof. Let A be the von Neumann algebra generated by a and b. As $\omega^j \neq \omega^k$ for $j, k \in \mathbb{Z}/n\mathbb{Z}, j \neq k$, the only operators which commute with b are the diagonal operators. Among diagonal operators, the only ones which commute with a are clearly the scalar multiples of the identity. Hence $A' = \mathbb{C}id_V$, so that $A = A'' = \mathcal{B}(V)$. □

2.34. Remarks. (i) It is easy to check that the abstract complex algebra $\mathcal{A}$ presented by two generators α, β and three relations

$$\alpha^n = 1 \ , \ \beta^n = 1 \ , \ \alpha\beta = \omega\beta\alpha$$

is of dimension $\leq n^2$. It follows from Example 2.33 that $\mathcal{A} \approx M_n(\mathbb{C})$.

(ii) Let ω be a primitive m^{th} root of 1 for some divisor m of n. Define $a, b \in M_n(\mathbb{C})$ by the same formulas as above. One may show that the von Neumann algebra on V generated by a and b is isomorphic to $M_m(\mathbb{C}) \oplus ... \oplus M_m(\mathbb{C})$, with n/m factors.

(iii) Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $(v_j)_{j\in\mathbb{Z}}$ be an orthonormal basis indexed by the infinite cyclic group. Choose an irrational number $\theta \in]0, 1[$ and set $\omega = e^{i2\pi\theta}$. Define two unitary operators a, b on $\mathcal{H}$ by

$$a(v_j) = v_{j+1}$$
$$b(v_j) = \omega^j v_j$$

for all $j \in \mathbb{Z}$. Observe that

$$ba = \omega ab.$$

The same argument as in the previous example shows that the von Neumann algebra generated by a and b is $\mathcal{B}(\mathcal{H})$ itself. In other words, the so-called *irrational rotation algebra* A_θ, namely the C*-algebra of operators on $\mathcal{H}$ generated by a and b, acts irreducibly on $\mathcal{H}$ (because its commutant A'_θ is reduced to $\mathbb{C}$).

2.35. Example. *Let* V *be a Hilbert space of dimension* $n \geq 2$. *Let* $v_1, ..., v_n$ *be an orthonormal basis of* V. *The tensor product* $V \otimes V$ *is naturally a Hilbert space which has an orthonormal basis* $(v_j \otimes v_k)_{1\leq j,k\leq n}$. *Consider the three projections*

e_1 *onto the subspace* $\mathbb{C}(v_1 + ... + v_n) \otimes V$ *of* $V \otimes V$
e_2 *onto the subspace* $\oplus_{1\leq j\leq n} \mathbb{C}(v_j \otimes v_j)$ *of* $V \otimes V$
e_3 *onto the subspace* $V \otimes \mathbb{C}(v_1 + ... + v_n)$ *of* $V \otimes V$.

Then one has

$$\begin{aligned} e_1e_2e_1 &= \frac{1}{n}e_1 & e_2e_1e_2 &= \frac{1}{n}e_2 \\ e_2e_3e_2 &= \frac{1}{n}e_2 & e_3e_2e_3 &= \frac{1}{n}e_3 \\ & e_1e_3 = e_3e_1. \end{aligned}$$

Moreover the von Neumann algebra on $V \otimes V$ *generated by* e_1, e_2 *is isomorphic to*

$$\begin{cases} \mathbb{C} \oplus M_2(\mathbb{C}) & \textit{if} \quad n \geq 3 \\ M_2(\mathbb{C}) & \textit{if} \quad n = 2. \end{cases}$$

Proof. For all $j, k \in \{1, ..., n\}$, one computes successively

$$\begin{aligned} e_1(v_j \otimes v_k) &= \frac{1}{n} \sum_{1 \leq l \leq n} v_l \otimes v_k \\ e_2(v_j \otimes v_k) &= \delta_{j,k}\ v_j \otimes v_k \\ e_1e_2e_1(v_j \otimes v_k) &= \frac{1}{n^2} \sum_l v_l \otimes v_k = \frac{1}{n} e_1(v_j \otimes v_k) \\ e_2e_1e_2(v_j \otimes v_k) &= \frac{1}{n} \delta_{j,k} v_j \otimes v_k = \frac{1}{n} e_2(v_j \otimes v_k) \\ e_1e_3(v_j \otimes v_k) &= \frac{1}{n^2} \sum_{l,m} v_l \otimes v_m = e_3e_1(v_j \otimes v_k). \end{aligned}$$

The formulae for the products of the e_j 's follow.

[A digression on the braid group on n strings ($n \leq 4$) could be in order here.]

Let A_0 be the unital subalgebra of $\mathcal{B}(V \otimes V)$ generated by e_1 and e_2. As any word in e_1, e_2 can be written as a scalar multiple of one of the words

(*) $\qquad 1 = \text{empty word} \quad , \quad e_1 \quad , \quad e_2 \quad , \quad e_1e_2 \quad , \quad e_2e_1 \quad ,$

one has $dim_{\mathbb{C}}(A_0) \leq 5$. As

$$e_2e_1(v_1 \otimes v_2) = \frac{1}{n} v_2 \otimes v_2 \neq 0 = e_1e_2(v_1 \otimes v_2),$$

the algebra A_0 is not commutative. Hence A_0 contains a two-sided ideal isomorphic to $M_2(\mathbb{C})$. If $n \geq 3$, we leave it to the reader to check that the five words in $(*)$ are linearly independent in $\mathcal{B}(V \otimes V)$. (Hint: evaluate a linear combination of these on the vectors $v_1 \otimes v_2$ and $v_2 \otimes v_2$.) It follows that

$$n \geq 3 \quad \Longrightarrow \quad A_0 \approx \mathbb{C} \oplus M_2(\mathbb{C}).$$

If $n = 2$, one checks that $2(e_1 + e_2 - e_1e_2 - e_2e_1) = id_{V\otimes V}$, so that $dim_{\mathbb{C}}A \leq 4$, and thus $A \approx M_2(\mathbb{C})$. $\square$

2.36. Example. *Let the notations be as in 2.35, and assume that $n \geq 3$. Then the von Neumann algebra on $V \otimes V$ generated by e_1, e_2 and e_3 is isomorphic to*

$$\mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

Proof. Let A be the unital subalgebra of $\mathcal{B}(V \otimes V)$ generated by e_1, e_2 and e_3. As any word in these can be written as a scalar multiple of one of the words

$$(**) \qquad \begin{array}{lllll} & & e_1e_2 & e_1e_2e_3 & \\ & e_1 & & & \\ & & e_1e_3 & e_1e_3e_2 & \\ & & e_2e_1 & e_2e_1e_3 & e_2e_1e_3e_2 \\ 1 & e_2 & & & \\ & & e_2e_3 & & \\ & & & & \\ & e_3 & e_3e_2 & e_3e_2e_1 & \end{array}$$

one has $dim_{\mathbb{C}}(A) \leq 14$.

Let $j, k \in \{1, ..., n\}$ be such that $|j - k| \geq 2$. The line spanned by

$$v_j \otimes v_k - v_{j+1} \otimes v_k - v_j \otimes v_{k+1} + v_{j+1} \otimes v_{k+1}$$

is in the kernel of e_1, e_2 and e_3. In particular, this line is invariant by A, and it follows that A contains a two-sided ideal isomorphic to $\mathbb{C}$.

Any unital homomorphism $\pi : A \to \mathbb{C} \approx \mathcal{B}(\mathbb{C})$ has to map e_1, e_2, e_3 onto 0 (because $e_1^2 = e_1 \Longrightarrow \pi(e_1) \in \{0, 1\}$ and $e_1e_2e_1 = \frac{1}{n}e_1 \Longrightarrow \pi(e_1) = 0$, and so on). Hence there is a unique unital homomorphism from A to $\mathbb{C}$, so that A contains a *unique* two-sided ideal of dimension 1.

Let W be the subspace of $V \otimes V$ spanned by

$$w_d = \sum_j v_j \otimes v_j \qquad \text{and} \qquad w_s = \sum_{j,k} v_j \otimes v_k.$$

One has

$$\begin{array}{ll} e_1(w_d) = \frac{1}{n}w_s & e_1(w_s) = w_s \\ e_2(w_d) = \frac{1}{n}w_d & e_2(w_s) = w_d \\ e_3(w_d) = \frac{1}{n}w_s & e_3(w_s) = w_s \end{array}$$

so that W is invariant by A. As W is clearly irreducible by A, it follows that A contains a two-sided ideal isomorphic to $M_2(\mathbb{C})$.

Consider the orthogonal $W^\perp$ of W in $V \otimes V$. On $W^\perp$, the projection e_1e_3 acts as 0, because the image of $V \otimes V$ by e_1e_3 is the subspace $\mathbb{C}w_s$ of W. We may choose a unit vector

$$\xi_2 \ \in \ W^\perp \cap Im(e_2)$$

(for example an appropriate scalar multiple of $v_1 \otimes v_1 - v_2 \otimes v_2$ would do). Set $\xi_1 = e_1(\sqrt{n}\xi_2)$ and $\xi_3 = e_3(\sqrt{n}\xi_2)$. One has

$$\langle \xi_1 | \xi_1 \rangle \ = \ n \langle e_1e_2(\xi_2) | e_1e_2(\xi_2) \rangle \ = \ n \langle \xi_2 | e_2e_1e_2(\xi_2) \rangle \ = \ 1$$

and similarly $\langle \xi_2 | \xi_2 \rangle = 1$. Let X be the subspace of $W^\perp$ spanned by the three unit vectors ξ_1, ξ_2, ξ_3. One has

$$\begin{aligned} \langle \xi_2 | \xi_1 \rangle \ &= \ \sqrt{n} \langle e_2(\xi_2) | e_1e_2(\xi_2) \rangle \ = \ \sqrt{n} \langle \xi_2 | e_2e_1e_2(\xi_2) \rangle \ = \ \frac{1}{\sqrt{n}} \\ \langle \xi_2 | \xi_3 \rangle \ &= \ \sqrt{n} \langle e_2(\xi_2) | e_3e_2(\xi_2) \rangle \ = \ \sqrt{n} \langle \xi_2 | e_2e_3e_2(\xi_2) \rangle \ = \ \frac{1}{\sqrt{n}} \\ \langle \xi_1 | \xi_3 \rangle \ &= \ n \langle e_1(\xi_2) | e_3(\xi_2) \rangle \ = \ n \langle \xi_2 | e_1e_3(\xi_2) \rangle \ = \ 0. \end{aligned}$$

As $n \geq 3$, it follows that $dim_{\mathbb{C}}(X) = 3$. Now X is clearly invariant by A, and one has consequently a morphism of $*$-algebras

$$\phi : A \longrightarrow \mathcal{B}(X)$$

such that $\phi(1) = 1$.

One has $e_2(\xi_2) = \xi_2$ by definition of e_2, one computes

$$e_2(\xi_1) \ = \ e_2e_1(\sqrt{n}e_2\xi_2) \ = \ \frac{1}{\sqrt{n}}\xi_2$$

and similarly $e_2(\xi_3) \ = \ \frac{1}{\sqrt{n}}\xi_2$. Hence $\phi(e_2)$ is the orthogonal projection of X onto $\mathbb{C}\xi_2$. Similarly $\phi(e_1)$ [respectively $\phi(e_3)$] is the orthogonal projection of X onto $\mathbb{C}\xi_1$ [resp. $\mathbb{C}\xi_3$]. It is then straightforward to check that the resulting representation of A in X is irreducible, so that the morphism ϕ is onto. It follows that A contains a two-sided ideal isomorphic to $M_3(\mathbb{C})$.

Now $dim_{\mathbb{C}} (\mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C})) \ = \ 14$, and the conclusion follows. $\square$

2.37. Exercice. The notation being as in the previous examples, set

$$\begin{aligned} f_1 &= e_2e_1e_3e_2 \\ f_2 &= e_1e_3 \end{aligned}$$

and let J be the sub-C*-algebra of A generated by f_1 and f_2.

(i) Check that $f_1^2 = \frac{1}{n} f_1, f_2^2 = f_2$ and that

$$f_1 f_2 f_1 = \frac{1}{n^2} f_1 \qquad f_2 f_1 f_2 = \frac{1}{n^2} f_2.$$

(ii) Show that

$$J \approx \mathbb{C} \oplus M_2(\mathbb{C}).$$

(Hint: mimick the argument used in 2.35 for A_0.)

(iii) Check that

$$\begin{array}{ccc} e_1 f_1 = f_2 f_1, & e_2 f_1 = f_1, & e_3 f_1 = f_2 f_1 \\ e_1 f_2 = f_2, & e_2 f_2 = n f_1 f_2, & e_3 f_2 = f_2 \end{array}$$

so that J is a two-sided ideal in A.

2.38. Exercice. Let Γ be a finite group. The group algebra $\mathbb{C}[\Gamma]$ is the space of functions $\Gamma \to \mathbb{C}$ for the convolution product defined by

$$(a \star b)(\gamma) \;=\; \sum a\left(\gamma'\right) b\left(\gamma''\right)$$

(summation over pairs $(\gamma', \gamma'') \in \Gamma \times \Gamma$ satisfying $\gamma'\gamma'' = \gamma$). It has a natural multiplicative basis $(\epsilon_\gamma)_{\gamma \in \Gamma}$, where $\epsilon_\gamma \in \mathbb{C}[\Gamma]$ denotes the characteristic function of $\{\gamma\}$.

(i) Let $\ell^2(\Gamma)$ denote the vector space of $\mathbb{C}[\Gamma]$ given together with the scalar product defined by

$$\langle a | b \rangle \;=\; \sum_{\gamma \in \Gamma} \overline{a(\gamma)} b(\gamma).$$

By convolution, one may view $\mathbb{C}[\Gamma]$ as a subalgebra of $\mathcal{B}\left(\ell^2(\Gamma)\right)$. Check that it is indeed an involutive subalgebra and that, for each $a \in \mathbb{C}[\Gamma]$, the adjoint of the left-multiplication operator $b \mapsto a \star b$ is the left-multiplication operator $b \mapsto a^* \star b$ where

$$a^*(\gamma) \;=\; \overline{a\left(\gamma^{-1}\right)}$$

for all $\gamma \in \Gamma$.

(ii) Let $\hat{\Gamma}$ denote the set of equivalence classes of irreducible unitary representations of Γ. The elementary theory of representations of finite groups shows that one has

$$\mathbb{C}[\Gamma] \;\approx\; \bigoplus_{\pi \in \hat{\Gamma}} M_{d(\pi)}(\mathbb{C})$$

where $d(\pi)$ denotes the degree of π. Write down explicitly the list of these numbers $d(\pi)$ for

an abelian group of order n $\quad (n \geq 1)$,

the dihedral group D_n of order $2n$ $\quad (n \geq 1)$,

the symmetric group $\mathcal{S}_n$ of n letters $\quad (n \leq 4)$,

the alternated group $\mathcal{A}_n$ of n letters $\quad (n \leq 4)$.

We will come back on C*-group-algebras in Number 4.1.

2.39. Exercice. Structure of the generic Temperley-Lieb algebras.

P. de la Harpe and V. Jones, July 1995.

CHAPTER 3. COMPACT AND HILBERT-SCHMIDT OPERATORS

Let $\mathcal{H}$ and $\mathcal{H}'$ denote Hilbert spaces.

3.A. Compact operators

3.1. Definition. A bounded operator $a : \mathcal{H} \to \mathcal{H}'$ is *finite rank* if the image $a(\mathcal{H})$ is finite dimensional.

The linear space of such operators is denoted by

$$\mathcal{F}(\mathcal{H}, \mathcal{H}').$$

We write $\mathcal{F}(\mathcal{H})$ instead of $\mathcal{F}(\mathcal{H}, \mathcal{H})$.

3.2. Lemma. *For any bounded operator $a : \mathcal{H} \to \mathcal{H}'$, one has*

$$\overline{Im(a^*)} = Ker(a)^{\perp}$$
$$Ker(a^*) = Im(a)^{\perp}.$$

Proof. It is straightforward to check that $Im(a^*) \subset Ker(a)^{\perp}$, and also that $(Im(a^*))^{\perp} \subset Ker(a)$, namely that $\overline{Im(a^*)} \supset Ker(a)^{\perp}$. The first equality follows. The second is obtained by exchanging a and a^*. □

3.3. Proposition. *The space $\mathcal{F}(\mathcal{H})$ is a self-adjoint two-sided ideal in the C^*-algebra $\mathcal{B}(\mathcal{H})$.*

Proof. The subspace $\mathcal{F}(\mathcal{H})$ is obviously a two-sided ideal. It is self-adjoint by the previous lemma. □

3.4. Recall of vocabulary. Let X be a topological space. A subset Y of X is *relatively compact* if its closure $\overline{Y}$ is compact. Assume moreover that X is a metric space with distance d. A subset Y of X is *precompact* if, for every real number $\epsilon > 0$, there exists a finite subset S of Y such that $Y \subset \cup_{x \in S} B(x, \epsilon)$, where $B(x, \epsilon)$ denotes the open ball of center x and of radius ϵ.

Inside a *complete* metric space, a subset is relatively compact if and only if it is precompact. See e.g. 3.17.5 in [Di1].

3.5. Definition. A bounded operator $a : \mathcal{H} \to \mathcal{H}'$ is *compact* if the image by a of the closed unit ball $\mathcal{H}(1)$ in $\mathcal{H}$ is relatively compact in $\mathcal{H}'$.

The linear space of all compact operators from $\mathcal{H}$ to $\mathcal{H}'$ is denoted by

$$\mathcal{K}(\mathcal{H}, \mathcal{H}').$$

We write $\mathcal{K}(\mathcal{H})$ instead of $\mathcal{K}(\mathcal{H}, \mathcal{H})$.

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3.6. Remark. *(This is in some sense for experts only, and can be ignored in a first reading.) Let $a : \mathcal{H} \to \mathcal{H}'$ be a bounded operator. Then*

$$a(\mathcal{H}(1)) \quad \textit{is relatively compact in} \quad \mathcal{H}' \quad \Longleftrightarrow \quad a(\mathcal{H}(1)) \quad \textit{is compact.}$$

Proof. (See also n^{os} 5.2 to 5.4 in [Dou].) We introduce the *weak topology* on $\mathcal{H}$, of which a basis of neighbourhoods of 0 consists of the finite intersections of the sets of the form

$$\mathcal{V}_\xi = \{ \ \eta \in \mathcal{H} \ \mid \ |\langle \xi \mid \eta \rangle| \ < \ 1 \ \}$$

where $\xi \in \mathcal{H}, \xi \neq 0$. It follows from Tychonoff's theorem that the unit ball $\mathcal{H}(1)$ is compact with respect to this weak topology. One checks that the linear map $a : \mathcal{H} \to \mathcal{H}'$, which is by hypothesis continuous for the norm topologies, is also continuous for the weak topologies (more on this in Problem 130 of [Hal]). Hence $a(\mathcal{H}(1))$ is weakly compact in $\mathcal{H}'$ (because it is the continuous image of a compact set !).

Assume now that $a(\mathcal{H}(1))$ is relatively compact in $\mathcal{H}'$; we claim that $a(\mathcal{H}(1))$ is closed in $\mathcal{H}'$ (these notions being with respect to the norm topology). For each η in the norm closure of $a(\mathcal{H}(1))$, there exists a (generalized) sequence $(\xi_j)_{j \geq 0}$ in $\mathcal{H}(1)$ such that $(a(\xi_j))_{j \geq 0}$ converges towards η (for the norm topology, and a fortiori also for the weak topology). Upon replacing $(\xi_j)_{j \geq 0}$ by a subsequence, one may assume that $(\xi_j)_{j \geq 0}$ converges weakly towards some $\xi \in \mathcal{H}(1)$. By weak continuity of a, one has then $\eta = a(\xi) \in a(\mathcal{H}(1))$. This proves the claim, and the implication $\Rightarrow$ of the remark.

The implication $\Leftarrow$ is obvious. □

3.7. Lemma. *(i) The space of compact operators $\mathcal{K}(\mathcal{H}, \mathcal{H}')$ is closed in the space $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ of all operators.*

(ii) The space of finite rank operators $\mathcal{F}(\mathcal{H}, \mathcal{H}')$ is dense in the space $\mathcal{K}(\mathcal{H}, \mathcal{H}')$.

Proof. (i) Let $(a_j)_{j \geq 0}$ be a sequence in $\mathcal{K}(\mathcal{H}, \mathcal{H}')$ which converges toward some $a \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$.

Choose a number $\epsilon > 0$. There exists an integer $n \geq 0$ such that $\|a - a_n\| < \epsilon$ and there exists a finite subset S in $\mathcal{H}(1)$ such that

$$a_n(\mathcal{H}(1)) \subset \bigcup_{\xi \in S} B(a_n(\xi), \epsilon)$$

because a_n is compact. Hence, for each $\eta \in \mathcal{H}(1)$, there exists $\xi \in S$ such that

$$\|a(\eta) - a(\xi)\| \ \leq \ \|a(\eta) - a_n(\eta)\| \ + \ \|a_n(\eta) - a_n(\xi)\| \ + \ \|a_n(\xi) - a(\xi)\| \ \leq \ 3\epsilon.$$

In other words one has

$$a(H(1)) \subset \bigcup_{\xi \in S} B(a(\xi), 3\epsilon).$$

It follows that $a(\mathcal{H}(1))$ is precompact, namely that $a \in \mathcal{K}(\mathcal{H}, \mathcal{H}')$.

(ii) Let $a \in \mathcal{K}(\mathcal{H}, \mathcal{H}')$. For all $\epsilon > 0$ there is a finite subset $\{\xi_1, ..., \xi_n\}$ in $\mathcal{H}(1)$ such that the closed balls of radius ϵ and of centers the $a(\xi_j)$'s cover $a(\mathcal{H}(1))$. Let E be the finite dimensional subspace of $\mathcal{H}'$ generated by the $a(\xi_j)$'s and let p denote the orthogonal projection of $\mathcal{H}'$ onto E. Then $pa \in \mathcal{F}(\mathcal{H}, \mathcal{H}')$. For all $\xi \in \mathcal{H}(1)$, there exists $j \in \{1, ..., n\}$ such that $\|a(\xi) - a(\xi_j)\| \leq \epsilon$. This shows that $\|a - pa\| \leq \epsilon$, and the proof is complete. □

3.8. Remark. The proof of Claim (ii) in the previous lemma relies essentially on the orthogonal projection p, and does *not* carry over to general Banach spaces. Indeed, P. Enflo showed in 1973 that there are Banach spaces for which the norm closure of finite rank operators is *strictly* contained in the space of compact operators (see [Enf] or [LiT, Section 1.e]).

3.9. Proposition. *The space $\mathcal{K}(\mathcal{H})$ is a closed self-adjoint two-sided ideal in the C^*-algebra $\mathcal{B}(\mathcal{H})$.*

In particular $\mathcal{K}(\mathcal{H})$ is a C^*-algebra of operators on $\mathcal{H}$.

Proof. It is straightforward to check that $\mathcal{K}(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, which is closed by Claim (i) of the previous lemma. Claim (ii) of the same Lemma implies that $\mathcal{K}(\mathcal{H})$ is self-adjoint, because $\mathcal{F}(\mathcal{H})$ is self-adjoint. □

3.10. Examples. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space given together with an orthonormal basis $(\xi_j)_{j\in\mathbb{N}}$ and let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers which converges to zero. Then the *diagonal operator* a defined by

$$a\xi_j = \lambda_j \xi_j$$

for all $j \in \mathbb{N}$ is compact, because it is a norm limit of operators of finite rank. We will show in Proposition 3.15 that any compact operator which is also self-adjoint is of this form (with real λ_j 's).

For each function $f \in L^2([0,1] \times [0,1])$, the operator a_f with *kernel* f is defined by

$$(a_f(\xi))(x) = \int_0^1 f(x,y)\xi(y)dy$$

for all $\xi \in L^2([0,1])$ and $x \in [0,1]$. It is a compact operator on $L^2([0,1])$. See Problem 135 in [Hal].

3.11. Lemma. *(i) The unit ball of the Hilbert space $\mathcal{H}$ is precompact if and only if $\mathcal{H}$ is finite dimensional.*

(ii) Let $a \in \mathcal{K}(\mathcal{H}, \mathcal{H}')$ and let E be a closed linear subspace of $\mathcal{H}'$ which is contained in the image of a. Then E is finite dimensional.

Proof. (i) Let us check that $\mathcal{H}(1)$ is not precompact if $\mathcal{H}$ is infinite dimensional. Let $(\xi_j)_{j\in\mathbb{N}}$ be an orthonormal subset of $\mathcal{H}$. Then the open subsets

$$\left\{ \xi \in \mathcal{H}(1) \mid \|\xi - \xi_j\| \leq \frac{1}{\sqrt{2}} \right\}$$

are non empty, pairwise disjoint and of constant diameters. The claim follows.

(ii) Let p denote the orthogonal projection of $\mathcal{H}'$ onto E. Then $pa : \mathcal{H} \to E$ is a bounded operator which is onto. It follows from the open mapping theorem 1.5 that there exists a non empty ball in E which is contained in $pa(\mathcal{H}(1))$. Hence $dim_{\mathbb{C}}(E) < \infty$ by (i). □

3.12. Examples. Let $\mathcal{H}$ be infinite dimensional. The identity operator on $\mathcal{H}$ is not compact. (This is a straightforward consequence of part (ii) in the previous lemma.) Consequently, any invertible operator on $\mathcal{H}$ is not compact.

The next target is Proposition 3.15, which constitutes an epsilon of spectral theory.

3.13. Observation. Let $\xi, \eta \in \mathcal{H}(1)$ be two unit vectors such that $|\langle \eta \mid \xi \rangle| = 1$. Then there exists $\theta \in \mathbb{R}$ such that $\xi = e^{i\theta}\eta$.

3.14. Lemma. *Let a be a compact self-adjoint operator on $\mathcal{H}$. Then one at least of the numbers $\|a\|, -\|a\|$ is an eigenvalue of a.*

Proof. We may assume without loss of generality that $\|a\| = 1$, hence that $\|a^3\| = 1$. (Indeed, it is obvious that $\|a^3\| \leq \|a\|^3 = 1$. If one had $\|a^3\| < 1$ one would also have $\|a^4\| \leq \|a^3\| \ \|a\| < 1 = \|a\|^4$, in contradiction with Corollary 1.12.) By Proposition 1.17, there exists a sequence $(\xi_j)_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} \langle \xi_j \mid a^3 \xi_j \rangle = \lim_{j \to \infty} \langle a\xi_j \mid a(a\xi_j) \rangle \in \{1, -1\}.$$

Upon replacing $(\xi_j)_{j \in \mathbb{N}}$ by a subsequence, one may assume by compacity of a that the sequence $(a(\xi_j))_{j \in \mathbb{N}}$ converges to some $\eta \in \mathcal{H}(1)$. Then

$$\langle \eta \mid a\eta \rangle \in \{1, -1\}$$

and the proof follows from the previous observation. □

3.15. Proposition. *Let a be a compact self-adjoint operator on $\mathcal{H}$. Then there exists an orthonormal basis $(\xi_j)_{j \in J}$ consisting of eigenvectors of a, and the corresponding sequence $(\lambda_j)_{j \in J}$ of eigenvalues converges to 0.*

Proof. (We assume for simplicity of the notations that $\mathcal{H}$ is infinite dimensional.) Set $a_0 = a$ and let $\eta_0 \in \mathcal{H}(1)$ be an eigenvector of a of eigenvalue $\lambda_0 \in \{\|a_0\|, -\|a_0\|\}$. Then a is an orthogonal sum of the operator $\lambda_0 id$ on $\mathbb{C}\eta_0$ and of a compact self-adjoint operator a_1 on the subspace $\mathcal{H}_1 = \eta_0^{\perp}$ of $\mathcal{H}$; moreover $\|a_1\| \leq \|a\|$.

By induction, one constructs in the same way an orthonormal sequence $(\eta_k)_{k \in K}$ of eigenvectors of a and a sequence $(\lambda_k)_{k \in K}$ of corresponding eigenvalues. It follows from Lemma 3.11.ii that the latter sequence converges to zero. Then a is an orthogonal sum of a diagonal operator (see Example 3.10) on the closed subspace $\mathcal{H}'$ of $\mathcal{H}$ spanned by the η_j 's and of the zero operator on $(\mathcal{H}')^{\perp}$. One manufactures an orthonormal basis $(\xi_j)_{j \in J}$ as in the proposition by concatenating $(\eta_k)_{k \in K}$ and an orthonormal basis of $(\mathcal{H}')^{\perp}$. □

3.16. Exercice. Assume $\mathcal{H}$ is a separable infinite dimensional Hilbert space.

(i) For any non trivial two-sided ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$, show that

$$\mathcal{F}(\mathcal{H}) \subset \mathcal{J} \subset \mathcal{K}(\mathcal{H})$$

(for the second inclusion, use Lemma 3.11.ii).

(ii) Show that the *Calkin algebra* $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is simple (more on this in [Har]).

(iii) For any $a \in \mathcal{K}(\mathcal{H})$, it is known that there exists a two-sided ideal $\mathcal{J}$ in $\mathcal{B}(\mathcal{H})$ such that

$$a \in \mathcal{J} \subsetneqq \mathcal{K}(\mathcal{H})$$

(see [Sal]).

3.B. Hilbert-Schmidt operators

3.17. Definition. A bounded operator $a : \mathcal{H} \to \mathcal{H}'$ is *Hilbert-Schmidt* if it is compact and if the series $(\mu_j)_{j\in J}$ of the eigenvalues of a^*a is summable. The *Hilbert-Schmidt norm* of such an operator is

$$\|a\|_2 = \sqrt{\sum_{j\in J} \mu_j}.$$

3.18. Lemma. *Let $a : \mathcal{H} \to \mathcal{H}'$ be a bounded operator. Let $(\xi_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$ and let $(\eta_k)_{k\in K}$ be an orthonormal basis of $\mathcal{H}'$. The three families of non negative real numbers*

$$\left(\|a\xi_j\|^2\right)_{j\in J} \quad , \quad \left(\|a^*\eta_k\|^2\right)_{k\in K} \quad , \quad \left(|\langle \eta_k \mid a\xi_j\rangle|^2\right)_{j\in J, k\in K}$$

are simultaneously summable or not. If they are summable, the three sums have the same value, which depends consequently only on a and not on the choosen basis.

Proof. By Parseval's equality, one has

$$\|a\xi_j\|^2 = \sum_{k'\in K} |\langle \eta_{k'} \mid a\xi_j\rangle|^2 \quad \text{and} \quad \|a^*\eta_k\|^2 = \sum_{j'\in J} |\langle \xi_{j'} \mid a^*\eta_k\rangle|^2$$

for all $j \in J$ and $k \in K$. If any of the families above is summable, one has

$$\sum_{j\in J} \|a\xi_j\|^2 = \sum_{j\in J, k\in K} |\langle \eta_k \mid a\xi_j\rangle|^2 = \sum_{j\in J, k\in K} |\langle \xi_j \mid a^*\eta_k\rangle|^2 = \sum_{k\in K} \|a^*\eta_k\|^2$$

and the proof is complete. $\square$

3.19. Proposition. *(i) Let $a \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$, let $(\xi_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$ and let $(\eta_k)_{k\in K}$ be an orthonormal basis of $\mathcal{H}'$. Let $(a_{k,j})_{k\in K, j\in J}$ be the resulting J-times-K matrix, where $a_{k,j} = \langle \eta_k \mid a\xi_j \rangle$. The three following conditions are equivalent*

$$a \text{ is a Hilbert-Schmidt operator,}$$
$$\sum_{j\in J} \|a\xi_j\|^2 < \infty,$$
$$\sum_{k\in K, j\in J} |a_{k,j}|^2 < \infty.$$

If they hold, then
$$\|a\|^2 = \sum_{j\in J} \|a\xi_j\|^2 = \sum_{k\in K, j\in J} |a_{k,j}|^2,$$

and the adjoint a^* of a is also a Hilbert-Schmidt operator such that

$$\|a^*\|_2 = \|a\|_2 \geq \|a\|.$$

Proof. Exercice. □

3.20. Proposition. *The set of all Hilbert-Schmidt operators form $\mathcal{H}$ to itself constitute a self-adjoint two-sided ideal in $\mathcal{B}(\mathcal{H})$ which is complete for the Hilbert-Schmidt norm $a \mapsto \|a\|_2$ and in which finite rank operators are dense.*

Moreover the ideal of Hilbert-Schmidt operators is itself a Hilbert space for the scalar product defined by
$$\langle a \mid b \rangle = \sum_{j\in J} \langle a\xi_j \mid b\xi_j \rangle$$
where $(\xi_j)_{j\in J}$ is an orthonormal basis of $\mathcal{H}$.

Proof. Exercice.

In case $\dim_{\mathbb{C}}(\mathcal{H}) < \infty$, observe that $\langle a|b \rangle = trace(a^*b)$; this carries over to the infinite dimensional case in terms of the canonical trace, defined on the appropriate two-sided ideal of *trace class* operators. □

3.21. Remark. For the theory of operator ideals in general and of Hilbert-Schmidt operators in particular, the classical book is that by R. Schatten [Sch]. For one (out of many) more recent exposition of Hilbert-Schmidt operators, see [Sim].

3.22. Example. For each $f \in L^2([0,1] \times [0,1])$, the operator $a_f \in \mathcal{B}\left(L^2([0,1])\right)$ defined in 3.10 by
$$(a_f(\xi))(x) = \int_0^1 f(x,y)\xi(y)dy$$

is a Hilbert-Schmidt operator; see Problems 135-6 in [Hal]. Indeed, it can be shown that any Hilbert-Schmidt operator on $L^2([0,1])$ is of this form (and this carries over from $[0,1]$ to an arbitrary σ-finite measure space); see Section II.2 in [Sch].

3.23. Example. Let $\mathbb{T}$ denote the unit circle in the complex plane, furnished with its usual measure $d\mu(e^{it}) = \frac{1}{2\pi}dt$ for $t \in [0, 2\pi]$. Let $(e_n)_{n\in\mathbb{Z}}$ denote the orthonormal basis of $L^2(\mathbb{T})$ used for Fourier series, defined by

$$e_n(t) = e^{int}$$

for all $t \in [0, 2\pi]$. The *Hardy space* $H^2(\mathbb{T})$ is defined by

$$H^2(\mathbb{T}) = \left\{ \xi \in L^2(\mathbb{T}) \;\middle|\; \langle e_{-n} \mid \xi\rangle = \frac{1}{2\pi}\int_0^1 e^{int}\xi(t)dt = 0 \ \text{ for all } \ n > 0 \right\}$$

and is a closed subspace of $L^2(\mathbb{T})$ (it is just the orthogonal of $\{e_{-1}, e_{-2}, ...\}$). We denote by p the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Recall that each $f \in L^\infty(\mathbb{T})$ defines a multiplication operator $M_f \in \mathcal{B}\left(L^2(\mathbb{T})\right)$. In case f is continuously differentiable, $f \in \mathcal{C}^1(\mathbb{T})$, we claim that

$$(*) \qquad pM_f - M_f p \quad \text{is a Hilbert-Schmidt operator.}$$

To prove this last statement, consider the matrices of the operators p and M_f with respect to the basis $(e_n)_{n\in\mathbb{Z}}$. On one hand one has

$$p \quad \leftrightsquigarrow \quad \begin{pmatrix} \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \end{pmatrix}$$

On the other hand, if $f = \sum_{n\in\mathbb{Z}} c_n e_n$,

$$M_f \quad \leftrightsquigarrow \quad \begin{pmatrix} \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ \cdots & c_0 & c_{-1} & c_{-2} & c_{-3} & c_{-4} & c_{-5} & \cdots \\ \cdots & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & c_{-4} & \cdots \\ \cdots & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & \cdots \\ \cdots & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & \cdots \\ \cdots & c_4 & c_3 & c_2 & c_1 & c_0 & c_{-1} & \cdots \\ \cdots & c_5 & c_4 & c_3 & c_2 & c_1 & c_0 & \cdots \\ \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \end{pmatrix}$$

It follows that

$$pM_f - M_f p \quad \leftrightsquigarrow \quad \begin{pmatrix} \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ \cdots & 0 & 0 & 0 & c_{-3} & c_{-4} & c_{-5} & \cdots \\ \cdots & 0 & 0 & 0 & c_{-2} & c_{-3} & c_{-4} & \cdots \\ \cdots & 0 & 0 & 0 & c_{-1} & c_{-2} & c_{-3} & \cdots \\ \cdots & c_3 & c_2 & c_1 & 0 & 0 & 0 & \cdots \\ \cdots & c_4 & c_3 & c_2 & 0 & 0 & 0 & \cdots \\ \cdots & c_5 & c_4 & c_3 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \end{pmatrix}$$

This is a Hilbert-Schmidt operator if and only if the matrix is square summable, namely if and only if f is in the so-called *Sobolev space*

$$H^{\frac{1}{2}}(\mathbb{T}) \;=\; \left\{ f = \sum_{n\in\mathbb{Z}} c_n e_n \;\middle|\; \sum_{n\in\mathbb{Z}} n|c_n|^2 \;<\; \infty \right\}.$$

(This notation is standard; do not confuse the "H" in the Hardy space $H^2(\mathbb{T})$ and the "H" in the Sobolev space above.) If $f \in \mathcal{C}^1(\mathbb{T})$, then $\sum_{n\in\mathbb{Z}} n^2|c_n|^2 \;<\; \infty$, and a fortiori $f \in H^{\frac{1}{2}}(\mathbb{T})$; this proves (*).

3.24. Exercise. Let A be the C*-algebra of operators on ℓ^2 generated by the unilateral shift of 1.19. Show that A contains the C*-algebra $\mathcal{K}(\ell^2)$ of all compact operators on ℓ^2.

P. de la Harpe and V. Jones, July 1995.

CHAPTER 4. ABSTRACT C*-ALGEBRAS AND FUNCTIONAL CALCULUS

4.A. Definition and first examples

4.1. C*-group algebras. We begin by a motivation for Definition 4.2 (see another one in 2.3).

Let G be a locally compact group and let μ denote a left invariant Haar measure on G. One introduces the Hilbert space $L^2(G,\mu)$, usually written $L^2(G)$ for simplicity. (Any other left invariant measure on G is of the form $c\mu$ for some $c \in \mathbb{R}_+^*$, and changing μ to $c\mu$ just amounts to multiplying all scalar products by c.) The *left regular representation* u of G in $L^2(G)$ is given by

$$(u_g\xi)(h) = \xi\left(g^{-1}h\right)$$

for all $g, h \in G$ and $\xi \in L^2(G)$. It is a *unitary representation* in the following sense: each u_g is a unitary operator on $L^2(G)$, the assignment $g \mapsto u_g$ is a homomorphism from the group G to the unitary group of $L^2(G)$, and the mapping $(g,\xi) \mapsto u_g(\xi)$ is continuous from $G \times \mathcal{H}$ to $\mathcal{H}$.

The *reduced C*-algebra* of G is the C*-algebra $C^*_{red}(G)$ of operators on $L^2(G)$ generated by the operators of the form $\int_G f(g)u_g dg$, for $f \in L^1(G)$. In particular, if the group G is discrete - we then rather write Γ instead of G and $\ell^2(\Gamma)$ instead of $L^2(G)$ - then $C^*_{red}(\Gamma)$ is generated by the unitaries u_γ 's for $\gamma \in \Gamma$.

For a group Γ endowed with the counting measure, the space $\ell^2(\Gamma)$ has a canonical basis $(\epsilon_\gamma)_{\gamma\in\Gamma}$ where $\epsilon_\gamma : \Gamma \to \mathbb{C}$ is the characteristic function of $\{\gamma\}$. We write e the unit element of Γ. The linear mapping

$$\Phi : \begin{cases} C^*_{red}(\Gamma) & \longrightarrow \quad \ell^2(\Gamma) \\ \qquad a & \longmapsto \quad a(\epsilon_e) \end{cases}$$

is injective. To see this, one introduces the right regular representation v of Γ in $\ell^2(\Gamma)$ defined by

$$(v_\gamma\xi)(\gamma') = \xi(\gamma'\gamma)$$

for all $\gamma, \gamma' \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. The operators u_γ and $v_{\gamma'}$ commute for all $\gamma, \gamma' \in \Gamma$. Consider $a, b \in C^*_{red}(\Gamma)$ such that $\Phi(a) = \Phi(b)$. As a and b commute with the v_γ 's, one has

$$a(\epsilon_\gamma) = av_\gamma(\epsilon_e) = v_\gamma a(\epsilon_e) = v_\gamma b(\epsilon_e) = bv_\gamma(\epsilon_e) = b(\epsilon_\gamma)$$

for all $\gamma \in \Gamma$, and this shows that $a = b$.

It follows that any $a \in C^*_{red}(\Gamma)$ can be written as a sum

$$a = \sum_{\gamma\in\Gamma} \lambda_\gamma u_\gamma.$$

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We leave it to the reader to check that $\|\Phi(a)\|_{\ell^2(\Gamma)} \leq \|a\|$ for all $a \in C^*_{red}(\Gamma)$. There isn't any simple condition on a family $(\lambda_\gamma)_{\gamma\in\Gamma}$ of complex numbers which is both necessary and sufficient for the sum $\sum_{\gamma\in\Gamma} \lambda_\gamma u_\gamma$ to represent an operator in $C^*_{red}(\Gamma.)$ A particular case of this statement is that there isn't any simple condition on a trigonometric series which is both necessary and sufficient for the series to be the Fourier series of a continuous function.

Suppose moreover that Γ is a finite group. Then $C^*_{red}(\Gamma)$ is nothing but the usual group algebra $\mathbb{C}[\Gamma]$, with the involution given by $(u_\gamma)^* = u_{(\gamma^{-1})}$ for all $\gamma \in \Gamma$. In particular, there is a bijective correspondance between unitary representations of Γ and unit-preserving *-representations of $C^*_{red}(\Gamma)$, as already recalled in Exercicr 2.38.

However, if Γ is infinite, this correspondence breaks down in general (and more precisely in the case Γ is not "amenable"). For example, it is known that the C*-algebra $C^*_{red}(\Gamma)$ is simple in case Γ is a non abelian free group [Pow] or a group $PSL(n,\mathbb{Z})$ for some $n \geq 2$ [BCH] (see also Section 5.C below). In particular, most unitary representations of such groups (e.g. those factoring via finite quotients of the groups) do *not* correspond to any *-representation of their reduced C*-algebra.

The way to keep a correspondence between unitary representation of a locally compact group G and *-representations of some group C*-algebra is to introduce the so-called *maximal C*-algebra* $C^*_{max}(G)$, which in general does *not* act naturally on $L^2(G)$.

This is a strong motivation to introduce abstract C*-algebras, not just algebras of operators on specific Hilbert spaces.

4.2. Definitions. A *C*-algebra* is an involutive algebra A endowed with a norm $a \mapsto \|a\|$ such that

(i) $\|ab\| \leq \|a\| \ \|b\|$ for all $a, b \in A$,
(ii) $\|a^*a\| = \|a\|^2$ for all $a \in A$,
(iii) A is complete for the given norm.

Corollary 1.12 shows that, if $\mathcal{H}$ is a Hilbert space, any closed *-algebra of $\mathcal{B}(\mathcal{H})$ is a C*-algebra. There is a converse, phrased below as Theorem 6.14.

A *sub-C*-algebra* of a C*-algebra A is an involutive subalgebra of A which is complete for the norm topology.

A *morphism* $\Phi : A \to B$ between two C*-algebras is a linear map such that

$$\Phi(ab) = \Phi(a)\Phi(b)$$
$$\Phi(a^*) = \Phi(a)^*$$

for all $a, b \in A$. Corollary 4.25 below shows that these conditions imply

$$\|\Phi(a)\| \leq \|a\|$$

for all $a \in A$. If moreover Φ is injective, Proposition 4.37 shows that $\|\Phi(a)\| = \|a\|$ for all $a \in A$. (See also n^{os} 1.3.7 and 1.8.1 in [DC*].)

Recall from n^o 1.13 that a *representation* of a C*-algebra A on a Hilbert space $\mathcal{H}$ is a morphism $A \to \mathcal{B}(\mathcal{H})$.

The following notions are defined as in Section 1.C:

$a \in A$ is *normal* if $a^*a = aa^*$;
$a \in A$ is *self-adjoint* if $a^* = a$;
$a \in A$ is *positive* if there exists $b \in A$ such that $a = b^*b$ (see also Section 4.E);
$p \in A$ is a *projection* if $p^* = p = p^2$;
$w \in A$ is a *partial isometry* if w^*w is a projection;
$u \in A$ is *unitary* (when A has a unit) if $uu^* = u^*u = 1$.

4.3. Direct products. Let $(A_\iota)_{\iota \in I}$ be a family of C*-algebras. The *product C*-algebra* is the subspace of the space of families $(a_\iota)_{\iota \in I}$, with $a_\iota \in A_\iota$ for all $\iota \in I$, such that $\sup_{\iota \in I} \|a_\iota\| < \infty$. In this product, the multiplication and the involution are defined componentwise, and the norm is defined by

$$\left\|(a_\iota)_{\iota \in I}\right\| = \sup_{\iota \in I} \|a_\iota\| .$$

4.4. Remark. Let A be a C*-algebra, let

$$\begin{cases} A & \longrightarrow & \mathcal{B}(A) \\ a & \longmapsto & (u_a : a' \mapsto aa') \end{cases}$$

be its left regular representation and set

$$\|u_a\| = \sup_{a' \in A, \|a'\| \leq 1} \|aa'\| .$$

Then $\|u_a\| = \|a\|$ for all $a \in A$. This is a straightforward consequence of the relations

$$\|aa'\| \;\leq\; \|a\| \; \|a'\| \qquad \text{and} \qquad \|aa^*\| \;=\; \|a\| \; \|a^*\| .$$

4.5. Adding a unit. Let A be an involutive algebra. Define a new involutive algebra $\tilde{A}$ as follows. As a vector space, set $\tilde{A} = \mathbb{C} \oplus A$. The involution is defined on $\tilde{A}$ by

$$(\lambda, a)^* = \left(\overline{\lambda}, a^*\right)$$

and the product by

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab).$$

In particular A is a two-sided ideal in $\tilde{A}$ which is invariant by the involution. The algebra $\tilde{A}$ has a unit $\tilde{e} = (1, 0)$. In case A itself has a unit e, then $\tilde{A}$ is the product of its two-sided ideals $\mathbb{C}(\tilde{e} - e)$ and A.

If A is moreover an involutive Banach algebra, then $\tilde{A}$ is also an involutive Banach algebra for the norm defined by $\|(\lambda, a)\| = |\lambda| + \|a\|$. However, this choice of a norm on $\tilde{A}$ is often not the best one: for example, if A is a C*-algebra, it does *not* make $\tilde{A}$ a C*-algebra.

Assume from now on that A is a C*-algebra. The definition of the appropriate norm on $\tilde{A}$ is more delicate. If A has a unit, $\tilde{A}$ can be viewed as a product C*-algebra as in 4.3.

Suppose now that A has no unit. For each $x = (\lambda, a) \in \tilde{A}$, set

$$\|x\| = \sup_{b \in A, \|b\| \leq 1} \|xb\| .$$

Then $x \mapsto \|x\|$ is a norm on $\tilde{A}$ which extends the original norm on A by Remark 4.4. Let us check for example that $\|x\| = 0$ implies $x = 0$. If $\lambda = 0$, this is because $a \mapsto \|a\|$ is a norm on A. If one had $\|x\| = 0$ for $x = (\lambda, a)$ with $\lambda \neq 0$, one would have $\lambda b + ab = 0$, hence also $\left(-\lambda^{-1}a\right) b = b$ for all $b \in A$, so that $-\lambda^{-1}a$ would be a left-unit for A; hence $-\overline{\lambda}^{-1} a^*$ would be a right-unit for A, and it would follow that A has a unit, in contradiction with the hypothesis.

The norm just defined makes $\tilde{A}$ complete, because $\tilde{A}$ contains a one-codimensional complete subspace A.

Let us finally check that

$$\|x^*x\| \ = \ \|x\|^2$$

for all $x = (\lambda, a) \in \tilde{A}$. We may assume without loss of generality that $\|x\| = 1$, and we have to check that $\|x^*x\| = 1$. For each real number $r < 1$, there exists $b \in A$ such that $\|b\| \leq 1$ and $\|xb\| \geq r$. As $xb \in A$, one has

$$\|x^*x\| \ \geq \ \|b^*\| \ \|x^*x\| \ \|b\| \ \geq \ \|(xb)^*xb\| \ = \ \|xb\|^2 \ \geq \ r^2.$$

As this holds for all $r < 1$ one has $\|x^*x\| \geq 1$, and consequently $\|x^*x\| = 1$.

In conclusion, $\tilde{A}$ is naturally a C*-algebra whenever A is a C*-algebra.

4.6. Example: finite dimensional C*-algebras. Let A be a finite dimensional algebra. We know from Theorem 2.31 that $\tilde{A}$ is a direct sum of full matrix algebras. It follows that the two-sided ideal A of $\tilde{A}$ is also a direct sum of full matrix algebras, and in particular that *any finite dimensional C*-algebra has a unit.* Thus Remark 2.24.i applies with the only change that the p_j 's have to be viewed now as the minimal *central* projections of A.

4.7. Example: C*-algebras of compact operators. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. The space $\mathcal{K}(\mathcal{H})$ of compact operator on $\mathcal{H}$ is a C*-algebra without unit by Example 3.12.

The C*-algebra obtained by adding a unit is the C*-algebra of those operators on $\mathcal{H}$ of the form $\lambda id_{\mathcal{H}} + a$ with $\lambda \in \mathbb{C}$ and $a \in \mathcal{K}(\mathcal{H})$.

4.8. Example: C*-algebras of continuous functions. Let X be a locally compact space. The algebra $\mathcal{C}_o(X)$ of continuous functions $X \to \mathbb{C}$ which vanish at infinity is a C*-algebra for the norm defined by

$$\|f\| = \sup_{x \in X} |f(x)|.$$

This algebra has a unit if and only if X is compact, in which case it is the C*-algebra $\mathcal{C}(X)$ of *all* continuous functions on X (see 2.3).

If X is not compact, the C*-algebra obtained by adding a unit to $\mathcal{C}_o(X)$ is the algebra of those continuous functions $X \to \mathbb{C}$ which have a limit at infinity. If X is compact, the algebra $\widetilde{\mathcal{C}(X)}$ obtained by the same construction is the algebra of continuous functions on a space $\tilde{X}$ which is the disjoint union of X and of a point.

4.9. Examples of Banach algebras which are not C*-algebras. On the algebra $A = \mathcal{C}([-1,1])$ of continuous functions from $[-1,1]$ to $\mathbb{C}$, consider the norm defined by $\|f\| = \sup_{|t|\leq 1}|f(t)|$ and the involutions defined by $f^{*!}(t) = \overline{f(-t)}$. Then A is a Banach algebra with an involution such that $\|f^{*!}\| = \|f\|$ for all $f \in A$. But A is not a C*-algebra; indeed, for f defined by $f(t) = 0$ for $t \leq 0$ and $f(t) = t$ for $t \geq 0$, one has $\|f\| = 1$ and $f^{*!}f = 0$.

On the convolution algebra $A = \ell^1(\mathbb{Z})$, consider the norm defined by $\|c\|_1 = \sum_{n\in\mathbb{Z}}|c(n)|$ and the involution defined by $c^*(n) = \overline{c(-n)}$. Then A is also a Banach algebra with an involution such that $\|c^*\|_1 = \|c\|_1$ for all $c \in A$. But A is not a C*-algebra; indeed, for c defined by $c(1) = c(0) = -c(-1) = 1$ and $c(n) = 0$ when $|n| \geq 2$, one has $\|c\|_1 = 3$ and $\|c^*c\|_1 = 5$.

More generally, for any infinite locally compact abelian group G, the convolution algebra $L^1(G)$ is not a C*-algebra.

4.B. Spectrum of an element in a Banach algebra

Part of spectral theory for C*-algebra holds for more general algebras. In the present section, we consider a Banach algebra A with unit (the definition has been recalled in 1.4).

4.10. Definition. For each $a \in A$, the *spectrum* of a is the subset

$$\sigma(a) = \{\ \lambda \in \mathbb{C} \mid \lambda - a \ \text{ is not invertible in } \ A\ \}$$

of the complex plane.

A better notation would be $\sigma_A(a)$ instead of $\sigma(a)$. As our main interest is C*-algebras, the notation $\sigma(a)$ will turn out to be non-ambiguous (see 4.26 below).

4.11. Lemma. *For each $a \in A$, the sequence $\left(\|a^n\|^{\frac{1}{n}}\right)_{n\geq 1}$ is convergent and*

$$\lim_{n\to\infty}\|a^n\|^{\frac{1}{n}} \ = \ \inf_{n\geq 1}\|a^n\|^{\frac{1}{n}} \ \leq \ \|a\|\,.$$

Proof. For $n \geq 1$, set $\alpha_n = \log\|a_n\|$. One has $\|a^{p+q}\| \ \leq \ \|a^p\| \ \|a^q\|$, hence

$$\alpha_{p+q} \leq \alpha_p + \alpha_q$$

for each $p, q \geq 1$. It is then a quite standard lemma that the subadditive sequence $(\alpha_n)_{n\geq 1}$ converges to its minimum (see e.g. [PoS], Problem 98 of Part I, page 23). We recall the proof.

Choose an integer $q \geq 1$. Write each integer $n \geq 1$ as $n = kq + r$ with $k \geq 0$ and $r \in \{0, 1, ..., q-1\}$. One has

$$\frac{\alpha_n}{n} = \frac{\alpha_{kq+r}}{kq+r} \leq \frac{\alpha_{kq}}{kq} + \frac{\alpha_r}{kq} \leq \frac{\alpha_q}{q} + \frac{\alpha_r}{kq}.$$

It follows that

$$\limsup_{n\to\infty} \frac{\alpha_n}{n} \leq \inf_{q\geq 1} \frac{\alpha_q}{q}.$$

As one has obviously $\inf_{q\geq 1} \frac{\alpha_q}{q} \leq \liminf_{n\to\infty} \frac{\alpha_n}{n}$, one has also

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = \inf_{q\geq 1} \frac{\alpha_q}{q}$$

and the proof is complete. $\square$

4.12. Definition. The *spectral radius* of $a \in A$ is the real number

$$\rho(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Equivalently, $\rho(a)^{-1}$ is the radius of convergence of the series $\sum_{n=0}^{\infty} \lambda^n a^n$.

Observe that one has $\rho(a) \leq \|a\|$.

4.13. Lemma. *(i) Let $a \in A$ and let $\lambda \in \mathbb{C}$.*

If $|\lambda| < \rho(a)^{-1}$, the element $1 - \lambda a$ is invertible in A.
If $|\lambda| > \rho(a)$, then $\lambda \notin \sigma(a)$.

(ii) The set A^{inv} of invertible elements in A is open and the map

$$\begin{cases} A^{\mathrm{inv}} & \longrightarrow & A^{inv} \\ a & \longmapsto & a^{-1} \end{cases}$$

is continuous.

Proof. (i) If $\rho(\lambda a) = |\lambda|\rho(a) < 1$, it follows from the definition above that the series $\sum_{n=0}^{\infty} \lambda^n a^n$ is convergent, and its limit is $(1-\lambda a)^{-1}$.

If $|\lambda| > \rho(a)$, the previous claim implies that $\lambda - a = \lambda\left(1 - (\lambda)^{-1}a\right)$ is invertible.

(ii) Let $a \in A^{\mathrm{inv}}$. For each $b \in A$ such that $\|b - a\| < \left\|a^{-1}\right\|^{-1}$ the element

$$b = a\left(1 - a^{-1}(a-b)\right)$$

is in A^{inv} because $\rho\left(a^{-1}(a-b)\right) \leq \left\|a^{-1}(a-b)\right\| < 1$. Hence A^{inv} is open. If $\|b-a\| \leq \frac{1}{2}\left\|a^{-1}\right\|^{-1}$, one has moreover

$$\begin{aligned}\left\|b^{-1}-a^{-1}\right\| &= \left\|\sum_{n=0}^{\infty}\left(a^{-1}(a-b)\right)^n a^{-1} \; - \; a^{-1}\right\| \; \leq \; \sum_{n=1}^{\infty}\left\|a^{-1}(a-b)\right\|^n \left\|a^{-1}\right\| \\ &\leq \frac{\left\|a^{-1}\right\|^2 \; \|a-b\|}{1-\left\|a^{-1}(a-b)\right\|} \; \leq \; 2\left\|a^{-1}\right\|^2 \; \|a-b\|\end{aligned}$$

and it follows that $a \mapsto a^{-1}$ is a continuous transformation of A^{inv}. □

4.14. Proposition. *For each $a \in A$, the spectrum $\sigma(a)$ is a non empty compact subset of $\mathbb{C}$ which is contained in the closed disc of radius $\rho(a)$ centered at the origin, and a fortiori in the closed disc of radius $\|a\|$ around the origin.*

Moreover the spectral radius of a is given by

$$\rho(a) = \sup\{\ r \geq 0 \mid \ \textit{there exists} \ \ \lambda \in \sigma(a) \ \ \textit{such that} \ \ |\lambda| = r\ \}$$

(as its name indicates).

Proof. The first part of the previous Lemma shows that $\sigma(a)$ is contained in the closed disc of radius $\rho(a)$ around the origin, and the second part shows in particular that $\sigma(a)$ is closed in $\mathbb{C}$. Hence the spectrum $\sigma(a)$ is compact.

The second part of this Lemma shows also that $(\lambda - a)^{-1}$ is given around any $\lambda_0 \in \mathbb{C} \setminus \sigma(a)$ by an entire series in $\lambda - \lambda_0$, namely that the *resolvant* of a

$$\begin{cases} \mathbb{C} \setminus \sigma(a) & \longrightarrow \quad A \\ \quad \lambda & \longmapsto (\lambda - a)^{-1} \end{cases}$$

is an analytic mapping. If $\sigma(a)$ were empty, the resolvant would be a non constant bounded holomorphic function defined on the whole of $\mathbb{C}$, in contradiction with Liouville's theorem. Hence the spectrum $\sigma(a)$ is non empty.

If $\rho(a) = 0$, it is clear that $\sigma(a) = \{0\}$. Assume now that $\rho(a) \neq 0$. If the spectrum $\sigma(a)$ were contained in some closed disc centered at the origin of radius $r < \rho(a)$, the resolvant $\lambda \mapsto (\lambda - a)^{-1} = \lambda^{-1}\left(1 - \lambda^{-1}a\right)^{-1}$ would be analytic in the domain defined by $|\lambda| > r$. Then the map $z \mapsto (1 - za)^{-1}$ would be defined and analytic in the open disc of radius r^{-1} around 0, and its Taylor series at the origin $\sum_{n=0}^{\infty} z^n a^n$ would have a radius of convergence $r^{-1} > \rho(a)^{-1}$. This would contradict the definition of $\rho(a)$; hence the proof is complete. (For a proof which does not use the theory of analytic mappings, see Theorems 7 and 8 in §5 of [BoD].) □

4.15. Gelfand-Mazur Theorem (1938). *A Banach algebra with unit in which all elements distinct from 0 are invertible is isomorphic to the field of complex numbers.*

Proof. For each a in such a Banach algebra A, there exists $\lambda \in \mathbb{C}$ such that $\lambda - a$ is non invertible, hence by hypothesis such that $\lambda - a = 0$, or $a \in \mathbb{C}$. □

4.16. Examples of spectra. Let $e \in A$ be an idempotent, $e^2 = e$. Then the spectrum of e is contained in $\{0, 1\}$. For each $\lambda \in \mathbb{C} \setminus \{0, 1\}$, the resolvent of e is given by

$$(\lambda - e)^{-1} = \frac{1 - \lambda - e}{\lambda(1 - \lambda)}.$$

Let $a \in A$ be an element such that $a^n = 1$ for some integer $n \geq 2$. Let C_n denote the cyclic group of n^{th} roots of 1. For each $\lambda \in C_n$, set

$$p_\lambda = \frac{1}{n} \sum_{0 \leq j \leq n-1} \lambda^{-j} a^j \in A.$$

Then it is straightforward to check that

$$\begin{aligned} p_\lambda p_\mu &= \delta_{\lambda,\mu} p_\lambda \quad \text{for all} \quad \lambda, \mu \in C_n, \\ \sum_{\lambda \in C_n} p_\lambda &= 1, \\ \sum_{\lambda \in C_n} \lambda p_\lambda &= a. \end{aligned}$$

It follows that one has

$$(z - a)^{-1} = \sum_{\lambda \in C_n} \frac{1}{z - \lambda} p_\lambda$$

for each $z \in \mathbb{C} \setminus C_n$, and that the spectrum of a is contained in C_n.

4.17. Spectra of multiplication operators. Let $c = (c_n)_{n \in \mathbb{N}} \in \ell^\infty$ be a bounded sequence of complex numbers and let M_c be the corresponding multiplication operator on ℓ^2. Then the spectrum of M_c is the closure in $\mathbb{C}$ of the set $\{ z \in \mathbb{C} \mid \text{there exists } n \in \mathbb{N} \text{ such that } z = c_n \}$. In particular, any compact subset of $\mathbb{C}$ is the spectrum of some bounded operator on ℓ^2.

More generally, let (X, μ) be a measure space and let $f \in L^\infty(X, \mu)$. The spectrum of the multiplication operator M_f on $L^2(X, \mu)$ is the closure of the *essential range*

$$\{ z \in \mathbb{C} \mid \mu\left(f^{-1}(\mathcal{V})\right) > 0 \text{ for every neighbourhood } \mathcal{V} \text{ of } z \text{ in } \mathbb{C} \}$$

of f. For details, see Problems 48 to 52 in [Hal].

4.18. Lemma. *Let $f \in \mathbb{C}[T]$ be a polynomial with complex coefficients. Then*

$$\sigma(f(a)) = f(\sigma(a))$$

for all $a \in A$.

Proof. The lemma holds for constant polynomials because spectra are not empty by the previous proposition. We may therefore assume that f is not constant. In the proof below, we will use repeatedly the following fact : if a is a product $a_1 ... a_n$ of *commuting* factors in an algebra, then a is invertible if and only if each a_j is invertible.

Let $\lambda_o \in \sigma(a)$. Let g be the polynomial such that $f(T) - f(\lambda_o) = (T - \lambda_o) g(T)$. As $a - \lambda_o$ is not invertible,

$$f(a) - f(\lambda_o) = (a - \lambda_o) g(a)$$

is not invertible. Hence $f(\lambda_o) \in \sigma(f(a))$.

Let $\mu_o \in \sigma(f(a))$. There exists complex numbers $c, \lambda_1, ..., \lambda_n$ such that $f(T) - \mu_0 = c(T - \lambda_1)...(T - \lambda_n)$, hence such that

$$f(a) - \mu_o = c(a - \lambda_1)...(a - \lambda_n).$$

As $f(a) - \mu_0$ is not invertible, there exists $j \in \{1, ..., n\}$ such that $a - \lambda_j$ is not invertible. As $f(\lambda_j) = \mu_0$, this shows that $\mu_o \in f(\sigma(a))$. $\square$

4.19. Remark. Let $\mathbb{K}$ be an algebraically closed field and let A be a $\mathbb{K}$-algebra with unit. One may mimick Definition 4.10 for the *spectrum* $\sigma(a) \subset \mathbb{K}$ of any element a in A. One has a natural morphism of algebras

$$\begin{cases} \mathbb{K}[T] & \longrightarrow & A \\ f & \longmapsto & f(a). \end{cases}$$

for which Lemma 4.18 and its proof hold without change.

Much of spectral theory has the following goal: under appropriate hypothesis on A and on a (say with $\mathbb{K} = \mathbb{C}$), show that Lemma 4.18 holds for more general functions.

For example, let A be a Banach algebra with unit and let $a \in A$. If f is a function defined and holomorphic in an open neighbourhood U of the spectrum $\sigma(a)$ and if γ is an appropriate curve in $U \setminus \sigma(a)$ surrounding $\sigma(a)$, then one may define

$$f(a) \;=\; \frac{1}{2i\pi} \int_\gamma f(z)\,(z-a)^{-1} dz$$

and one shows that $\sigma(f(a)) = f(\sigma(a))$. For this *holomorphic functional calculus*, see for example § I.4 of [Bou] or the end of Chapter 2 in [StZ].

Other examples include functions continuous on the spectrum of a normal element in a C*-algebra (see Theorems 4.24 and 4.35), and functions Borel measurable on the spectrum of a normal element in a von Neumann algebra (see Section 4.F).

4.c. Spectrum of an element in a C*-algebra

Let A be a C*-algebra with unit. We denote by $\mathbb{T}$ the unit circle of the complex plane.

4.20. Proposition. *(i) For each $a \in A$, the spectrum of a^* is $\overline{\sigma(a)}$.*

(ii) If $a \in A$ is self-adjoint, its spectrum is in $\mathbb{R}$.

(iii) If $u \in A$ is unitary, its spectrum is in $\mathbb{T}$.

Proof. (i) For $\lambda \in \mathbb{C}$, the element $\overline{\lambda} - a^*$ is invertible (say with inverse b), if and only if $\lambda - a$ is invertible (with inverse b^*).

(ii) Let $\lambda = x + iy \in \sigma(a)$, with $x, y \in \mathbb{R}$. For each $t \in \mathbb{R}$ the number $x + i(y+t)$ is in $\sigma(a + it)$. As

$$\|a + it\|^2 \;=\; \|(a - it)(a + it)\| \;=\; \left\|a^2 + t^2\right\| \;\leq\; \|a\|^2 + t^2,$$

Proposition 4.14 implies that

$$|x + i(y+t)|^2 = x^2 + (y+t)^2 \leq \|a\|^2 + t^2$$

and this inequality can also be written as

$$2yt \leq \|a\|^2 - x^2 - y^2.$$

As this has to hold for all $t \in \mathbb{R}$, one has $y = 0$.

(iii) Let $\lambda \in \sigma(u)$. Observe that $\lambda \neq 0$, because u is invertible, and that $\lambda^{-1} \in \sigma(u^{-1})$, because $\lambda^{-1} - u^{-1} = -\lambda^{-1}(\lambda - u)u^{-1}$ is not invertible. As $|\lambda| \leq \|u\| = 1$ and $|\lambda^{-1}| \leq \left\|u^{-1}\right\| = 1$ by Proposition 4.14, one has $\lambda \in \mathbb{T}$. $\square$

4.21. Corollary. *(i) Let $a \in A$ be a normal element. Then*

$$\rho(a) = \|a\|.$$

In particular, if $a \in A$ is self-adjoint, then one at least of $\|a\|, -\|a\|$ is in the spectrum $\sigma(a)$.

(ii) Let $\mathcal{H}$ be a Hilbert space and let $a \in \mathcal{B}(\mathcal{H})$ be a self-adjoint element. Set

$$m(a) = \inf_{\substack{\xi \in \mathcal{H} \\ \|\xi\| \leq 1}} \langle \xi | a\xi \rangle \qquad \text{and} \qquad M(a) = \sup_{\substack{\xi \in \mathcal{H} \\ \|\xi\| \leq 1}} \langle \xi | a\xi \rangle .$$

then

$$\sigma \;\subset\; [m(a), M(a)].$$

Proof. (i) Assume first that a is self-adjoint. From the definition of C*-algebras, one has

$$\left\|a^{2^k}\right\| = \|a\|^{2^k}$$

for all $k \geq 0$, hence

$$\rho(a) = \lim_{k \to \infty} \left\|a^{2^k}\right\|^{2^{-k}} = \|a\|.$$

It follows from Propositions 4.14 and 4.20.ii that at least one of the numbers $\|a\|, -\|a\|$ is in $\sigma(a)$.

Assume now that a is normal. Then

$$\begin{aligned}
\rho(a^2) &\leq \|a^2\| && \text{by Lemma 4.11}\\
&\leq \|a\|^2 = \|a^*a\| = \rho(a^*a) && \text{by the argument above}\\
&= \lim_{n\to\infty} \|(a^*a)^n\|^{1/n} && \text{by definition of } \rho\\
&= \lim_{n\to\infty} \|(a^*)^n a^n\|^{1/n} && \text{because } a \text{ is normal}\\
&\leq \lim_{n\to\infty} \|(a^*)^n\|^{1/n} \lim_{n\to\infty} \|a^n\|^{1/n} &&\\
&= \rho(a^*)\rho(a) && \text{by definition of } \rho\\
&= \rho(a)^2 && \text{by proposition 4.20.i}\\
&= \rho(a^2) && \text{by Lemma 4.18.}
\end{aligned}$$

Hence all inequalities are equalities, and $\rho(a)^2 = \|a\|^2$.

(ii) Recall from Proposition 1.17 that $\|a\| = \max\{-m(a), M(a)\}$. Set $\lambda = \frac{1}{2}(M(a) + m(a))$ and $r = \frac{1}{2}(M(a) - m(a))$, so that $m(a-\lambda) = -r$ and $M(a-\lambda) = r$. One has firstly $\|a-\lambda\| = r$ by Proposition 1.17, secondly $\sigma(a-\lambda) \subset [-r, r]$ by Propositions 4.14 and 4.20, and finally $\sigma \subset [m(a), M(a)]$ by (a trivial case of) Lemma 4.18. □

4.22. Corollary. *For any $a \in A$ one has*

$$\|a\|^2 \;=\; \rho(a^*a).$$

4.23. Remark. There are elements $a \in A$ such that $\rho(a) < \|a\|$. The simplest example is probably the nilpotent operator $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. Another example is the *Volterra integration operator* V defined on $L^2([0,1])$ by

$$(Vf)(x) = \int_0^x f(t)dt.$$

It is a *quasi-nilpotent* operator, namely one with spectral radius equal to zero. See Problem 80 and Solution 147 in [Hal].

4.24. Theorem (Continuous functional calculus for bounded self-adjoint operators). *Let A be a C*-algebra with unit, let $a \in A$ be a self-adjoint element, and let $\mathcal{C}(\sigma(a))$ be the C*-algebra of continuous functions on the spectrum of a (as in 4.8). Then there exists a unique morphism of C*-algebra*

$$\begin{cases} \mathcal{C}(\sigma(a)) & \longrightarrow & A \\ \qquad f & \longmapsto & f(a) \end{cases}$$

which maps the constant function 1 *[respectively the inclusion of* $\sigma(a)$ *in* $\mathbb{C}$*] to the operator* $id_{\mathcal{H}}$ *[resp. to* a*]. Moreover, one has*

$$\sigma(f(a)) = f(\sigma(a))$$

for all $f \in (\sigma(a))$*.*

Proof. Let $\mathcal{P}(\sigma(a))$ denote the involutive subalgebra of $\mathcal{C}(\sigma(a))$ consisting of the restrictions to $\sigma(a)$ of the polynomial functions $\mathbb{R} \to \mathbb{C}$, and let $\mathbb{C}[a]$ denote the involutive subalgebra of A consisting of the elements $f(a)$ with $f \in \mathbb{C}[T]$ a complex polynomial in one variable. It is a straightforward consequence of Lemma 4.18 and of Corollary 4.21 that $\|f(a)\| = \sup\{\ |f(\lambda)| \mid \lambda \in \sigma(a)\ \}$ for every $f \in \mathbb{C}[T]$, namely that the obvious morphism

$$\begin{cases} \mathcal{P}(\sigma(a)) & \longrightarrow & \mathbb{C}[a] \\ \qquad f & \longmapsto & f(a) \end{cases}$$

is well defined and isometric (and a fortiori injective !). By the Weierstrass Approximation Theorem, this has an isometric extension from $\mathcal{C}(\sigma(a))$ onto the sub-C*-algebra $C^*(a)$ of A generated by a (which is also the closure of $\mathbb{C}[a]$ in A). We fix now a function $f \in \mathcal{C}(\sigma(a))$, and we have to check that $\sigma(f(a)) = f(\sigma(a))$.

Consider first $\mu \in f(\sigma(a))$. Choose $\lambda \in \sigma(a)$ such that $\mu = f(\lambda)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}(\sigma(a))$ with limit f. Then $(f_n(\lambda) - f_n(a))_{n\in\mathbb{N}}$ converges to $\mu - f(a)$. As $f_n(\lambda) - f_n(a)$ is not invertible for each $n \in \mathbb{N}$ by Lemma 4.18, it follows from Lemma 4.13.ii (the set of non invertible elements in A is closed) that $\mu \in \sigma(f(a))$. Hence $f(\sigma(a)) \subset \sigma(f(a))$.

Consider then $\mu \in \mathbb{C} \setminus f(\sigma(a))$. The function g defined by $g(t) = (\mu - f(t))^{-1}$ is in $\mathcal{C}(\sigma(a))$ and $g(a) = (\mu - f(a))^{-1}$, so that $\mu \notin \sigma(f(a))$. Hence $\sigma(f(a)) \subset f(\sigma(a))$. □

This Theorem carries over to normal operators : see Theorem 4.35 below.

4.25. Corollary. *Let* A *be a Banach involutive algebra (see 1.13), let* B *be a* C^**-algebra and let* $\Phi : A \to B$ *be a linear map such that* $\Phi(aa') = \Phi(a)\Phi(a')$ *and* $\Phi(a^*) = \Phi(a)^*$ *for all* $a, a' \in A$*. Then*

$$\|\Phi(a)\| \ \leq \ \|a\|$$

for all $a \in A$*.*

Proof. Suppose first that A and B have units and that $\Phi(1) = 1$. For each $x \in A$, one has obviously $\sigma(\Phi(x)) \subset \sigma(x)$. In case $x = a^*a$ for some $a \in A$, this implies

$$\begin{aligned} \|\Phi(a)\|^2 \ = \ \|\Phi(x)\| \ &= \ \sup\{\ r > 0 \mid r \in \sigma(\Phi(x))\ \} \\ &\leq \ \sup\{\ r > 0 \mid r \in \sigma(x)\ \} \ \leq \ \|x\| \ \leq \ \|a\|^2 . \end{aligned}$$

For the general case, the Corollary follows from the same argument applied to the natural morphism

$$\tilde{\Phi} : \begin{cases} \quad \tilde{A} & \longrightarrow & \tilde{B} \\ (\lambda, a) & \longmapsto & (\lambda, \Phi(a)) \end{cases}$$

(see 4.5). □

4.26. Proposition. *Let B be a C*-algebra with unit, let $A \subset B$ be a sub-C*-algebra containing the unit and let $a \in B$. Then a is invertible in A if and only if it is invertible in B. More generally, the spectra*

$$\sigma_A(a) = \{ \ \lambda \in \mathbb{C} \mid \lambda - a \ \text{ is not invertible in } \ A \ \}$$

and

$$\sigma_B(a) = \{ \ \lambda \in \mathbb{C} \mid \lambda - a \ \text{ is not invertible in } \ B \ \}$$

coïncide.

Proof. It is obvious that $\sigma_A(a) \supset \sigma_B(a)$.

Let now $\lambda \in \sigma_A(a)$ be a *boundary point* of $\sigma_A(a)$. We claim that $\lambda \in \sigma_B(a)$. Indeed, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C} \setminus \sigma_A(a)$ which converges to λ. Set $x_n = (\lambda_n - a)^{-1} \in A$ for each $n \in \mathbb{N}$. If $\lambda - a$ were invertible in B, say with inverse x, then x would be the limit of the x_n 's by Lemma 4.13.ii, so that x would be an inverse of $\lambda - a$ in A, in contradiction with the hypothesis $\lambda \in \sigma_A(a)$. This proves that $\lambda \in \sigma_B(a)$, as claimed. (For the argument so far, it is enough to assume that A and B are Banach algebras with unit. They need not be C*-algebras.)

In particular, if $a \in A$ is self-adjoint, all points in $\sigma_A(a)$ are boundary points because $\sigma_A(a) \subset \mathbb{R}$ by Proposition 4.20.ii, and thus $\sigma_A(a) = \sigma_B(a)$.

Let $a \in A$ be an arbitrary element and let $\lambda \in \mathbb{C}$. If $\lambda - a$ is invertible in B, then $(\overline{\lambda} - a^*)(\lambda - a)$ and $(\lambda - a)(\overline{\lambda} - a^*)$ are both self-adjoint elements in A which are invertible in B. Hence they are invertible in A by the previous argument. Thus $\lambda - a$ is both right-invertible and left-invertible, namely invertible in A. It follows that $\sigma_A(a) = \sigma_B(a)$. □

4.27. Corollary on polar decomposition. *Let $\mathcal{H}, \mathcal{H}'$ be two Hilbert spaces and let $a : \mathcal{H} \to \mathcal{H}'$ be a bounded operator. Then there exists a pair (w, p) where*

w is a partial isometry from $\mathcal{H}$ to $\mathcal{H}'$,
p is a positive operator on $\mathcal{H}$

such that $a = wp$.

Moreover, there is a unique such pair (w, p) for which $Ker(w) = Ker(p)$, and it satisfies $a^*a = p^2$. This is called the **polar decomposition** *of a.*

Proof. The positive part of the polar decomposition of an operator a is given by $p = \sqrt{a^*a}$. For more details, see Problem 105 of [Hal], or almost any other book dealing with operators on Hilbert spaces.

Alternatively, one may define an appropriate bounded increasing sequence of self-adjoint operators with strong limit $\sqrt{a^*a}$ (see Proposition 4.44 below, as well as Problems 94 and 95 in [Hal]). □

4.28. Exercise. Let A be a C*-algebra with unit and let $e_0, e_1 \in A$ be two projections.

(i) If $\|e_1 - e_0\|$ is small enough, show that there exists a unitary element $u \in A$ such that $e_1 = ue_0u^*$.

(ii) If there exists a continuous path of projections $\begin{cases} [0,1] \to A \\ t \mapsto e_t \end{cases}$, show that the same conclusion holds.

(iii) Suppose moreover that there exists a tower $A_1 \subset A_2 \subset ...$ of sub-C*-algebras of A such that $\mathcal{A}_\infty = \bigcup_{n\geq 0} A_n$ is dense in A (examples will be discussed in Chapter 6). For each projection $e \in A$, show that there exists a unitary element $u \in A$ such that $ueu^* \in \mathcal{A}_\infty$.

(iv) Suppose moreover that $1 \in A_1$. For each unitary $u \in A$ and for each $\epsilon > 0$, show that there exists a unitary $v \in \mathcal{A}_\infty$ such that $\|v - u\| < \epsilon$.

[Indications. (i) Set first $g = 1 - e_0 - e_1 + 2e_0e_1$. Check that $e_0 g = g e_1$, and that g is invertible if $\|e_1 - e_0\|$ is small enough. Use functional calculus to define $u = g(g^*g)^{-1/2}$, check it solves (i), and that (ii) follows.

(iii) Let $x \in \mathcal{A}_\infty$ be such that $x^* = x$ and such that $\|e - x\|$ is small enough, and let $n \geq 1$ be such that $x \in A_n$. Using functional calculus in A_n, one finds a projection $f \in A_n$ such that $\|f - e\|$ is small, so that (iii) follows from (i).

(iv) Choose $\eta > 0$ (to be precised later), let $x \in \mathcal{A}_\infty$ be such that $\|x - u\| < \eta$ and let $n \geq 1$ be such that $x \in A_n$. By polar decomposition in A_n one has a unitary $v \in A_n$ such that $x = v\sqrt{x^*x}$, and $\|v - u\| < f(\eta)$ for some function f such that $\lim_{\eta\to 0} f(\eta) = 0$. Thus one may choose η such that $\|v - u\| < \epsilon$.]

The results of this exercise are used in Example 5.10 and in Exercice 6.33.

4.D. Gelfand-Naimark Theorem

4.29. Characters of abelian algebra. Let A be a commutative complex algebra. A *character* on A is a linear map $\chi : A \to \mathbb{C}$ *distinct from zero* such that

$$\chi(ab) = \chi(a)\chi(b)$$

for all $a, b \in A$. The set of all characters on A is denoted by $X(A)$. If A has a unit, observe that $\chi(1) = 1$ for any $\chi \in X(A)$. If A has no unit, any character χ on A extends uniquely to a character $\tilde{\chi}$ on $\tilde{A}$, defined by $\tilde{\chi}(\lambda, a) = \lambda + \chi(a)$ for all $(\lambda, a) \in \tilde{A}$. Moreover $X(\tilde{A})$ is naturally identified to the union of $X(A)$ and of the character $(\lambda, a) \mapsto \lambda$ of $\tilde{A}$.

If A is a commutative Banach algebra with unit, any character χ on A satisfies

$$(*) \qquad \sup_{a\in A, \|a\|\leq 1} |\chi(a)| \leq 1$$

for all $a \in A$, and in particular any character on A is continuous. Indeed, for such a χ and for all $a \in A$, one has obviously $\chi(a) \in \sigma(a)$, hence $|\chi(a)| \leq \|a\|$ by Proposition 4.14. If moreover A is a commutative C*-algebra with unit, any character χ on A satisfies also

$$(**) \qquad \chi(a^*) = \overline{\chi(a)}$$

for all $a \in A$. Indeed, if $a^* = a$ then $\chi(a) \in \sigma(a) \subset \mathbb{R}$ by Proposition 4.20. Hence, for any $a \in A$, one has

$$\chi(a^*) = \chi\left(\frac{a + a^*}{2} + i\frac{ia - ia^*}{2}\right) = \chi\left(\frac{a + a^*}{2}\right) + i\chi\left(\frac{ia - ia^*}{2}\right) = \overline{\chi(a)}$$

as claimed in (**). Properties (*) and (**) of characters hold also in a C*-algebra A without unit (consider the extension of characters from A to $\tilde{A}$).

For any commutative complex algebra A, one defines on $X(A)$ the topology of pointwise convergence. If A is a commutative Banach algebra with unit, (*) shows that there is a natural injection of $X(A)$ into the product of copies of the closed unit disc indexed by the unit ball of A; it follows then essentially from Tychonoff's Theorem that $X(A)$ is a compact space. If A is a commutative Banach algebra without unit, $X(A)$ is a locally compact space with one-point-compactification canonically identified with $X(\tilde{A})$. In all cases (with or without unit), the *Gelfand transform* is the homomorphism

$$\mathcal{G} : A \longrightarrow \mathcal{C}_o(X(A))$$

defined by $\mathcal{G}(a)(\chi) = \chi(a)$ for all $a \in A$ and $\chi \in X(A)$ (the notation $\mathcal{C}_o$ is that of 4.8).

It is known that the locally compact space $X(A)$ is metrizable if and only if the commutative C*-algebra A is separable.

4.30. Lemma. *Let A be a commutative Banach algebra with unit and let $\mathcal{G} : A \to \mathcal{C}(X(A))$ be the corresponding Gelfand transform. Then*

$$\sigma(\mathcal{G}(a)) = \sigma(a)$$

for all $a \in A$.

Proof. Let $\lambda \in \sigma(\mathcal{G}(a))$. There exists $\chi \in X(A)$ such that

$$(\lambda - \mathcal{G}(a))(\chi) = \chi(\lambda - a) = 0.$$

Hence $\lambda - a$ is not invertible, and $\lambda \in \sigma(a)$.

Let $\lambda \in \sigma(a)$. By Zorn's Lemma, there exists a maximal ideal $\mathcal{J}$ in A which contains $\lambda - a$; observe that $\mathcal{J}$ is closed in A (see Lemma 4.14.ii). Then $A/\mathcal{J}$ is both a Banach algebra and a field, and thus is isomorphic to the field of complex numbers by Gelfand-Mazur Theorem 4.15. The canonical projection $A \to A/\mathcal{J}$ can be viewed as a character $\chi \in X(A)$ such that $\chi(\lambda - a) = 0$. Hence $(\lambda - \mathcal{G}(a))(\chi) = 0$, and $\lambda \in \sigma(\mathcal{G}(a))$. □

4.31. Gelfand-Naimark Theorem (1943). *Let A be a commutative C*-algebra. Then the Gelfand transform*

$$\mathcal{G} : A \to \mathcal{C}_o(X(A))$$

defined in 4.29 is an isometrical isomorphism (the norm on the right-hand side being as in Example 4.8).

Proof. The image of $\mathcal{G}$ separates points of $X(A)$; indeed, if $\chi_1, \chi_2 \in X(A)$ are distinct characters, there exists $a \in A$ such that $\chi_1(a) \neq \chi_2(a)$, namely such that $\mathcal{G}(a)(\chi_1) \neq \mathcal{G}(a)(\chi_2)$. Observe also that

$$\mathcal{G}(a^*) = \mathcal{G}(a)^*$$

for all $a \in A$ by (**) of 4.29. Moreover, if A has a unit, the image of $\mathcal{G}$ contains the constants. It follows from Weierstrass Approximation Theorem that the image of the Gelfand transform is a dense subalgebra of $\mathcal{C}_o(X(A))$.

Moreover, one has

$$\begin{aligned}\|\mathcal{G}(a)\|^2 &= \|\mathcal{G}(a^*a)\| \\ &= \rho(\mathcal{G}(a^*a)) && \text{by Corollary 4.21} \\ &= \rho(a^*a) && \text{by Lemma 4.30} \\ &= \|a\|^2 && \text{by Corollary 4.21 again}\end{aligned}$$

for all $a \in A$. Hence $\mathcal{G}$ is isometric and this ends the proof. □

4.32. Corollary. *Let A be a commutative C*-algebra with unit and let $a \in A$. Assume that a generates A. Then the mapping*

$$\begin{cases} X(A) & \longrightarrow & \sigma(a) \\ \chi & \longmapsto & \chi(a) \end{cases}$$

is an homeomorphism.

Proof. The mapping is continuous, and its image is $\sigma(a)$ by the proof of Lemma 4.30.

To show that the mapping is injective, consider $\chi_1, \chi_2 \in X(A)$ such that $\chi_1(a) = \chi_2(a)$. The set $\{\ b \in A \mid \chi_1(b) = \chi_2(b)\ \}$ is a sub-C*algebra of A which contains 1; it is therefore A itself, and $\chi_1 = \chi_2$.

Hence the mapping is a homeomorphism. □

4.33. Ideals. Let X be a locally compact space and let $A = \mathcal{C}_o(X)$ be the corresponding abelian C*-algebra. For any ideal $\mathcal{J}$ of A, let

$$V(\mathcal{J}) = \Big\{ x \in X \ \Big|\ f(x) = 0 \ \text{ for all } \ f \in \mathcal{J} \Big\}$$

be the "variety of $\mathcal{J}$", which is a closed subset of X. For any subset Y of X, let

$$\mathcal{I}(Y) = \Big\{ f \in A \ \Big|\ f(y) = 0 \ \text{ for all } \ y \in Y \Big\}$$

be the "ideal of Y", which is a closed ideal of A. One has obvious inclusions $\overline{\mathcal{J}} \subset \mathcal{I}(V(\mathcal{J}))$ and $\overline{Y} \subset V(\mathcal{I}(Y))$. Indeed, it is not difficult to show that

$$\overline{\mathcal{J}} = \mathcal{I}(V(\mathcal{J})) \quad \text{ and } \quad \overline{Y} = V(\mathcal{I}(Y))$$

for any $\mathcal{J}$ and Y as above; see e.g. [Bou], chap. I, § 3, n^o 2, Proposition 1. (The statement compares with the Hilbert's Nullstellensatz, but is not nearly as deep !)

This establishes a bijective correspondance between closed ideals in A and closed subspaces of the character space $X(A)$. Here is another way to describe the same bijection. In one direction, any C*-algebra quotient $A \to B = A/\mathcal{J}$ provides inside $X(A)$ the closed subspace of those characters on A which factor through B. In the other direction, one may identify A to $\mathcal{C}_0(X(A))$, and each closed subspace Y of $X(A)$ provides the quotient of A consisting of the restrictions to Y of the continuous functions on $X(A)$.

In particular, maximal ideals in A (which are necessarily closed) correspond bijectively to characters on A, and also to points in $X(A)$.

4.34. Exercise. In the C*-algebra $\mathcal{C}([0,1])$, show that the principal ideal $\mathcal{J}$ generated by the function "t" is *not* closed.

Hint: the function $t \mapsto \sqrt{t}$ is not in $\mathcal{J}$, but the sequence of nth term

$$t \longmapsto \begin{cases} nt & \text{if} \quad t \leq \dfrac{1}{n^2} \\ \sqrt{t} & \text{if} \quad t \geq \dfrac{1}{n^2} \end{cases}$$

shows that it is in $\overline{\mathcal{J}}$. For other examples of ideals which are not closed, see Exercise IX.4.2 in [Dun].

4.35. Theorem (Continuous functional calculus for normal operators). *Let A be a C^*-algebra with unit, let $a \in A$ be a normal element and let $C^*(a)$ be the C^*-subalgebra of A generated by a. Then there exists a unique morphism of C^*-algebra*

$$\begin{cases} \mathcal{C}(\sigma(a)) & \rightarrow & C^*(a) \\ f & \mapsto & f(a) \end{cases}$$

such that $1(a) = 1$ and $\iota(a) = a$ (where $\iota \in \mathcal{C}(\sigma(a))$ denotes the inclusion of $\sigma(a)$ in $\mathbb{C}$). Moreover one has

$$\sigma(f(a)) \;=\; f(\sigma(a))$$

for all $f \in \mathcal{C}(\sigma(a))$.

Proof. The space of polynomials in z and $\overline{z}$ is dense in $\mathcal{C}(\sigma(a))$. As any morphism $\mathcal{C}(\sigma(a)) \rightarrow C^*(a)$ is continuous by Corollary 4.25, the claim of unicity follows.

Let X denote the spectrum of the algebra $C^*(a)$ and let

$$\mathcal{G} : \mathcal{C}(X) \longrightarrow C^*(a)$$

denote the Gelfand isomorphism of Theorem 4.31. By Corollary 4.32, we may identify X with $\sigma(a)$, and this shows the existence of the morphism $\mathcal{C}(\sigma(a)) \rightarrow C^*(a)$.

Let $f \in \mathcal{C}(\sigma(a))$. One has $f(a) = \mathcal{G}(f)$. The spectrum of $f(a)$ in A coincides with the spectrum of $f(a)$ in $C^*(a)$ by Proposition 4.26, hence with the spectrum of f in $\mathcal{C}(\sigma(a))$ because $\mathcal{G}$ is an isomorphism, hence with $f(\sigma(a))$ by Example 4.17. □

4.36. Lemma. *Let A, B be commutative C^*-algebras with units, let $\Phi : A \hookrightarrow B$ be an injective morphism such that $\Phi(1_A) = 1_B$ and let $\phi : X(B) \rightarrow X(A)$ be the map defined on the character spaces by $\phi(\chi) = \chi \circ \Phi$ for all $\chi \in X(B)$. Then ϕ is continuous and onto.*

Proof. The continuity of ϕ is a straightforward consequence of the continuity of Φ, proved in Corollary 4.25. Thus the image of ϕ is a compact subset of $X(A)$, say K.

Suppose ab absurdo that ϕ is not onto. Identifying A and $\mathcal{C}(X(A))$, we may choose continuous functions $a_1, a_2 \in \mathcal{C}(X(A))$ such that $a_1 \neq 0$, $a_2(x) = 1$ for all $x \in K$ and $a_1 a_2 = 0$. Then $\chi(\Phi(a_2)) = 1$ for all $\chi \in X(B)$, which implies that $\Phi(a_2)$ is invertible in B. But this is absurd because $\Phi(a_1) \neq 0$ by injectivity of Φ and $\Phi(a_1)\Phi(a_2) = \Phi(a_1 a_2) = 0$. □

4.37. Proposition. *Let A, B be C*-algebras and let $\Phi : A \hookrightarrow B$ be an injective morphism. Then*

$$\|\Phi(a)\| \;=\; \|a\|$$

for all $a \in A$. In particular, the image of Φ is closed in B, and is indeed a sub-C-algebra of B.*

Proof. Suppose first that $a^* = a$. Upon restricting Φ to the sub-C*-algebra of A generated by a, one may assume that A and B are both abelian. Without loss of generality, one may also assume that A and B have units and that $\Phi(1_A) = 1_B$. In this case, we denote by X the character space of A, by Y that of B and by $\phi : Y \to X$ the restriction map of the previous lemma. For all $a \in A$, one has

$$\|a\| \;=\; \sup_{\chi \in X} |\chi(a)| \;=\; \sup_{\psi \in Y} |\phi(\psi)(a)| \;=\; \sup_{\psi \in Y} |\psi(\Phi(a))| \;=\; \|\Phi(a)\|\,.$$

Going back to the general case of arbitrary C*-algebras and of a non necessary self-adjoint element a, one has then

$$\|\Phi(a)\| \;=\; \sqrt{\|\Phi(a^*a\|} \;=\; \sqrt{\|a^*a\|} \;=\; \|a\|$$

and the proof is complete. □

4.38. Sub-C*-algebras of abelian C*-algebras. Let B be a commutative C*-algebra with unit and let Y denote its character space.

Let A be a sub-C*-algebra of B containing the unit and let X denote the character space of A. One has by restriction of characters a continuous map

$$\phi : Y \longrightarrow X$$

which is onto by Lemma 4.36.

Conversely, let $\phi : Y \to X$ be a surjective continuous map of Y onto some compact space X. Then

$$A_X \;=\; \left\{\; b \in \mathcal{C}(Y) \;\middle|\; \text{there exists } a \in \mathcal{C}(X) \text{ such that } \; b = a \circ \phi \;\right\}$$

is a sub-C*-algebra of B containing the unit.

Thus there is a bijective correspondance between unital sub-C*-algebras of B and compact quotients of Y.

4.39. Proposition. *Let A, B be two commutative C*-algebras with units and let $\Phi : A \to B$ be a morphism of C*-algebras such that $\Phi(1_A) = 1_B$. Then there exists a closed subset Z of the character space $X(A)$ of A and a surjective continuous map $\phi : X(B) \to Z$ such that the diagram*

$$\begin{array}{ccccc} A & \longrightarrow & \mathcal{C}(X(A)) & \qquad & X(A) \\ \downarrow & & \downarrow & & \uparrow \\ A/Ker\Phi & \longrightarrow & \mathcal{C}(Z) & & Z \\ \downarrow & & \downarrow & & \uparrow \\ B & \longrightarrow & \mathcal{C}(X(B)) & & X(B) \end{array}$$

commutes, where horizontal arrows denote Gelfand isomorphisms and where the top [respectively bottom] right vertical arrow is given by the restriction of functions from $X(A)$ to Z [resp. by the composition of functions with the quotient map $X(B) \to Z$].

Proof. The image of the inclusion $A/Ker\Phi \hookrightarrow B$ is a sub-C*-algebra of B, because this image is closed by Proposition 4.37. Thus the proposition follows from Numbers 4.33 and 4.38. $\square$

4.E. The positive cone of a C*-algebra

4.40. Notations. Let A be a C*-algebra. We denote by

$$A_{sa} = \left\{ \, a \in A \mid a^* = a \, \right\}$$

the Banach space of its self-adjoint elements and we let

$$A_+ = \left\{ \, a \in A_{sa} \mid \sigma(a) \subset \mathbb{R}_+ \, \right\}$$

denote the *positive cone* of A (this terminology is justified by Proposition 4.43 below). In case A has no unit, the spectrum $\sigma(a)$ of $a \in A$ is understood in the algebra $\tilde{A}$ obtained from A by adding a unit.

For $a, b \in A_{sa}$, we write $a \geq 0$ if $a \in A_+$ and $a \geq b$ if $a - b \geq 0$.

4.41. Lemma. *Let A be a C*-algebra with unit.*
(i) For $a \in A_{sa}$ such that $\|a\| \leq 1$, one has

$$a \in A_+ \Longleftrightarrow \|1 - a\| \leq 1.$$

(ii) The set A_+ is closed in A_{sa}.

Proof. (i) If $a \in A_+$, one has $\sigma(a) \subset [0, \|a\|] \subset [0,1]$. Thus $\sigma(1-a)$ is again in $[0,1]$, so that $\|1-a\| \leq 1$ by Corollary 4.21.

If $\|1-a\| \leq 1$, one has $\sigma(a-1) \subset [-1,1]$ by Propositions 4.20 and 4.14. Then $\sigma(a) = 1 + \sigma(a-1) \subset [0,2] \subset \mathbb{R}_+$ so that $a \in A_+$.

(ii) For $a \in A_{sa}$ one has $a \in A_+ \Longleftrightarrow \|\|a\| - a\| \leq \|a\|$ by (i), so that A_+ is closed in A_{sa}. $\square$

4.42. Lemma. *Let A be a complex algebra with unit and let $a, b \in A$. Then*

$$\sigma(ab) \cup \{0\} \;=\; \sigma(ba) \cup \{0\}.$$

In particular, if A is a C^*-algebra and if $b \in A$, then $\sigma(bb^*) \subset \mathbb{R}_+$ if and only if $\sigma(b^*b) \subset \mathbb{R}_+$.

Proof. Let $\lambda \in \mathbb{C}^*$ be such that $\lambda - ab$ has an inverse, say x. Set $y = \lambda^{-1}(1 + bxa)$. Then

$$\begin{aligned}(\lambda - ba)y &= 1 + bxa - ba\lambda^{-1} - b\lambda^{-1}(abx)a \\ &= 1 + bxa - ba\lambda^{-1} - b\lambda^{-1}(\lambda x - 1)a = 1\end{aligned}$$

and similarly $y(\lambda - ba) = 1$. It follows that $\lambda - ba$ is invertible with inverse y. □

4.43. Proposition. *Let A be a C^*-algebra. For each $a \in A_{sa}$, the three following properties are equivalent*

(i) $\sigma(a) \subset \mathbb{R}_+$, *namely* $a \in A_+$,
(ii) *there exists* $b \in A$ *such that* $a = b^*b$,
(iii) *there exists* $b \in A_{sa}$ *such that* $a = b^2$.

Moreover

(iv) A_+ *is a closed convex cone in* A_{sa},
(v) $A_+ \cap (-A_+) = \{0\}$.

Proof. Let $a \in A_{sa}$. The implication (iii) $\Longrightarrow$ (i) follows from Proposition 4.20.ii, which implies that $\sigma(b) \subset \mathbb{R}$, and from Lemma 4.18, which implies that any $\mu \in \sigma(a)$ is of the form $\mu = \lambda^2$ for some $\lambda \in \sigma(b)$. The implication (i) $\Longrightarrow$ (iii) follows from Theorem 4.24 because ony may set $b = \sqrt{a}$. The implication (iii) $\Longrightarrow$ (ii) is obvious.

Assume that A has a unit. Then A_+ is closed by Lemma 4.41, and λa is obviously in A_+ whenever $\lambda \in \mathbb{R}_+$ and $a \in A_+$. Let $a, b \in A_+$ be such that $\|a\| \le 1$ and $\|b\| \le 1$. One has $\|1 - a\| \le 1$ and $\|1 - b\| \le 1$ by Lemma 4.41, so that

$$\left\|1 - \frac{1}{2}(a+b)\right\| \;\le\; \frac{1}{2}\|1-a\| + \frac{1}{2}\|1-b\| \;\le\; 1$$

and $\frac{1}{2}(a+b) \in A_+$ by the same Lemma. It follows that (iv) holds. Let $a \in A_+ \cap (-A_+)$; then $\sigma(a) = \{0\}$, so that $a = 0$ by Corollary 4.21. This shows (v).

If A has no unit, claims (iv) and (v) follow from the case with unit by 4.5.

Let us finally prove the implication (ii) $\Longrightarrow$ (iii). Let $f_+, f_- \in \mathcal{C}(\mathbb{R})$ be defined by

$$f_+(t) = sup(t, 0) \qquad f_-(t) = sup(-t, 0)$$

for all $t \in \mathbb{R}$. For $a = b^*b$ as in (ii), set

$$\begin{aligned} a_+ &= f_+(a) \in A_{sa} & \qquad a_- &= f_-(a) \in A_{sa} \\ x_+ &= \sqrt{f_+}(a) \in A_{sa} & \qquad x_- &= \sqrt{f_-}(a) \in A_{sa}\end{aligned}$$

and observe that

$$a = a_+ - a_- = x_+^2 - x_-^2 \qquad x_+x_- = 0$$

because similar relations hold in $\mathcal{C}(\mathbb{R})$. On one hand, one has

$$(*) \qquad -(x_-b^*)(x_-b^*)^* = -x_-ax_- = -x_-x_+^2x_- + x_-^4 = x_-^4 \in A_+$$

(the inclusion holds because (iii) $\Longrightarrow$ (i)). On the other hand, writing $x_-b^* = s + it$ with $s, t \in A_{sa}$, one has

$$\begin{aligned}(x_-b^*)^*(x_-b^*) &= -(x_-b^*)(x_-b^*)^* + (s+it)(s-it) + (s-it)(s+it) \\ &= -(x_-b^*)(x_-b^*)^* + 2s^2 + 2t^2 \in A_+\end{aligned}$$

(the inclusion holds because $-(x_-b^*)(x_-b^*)^* \in A_+$, as shown above, and because $2s^2 + 2t^2 \in A_+$). It follows from (v) and from Lemma 4.42 that

$$(**) \qquad (x_-b^*)(x_-b^*)^* \in A_+.$$

Finally (*) and (**) imply $x_-^4 = 0$, so that $x_- = 0$ and $a = x_+^2$. □

4.F. Borel functional calculus

In this Section 4.F, we do not give full proofs and we refer, among many other good sources, to [Bea], [ReS] and [StZ]. We denote by $\mathcal{H}$ a complex Hilbert space. Recall that the *strong topology* on $\mathcal{B}(\mathcal{H})$ has been defined in Section 2.B.

4.44. Proposition. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive operators in $\mathcal{B}(\mathcal{H})$ and let $b \in \mathcal{B}(\mathcal{H})$ be such that*

$$0 \le a_n \le a_{n+1} \le b$$

for all $n \in \mathbb{N}$. Then there exists a positive operator $a \in \mathcal{B}(\mathcal{H})$ which is the strong limit of the a_n 's, namely which is such that

$$a\xi = \lim_{n\to\infty} a_n\xi$$

for all $\xi \in \mathcal{H}$; moreover $a_n \le a \le b$ for all $n \in \mathbb{N}$. We write

$$a_n \uparrow a.$$

Proof (sketch). For each $n \in \mathbb{N}$ and $\xi \in \mathcal{H}$, set $q_n(\xi) = \langle \xi | a_n\xi \rangle$ and

$$q(\xi) = \lim_{n\to\infty} q_n(\xi) = \sup_{n\in\mathbb{N}} q_n(\xi).$$

Observe that

$$\langle \xi | a_n\eta \rangle = \frac{1}{4}\Big\{ q_n(\xi+\eta) - q_n(\xi-\eta) - iq_n(\xi+i\eta) + iq_n(\xi-i\eta) \Big\}$$

and define a sesquilinear form s on $\mathcal{H}$ by

$$s(\xi,\eta) \;=\; \frac{1}{4}\Big\{q(\xi+\eta)-q(\xi-\eta)-iq(\xi+i\eta)+iq(\xi-i\eta)\Big\}.$$

Using Riesz Theorem 1.3, one obtains an operator a such that $s(\xi,\eta) = \langle\xi|a\eta\rangle$ for all $\xi,\eta\in\mathcal{H}$, and it is obvious that a is the weak limit of $(a_n)_{n\in\mathbb{N}}$. One may also check that a is the strong limit of $(a_n)_{n\in\mathbb{N}}$; see Solution 94 of [Hal] for the details. □

4.45. Example. Consider the Hilbert space $L^2([0,1])$. For each $t\in[0,1]$, let P_t be the multiplication operator by the characteristic function of $[0,t]$. If $(t_n)_{n\in\mathbb{N}}$ is a sequence which converges to 1, then $P_{t_n}\uparrow id_{\mathcal{H}}$.

4.46. Notations. Let $a\in\mathcal{B}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$. For each $t\in\mathbb{R}$, let $e_t:\sigma(a)\to\mathbb{R}$ be the function defined by $e_t(s)=1$ if $s<t$ and $e_t(s)=0$ if $s\geq t$. Choose a sequence $(f_{t,n})_{n\geq1}$ in $\mathcal{C}(\sigma(a))$ such that $0\leq f_{t,n}(s)\leq f_{t,n+1}(s)\leq e_t(s)$ for all $n\geq1$ and $s\in\sigma(a)$, and also such that $\lim_{n\to\infty}f_{t,n}(s)=e_t(s)$ for all $s\in\sigma(a)$. By Proposition 4.44, the sequence $(f_{t,n}(a))_{n\geq1}$ converges strongly to an operator in $\mathcal{B}(\mathcal{H})$ that we denote by E_t.

4.47. Spectral measure of a self-adjoint operator. Let $a\in\mathcal{B}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$. Set $\alpha=\inf_{\xi\in\mathcal{H},\ \|\xi\|\leq1}\langle\xi|a\xi\rangle$ and $\beta=\sup_{\xi\in\mathcal{H},\ \|\xi\|\leq1}\langle\xi|a\xi\rangle$. For each $t\in\mathbb{R}$, let E_t be defined as in 4.46. Then one has:

(i) each E_t is a projection on $\mathcal{H}$,
(ii) $E_s\leq E_t$ whenever $s\leq t$,
(iii) $E_s\uparrow E_t$ if $s\uparrow t$ (notations as in 4.44),
(iv) $t\leq\alpha\ \Rightarrow\ E_t=0$,
(v) $t>\beta\ \Rightarrow\ E_t=1$,
(vi) for $b\in\mathcal{B}(\mathcal{H})$, one has $ba=ab$ if and only if $bE_t=E_tb$ for all $t\in\mathbb{R}$

The family $(E_t)_{t\in\mathbb{R}}$ is called the *spectral measure* of the self-adjoint operator a.

Conversely, any family $(E_t)_{t\in\mathbb{R}}$ satisfying (i) to (v) above (for some $\alpha,\beta\in\mathbb{R}$ with $\alpha\leq\beta$) defines a self-adjoint operator

$$a=\int_\alpha^{\beta+0}t\,dE_t$$

with spectral measure $(E_t)_{t\in\mathbb{R}}$.

4.48. Notations. Let σ be a compact metric space. We denote by $\mathcal{B}(\sigma)$ the *-algebra of bounded complex-valued Borel functions on σ. By a theorem of Baire, it is also the smallest family of complex-valued functions on σ which contains the continuous functions and which is closed under pointwise limits (see the indications for Exercise 11.46 in [HeS]). One may define a norm by $\|f\|=sup_{t\in\sigma}|f(t)|$ and an involution by $f^*(t)=\overline{f(t)}$; then $\mathcal{B}(\sigma)$ becomes what has to be called a *pre-C*-algebra* (i.e. satisfies all conditions to be a C*-algebra, but completeness), and $\mathcal{C}(\sigma)$ a sub-C*-algebra of $\mathcal{B}(\sigma)$.

However, in most cases and for example for $\sigma=[0,1]$, observe firstly that $\mathcal{B}(\sigma)$ is not complete, and secondly that its completion is a C*-algebra which is not separable (see the discussion in 2.4).

For $\sigma \subset \mathbb{R}$, we denote by $\mathcal{P}(\sigma)$ the involutive subalgebra of $\mathcal{B}(\sigma)$ consisting of the restrictions to σ of the polynomial functions $\mathbb{R} \to \mathbb{C}$. (Observe that $dim_{\mathbb{C}}\mathcal{P}(\sigma) < \infty$ in case σ is a finite subset of $\mathbb{R}$, and that $\mathcal{P}(\sigma)$ is isomorphic to $\mathbb{C}[T]$ if σ is an infinite set.)

4.49. Theorem (Borel functional calculus). *Let $a \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$. Then the natural map $\mathcal{P}(\sigma(a)) \to \mathcal{B}(\mathcal{H})$ has a unique extension*

$$\Gamma : \mathcal{B}(\sigma(a)) \to \mathcal{B}(\mathcal{H})$$

such that the following holds:

if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(\sigma(a))$ such that $\sup_{n\in\mathbb{N}} \sup_{s\in\sigma(a)} |f_n(s)| < \infty$ and which tends pointwise to a function $f \in \mathcal{B}(\sigma(a))$, then $\Gamma(f_n)$ tends strongly to $\Gamma(f)$ in $\mathcal{B}(\mathcal{H})$.

Moreover the restriction of Γ to $\mathcal{C}(\sigma(a))$ coincides with the morphism of continuous functional calculus (4.24).

On the proof. For details, see Chapter 2 in [StZ], as well as Proposition 2.3 in [Bea], or n^o 106 in [RiN].

The uniqueness is a straightforward consequence of the Theorem of Baire recalled in 4.48.

The proof of the existence of Γ uses tools of classical analysis such as the Lebesgue-Stieltjes integral and the Lebesgue dominated convergence theorem.

For the last claim, one may use the following theorem of Dini (see n^o 7.2.2 in [Di1]): let σ be a compact metric space and let $(f_n)_{n\geq 1}$ be a pointwise increasing sequence of continuous functions $\sigma \to \mathbb{R}$ which converges pointwise to a continuous function $f : \sigma \to \mathbb{R}$. Then $(f_n)_{n\geq 1}$ converges to f in the norm topology of $\mathcal{C}(\sigma)$. □

4.50. Corollary (Schur's Lemma). *Let $\mathcal{H}$ be a Hilbert space and let S be a selfadjoint subset of $\mathcal{B}(\mathcal{H})$. The following are equivalent.*

(i) S is topologically irreducible. More precisely, the only S-invariant closed subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$.

(ii) The commutant of S is trivial. More precisely, the only operators $a \in \mathcal{B}(\mathcal{H})$ such that $as = sa$ for all $s \in S$ are the homothecies $a \in \mathbb{C}id_{\mathcal{H}}$.

Proof. (i) $\Rightarrow$ (ii) Let $a \in \mathcal{B}(\mathcal{H})$ be an operator which commutes with S. If a is self-adjoint, then $a \in \mathbb{C}id_{\mathcal{H}}$ by the argument below. The general case follows because any $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of two self-adjoint operators.

Suppose that a is self-adjoint, commutes with S, and is not a homothecy. We shall reach a contradiction. There exist two distinct points λ, μ in the spectrum of a. Let $f, g \in \mathcal{C}(\sigma(a))$ be such that

$$f(\lambda) \neq 0 \quad g(\mu) \neq 0 \quad fg = 0.$$

Define $\mathcal{H}_f$ to be the closure in $\mathcal{H}$ of the image of $f(a)$. For each $s \in S$, one has $sf(a) = f(a)s$, hence $s(\mathcal{H}_f) \subset \mathcal{H}_f$. As $f(a) \neq 0$, one has $\mathcal{H}_f \neq \{0\}$, and thus $\mathcal{H}_f = \mathcal{H}$ by the irreducibility assumption (i). Hence

$$g(a)(\mathcal{H}) = g(a)\overline{f(a)\mathcal{H}} \subset \overline{g(a)f(a)\mathcal{H}} = \overline{(gf)(a)\mathcal{H}} = \{0\}$$

because $gf = 0$, and this is absurd because $g(a) \neq 0$.

(ii) $\Rightarrow$ (i) The assumption (ii) implies that the only projections in $\mathcal{B}(\mathcal{H})$ which commute with S are 0 and $id_{\mathcal{H}}$, which is another phrasing of (i); see Lemma 2.11. □

4.51. Proposition. *Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of C^*-algebra A on a Hilbert space $\mathcal{H}$. The following are equivalent:*

(i) the only A-invariant closed subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$,
(ii) the only projection $p \in \mathcal{B}(\mathcal{H})$ which commute with $\pi(A)$ are 0 and 1,
(iii) the commutant $\pi(A)' = \{\ b \in \mathcal{B}(\mathcal{H}) \mid b\pi(a) = \pi(a)b \ \text{ for all } \ a \in A\ \}$ is reduced to $\mathbb{C}$.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) follows from Lemma 2.11 and (ii) $\Longleftrightarrow$ (iii) is a particular case of Schur's Lemma.

For other equivalent conditions, see Proposition 6.21 below. □

4.52. Definition. A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ of a C*-algebra A on a Hilbert space $\mathcal{H}$ is *irreducible* if it satisfies the conditions of the previous proposition.

4.53. Remark. Observe that the morphism Γ of Theorem 4.49 needs not be an injection. Indeed, let $t \in \sigma(a)$ be a spectral value of a which is not an eigenvalue of a, and let $\delta_t \in \mathcal{B}(\sigma(a))$ denote the characteristic function of $\{t\}$. Then $\delta_t \neq 0$ in $\mathcal{B}(\sigma(a))$, but $\Gamma(\delta_t) = 0$ in $\mathcal{B}(\mathcal{H})$. For example, if $a \in \mathcal{B}\left(L^2([0,1])\right)$ is defined by $(a\xi)(s) = s\xi(s)$, one has $\Gamma(\delta_t) = 0$ for all $t \in [0,1] = \sigma(a)$.

4.54. Towards a "Lebesgue functional calculus". Let $a \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$. For each pair ξ, η of vector in $\mathcal{H}$, the linear map

$$\begin{cases} \mathcal{C}\big(\sigma(a)\big) & \longrightarrow & \mathbb{C} \\ f & \longmapsto & \langle \xi | f(a)\eta \rangle \end{cases}$$

defines a bounded measure $\mu_{\xi,\eta}$ on $\sigma(a)$. It can be shown that there exists a measure μ on $\sigma(a)$ such that

$$\mu_{\xi,\eta} \prec \mu \quad \text{for all} \quad \xi, \eta \in \mathcal{H}$$

(where $\prec$ indicates absolute continuity), such that any measure ν satisfying $\mu_{\xi,\eta} \prec \nu$ for all $\xi, \eta \in \mathcal{H}$ satisfies also $\mu \prec \nu$, and that the measure class of μ is well defined by these requirements. The Gelfand isomorphism $\mathcal{C}(\sigma(a)) \to C^*(a)$ has then a natural extension

$$L^\infty\big(\sigma(a), \mu\big) \quad \longrightarrow \quad W^*(a)$$

which is an isomorphism of the von Neumann algebra $L^\infty\big(\sigma(a), \mu\big)$ onto the von Neumann algebra generated by a in $\mathcal{B}(\mathcal{H})$. See [DvN, Appendice I] or Corollary X.2.9 in [DuS].

P. de la Harpe and V. Jones, July 1995.

CHAPTER 5. TWO FAMILIES OF EXAMPLES : AF-ALGEBRAS AND REDUCED C*-GROUP-ALGEBRAS

A separable C*-algebra A is *approximately finite dimensional*, or more simply AF, if there exists a nested sequence $A_1 \subset A_2 \subset \ldots$ of finite dimensional sub-C*-algebras of A of which the union is dense in A. Before giving in Section 5.B examples of AF-algebras, we need some preliminaries on the inclusions $A_n \subset A_{n+1}$, and this is exposed in Section 5.A. Group algebras are very briefly discussed in Section 5.C.

5.A. Pairs and towers of finite dimensional C*-algebras, and their Bratteli diagrams

5.1. Examples of Bratteli diagrams. We know from Section 2.B how to describe finite dimensional C*-algebras. A good device to describe an *inclusion* of one such an algebra in another is its Bratteli diagram. Before the actual definition, we give two examples.

The mapping

$$\begin{cases} \mathbb{C} \oplus M_2(\mathbb{C}) & \longrightarrow \qquad M_3(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_2(\mathbb{C}) \\ (x,y) & \longmapsto \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix}, y \right) \end{cases}$$

defines a pair of finite dimensional C*-algebras with Bratteli diagram as in Figure 1.i. Observe that the inclusion is unital, and correspondingly that each weight on the right of the diagram is the sum (with multiplicities) of the related weights on the left : $3 = 1 + 2$, $5 = 1 + 2{\cdot}2$, and $2 = 2$.

The mapping

$$\begin{cases} M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) & \longrightarrow \qquad M_9(\mathbb{C}) \oplus M_7(\mathbb{C}) \\ (x,y,z) & \longmapsto \left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \right) \end{cases}$$

defines a pair of finite dimensional C*-algebras with Bratteli diagram as in Figure 1.ii. Observe that the inclusion is not unital, and correspondingly that $2{\cdot}3 < 9$.

Figure 1, (i) and (ii).

5.2. Definition of Bratteli diagrams for pairs. Let $A \subset B$ be a pair of finite dimensional C*-algebras. We know from Section 2.B that there are integers

$$m_1, ..., m_r, n_1, ..., n_s \geq 1$$

such that

$$\bigoplus_{j=1}^{r} M_{m_j}(\mathbb{C}) \ \approx \ A \ \subset \ B \ \approx \ \bigoplus_{k=1}^{s} M_{n_k}(\mathbb{C}).$$

Denote by $e_1, ..., e_r$ the minimal central projections in A and by $f_1, ..., f_s$ the minimal central projections in B. One has

$$e_j A \approx M_{m_j}(\mathbb{C}) \quad \text{and} \quad f_k B \approx M_{n_k}(\mathbb{C})$$

for all $j \in \{1, ..., r\}$ and $k \in \{1, ..., s\}$.

Given $j \in \{1, ..., r\}$ and $k \in \{1, ..., s\}$, define the *multiplicity* $\lambda_{k,j} \in \mathbb{N}$ as follows. If p_j is a minimal projection in $e_j A$, then $f_k p_j$ is a sum of $\lambda_{k,j}$ minimal projections in $f_k B$ (one may say that $\lambda_{k,j}$ is the *rank* of $f_k p_j$ in $f_k B$). Then the data

$$\vec{m} = (m_j)_{1 \leq j \leq r} \quad , \quad \vec{n} = (n_k)_{1 \leq k \leq s} \quad , \quad \Lambda = (\lambda_{k,j})_{1 \leq k \leq s, 1 \leq j \leq r}$$

determine completely (up to $*$-isomorphism) the pair $A \subset B$. The matrix Λ is known as the *inclusion matrix* of the pair $A \subset B$.

The *Bratteli diagram* of the inclusion $A \subset B$ is a bipartite graph, which has weights on vertices and multiplicities on edges, and which has

r vertices with respective weights $m_1, ..., m_r$,
s vertices with respective weights $n_1, ..., n_s$,
one edge of multiplicity $\lambda_{k,j}$ between the vertex of weight m_j
and the vertex of weight n_k

(of course edges of multiplicity zero are "nonedges"!). Observe that one has

$$\Lambda \vec{m} \ \leq \ \vec{n},$$

by which we mean $\sum_{j=1}^{r} \lambda_{k,j} m_j \leq n_k$ for all $k \in \{1, ..., s\}$. Moreover the pair $A \subset B$ is *unital* (i.e. the unit in A coïncides with the unit in B) if and only if

$$\Lambda \vec{m} = \vec{n}.$$

5.3. Remark. The notations being as in 5.2, one may show that the following are equivalent:

the rank $\lambda_{k,j}$ of $f_k p_j$ in $f_k B$,
the number of A-simple modules in the restriction to $e_j A$
of the simple B-module corresponding to f_k,
the number of simple modules in the restriction to B_k of the induction from A to B

of the simple A-module corresponding to e_j.

Moreover, the matrix Λ is also the matrix with respect to the canonical basis of the map

$$K_0(A) \approx \mathbb{Z}^r \longrightarrow \mathbb{Z}^s \approx K_0(B)$$

induced by the inclusion $A \hookrightarrow B$ on the corresponding Grothendieck groups.

5.4. Examples associated to pairs of finite groups. Let H be a finite group and let G be a subgroup of H. The group algebras provide an example

$$\mathbb{C}[G] \subset \mathbb{C}[H]$$

of unital pair of finite dimensional C*-algebras. The corresponding Bratteli diagram encodes standard informations on the irreducible complex representations of G and H. As a sample of examples, we give in Figure 2 the Bratteli diagrams for the inclusions

$$\begin{array}{ccc} \mathcal{S}_2 \subset \mathcal{S}_3 & \qquad & \mathcal{A}_3 \subset \mathcal{S}_3 \\ \mathcal{S}_3 \subset \mathcal{S}_4 & \qquad & \mathcal{A}_4 \subset \mathcal{S}_4 \end{array}$$

where $\mathcal{S}_n$ [respectively $\mathcal{A}_n$] denotes the symmetric [resp. alterating] group on n letters.

Figure 2.

5.5. Bratteli diagrams for towers. A *tower of C*-algebras* is a nested sequence

$$A_0 \subset A_1 \subset ... \subset A_n \subset A_{n+1} \subset ...$$

of C*-algebras, each included in the next. The *Bratteli diagram* of a tower of finite dimensional C*-algebras is obtained by concatenation of the Bratteli diagrams for the pairs $A_0 \subset A_1$, $A_1 \subset A_2$, ...

As a first example, consider a separable Hilbert space $\mathcal{H}$. Choose a nested sequence $V_1 \subset V_2 \subset ...$ of subspaces of $\mathcal{H}$ such that $dim_{\mathbb{C}} V_n = n$ for each $n \geq 1$ and such that the union of the V_n 's is dense in $\mathcal{H}$. For each $n \geq 1$, identify $\mathcal{B}(V_n)$ with the sub-C*-algebra of $\mathcal{B}(\mathcal{H})$ of those operators mapping V_n into itself and $V_n^{\perp}$ onto 0. Then

$$\mathcal{B}(V_1) \subset \mathcal{B}(V_2) \subset ... \subset \mathcal{B}(V_n) \subset \mathcal{B}(V_{n+1}) \subset ...$$

is a tower of finite-dimensional C*-algebras; its Bratteli diagram is the half-line $[1, \infty[$ with vertices at the integers $1, 2, 3, ...$ (Figure 3). Adding units, one finds the tower

$$\mathbb{C}id_{\mathcal{H}} + \mathcal{B}(V_1) \subset \mathbb{C}id_{\mathcal{H}} + \mathcal{B}(V_2) \subset ... \subset \mathbb{C}id_{\mathcal{H}} + \mathcal{B}(V_n) \subset \mathbb{C}id_{\mathcal{H}} + \mathcal{B}(V_{n+1}) \subset ...$$

and the second Bratteli diagram of Figure 3.

Figure 3.

As a second example, consider the Cantor ternary set X. For each $n \in \mathbb{N}$, define an algebra A_n of continuous functions on X as follows:

A_0 is the algebra of constant functions,
A_1 is the algebra of functions constant on $X \cap [0, \frac{1}{3}]$ and $X \cap [\frac{2}{3}, 1]$,
....
A_n is the algebra of functions constant on $X \cap [\frac{j}{3^n}, \frac{j+1}{3^n}]$ for each $j \in \{0, 1, ..., 3^n - 1\}$.
....

Then

$$A_0 \subset A_1 \subset ... \subset A_n \subset A_{n+1} \subset ...$$

is a tower of finite dimensional abelian C*-algebras with Bratteli diagram a regular rooted tree where each vertex is of weight 1 and has two successors, as in Figure 4.

Figure 4.

5.b. AF-algebras

For simplicity, all C*-algebras appearing in this section are assumed to be **separable**.

5.6. Definition. A separable C*-algebra A is said to be an *approximately finite dimensional C*-algebra,* or simply an *AF-algebra,* if there exists a tower $A_0 \subset A_1 \subset ...$ of finite dimensional sub-C*-algebras of A such that the union $\cup_{n \in \mathbb{N}} A_n$ is dense in A.

It is sometimes self-understood that an AF-algebra is infinite dimensional.

5.7. First examples. The algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable infinite dimensional Hilbert space is an AF-algebra; the same holds for the algebra $\mathbb{C}1_{\mathcal{H}} + \mathcal{K}(\mathcal{H})$. Also, the algebra of continuous functions on the Cantor ternary set is a commutative AF-algebra. All this follows from 5.5 above.

On the other hand, if X is connected compact space which is neither empty nor reduced to one point, then the only finite-dimensional sub-C*-algebra of $\mathcal{C}(X)$ is the algebra of constant functions. In particular, $\mathcal{C}(X)$ is *not* an AF-algebra.

If X is a secound countable compact space, then $\mathcal{C}(X)$ is an AF-algebra if and only if X is totally disconnected (this is part of Proposition 3.1 in [Br2]), namely if and only if X is zero dimensional (this is standard dimension theory - see e.g. [HuW]). Recall that a compact space which is totally disconnected and secound countable is always homeomorphic to a closed subset of the unit interval.

5.8. Inductive limits. Let $A_0 \subset A_1 \subset \ldots$ be a tower of C*-algebras, and let $\mathcal{A}_\infty$ denote the union of the A_n 's. Then $\mathcal{A}_\infty$ is an involutive algebra (see the definition in 1.13). Moreover, each $a \in \mathcal{A}_\infty$ has a norm

$$\|a\| = \sqrt{\text{spectral radius of } a^* a}$$

where either the norm or the spectral radius may be computed in any A_n such that $a \in A_n$. If A denotes the completion of $\mathcal{A}_\infty$ with respect to this norm, it is obvious that A is a C*-algebra, called the *inductive limit* of the tower $(A_n)_{n \in \mathbb{N}}$.

Observe that, in case all A_n 's have units and all inclusions $A_n \hookrightarrow A_{n+1}$ are unital, then A has a unit. Conversely, if A has a unit, upon replacing whenever necessary each A_n by $\mathbb{C}1 + A_n$ (and this is easily seen to be necessary for finitely many n 's only), then one may assume that all A_n 's and all inclusions are unital.

We will apply this to towers of finite dimensional C*-algebras: each such tower defines an AF-algebra.

5.9. UHF-algebras. Let $(k_j)_{j \geq 1}$ be a sequence of integers, with $k_j \geq 2$. For each $n \geq 1$, the algebra

$$A_n = \bigotimes_{j=1}^{n} M_{k_j}(\mathbb{C}) \approx M_{k_1 k_2 \ldots k_n}(\mathbb{C})$$

is a full matrix algebra, and the assignment

$$x \mapsto x \otimes 1_{k_{n+1}}$$

(where $1_{k_{n+1}}$ denotes the unit in $M_{k_{n+1}}(\mathbb{C})$) defines a pair $A_n \subset A_{n+1}$. The resulting tower is described by its Bratteli diagram, indicated in Figure 5. The resulting inductive limit C*-algebra

$$\bigotimes_{j=1}^{\infty} M_{k_j}(\mathbb{C})$$

is an AF-algebra known as a *UHF-algebra* (for Uniformly HyperFinite), or sometimes as a *Glimm algebra* [Gli].

Figure 5.

5.10. Example. Consider the UHF-algebras

$$A = \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})_j \quad \text{and} \quad B = \bigotimes_{j=1}^{\infty} M_3(\mathbb{C})_j$$

where $M_k(\mathbb{C})_j$ denotes a copy of $M_k(\mathbb{C})$ for each $j \geq 1$ and for $k \in \{2, 3\}$. These two UHF-algebras are known to be non-isomorphic.

Here is a sketch of the proof. Let $tr : A \to \mathbb{C}$ be a normalised *trace* on A, namely a linear map such that $tr(1) = 1$, $tr(a^*a) \geq 0$ and $tr(aa') = tr(a'a)$ for all $a, a' \in A$. It follows from the unicity of the trace on a matrix algebra that there is a unique trace on A. Let now D_A be the subgroup of the additive group $\mathbb{R}$ generated by the numbers $tr(e)$, where e is a projection in A. Using Exercise 4.28, one may check that

$$D_A \approx \mathbb{Z}\left[\frac{1}{2}\right]$$

is the ring of 2-adic integers.

The same consideration for B gives rise to the group

$$D_B \approx \mathbb{Z}\left[\frac{1}{3}\right]$$

which is not isomorphic to D_A (because D_B is 3-divisible and D_A is not). Thus the C*-algebra B is not isomorphic to A.

(In a more sophisticated terminology, D_A is the image $tr(K_0(A))$ of the Grothendieck group of A by the canonical map $K_0(A) \to \mathbb{C}$ induced by *the* trace.)

More generally, given any set S of prime numbers, let $(k_j)_{j\geq 1}$ be a sequence of integers with $k_j \in S$ for each $j \geq 1$ and $|\{j \geq 1 \mid k_j = s\}| = \infty$ for each $s \in S$. Then the associated UHF-algebra gives rise to the group

$$D^{(S)} \approx \mathbb{Z}\left[\left(\frac{1}{s}\right)_{s\in S}\right].$$

For two sets S, S' of prime numbers, one has $D^{(S)} \approx D^{(S')}$ if and only if $S = S'$. It follows that there are uncountably many pairwise nonisomorphic UHF-algebras.

Similar considerations provide a complete classification of UHF-algebras, firstly established in J. Glimm's thesis; see [Gli], Theorem 6.4.6 in [Ped] and n^o 6.1 in [Ell]. The groups which appear in the classification are torsion-free abelian groups of rank 1; they have been classified by R. Baer (1937); see § 85 in Volume II of [Fuc].

5.11. Example (Bratteli). The two diagrams of Figure 6 define two algebras which are isomorphic.

Figure 6.

Indeed, jumping through the even-numbered floors of the left-hand diagram of Figure 6, one obtains firstly the left-hand diagram of Figure 7. Building new intermediate floors, one obtains secondly the right-hand diagram of Figure 7. Jumping again through every other floor, one obtains finally the right-hand diagram of Figure 6.

Figure 7.

5.12. Isomorphisms of AF-algebras. Let A, B be two AF-algebras. Let $A_0 \subset A_1 \subset \ldots$ [respectively $A_0 \subset A_1 \subset \ldots$] be a tower of finite dimensional sub-C*-algebras of A [resp. B] with union $\mathcal{A}_\infty$ dense in A [resp. $\mathcal{B}_\infty$ dense in B].

The two towers are *isomorphic* if there exists isomorphisms $A_n \approx B_n$ which commute with the tower inclusions. If the towers are isomorphic, then A and B are isomorphic (this is easy, but *does* require some argument !); the converse does *not* hold, as Example 5.11 shows dramatically.

It is a theorem of Bratteli that the C*-algebras A and B are isomorphic if and only the involutive algebras $\mathcal{A}_\infty$ and $\mathcal{B}_\infty$ are isomorphic. Thus, the study of AF-algebras is in some sense equivalent to that of involutive algebras of countable complex dimensions which are *locally finite dimensional,* i.e. such that any finite subset is contained in a finite dimensional sub-involutive-algebra.

5.13. Another definition. A separable C*-algebra A is a AF-algebra if and only if the following holds:

For any finite subset $\{a_1, \ldots, a_n\}$ of A and for any $\epsilon > 0$,
there exist a finite dimensional sub-C*-algebra B of A and a subset $\{b_1, \ldots, b_n\}$ of B such that $\|a_j - b_j\| < \epsilon$ for $j \in \{1, \ldots, n\}$.

This is Theorem 2.2 of [Bra].

5.14. Stability. The class of AF-algebras has remarkable stability properties. For example, it is easy to check that

closed two-sided ideals of AF-algebras are AF-algebras,
quotient C*-algebras of AF-algebras are AF-algebras,
C*-tensor products of AF-algebras of AF-algebras are AF-algebras.

A much deeper fact is the following result of L.G. Brown : if

$$0 \rightarrow \mathcal{J} \rightarrow A \rightarrow B \rightarrow 0$$

is a short exact sequence of C*-algebras such that $\mathcal{J}$ and B are AF-algebras, then A is also a AF-algebra. See [Bro], and Theorem 9.9 in [Eff].

A sub-C*-algebra of a AF-algebra needs not be a AF-algebra. Indeed, a famous isomorphism problem about the so-called *irrational rotation algebras* (which are far from being AF-algebras) has been solved by embedding them in appropriate AF-algebras [PiV].

For an easier example, consider a connected separable compact metric space Y and a continuous map p from the Cantor ternary set X onto Y [HeS, Exercice 6.100]. One may identify via p the C*-algebra $\mathcal{C}(Y)$, which is not AF, to a sub-C*-algebra of $\mathcal{C}(X)$, which is AF (see 5.7).

Let G be a finite group of automorphisms of a AF-algebra and let A^G denote the corresponding sub-C*-algebra of fixed points. It has been an open question for quite some time to know whether A^G is necessarily AF. The answer is no, because B. Blackadar [Bl2] has constructed an automorphism of order 2 on the CAR-algebra (see Chapter 7), namely an action of $\mathbb{Z}/2\mathbb{Z}$ on $CAR(\mathcal{H})$, such that $CAR(\mathcal{H})^{\mathbb{Z}/2\mathbb{Z}}$ is not AF (it is a C*-algebra with K_1-group not reduced to zero).

5.15. On the classification of AF-algebras. Example 5.10 shows that there are large classes of AF-algebras, and Example 5.11 shows that it is not always easy to decide whether two AF-algebras given by two inductive systems are isomorphic or not. To cope with these problems, G. Elliott has made good use of K-theory.

Let A be a C*-algebra. The Grothendieck group $K_0(A)$ is, as always, the abelian group of projective A-modules of finite type, up to stable isomorphisms. But C*-algebraists rather view it as defined via appropriate equivalence classes of projections in the C*-algebra $A \otimes \mathcal{K}(\mathcal{H})$, where $\mathcal{H}$ denotes an infinite dimensional separable Hilbert space. If A has a unit, we denote its class in $K_0(A)$ by $[1_A]$ (this can also be viewed as the class of the free A-module of rank one.)

The positive part $K_0(A)_+$ of $K_0(A)$ is the set of classes in $K_0(A)$ which can be represented by actual projections (or actual A-modules), rather than by formal differences of these. In general, the pair $(K_0(A), K_0(A)_+)$ is far from being an ordered group. For example, for the so-called "Cuntz algebra" 0_n, where $n \geq 2$ is an integer, one has $K_0(O_n)_+ = K_0(O_n) \approx \mathbb{Z}/(n-1)\mathbb{Z}$. But there are important classes of C*-algebras, including AF-algebras, for which $(K_0(A), K_0(A)_+)$ is an *ordered group,* namely for which one has

$$K_0(A)_+ - K_0(A)_+ = K_0(A) \quad \text{and} \quad K_0(A)_+ \cap (-K_0(A)_+) = \{0\}.$$

For details, see Chapter III in [Bl1].

It is a remarkable result, due to Elliott (1976), that AF-algebras can be classified in terms of K-theory. In particular, let A, B be two AF-algebras with units. If there exists an isomorphism of abelian groups $\phi : K_0(A) \to K_0(B)$ such that $\phi([1_A]) = [1_B]$ and $\phi(K_0(A)_+) = K_0(B)_+$, then A and B are isomorphic as C*-algebras.

Ordered groups coming from AF-algebras are countable (because AF-algebras are separable) and have two special properties: they are so-called "unperforated groups" which have the "Riesz Interpolation Property". It is another remarkable result, due to Effros, Handelman and Shen (1980), that any countable ordered group with these two properties is the Grothendieck group of an AF-algebra.

For all this, we refer to the original papers and to [Eff].

5.16. Example. Consider the tower

$$B_1 = M_2(\mathbb{C}) \;\subset \ldots \subset\; B_n = \bigotimes_{j=1}^{n} M_2(\mathbb{C})_j \;\subset\; B_{n+1} = \bigotimes_{j=1}^{n+1} M_2(\mathbb{C})_j \;\subset \ldots \subset\; B = \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})_j$$

as in 5.10 above, and let $\pi : \mathcal{S}_3 \to U(2) \subset M_2(\mathbb{C})$ be an irreducible unitary representation of the symmetric group on 3 letters. For each $n \geq 1$, the group $\mathcal{S}_3$ acts by automorphisms on B_n according to

$$g(x) = \left(\bigotimes_{j=1}^{n} \pi(g) \right) x \left(\bigotimes_{j=1}^{n} \pi(g) \right)^{-1}$$

for each $g \in \mathcal{S}_3$ and $x \in B_n$. The action extends to $\bigcup_{n=1}^{\infty} B_n$ and to the C*-algebra B. For each $n \geq 1$, let $A_n = B_n^{\mathcal{S}_3}$ denote the sub-C*-algebra of elements in B_n fixed by $\mathcal{S}_3$. Our purpose is to indicate why the Bratteli diagram for the tower

$$A_1 \subset \dots \subset A_n \subset A_{n+1} \subset \dots \subset \overline{\bigcup_{n=1}^{\infty} A_n}$$

is as indicated in Figure 8.

Figure 8.

Let us first recall that $\mathcal{S}_3$ has three irreducible representations (over $\mathbb{C}$) which we denote by π (as above), ϵ (the identity) and σ (the signature). The corresponding table of characters is shown in Figure 9.

Figure 9.

It follows that tensor products by π of the irreducible representations of $\mathcal{S}_3$ decompose as

$$(*) \qquad \epsilon \otimes \pi \approx \pi \qquad \pi \otimes \pi \approx \epsilon \oplus \pi \oplus \sigma \qquad \sigma \otimes \pi \approx \pi.$$

One may encode these information about the pair $(\mathcal{S}_3, \pi)$ in the graph of Figure 10.

Figure 10.

More generally, given a finite group G and a representation π of G, information on decompositions of tensor products by π may be encoded in the associated *McKay representation graph* defined as follows: the set of vertices is the set $\hat{G}$ of irreducible complex representations of G; and there is one directed edge of multiplicity $m_{j,k}$ from $\rho_j \in \hat{G}$ to $\rho_k \in \hat{G}$, where the $m_{j,k}$'s are given by

$$\rho_j \otimes \pi \approx \bigoplus_{\rho_k \in \hat{G}} m_{j,k} \rho_k.$$

If $m_{j,k} = m_{k,j}$ for all paire (ρ_j, ρ_k), one draws usually the undirected graph corresponding to the graph just defined, as we have shown in Figure 10. More on this in [FoM] and [McK].

Now, for each $n \geq 1$, the algebra $A_n = B_n^{\mathcal{S}_3}$ of elements in B_n fixed by $\mathcal{S}_3$ is precisely the commutant in B_n of the representation

$$\pi^{\otimes n} : \mathcal{S}_3 \longrightarrow \bigotimes_{j=1}^{n} M_2(\mathbb{C})_j .$$

Thus, if a_n, b_n, c_n are the integers defined by

$$\pi^{\otimes n} \;\approx\; a_n \epsilon \oplus b_n \pi \oplus c_n \sigma$$

one has

$$A_n \;\approx\; M_{a_n}(\mathbb{C}) \oplus M_{b_n}(\mathbb{C}) \oplus M_{c_n}(\mathbb{C}).$$

It is clear that these integers are recursively defined by (*) above, namely by Figure 10, or more transparently by Figure 8.

Observe in Figure 8 that each edge is the symmetrical image of an edge on the previous floor, but for five initial edges which constitute a so-called *affine Coxeter graph* of type $\tilde{D}_5$. It is an observation of John McKay that finite subgroups of $SU(2)$ provide in the same way the following list:

binary dihedral group of order $4n$	$\leftrightsquigarrow$	$\tilde{D}_{n+2}$	$(n \geq 2)$
binary tetrahedral group of order 24	$\leftrightsquigarrow$	$\tilde{E}_6$	
binary octahedral group of order 48	$\leftrightsquigarrow$	$\tilde{E}_7$	
binary tetrahedral group of order 120	$\leftrightsquigarrow$	$\tilde{E}_8$.	

The graphs $\tilde{A}_n$ appear also naturally in McKay's list, but the presentation above has to be slightly modified for them, because the analogue of π above is reducible for cyclic subgroups of $SU(2)$.

5.17. Exercise. Let $A_0 \subset A_1 \subset ...$ be a tower of C*-algebras and let A be the inductive limit C*-algebra, as in 5.8.

(i) Let $\mathcal{J}$ be a closed two-sided ideal of A, and set $\mathcal{J}_n = \mathcal{J} \cap A_n$ for each $n \geq 0$. Show that

$$\mathcal{J} \;=\; \overline{\bigcup_{n \geq 0} \mathcal{J}_n}.$$

(ii) Show that A has no non trivial closed two-sided ideal if and only if, for each $m \geq 0$ and for each non zero $a \in A_m$, there exists $n \geq m$ such that the closed two-sided ideal of A_n generated by a is A_n itself.

(iii) If the A_k 's are all finite dimensional (so that A is AF), state a condition equivalent to those of (ii) in terms of the Bratelli diagram of the tower $A_0 \subset A_1 \subset ....$

(iv) Show that a UHF-algebra A with unit is simple, namely that the only two-sided ideals of A are $\{0\}$ and A itself.

[Indication for the implication $a \in A$, $a \notin \overline{\cup_n \mathcal{J}_n} \Rightarrow a \notin \mathcal{J}$ of (i). Consider the canonical projections $\pi : A \to A/\mathcal{J}$ and $\pi_* : A \to A/\left(\overline{\cup_n \mathcal{J}_n}\right)$. Set $\epsilon = \|\pi_*(a)\| > 0$. For each n large enough, there exists $a_n \in A_n$ such that $\|a_n - a\| \leq \epsilon/2$. If $\pi_n : A_n \to A_n/\mathcal{J}_n$ denotes the canonical projection at level n, one has

$$\|\pi_n(a_n)\| \;=\; \inf_{b \in \mathcal{J}_n} \|a_n - b_n\| \;\geq\; \inf_{b \in \overline{\cup_n \mathcal{J}_n}} \|a_n - b\| \;=\; \|\pi_*(a_n)\| \;\geq\; \epsilon/2.$$

As the canonical inclusion $A_n/\mathcal{J}_n \hookrightarrow A/\mathcal{J}$ is an isometry, one has also

$$\|\pi(a)\| = \lim_{n\to\infty} \|\pi_n(a_n)\| \geq \epsilon/2.$$

Thus $a \notin \mathcal{J}$.

Claim (iv) is a consequence of (iii) because, in a C*-algebra with unit A, the only two-sided ideal which is dense in A is A itself. But it can also be checked as follows.

If $A = \otimes_{j=1}^{\infty} M_{k_j}(\mathbb{C})$ is a UHF-algebra, every quotient $A \to A/\mathcal{J}$ is faithful on the simple algebra $\otimes_{j=1}^{n} M_{k_j}(\mathbb{C})$ for all $n \geq 1$, hence is faithful on A itself.]

5.18. Exercise. Let A be an AF C*-algebra and let B be a finite dimensional sub-C*-algebra of A. Show that the *relative commutant*

$$B' \cap A \;=\; \left\{ a \in A \mid ab = ba \;\text{ for all }\; b \in B \right\}$$

is an AF algebra. [If necessary, see the solution in Lemma 3.1 of [HeR].]

5.19. Exercise. Let $A = \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})_j$ be as in Example 5.10 and let

$$\mathcal{A}_\infty = \bigcup_{n\geq 1} \left(\bigotimes_{j=1}^{n} M_2(\mathbb{C})_j \right) \subset A,$$

so that any $x \in \mathcal{A}_\infty$ has finite spectrum. Write down elements of A with infinite spectra.

5.c. On reduced group C*-algebras

In this section, we consider a countable group Γ endowed with the counting measure, the Hilbert space $\ell^2(\Gamma)$ together with its usual orthonormal basis $(\epsilon_\gamma)_{\gamma\in\Gamma}$, the reduced C*-algebra $C^*_{red}(\Gamma)$ and the linear injection

$$\phi \;:\; \begin{cases} C^*_{red}(\Gamma) \longrightarrow \ell^2(\Gamma) \\ \qquad a \longmapsto a(\epsilon_e) \end{cases}$$

defined in number 4.1.

5.20. Proposition. *The reduced C^*-algebra of the infinite cyclic group is isomorphic to the algebra of continuous functions on the unit circle of the complex plane:*

$$C^*_{red}(\mathbb{Z}) \;\approx\; \mathcal{C}(\mathbb{T}).$$

Proof. Let $d\mu = \frac{1}{2\pi}d\theta$ denote the usual Lebesgue (or Haar!) measure on

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

The "Fourier transform" is an invertible isometry

$$\mathcal{F} : \ell^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{T}, d\mu)$$

by Plancherel theorem (also called here Parseval theorem).

Let $\mathcal{T}(\mathbb{T})$ denote the subspace of $L^2(\mathbb{T}, d\mu)$ consisting of trigonometric polynomials. There is by definition a bijection

$$\begin{cases} \mathbb{C}[\mathbb{Z}] \longrightarrow \mathcal{T}(\mathbb{T}) \\ \quad a \longmapsto \hat{a} \end{cases}$$

defined by $\hat{a}(z) = \sum_{n\in\mathbb{Z}}^{finite} a(n)z^n$ for all $z \in \mathbb{T}$. (The algebra $\mathbb{C}[\mathbb{Z}]$ of Laurent polynomials is often denoted by $\mathbb{C}[z, z^{-1}]$.) For each $a \in \mathbb{C}[\mathbb{Z}]$, let

$$\lambda(a) : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$$

denote as in 4.1 the convolution by a, and let

$$M_{\hat{a}} : L^2(\mathbb{T}, d\mu) \longmapsto L^2(\mathbb{T}, d\mu)$$

denote as in 1.15 the multiplication by $\hat{a}$. A straightforward computation shows that one has $M_{\hat{a}} = \mathcal{F}\lambda(a)\mathcal{F}^{-1}$ for all $a \in \mathbb{C}[\mathbb{Z}]$. In other terms, the diagram

$$\begin{array}{ccc} \mathbb{C}[\mathbb{Z}] & \overset{\lambda}{\hookrightarrow} & \mathcal{B}(\ell^2(\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{T}(\mathbb{T}) & \overset{M}{\hookrightarrow} & \mathcal{B}\left(L^2(\mathbb{T}, d\mu)\right) \end{array}$$

is commutative, where the left-hand side vertical arrow is the bijection $a \mapsto \hat{a}$ and where the right-hand side vertical arrow is the bijection $x \mapsto \mathcal{F}x\mathcal{F}^{-1}$.

Now $C^*_{red}(\mathbb{Z})$ is by definition the norm closure of $\lambda(\mathbb{C}[\mathbb{Z}])$ in $\mathcal{B}\left(\ell^2(\mathbb{Z})\right)$, and it follows from the Weierstrass approximation theorem that $\mathcal{C}(\mathbb{T})$ is isomorphic (via multiplication operators) to the norm closure of $M\left(\mathcal{T}(\mathbb{T})\right)$ in $\mathcal{B}\left(L^2(\mathbb{T}, d\mu)\right)$. Thus the isomorphism $\mathbb{C}[\mathbb{Z}] \to \mathcal{T}(\mathbb{T})$ extends to an isomorphism $C^*_{red}(\mathbb{Z}) \to \mathcal{C}(\mathbb{T})$. $\square$

5.21. Proposition. *Let G be a locally compact abelian group and let $\hat{G}$ denote its Pontryagin dual. Then one has*

$$C^*_{red}(G) \;\approx\; \mathcal{C}_0(\hat{G}).$$

Proof. The argument of the previous proof carries over, modulo Pontryagin theory of duality for locally compact abelian groups. □

5.22. Proposition (Powers). *Let Γ be a non abelian free group. Then $C^*_{red}(\Gamma)$ is a simple C^*-algebra.*

Proof. We refer to [Po2]. □

5.23. Proposition. *Let Γ be a group $PSL(n,\mathbb{Z})$ for some $n \geq 2$, or more generally a lattice in a connected real semi-simple Lie group without compact factors and with center reduced to $\{1\}$. Then $C^*_{red}(\Gamma)$ is a simple C^*-algebra.*

Proof. We refer to [BCH1] and [BCH2]. □

5.24. Remark. As already mentioned in 4.1, the two previous propositions show that, in general, a unitary representation $\Gamma \to \mathcal{U}(\mathcal{H})$ *does not* correspond to any representation $C^*_{red}(\Gamma) \to \mathcal{B}(\mathcal{H})$. There is another C*-algebra $C^*_{max}(\Gamma)$ of which the representation *do* correspond to the unitary representations of Γ, and for which we refer to [DC*].

There is a canonical morphism

$$C^*_{max}(\Gamma) \;\longrightarrow\; C^*_{red}(\Gamma)$$

which is always onto and which is an isomorphism if and only if Γ is *amenable*. Any group Γ for which there exists a short exact sequence

$$1 \longrightarrow F \longrightarrow \Gamma \longrightarrow S \longrightarrow 1$$

with F finite and S solvable is amenable. For *linear groups*, i.e. for groups having faithful representations in $GL(n,\mathbb{C})$ for some $n \geq 1$, the converse holds : if such a group is amenable, then there exists a short exact sequence as above [Tit].

P. de la Harpe and V. Jones, July 1995.

CHAPTER 6. STATES AND THE GNS-CONSTRUCTION

The letters GNS refer to I.M. Gelfand, M.A. Naimark [GeN, 1943] and I. Segal [Seg, 1947].

6.A. States

6.1. Definition. Let A be a C*-algebra. A linear form $\phi : A \to \mathbb{C}$ is *positive* if $\phi(a^*a) \geq 0$ for all $a \in A$.

Observe that there is a Cauchy-Schwarz inequality for a positive linear form ϕ :

$$|\phi(b^*a)|^2 \leq \phi(b^*b)\phi(a^*a)$$

for all $a, b \in A$.

6.2. Proposition. *Let A be a C^*-algebra with unit.*
(i) A positive linear form ϕ on A is bounded, and

$$\|\phi\| = \phi(1).$$

(ii) A bounded linear form ϕ on A such that $\|\phi\| = \phi(1)$ is positive.

Proof. (i) For each $a \in A$ one has $\|a^*a\| 1 - a^*a \geq 0$ by Lemma 4.41, hence

$$\phi(a^*a) \leq \|a^*a\| \phi(1).$$

Using Cauchy-Schwarz inequality (6.1), we have consequently

$$|\phi(a)| \leq \phi(1)^{1/2}\phi(a^*a)^{1/2} \leq \|a^*a\|^{1/2} \phi(1) = \|a\| \phi(1).$$

It follows that $\|\phi\| \leq \phi(1)$.

(ii) Upon replacing ϕ by $\phi / \|\phi\|$, we may assume that $\phi(1) = 1$.

Let us first show that ϕ takes real values on self-adjoint elements. Choose $a \in A$, $a^* = a$, and let $\alpha, \beta \in \mathbb{R}$ be such that $\phi(a) = \alpha + i\beta$. For each $\lambda \in \mathbb{R}$ one has

$$\|a + i\lambda\|^2 \;=\; \|(a - i\lambda)(a + i\lambda)\| \;=\; \left\|a^2 + \lambda^2\right\| \;=\; \|a\|^2 + \lambda^2$$

(the last equality because of Corollary 4.21) and consequently

$$\beta^2 + 2\lambda\beta + \lambda^2 \;\leq\; |\alpha + i(\beta + \lambda)|^2 \;=\; |\phi(a + i\lambda)|^2 \;\leq\; \|a\|^2 + \lambda^2;$$

this implies $\beta = 0$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let us show that ϕ takes positive values on positive elements. Choose $h \in A$, $0 \le h \le 1$. Then

$$|1 - \phi(h)| \;=\; |\phi(1-h)| \;\le\; \|1-h\| \;\le\; 1$$

and consequently $\phi(h) \ge 0$. □

6.3. Definitions. A *state* on a C*-algebra A is a linear form on A which is positive and of norm 1. The *state space* S_A of A is the set of all states on A.

If A has a unit, it follows straightforwardly from Proposition 6.2 that S_A is a *convex* subset of the dual of A. The same fact holds for C*-algebras without units (but the proof is not completely trivial: see e.g. n^{os} 2.1.5 and 2.1.6 in [DC*]).

A state ϕ on a C*-algebra A is *extreme,* or *pure,* if it has the following property: if ϕ_0, ϕ_1 are states on A and if $t \in]0,1[$ is a real number such that $\phi = (1-t)\phi_0 + t\phi_1$, then $\phi_0 = \phi_1$.

6.4. Etymology. In quantum mechanics, there are models where the universe (??? or the system to analyze ???) is described by some complex Hilbert space $\mathcal{H}$. An "observable" is a (possibly unbounded) self-adjoint operator a on $\mathcal{H}$, and the spectrum $\sigma(a)$ is the set of possible outcomes of a measure of a.

To each *local* system corresponds a C*-algebra A of bounded operators on $\mathcal{H}$. If such a system is in a *state* ϕ, the average of many measures of the observable a is a number $\phi(a)$. There are abstract considerations which justify that the assignment $a \mapsto \phi(a)$ should be linear and positive.

The superposition principle of quantum mechanics gives rise to the notion of *pure* state, one which cannot be obtained as a superposition of other states.

6.5. Examples. (i) Let A be a C*-algebra of operators on a Hilbert space $\mathcal{H}$ which contains $id_{\mathcal{H}}$ and let $\xi \in \mathcal{H}(1)$ be a vector of norm 1. Then the linear form

$$\omega_\xi \begin{cases} A & \longrightarrow & \mathbb{C} \\ a & \longmapsto & \langle \xi | a\xi \rangle \end{cases}$$

is positive. One has $\omega_\xi = 1$ if moreover $id_{\mathcal{H}} \in A$ (or more generally if moreover $\overline{A\mathcal{H}} = \mathcal{H}$ - see [DC*, 2.4.3]) and in this case ω_ξ is called a *vector state.* Observe that

$$\omega_{\xi} = \omega_{e^{it}\xi}$$

for all $t \in \mathbb{R}$.

Proposition 6.11 below shows that these vector states are in some sense the only states on C*-algebras.

(ii) Let X be a compact space. For any probability measure μ on X, the map

$$\phi_\mu : \begin{cases} \mathcal{C}(X) & \longrightarrow & \mathbb{C} \\ f & \longmapsto & \displaystyle\int_X f(x)d\mu(x) \end{cases}$$

is a state on the abelian C*-algebra $\mathcal{C}(X)$. All states on $\mathcal{C}(X)$ are of this form by a theorem of F. Riesz (see Theorem 2.14 in [Ru1], and do not confuse this theorem of Riesz with the one recalled in 1.3). Pure states correspond bijectively to Dirac measures on X.

(iii) Let $\mathcal{A}$ be the dense $*$-subalgebra of $\mathcal{C}([0,1])$ of polynomial functions $\mathbb{R} \to \mathbb{C}$ and let $\phi : \mathcal{A} \to \mathbb{C}$ be the linear form defined by $\phi(f) = f(2)$. Then one has

$$\phi(f^* f) \geq 0$$

for all $f \in \mathcal{A}$ but ϕ is *not continuous* with respect to the norm $\|f\| = \sup_{0 \leq t \leq 1} |f(t)|$. Indeed, if $(f_n)_{n \geq 1}$ is the sequence in $\mathcal{A}$ defined by $f(t) = t^n$, then $\|f_n\| = 1$ and $\phi(f_n) = 2^n$ for all $n \geq 1$.

6.6. Remark. Let ϕ be a state on a finite dimensional C*-algebra A. We know from § 2.B that $A = \bigoplus_{j=1}^r A_j$ where each A_j is isomorphic to a full matrix algebra. For each $j \in \{1, ..., r\}$, let $\phi_j : A_j \to \mathbb{C}$ be the composition of the canonical inclusion $A_j \subset A$ and of ϕ. Then ϕ_j is a state on A_j and ϕ is in the appropriate sense the sum of the ϕ_j 's. Thus, to understand states on finite dimensional C*-algebras, it is essentially sufficient to understand states on full matrix algebras.

For the definition of the positive cone $M_n(\mathbb{C})_+$ in the next proposition, see 4.40.

6.7. Proposition. *Consider an integer $n \geq 1$ and a state ϕ on the C*-algebra $M_n(\mathbb{C})$. Then there exists a positive matrix $h \in M_n(\mathbb{C})_+$ such that $trace(h) = 1$ and*

$$\phi(a) = trace(ha)$$

for all $a \in M_n(\mathbb{C})$.

The state ϕ is pure if and only if h is a projection of rank 1.

Proof. The bilinear form defined on $M_n(\mathbb{C})$ by $(a, b) \mapsto trace(ab)$ is non degenerate, because $trace(a^* a) = \sum_{j,k=1}^n |a_{j,k}|^2$ for all $a = (a_{j,k})_{1 \leq j,k \leq n}$. Hence, for every linear form ϕ on $M_n(\mathbb{C})$, there exists $h \in M_n(\mathbb{C})$ such that $\phi(a) = trace(ha)$ for all $a \in M_n(\mathbb{C})$. Assuming now that ϕ is a state, we have to check that h is positive and of trace 1.

For all $a \in A$, one has

$$trace\big((h - h^*)a\big) = trace(ha) - \overline{trace(ha^*)} = \phi(a) - \overline{\phi(a^*)} = 0.$$

It follows that $h = h^*$. Thus there are orthogonal minimal projections $p_1, ..., p_n \in M_n(\mathbb{C})$ and real numbers $t_1, ..., t_n$ such that $h = \sum_{k=1}^n t_k p_k$. For all $j \in \{1, ..., n\}$, one has $t_j = trace(hp_j) = \phi(p_j) \geq 0$. In other words, the matrix h is positive. Also $trace(h) = \phi(1) = 1$. The last statement is now straightforward. □

6.8. Remark. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. There is a notion of *trace-class* operator on $\mathcal{H}$: they are compact operators h on $\mathcal{H}$ such that

$$\sum_{\iota \in I} \langle e_\iota | h e_\iota \rangle < \infty$$

for all orthonormal basis $(e_\iota)_{\iota\in I}$ of $\mathcal{H}$, and they have a *trace* defined by

$$trace(h) = \sum_{\iota\in I} \langle e_\iota | h e_\iota \rangle$$

(this number $trace(h)$ is indeed independent of the choice of the orthonormal basis). For such a trace-class operator h which is moreover positive and of trace 1, the linear form $a \mapsto trace(ha)$ is a state on $\mathcal{B}(\mathcal{H})$. It is known that any state on $\mathcal{B}(\mathcal{H})$ which is *normal* is of this form: see e.g. Theorem 1.15.3 in [Sak]. (A linear form is said to be normal if it is continuous with respect to the so-called ultra-weak topology, or equivalently to the so-called ultra-strong topology - see e.g. [StZ], Corollary 1.6 of Chapter 1.)

Exercise: find the positive trace-class operators h of trace 1 for which the state $a \mapsto trace(ha)$ is pure.

There are states on $\mathcal{B}(\mathcal{H})$ which are *not* normal. For example, consider a representation

$$\pi : \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{L})$$

of the *Calkin algebra* of $\mathcal{H}$ in some Hilbert space $\mathcal{L}$, and let $p : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the canonical projection. For any vector $\xi \in \mathcal{L}$, the state ω defined on $\mathcal{B}(\mathcal{H})$ by $\omega(a) = \langle \xi | \pi(p(a))\xi \rangle$ vanishes on $\mathcal{K}(\mathcal{H})$ and consequently is not normal. For the existence of representations such as π, see Theorem 6.11 below, or more constructively Calkin's original paper. It is known that π is necessarily faithful (because the Calkin algebra is simple) and that $\mathcal{L}$ is necessarily not separable [Cal].

6.9. Proposition. *Let A be a C^*-algebra and let $a \in A$, $a \neq 0$. Then there exists a state ϕ on A such that $\phi(a^*a) > 0$.*

Proof. We know from Proposition 4.43 that A_+ is a closed convex cone in A_{sa}. As $-a^*a \notin A_+$, the Hahn-Banach theorem shows that there exists a linear form $\tilde{\phi} : A_{sa} \to \mathbb{R}$ of norm 1 which is positive on A_+ and strictly negative on $-a^*a$. The $\mathbb{C}$-linear extension ϕ of $\tilde{\phi}$ to A is a state such that $\phi(-a^*a) < 0$. □

6.10. Topology on the state space. Let A be a C*-algebra. We denote by B_A the space of positive linear form on A of norm ≤ 1, together with the topology of pointwise convergence; this is obviously a convex subset in the dual of A, and it is a compact space (a consequence of Tychonoff theorem). As already observed in 6.3, the state space S_A is a convex subset of B_A. The set of extreme points of B_A is clearly the union of $\{0\}$ and of the space $P(A)$ of pure states, because a form $\phi \in B_A$ such that $0 < \|\phi\| < 1$ cannot be an extreme point of B_A (indeed $\phi = (1-t)0 + t\phi\,\|\phi\|$ for $t = \|\phi\|$).

If A has a unit, S_A is the intersection of B_A with the closed affine hyperplane of equation $\phi(1) = 1$, so that S_A is a compact space.

If A has no unit, S_A is a locally compact space which is non compact ([BrR], Theorem 2.3.15).

If $A = \mathcal{C}_0(X)$ is an abelian C*-algebra, we have already observed in Example 6.5 that $P(A)$ is in bijective correspondance with X (via Dirac measures), and one may check that this bijection is an homeomorphism. If $A = M_n(\mathbb{C})$ for some integer $n \geq 1$, Proposition

6.7 shows that $P(A)$ is in natural bijection with the projective space $\mathbb{P}^{n-1}(\mathbb{C})$ of lines in $\mathbb{C}^n$ (i.e. of orthogonal projections of rank 1 in $M_n(\mathbb{C})$), and again this bijection is a homeomorphism. These examples show that $P(A)$ is an interesting topological space.

The quotient space of $P(A)$ by the relation of equivalence for states (see Definition 6.23 below) is the *dual* $\hat{A}$ of A, and the quotient topology is the *Jacobson topology* on $\hat{A}$ [DC*, § 3].

6.B. The GNS-construction

Recall from 4.2 that a representation of a C*-algebra A on a Hilbert space $\mathcal{H}$ is a linear map $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^*$ for all $a, b \in A$, from Corollary 4.25 that these conditions imply $\|\pi(a)\| \leq \|a\|$ for all $a \in A$, and from Proposition 4.37 that a faithful (= injective) representation satisfies moreover $\|\pi(a)\| = \|a\|$ for all $a \in A$.

6.11. Theorem (GNS construction). *Let A be a C*-algebra with unit and let $\phi : A \to \mathbb{C}$ be a state.*

(i) Then there exist

→ a Hilbert space $\mathcal{H}_\phi$,
→ a representation $\pi_\phi : A \to \mathcal{B}(\mathcal{H}_\phi)$,
→ a vector $\xi_\phi \in \mathcal{H}_\phi$ of norm 1

such that

$$\phi(a) = \langle \xi_\phi \mid \pi(a)\xi_\phi \rangle$$

for all $a \in A$ and such that ξ_ϕ is cyclic for π_ϕ (namely such that $\overline{\pi_\phi(A)\xi_\phi} = \mathcal{H}_\phi$).

(ii) The triple $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ is unique up to isomorphism in the following sense. Let $\mathcal{H}$ be a Hilbert space, let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation and let $\xi \in \mathcal{H}$ be a unit vector such that $\phi(a) = \langle \xi | \pi(a)\xi \rangle$ for all $a \in A$ and such that ξ is cyclic for π. Then there exists a unitary isomorphism $u : \mathcal{H}_\phi \to \mathcal{H}$ such that $\pi(a) = u\pi_\phi(a)u^*$ for all $a \in A$ and such that $u(\xi_\phi) = \xi$.

Proof. (i) Set $V_\phi = \{ a \in A \mid \phi(a^*a) = 0 \}$. For $a \in V_\phi$ and $b \in A$, one has also $\phi(b^*a) = 0$ by Cauchy-Schwarz inequality. Thus

$$V_\phi = \{ a \in A \mid \phi(b^*a) = 0 \ \text{ for all } \ b \in A \}$$

and V_ϕ is a closed left ideal in A. The positive sesquilinear form $(b, a) \mapsto \phi(b^*a)$ on A defines a positive sesquilinear form on the quotient A/V_ϕ given by

$$\langle b + V_\phi \mid a + V_\phi \rangle = \phi(b^*a)$$

for all $a, b \in A$. This makes A/V_ϕ a prehilbert space. We define $\mathcal{H}_\phi$ to be its completion, and $\xi_\phi \in \mathcal{H}_\phi$ to be the vector $1 + V_\phi \in A/V_\phi \subset \mathcal{H}_\phi$.

For each $a \in A$, let $L_a : A/V_\phi \to A/V_\phi$ denote the left multiplication $b + V_\phi \mapsto ab + V_\phi$. To compute $\|L_a\|$, consider the positive linear form defined on A by $a \mapsto \phi(b^*ab)$, which is of norm $\phi(b^*b)$ by Proposition 6.2.i. One has

$$\langle L_a(b+V_\phi) \mid L_a(b+V_\phi)\rangle \;=\; \phi(b^*a^*ab) \;\leq\; \|a\|^2\,\phi(b^*b) \;=\; \|a\|^2\,\langle b+V_\phi \mid b+V_\phi\rangle$$

for all $a \in A$, so that $\|L_a\| \leq \|a\|$. Thus L_a extends to a bounded operator on $\mathcal{H}_\phi$ that we denote by $\pi_\phi(a)$. The map $\pi_\phi : A \to \mathcal{B}(\mathcal{H}_\phi)$ is clearly a representation such that $\phi(a) = \langle \xi_\phi | \pi_\phi(a)\xi_\phi\rangle$ for all $a \in A$ and such that $\overline{\pi_\phi(A)\xi_\phi} = \overline{A/V_\phi} = \mathcal{H}_\phi$.

(ii) For all $a, b \in A$, one has

$$\langle \pi_\phi(b)\xi_\phi \mid \pi_\phi(a)\xi_\phi\rangle \;=\; \langle \xi_\phi \mid \pi_\phi(b^*a)\xi_\phi\rangle \;=\; \langle \xi \mid \pi(b^*a)\xi\rangle \;=\; \langle \pi(b)\xi \mid \pi(a)\xi\rangle\,.$$

As the $\pi_\phi(a)\xi_\phi$'s [respectively the $\pi(a)\xi$'s] are dense in $\mathcal{H}_\phi$ [resp. in $\mathcal{H}$], there exists an isomorphism $u : \mathcal{H}_\phi \to \mathcal{H}$ such that $u\pi_\phi(a)\xi_\phi = \pi(a)\xi$ for all $a \in A$. We leave it to the reader to finish the proof. (See [DC*, 2.4.1] if necessary.) □

6.12. Example. Let us revisit the previous proof in case $A = M_n(\mathbb{C})$ and $\phi : A \to \mathbb{C}$ given by $\phi(a) = trace(ap)$, where

$$p = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

is a projection of rank one. One has

$$V_\phi = \left\{ \begin{pmatrix} 0 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & * & \dots & * \end{pmatrix} \right\} \;\subset\; M_n(\mathbb{C}).$$

Thus, if e_1 denotes the first vector of the canonical basis of $\mathbb{C}^n$, the map $A \to \mathbb{C}^n$ defined by $a \mapsto a(e_1)$ factors as an isomorphism $A/V_\phi \to \mathbb{C}^n$. The scalar product defined via ϕ on this $A/V_\phi \approx \mathbb{C}^n$ is given by

$$\begin{aligned} \langle b \mid a\rangle \;&=\; \phi\left(\begin{pmatrix} b_1 & * & \dots & * \\ b_2 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ b_n & * & \dots & * \end{pmatrix}^* \begin{pmatrix} a_1 & * & \dots & * \\ a_2 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ a_n & * & \dots & * \end{pmatrix} \right) \\ &=\; \phi \begin{pmatrix} \sum \overline{b_j} a_j & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ * & * & \dots & * \end{pmatrix} \;=\; \sum_{j=1}^{n} \overline{b_j} a_j \end{aligned}$$

and is thus nothing but the standard scalar product on $\mathbb{C}^n$. The representation π_ϕ is the tautological representation of $M_n(\mathbb{C})$ on $\mathbb{C}^n$.

6.13. Proposition. *Let A be a C^*-algebra with unit, let ϕ be a state on A and let $\pi_\phi : A \to \mathcal{B}(\mathcal{H}_\phi)$ be the representation obtained by the GNS-construction.*

Then the representation π_ϕ is irreducible if and only if the state ϕ is pure.

Proof. Suppose first that π_ϕ is reducible. There exist two orthogonal projections $p, q \in \pi_\phi(A)'$, both distinct from 0, such that $p + q = 1$. Observe that $p\xi_\phi \neq 0$, because

$$\overline{\pi_\phi(A)p\xi_\phi} = p\overline{\pi_\phi(A)\xi_\phi} = p\mathcal{H}_\phi \neq \{0\},$$

and similarly that $q\xi_\phi \neq 0$. Define two states ϕ_0 and ϕ_1 on A by

$$\begin{aligned} \phi_0(a) &= \frac{\langle p\xi_\phi \mid \pi_\phi(a)p\xi_\phi\rangle}{\|p\xi_\phi\|^2}, \\ \phi_1(a) &= \frac{\langle q\xi_\phi \mid \pi_\phi(a)q\xi_\phi\rangle}{\|q\xi_\phi\|^2} \end{aligned}$$

so that $\phi = \|p\xi_\phi\|^2\,\phi_0 + \|q\xi_\phi\|^2\,\phi_1$.

To prove that ϕ is not pure, it remains to check that $\phi_0 \neq \phi_1$. But this is clear because one has $\phi_0(p) = 1$ and, as $qp = 0$, one has $\phi_1(p) = 0$.

Suppose now that ϕ is not pure, namely that there exist two distinct states ϕ_0, ϕ_1 on A and a number $t \in]0,1[$ such that $\phi = (1-t)\phi_0 + t\phi_1$. The sesquilinear form $(a, b) \mapsto \phi_0(a^*b)$ on A defines a sesquilinear form on $\mathcal{H}_\phi$, let's denote it by $(\xi, \eta) \mapsto [\xi|\eta]$, and it follows from Riesz Theorem 1.3 that there exists a bounded operator (indeed a positive one) h on $\mathcal{H}_\phi$ such that $[\xi|\eta] = \langle\xi|h\eta\rangle$ for all $\xi, \eta \in \mathcal{H}_\phi$. A straightforward computation shows that $h \in \pi_\phi(A)'$.

To prove that π_ϕ is reducible, it remains to check that $h \notin \mathbb{C}id_{\mathcal{H}}$. But this is clear, because $h \in \mathbb{C}id_{\mathcal{H}}$ would imply $\phi_0 = \phi$. □

6.14. Theorem (Gelfand-Naimark, 1943). *Let A be a C^*-algebra.*

(i) There exists a Hilbert space $\mathcal{H}$ and a faithful representation π of A on $\mathcal{H}$. If A is separable, $\mathcal{H}$ can be chosen separable.

(ii) For each $a \in A$, there exists an irreducible representation π of A such that $\pi(a) \neq 0$.

Sketch of proof. (i) Let $a \in A, a \neq 0$. Let ϕ_a be as in Proposition 6.9 a state on A such that $\phi_a(a^*a) > 0$. Let $\pi_a : A \to \mathcal{B}(\mathcal{H}_a)$ be the corresponding GNS representation, with cyclic vector denoted by ξ_a. Then $\|\pi_a(a)\xi_a\|^2 = \phi_a(a^*a) > 0$, so that $\pi_a(a) \neq 0$. The Hilbert sum of the π_a 's over a set of a 's which is dense in A provides a faithful representation of A.

(ii) This follows from the Krein-Milman theorem, which shows in this context that the set of all states is the weakly closed convex hull of the set of pure states.

For more on this, see e.g. Théorème 2.7.3 in [DC*]. □

6.15. Corollary (Gelfand-Raikov, 1943). *For any locally compact group G and for any $g \in G$ distinct from the unit element, there exists an irreducible continuous unitary representation π of G such that $\pi(g) \neq 1$.*

Proof. See e.g. Corollaire 13.6.6 in [DC*]. □

6.16. Proposition. *Let A be a C^*-algebra, let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of A, let $\xi \in \mathcal{H}$ be a unit vector and let ϕ denote the state on A defined by*

$$\phi(a) = \langle \xi | \pi(a)\xi \rangle \, .$$

Then the GNS-representation π_ϕ is a subrepresentation of π. In particular, if π is irreducible, then π_ϕ and π are unitarily equivalent.

Proof. This is a straightforward consequence of the unicity part of Theorem 6.11. □

6.c. The Kaplansky density theorem, and some applications.

6.17. Comment. Let A be an involutive subalgebra of $\mathcal{H}$ containing $id_{\mathcal{H}}$ and let $a \in A''$. Von Neumann Density Theorem (2.17) shows that there exists a generalized sequence in A which converges strongly to a, but it gives no information on the norms. For this, the following is useful.

Whereas the theorem of von Neumann is strictly about von Neumann algebras, the theorem of Kaplansky mixes the C*-algebra structure and the von Neumann algebra structure.

6.18. Kaplansky Density Theorem. *Let A, B be two involutive algebras of operators on a Hilbert space $\mathcal{H}$ such that*

$$id_{\mathcal{H}} \;\in\; A \;\subset\; B \;\subset\; \mathcal{B}(\mathcal{H})$$

and such that A is strongly dense in B. Then the unit ball of A is strongly dense in the unit ball of B.

6.19. Lemma. *In the hypothesis of the previous theorem, the self-adjoint part A_{sa} of A is strongly dense in the self-adjoint part B_{sa} of B.*

Proof of Lemma. If A is strongly dense in B, then A is a fortiori weakly dense in B. As the mapping

$$\begin{cases} \mathcal{B}(\mathcal{H}) & \longrightarrow & \mathcal{B}(\mathcal{H}) \\ x & \longmapsto & \dfrac{1}{2}(x + x^*) \end{cases}$$

is weakly continuous, the self-adjoint part $A_{sa} = \{ \, a \in A | a^* = a \, \}$ of A is weakly dense in B_{sa}. Thus the lemma follows from Proposition 2.16.iii, according to which a convex subset of $\mathcal{B}(\mathcal{H})$ which is weakly closed is automatically strongly closed.

Observe that $x \mapsto \frac{1}{2}(x + x^*)$ needs not be strongly continuous, so that the argument using the weak topology cannot be avoided. □

6.20. Proof of Theorem 6.18. We assume for simplicity that A and B are C*-algebras of operators in $\mathcal{H}$, and we leave it to the reader to check that this hypothesis is harmless (because the norm topology is stronger than the strong topology).

Consider $b \in B$ such that $\|b\| \leq 1$. We have to show that any basic strong neighbourhood $\mathcal{V}$ of b in $\mathcal{B}(\mathcal{H})$ has a non empty intersection with the unit ball of A. As the case $b = 0$ is clear, there is no loss of generality in assuming from now on that $\|b\| = 1$.

Assume firstly that $b^* = b$. Let $f : [-1,1] \to [-1,1]$ be the function defined by $f(t) = 2t/(1+t^2)$. Calculus shows that f is a homeomorphism; let g denote the inverse homeomorphism. By continuous functional calculus, one may define $y = g(b) \in B$, and one has

$$b = \frac{2y}{1+y^2}.$$

Consider a finite sequence of vectors $\xi_1, ..., \xi_n \in \mathcal{H}$, a number $\epsilon > 0$ and the strong neighbourhood

$$\mathcal{V} = \left\{ c \in \mathcal{B}(\mathcal{H}) \;\middle|\; \|(c-b)\xi_j\| < \epsilon \quad \text{for} \quad j \in \{1, ..., n\} \right\}$$

of b in $\mathcal{B}(\mathcal{H})$. We have to show that there exists $a \in A \cap \mathcal{V}$ such that $\|a\| \leq 1$.

By the previous lemma, there exists a self-adjoint element $x \in A_{sa}$ such that

$$\begin{aligned} \|(x-y)b\xi_j\| &< \frac{\epsilon}{2} &\quad \text{for each} \quad & j \in \{1, ..., n\} \\ \left\|(x-y)(1+y^2)^{-1}\xi_j\right\| &< \frac{\epsilon}{4} &\quad \text{for each} \quad & j \in \{1, ..., n\}. \end{aligned}$$

One sets $a = 2x(1+x^2)^{-1} \in A_{sa}$. As $t \mapsto 2t(1+t^2)^{-1}$ maps the whole of $\mathbb{R}$ onto $[-1,1]$, one has $\|a\| \leq 1$. One computes

$$\begin{aligned} a - b &= \frac{2x}{1+x^2} - \frac{2y}{1+y^2} = 2\frac{1}{1+x^2}\Big(x(1+y^2) - (1+x^2)y\Big)\frac{1}{1+y^2} \\ &= 2\frac{1}{1+x^2}(x-y)\frac{1}{1+y^2} + 2\frac{x}{1+x^2}(y-x)\frac{y}{1+y^2} \\ &= 2\frac{1}{1+x^2}(x-y)\frac{1}{1+y^2} + \frac{1}{2}a(y-x)b. \end{aligned}$$

It follows that

$$\|(a-b)\xi_i\| \leq 2\left\|\frac{1}{1+x^2}\right\| \left\|(x-y)\frac{1}{1+y^2}\xi_j\right\| + \frac{1}{2}\|a\| \; \|(x-y)b\xi_j\| < \epsilon$$

namely that $a \in \mathcal{V}$.

Consider now the general case (b not necessarily self-adjoint). The operator

$$\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} \in M_2(B)$$

is self-adjoint, and its norm is the same as that of b. As $M_2(A)$ is strongly dense in $M_2(B)$, the previous argument applies and there exists a self-adjoint element

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in A_{sa}$$

of norm at most 1 which approximates $\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$ in the strong topology. In particular $a_{1,2} \in A$ is of norm at most one and approximates b in the strong topology. $\square$

6.21. Proposition. *Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of a C*-algebra A on a Hilbert space $\mathcal{H}$. The following are equivalent:*

(i) the only A-invariant closed subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$,
(ii) the only A-invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$,
(iii) for all finite sequences $\xi_1, ..., \xi_n$ in $\mathcal{H}$ and for all $u \in \mathcal{U}(\mathcal{H})$, there exists a unitary element v in A such that $u(\xi_j) = \pi(v)(\xi_j)$ for all $j \in \{1, ..., n\}$.

(If A has no unit or if $\pi(1) \neq id_{\mathcal{H}}$, statement (iii) should be understood for $v \in \tilde{A}$ - see 4.5 - and for the canonical extension of π to $\tilde{A}$.)

Proof. We refer to [DC*,théorème 2.8.3] or to [Ped, Theorems 2.7.5 and 3.13.2]. The proof depends strongly on Kaplansky's density theorem. □ (For other equivalent conditions, see Proposition 4.51.)

6.22. Remark. The equivalence (i) $\Longleftrightarrow$ (ii) is sometimes expressed as follows: a representation of A is topologically irreducible if and only if it is algebraically irreducible.

6.23. Definition. Let A be a C*-algebra. Two representations $\pi : A \to \mathcal{B}(\mathcal{H}_\pi)$ and $\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ are *equivalent* if there exists a surjective isometry $u : \mathcal{H}_\pi \to \mathcal{H}_\rho$ such that $\rho(a) = u\pi(a)u^*$ for all $a \in A$.

Similarly, two states ϕ and ψ on A are *equivalent* if the GNS-representations π_ϕ and π_ψ of A are equivalent.

6.24. Examples. (i) Let ϕ_1, ϕ_2 be two states on a C*-matrix algebra $M_n(\mathbb{C})$, and let $h_1, h_2 \in M_n(\mathbb{C})_+$ be the corresponding positive matrices, as in Proposition 6.7. Then ϕ_1 and ϕ_2 are equivalent if and only if h_1 and h_2 have the same rank.

(ii) Let X be a compact space, let ϕ_1, ϕ_2 be two states on the abelian C*-algebra $\mathcal{C}(X)$ and let μ_1, μ_2 be the corresponding probability measures on X. Then ϕ_1 and ϕ_2 are equivalent states if and only if μ_1 and μ_2 are equivalent measures. The proof involves essentially the Radon-Nikodym theorem; see Theorem 2.2.2 in [Arv] for details.

6.25. Proposition. *Let A be a C*-algebra with unit and let ϕ_1, ϕ_2 be two pure states on A. Then ϕ_1 and ϕ_2 are equivalent if and only if there exists a unitary element $v \in A$ such that*

$$\phi_2(a) \;=\; \phi_1(vav^*)$$

for all $a \in A$.

Proof. For $j \in \{1, 2\}$, let $(\mathcal{H}_j, \pi_j, \xi_j)$ denote the GNS data associated to the state ϕ_j as in Theorem 6.11.

Suppose that there exists a unitary $v \in A$ such that $\phi_2(a) = \phi_1(vav^*)$ for all $a \in A$. Set $\xi_1' = \pi_1(v^*)\xi_1$. Then

$$\langle \xi_2 \mid \pi_2(a)\xi_2 \rangle \;=\; \phi_2(a) \;=\; \phi_1(vav^*) \;=\; \langle \xi_1 \mid \pi_1(vav^*)\xi_1 \rangle \;=\; \langle \xi_1' \mid \pi_1(a)\xi_1' \rangle$$

for all $a \in A$. Hence π_1 and π_2 are equivalent by the unicity part of Theorem 6.11.

Suppose conversely that ϕ_1 and ϕ_2 are equivalent. There exists an isomorphism $u : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(a) = u\pi_1(a)u^*$ for all $a \in A$. As

$$\|u^*(\xi_2)\| \;=\; \|\xi_1\| \;=\; 1$$

there exists a unitary operator on $\mathcal{H}$ which maps $u^*(\xi_2)$ to ξ_1, hence there exists by Proposition 6.21 a unitary $v \in A$ such that $\pi_1(v)u^*(\xi_2) = \xi_1$. Then

$$\begin{aligned}\phi_2(a) &= \langle \xi_2 \mid \pi_2(a)\xi_2\rangle \;=\; \langle u^*\xi_2 \mid \pi_1(a)u^*\xi_2\rangle \\ &= \langle \pi_1(v^*)\xi_1 \mid \pi_1(a)\pi_1(v^*)\xi_1\rangle \;=\; \langle \xi_1 \mid \pi_1(vav^*)\xi_1\rangle \\ &= \phi_1(vav^*)\end{aligned}$$

for all $a \in A$. □

Let ϕ, ψ be two states on a C*-algebra. As states are by definition of norm 1, one has $\|\phi - \psi\| \leq 2$. The next result appeared in [GlK].

6.26. Theorem (Glimm-Kadison). *Let ϕ, ψ be two pure states on a C*-algebra A such that $\|\phi - \psi\| < 2$. Then ϕ and ψ are equivalent.*

Proof. Denote by $\pi_\phi : A \to \mathcal{B}(\mathcal{H}_\phi)$ and $\pi_\psi : A \to \mathcal{B}(\mathcal{H}_\psi)$ the GNS-representations defined by ϕ and ψ, and let $\rho : A \to \mathcal{B}(\mathcal{H}_\phi \oplus H_\psi)$ denote the direct sum $\pi_\phi \oplus \pi_\psi$.

Let $x = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathcal{B}(\mathcal{H}_\phi \oplus H_\psi)$ be an element in the commutant of $\rho(A)$, namely be such that

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} \pi_\phi(a) & 0 \\ 0 & \pi_\psi(a) \end{pmatrix} \;=\; \begin{pmatrix} \pi_\phi(a) & 0 \\ 0 & \pi_\psi(a) \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

for all $a \in A$. One has $t\pi_\phi(a) = \pi_\psi(a)t$ for all $a \in A$, hence also

$$\begin{aligned} t^*t\pi_\phi(a) &= \pi_\phi(a)t^*t \qquad \text{for all} \qquad a \in A, \\ tt^*\pi_\psi(a) &= \pi_\psi(a)tt^* \qquad \text{for all} \qquad a \in A. \end{aligned}$$

Similarly s^*s commutes with $\pi_\psi(a)$ and ss^* commutes with $\pi_\phi(a)$ for all $a \in A$. As π_ϕ, π_ψ are irreducible (Proposition 6.16), Schur's lemma implies that t^*t and ss^* [respectively tt^* and s^*s] are scalar multiples of the identity on $\mathcal{H}_\phi$ [resp. $\mathcal{H}_\psi$]. It follows that there exist constants $\lambda \geq 0, \mu \geq 0$ and unitary isomorphisms $\tilde{t} : \mathcal{H}_\phi \to \mathcal{H}_\psi$, $\tilde{s} : \mathcal{H}_\psi \to \mathcal{H}_\phi$ such that $t = \lambda\tilde{t}$, $s = \mu\tilde{s}$.

Let us now assume that ϕ and ψ are not equivalent, so that we have to prove that $\|\phi - \psi\| = 2$. As $t = \lambda\tilde{t}$ satisfies $t\pi_\phi(a) = \pi_\psi(a)t$ for all $a \in A$, one has necessarily $t = 0$. Similarly, $s = 0$. Hence any operator in the commutant of $\rho(A)$ is of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, and one has

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;\in\; \rho(A)'' \;\subset\; \mathcal{B}\left(\mathcal{H}_\phi \oplus \mathcal{H}_\psi\right).$$

By Kaplansky Density Theorem, there exists a generalized sequence $(x_\iota)_{\iota \in I}$ in the unit ball of A such that

$$\rho(x_\iota) \;=\; \begin{pmatrix} \pi_\phi(x_\iota) & 0 \\ 0 & \pi_\psi(x_\iota) \end{pmatrix} \;\longrightarrow\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(strong convergence). In particular, if $\xi_\phi \in \mathcal{H}_\phi$ and $\xi_\psi \in \mathcal{H}_\psi$ are the GNS cyclic vectors,

$$\left\| \begin{pmatrix} \pi_\phi(x_\iota) & 0 \\ 0 & \pi_\psi(x_\iota) \end{pmatrix} \begin{pmatrix} \xi_\phi \\ 0 \end{pmatrix} - \begin{pmatrix} \xi_\phi \\ 0 \end{pmatrix} \right\| \;\longrightarrow\; 0$$
$$\left\| \begin{pmatrix} \pi_\phi(x_\iota) & 0 \\ 0 & \pi_\psi(x_\iota) \end{pmatrix} \begin{pmatrix} 0 \\ \xi_\psi \end{pmatrix} + \begin{pmatrix} 0 \\ \xi_\psi \end{pmatrix} \right\| \;\longrightarrow\; 0$$

and this implies

$$\begin{aligned} \langle \xi_\phi \mid \pi_\phi(x_\iota)\xi_\phi \rangle \;&\longrightarrow\; 1 \qquad i.e. \qquad \phi(x_\iota) \;\to\; 1 \\ \langle \xi_\psi \mid \pi_\psi(x_\iota)\xi_\psi \rangle \;&\longrightarrow\; -1 \qquad i.e. \qquad \psi(x_\iota) \;\to\; -1\;. \end{aligned}$$

This shows that

$$|(\phi - \psi)(x_\iota)| \;\to\; 2$$

and thus, because $\|x_\iota\| \leq 1$ for all $\iota \in I$,

$$\|\phi - \psi\| \;=\; 2$$

as was to be proved. □

6.26$^{\text{bis}}$. Exercice. Two representations $\pi_1 : A \to \mathcal{B}(\mathcal{H}_1)$ and $\pi_2 : A \to \mathcal{B}(\mathcal{H}_2)$ of a C*-algebra A are said to be *quasi-equivalent* if there exists an isomorphism of von Neumann algebras $\Phi : \pi_1(A)'' \to \pi_2(A)''$ such that $\Phi(\pi_1(a)) = \pi_2(a)$ for all $a \in A$ (equivalent definitions in [DvN, n^0 5.3]).

Let ϕ, ψ be two states on A such that $\|\phi - \psi\| < 2$. If ϕ is pure, show that the corresponding GNS-representations π_ϕ and π_ψ are quasi-equivalent.

Let $(\phi_n)_{n \geq 1}$ be a sequence of *pure* states on A which converge to a state ψ on A in the norm : $\lim_{n\to\infty} \|\psi - \phi_n\| = 0$. Show that ψ is pure, and equivalent to ϕ_n for $n \gg 1$. [This is Corollary 4.8 in [Kad].]

6.27. Proposition. *Let A be a C*-algebra of operators on a Hilbert space $\mathcal{H}$, let $\xi, \eta \in \mathcal{H}(1)$ be two vectors of norm 1 and let ω_ξ, ω_η be the corresponding vector states, defined as in Example 6.5.*

(i) One has

$$\begin{aligned} \|\omega_\xi - \omega_\eta\| \;&\leq\; \|\xi + \eta\| \;\, \|\xi - \eta\|\,, \\ \|\omega_\xi - \omega_\eta\| \;&\leq\; 2\sqrt{1 \;-\; |\langle \xi \mid \eta \rangle|^2}. \end{aligned}$$

(ii) Assume moreover that A is irreducible on $\mathcal{H}$. Then

$$\|\omega_\xi - \omega_\eta\| \ \geq \ 1 \ - \ |\langle \xi \mid \eta \rangle|^2 .$$

Proof. (i) For each $a \in A$, one has

$$\begin{aligned} |\langle \xi \mid a\xi \rangle \ - \ \langle \eta \mid a\eta \rangle| \ &= \ \frac{1}{2}\big|\langle \xi + \eta \mid a(\xi - \eta)\rangle \ + \ \langle \xi - \eta \mid a(\xi + \eta)\rangle\big| \\ &\leq \ \|a\| \ \|\xi + \eta\| \ \|\xi - \eta\| \end{aligned}$$

by the Cauchy-Schwarz inequality, and the first inequality follows. For the second one, upon replacing ξ by $e^{it}\xi$ for some $t \in \mathbb{R}$, we may assume that $\langle \xi \mid \eta \rangle$ is real. Then

$$\begin{aligned} \|\xi + \eta\|^2 \ &= \ \langle \xi \mid \xi \rangle + 2\langle \xi \mid \eta \rangle + \langle \eta \mid \eta \rangle \ = \ 2\,(1 + \langle \xi \mid \eta \rangle) \\ \|\xi - \eta\|^2 \ &= \ \langle \xi \mid \xi \rangle - 2\langle \xi \mid \eta \rangle + \langle \eta \mid \eta \rangle \ = \ 2\,(1 - \langle \xi \mid \eta \rangle) \end{aligned}$$

and the inequalities of (i) follow.

(ii) Consider the projection p_ξ of $\mathcal{H}$ onto the line $\mathbb{C}\xi$, given by

$$p_\xi(\zeta) \ = \ \langle \xi \mid \zeta \rangle \, \xi$$

for all $\zeta \in \mathcal{H}$. By Kaplanski density theorem, there exists a sequence $(a_n)_{n \geq 1}$ in the unit ball of A such that

$$a_n\xi \ \longmapsto \ p_\xi(\xi) \qquad\qquad \text{and} \qquad\qquad a_n\eta \ \longmapsto \ p_\xi(\eta)$$

when $n \mapsto \infty$. One has

$$\begin{aligned} \lim_{n \to \infty} \omega_\xi\,(a_n) \ &= \ \lim_{n \to \infty} \langle a_n\xi \mid \xi \rangle \ = \ 1 \\ \lim_{n \to \infty} \omega_\eta\,(a_n) \ &= \ \lim_{n \to \infty} \langle a_n\eta \mid \eta \rangle \ = \ |\langle \xi \mid \eta \rangle|^2 \\ \lim_{n \to \infty} (\omega_\xi - \omega_\eta)\,(a_n) \ &= \ 1 \ - \ |\langle \xi \mid \eta \rangle|^2 \end{aligned}$$

and Claim (ii) follows. □

6.D. Limit states on AF-algebras

6.28. States as limit states. Let A be an AF-algebra with unit, and let $A_0 \subset A_1 \subset ...$ be a tower of finite dimensional sub-C*-algebras of A such that the identity of A is in A_0 and such that

$$A = \overline{\bigcup_{n \geq 0} A_n}.$$

For each $n \geq 0$, let ϕ_n be a state on A_n; we assume that the restriction to A_n of ϕ_{n+1} coincides with ϕ_n. Let $\phi_\infty : \bigcup_{n \geq o} A_n \to \mathbb{C}$ be the resulting linear form. As $\|\phi_n\| = 1$ for all $n \geq 0$, the form ϕ_∞ extends to a state $\phi : A \to \mathbb{C}$ called the *limit* of the ϕ_n 's.

Any state ϕ on A is the limit of the restrictions $\phi|A_n$'s.

6.29. Proposition. *Let $A = \overline{\bigcup_{n\geq 0} A_n}$ be an AF-algebra and let $\phi = \lim_{n\to\infty} \phi_n$ be a state on A as in the previous number. If ϕ_n is pure for each $n \geq 0$ then ϕ is pure.*

Proof. Let ϕ_0, ϕ_1 be states on A and let $t \in]0,1[$ be such that $\phi = (1-t)\phi_0 + t\phi_1$. For each $n \geq 0$ one has $\phi_n = (1-t)(\phi_0|A_n) + t(\phi_1|A_n)$ by restriction to A_n, hence $\phi_0|A_n = \phi_1|A_n$ by purity of ϕ_n. It follows that $\phi_0 = \phi_1$. □

6.30. Product states on tensor products of two C*-algebras. Let A, B be two C*-algebras and let $\phi : A \to \mathbb{C}$, $\psi : B \to \mathbb{C}$ be two states. To avoid technicalities on tensor products, we assume here that A and B are finite dimensional, so that the tensor product $A \otimes B$ is obviously a C*-algebra. Then

$$\phi \otimes \psi \ : \ \begin{cases} A \otimes B & \longrightarrow \quad \mathbb{C} \\ \sum_i a_i \otimes b_i & \longmapsto \quad \sum_i \phi(a_i)\psi_(b_i) \end{cases}$$

is a state. Indeed, the GNS construction provides data $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ and $(\pi_\psi, \mathcal{H}_\psi, \xi_\psi)$ associated to ϕ and ψ, hence also a representation

$$\pi_\phi \otimes \pi_\psi \ : \ A \otimes B \longrightarrow \mathcal{B}(\mathcal{H}_\phi \otimes \mathcal{H}_\psi)$$

and a unit vector $\xi_\phi \otimes \xi_\psi \in \mathcal{H}_\phi \otimes \mathcal{H}_\psi$. As

$$\begin{aligned} (\phi \otimes \psi)\left(\sum_i a_i \otimes b_i\right) &= \sum_i \phi(a_i)\psi(b_i) = \sum_i \langle \xi_\phi | \pi_\phi(a_i)\xi_\phi\rangle \ \langle \xi_\psi | \pi_\psi(a_i)\xi_\psi\rangle \\ &= \left\langle \xi_\phi \otimes \xi_\psi \ \middle| \ \left(\sum_i \pi_\phi(a_i) \otimes \pi_\psi(b_i)\right) \xi_\phi \otimes \xi_\psi \right\rangle \end{aligned}$$

for all $\sum_i a_i \otimes b_i \in A \otimes B$, the linear form $\phi \otimes \psi$ is indeed a state, as claimed.

States of this form are called *product states* on $A \otimes B$.

All this carries over to arbitrary C*-algebras as long as one deals with the so-called *maximal tensor product* $A \otimes_{max} B$.

States on tensor products are far from all being product states. This is intuitively clear from Proposition 6.8: given two integers $k, l \geq 1$, the state space of $M_k(\mathbb{C})$ [respectively of $M_l(\mathbb{C})$, of $M_k(\mathbb{C}) \otimes M_l(\mathbb{C}) \approx M_{kl}(\mathbb{C})$] is of dimension $\frac{k(k+1)}{2} - 1$ [resp. $\frac{l(l+1)}{2} - 1$, $\frac{kl(kl+1)}{2} - 1$] and

$$\frac{k(k+1)}{2} - 1 \ + \ \frac{l(l+1)}{2} - 1 \ < \ \frac{kl(kl+1)}{2} - 1$$

as soon as $k \geq 2$ and $l \geq 2$. See also the exercice below.

6.31. Exercice. Let ω be the linear form defined on $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ by

$$\omega\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \ = \ \frac{1}{2}(aa' + dd').$$

Check that ω is a state which is not a product state.
[Indication. The two linear forms mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

to aa' and dd' are product states on $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, so that ω is indeed a state.

Let ω_p be a product state, given by an expression of the form

$$\omega_p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \\ \operatorname{trace}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & \overline{s} \\ s & 1-r \end{pmatrix}\right) \operatorname{trace}\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} r' & \overline{s'} \\ s' & 1-r' \end{pmatrix}\right)$$

where $\begin{pmatrix} r & \overline{s} \\ s & 1-r \end{pmatrix}$ and $\begin{pmatrix} r' & \overline{s'} \\ s' & 1-r' \end{pmatrix}$ positive matrices. Suppose (ab absurdo) that $\omega = \omega_p$, evaluate on tensor products of elementary 2-by-2-matrices, and see that one arrives at a contradiction.]

6.32. Product states on UHF-algebras. Let A be a UHF-algebra and let $(k_j)_{j\geq 1}$ be a sequence of integers, with $k_j \geq 2$ for all j, such that

$$A = \bigotimes_{j=1}^{\infty} M_{k_j}(\mathbb{C})$$

(see 5.9). For each $j \geq 1$, choose a positive matrix $h_j \in M_{k_j}(\mathbb{C})$ of trace 1. For each $n \geq 1$, the linear form

$$\phi_n : \bigotimes_{j=1}^{n} M_{k_j}(\mathbb{C}) \approx M_{k_1 k_2 \dots k_n}(\mathbb{C}) \longrightarrow \mathbb{C}$$

defined by $\phi_n(x) = tr\left(\left(\bigotimes_{j=1}^{n} h_j\right) x\right)$ is a state. The resulting state ϕ on A is called a *product state.* We write

$$\phi(x) = tr\left(\left(\bigotimes_{j=1}^{\infty} h_j\right) x\right)$$

for all $x \in A$.

6.33. Exercice. Consider the UHF algebra

$$A = \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})_j$$

of Example 5.16. Given a sequence $\Lambda = (\lambda_j)_{j\geq 1}$ of real numbers in $[0,1]$, define a product state ϕ_Λ on A by

$$\phi_\Lambda \;=\; tr\left(\left(\bigotimes_{j=1}^{\infty}\begin{pmatrix}\lambda_j & 0\\ 0 & 1-\lambda_j\end{pmatrix}\right)\; x\right)$$

for all $a \in A$.

(i) Check that ϕ_Λ is the unique *tracial* state (see 5.10) on A if and only if $\lambda_j = \frac{1}{2}$ for all $j \geq 1$.

(ii) Show that ϕ_Λ is pure if and only if $\lambda_j \in \{0,1\}$ for all $j \geq 1$.

(iii) Let $\Lambda = (\lambda_j)_{j\geq 1}$ and $\Lambda' = (\lambda'_j)_{j\geq 1}$ be two sequences of 0 's and 1 's. Show that the states ϕ_Λ and $\phi_{\Lambda'}$ are equivalent if and only if there exists $n \geq 1$ such that $\lambda_j = \lambda'_j$ for all $j \geq n$.

(iv) Deduce from (iii) that A has uncountably many pairwise inequivalent irreducible representations.

[Indication. (ii) If $0 < \lambda_j < 1$ for some $j \geq 1$, it is easy to write ϕ_Λ as a non trivial convex combination of two distinct states. Conversely, suppose $\lambda_j \in \{0,1\}$ for all $j \geq 1$. Let ϕ', ϕ'' be two states on A and let $t \in]0,1[$ be such that $\phi_\Lambda = (1-t)\phi' + t\phi''$. Check that the restrictions of $\phi_\Lambda, \phi', \phi''$ to $\bigotimes_{j=1}^{\infty} M_2(\mathbb{C})_j$ coincide for all $n \geq 1$, so that one has $\phi_\Lambda = \phi' = \phi''$.

(iii) If $\lambda_j = \lambda'_j$ for $j \geq n$, use the fact that $\bigotimes_{j=1}^{n} M_2(\mathbb{C})_j$ has a unique irreducible representation (up to equivalence).

Suppose conversely that ϕ_Λ and $\phi_{\Lambda'}$ are equivalent. By Proposition 6.25 and Exercice 4.28, there exists an integer $n \geq 1$ and a unitary element $v \in A_n$ such that

$$|\phi_{\Lambda'}(x) \;-\; \phi_\Lambda(vxv^*)| \;<\; 1$$

for all $x \in A$, $\|x\| \leq 1$. For any $j > n$, let $e_j = 1\otimes ...\otimes 1 \otimes \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes 1 \otimes ... \;\in\; A$, where the matrix $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ appears in the j^{th} place; as $\phi_{\Lambda'}(e_j) = \lambda'_j$ and $\phi_\Lambda(ve_jv^*) = \phi_\Lambda(e_j) = \lambda_j$, one has $|\lambda'_j - \lambda_j| < 1$, namely $\lambda'_j = \lambda_j$.]

6.34. Remarks on Powers' factors. Let A be as in the previous exercice. For each $\alpha \in]0, \frac{1}{2}[$, set $\lambda = \frac{\alpha}{1-\alpha} \in]0,1[$, let ϕ_Λ be the state on A associated to the constant sequence $\Lambda = (\alpha, \alpha, \alpha, ...)$, and let $\pi_\Lambda : A \to \mathcal{B}(\mathcal{H}_\lambda)$ be the corresponding GNS representation. The von Neumann algebras

$$R_\lambda \;=\; \pi_\Lambda(A)''$$

generated by the image of π_Λ is known to be a factor, indeed a factor of type III, and the R_λ 's are known to be pairwise nonisomorphic factors of type III [Po1]. They are the *Powers' factors.* This family of factors has played a central rôle in the theory of von Neumann algebras (works of Pukanzky and Glimm before [Po1], works of Araki, Woods, Krieger and Connes after [Po1], to quote but a few).

P. de la Harpe and V. Jones, July 1995.

CHAPTER 7. THE ALGEBRA OF CANONICAL ANTICOMMUTATION RELATIONS

7.a. The full Fock space.

7.1. Tensor products of Hilbert spaces. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. We denote by $\mathcal{H}_1 \odot \mathcal{H}_2$ the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ viewed as complex vector spaces.

Recall that, by definition of the tensor product, any bilinear map $\mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ gives rise to a linear form on $\mathcal{H}_1 \odot \mathcal{H}_2$; similarly, any $\mathbb{R}$-bilinear map $\beta : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ such that $\beta(i\xi_1, \xi_2) = \beta(\xi_1, i\xi_2) = -i\beta(\xi_1, \xi_2)$ for all $(\xi_1, \xi_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ gives rise to an antilinear form on $\mathcal{H}_1 \odot \mathcal{H}_2$. It follows that the form

$$\begin{cases} \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_1 \times \mathcal{H}_2 & \longrightarrow \qquad \mathbb{C} \\ \qquad (\xi_1, \xi_2, \eta_1, \eta_2) & \longmapsto \langle \xi_1 | \eta_1 \rangle \langle \xi_2 | \eta_2 \rangle \end{cases}$$

gives rise to a sesquilinear form $\langle ... | ... \rangle$ on $\mathcal{H}_1 \odot \mathcal{H}_2$ such that

$$\langle \xi_1 \otimes \xi_2 | \eta_1 \otimes \eta_2 \rangle = \langle \xi_1 | \eta_1 \rangle \langle \xi_2 | \eta_2 \rangle$$

for all $\xi_1, \eta_1 \in \mathcal{H}_1$ and $\xi_2, \eta_2 \in \mathcal{H}_2$.

Any $\xi \in \mathcal{H}_1 \odot \mathcal{H}_2$ may be written $\xi = \sum_{i=1}^n \xi_{1,i} \otimes \xi_{2,i}$. The Gram-Schmidt orthogonalization process shows that there is no loss of generality in assuming that the sequence $(\xi_{2,i})_{1 \leq i \leq n}$ is orthogonal. Then $\langle \xi | \xi \rangle = \sum_{i=1}^n \|\xi_{1,i}\|^2 \|\xi_{2,i}\|^2$. It follows that the sesquilinear form defined above is positive definite on $\mathcal{H}_1 \odot \mathcal{H}_2$.

The completion of $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to this scalar product is a Hilbert space which is called the *tensor product* of $\mathcal{H}_1$ and $\mathcal{H}_2$, and which is denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$.

If $(e_{1,\iota})_{\iota \in I}$ and $(e_{2,\kappa})_{\kappa \in K}$ are respectively orthonormal basis in $\mathcal{H}_1$ and $\mathcal{H}_2$, it is easy to check that $(e_{1,\iota} \otimes e_{2,\kappa})_{(\iota,\kappa) \in I \times K}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\mathcal{H}_1 = L^2(X_1, \mu_1)$ and $\mathcal{H}_2 = L^2(X_2, \mu_2)$ for measure spaces (X_1, μ_1) and (X_2, μ_2), one may check that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is isomorphic to $L^2(X_1 \times X_2, \mu_1 \times \mu_2)$.

Tensor products $\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n$ of $n \geq 2$ Hilbert spaces are defined similarly.

Caution. The reader may remember that the "algebraic" tensor product can be defined by a universal property summed up in the canonical isomorphism

$$Lin(\mathcal{H}_1 \odot \mathcal{H}_2; \mathcal{H}_3) \approx Bil(\mathcal{H}_1, \mathcal{H}_2; \mathcal{H}_3)$$

(see e.g. Bourbaki, Algèbre, Chapitre II, page II.51). There is a *Banach space* $\mathcal{H}_1 \otimes_\pi \mathcal{H}_2$, which is a completion of $\mathcal{H}_1 \odot \mathcal{H}_2$, such that the space of *bounded* linear operators $\mathcal{H}_1 \otimes_\pi \mathcal{H}_2 \to \mathcal{H}_3$ is canonically isomorphic to the space of *bounded* bilinear operators

$\mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_3$; but this so-called *projective tensor product* is not in general isomorphic to the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ defined here.

7.2. The full Fock space of a Hilbert space. Let $\mathcal{H}$ be a Hilbert space. For each integer $n \geq 0$, set

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \ldots \otimes \mathcal{H} \qquad (n \text{ copies})$$

with the convention $\mathcal{H}^{\otimes 0} = \mathbb{C}$. The *full Fock space* or the *exponential* of $\mathcal{H}$ is the Hilbert space direct sum

$$EXP(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}.$$

Observe that there is a canonical inclusion of the tensor algebra $\bigoplus_{n\geq 0}^{alg} \mathcal{H}^{\odot n}$ in $EXP(\mathcal{H})$, with dense image.

If $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert spaces, there is a natural isomorphism

$$EXP(\mathcal{H}_1 \oplus \mathcal{H}_2) \; \approx \; EXP(\mathcal{H}_1) \otimes EXP(\mathcal{H}_2)$$

which motivates the notation. For each $\xi \in \mathcal{H}$, one may define

$$EXP(\xi) \;=\; \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{(n!)}}\, \xi^{\otimes n};$$

one has then $\langle EXP(\xi) | EXP(\eta) \rangle \;=\; exp\langle \xi | \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. If $\xi_1, \ldots, \xi_k$ are pairwise distinct vectors in $\mathcal{H}$, it can be shown that $EXP(\xi_1), \ldots, EXP(\xi_k)$ are linearly independent vectors in $EXP(\mathcal{H})$ [Gui, Proposition 2.2].

One may view EXP as a functor from the category of Hilbert spaces and contractions (namely operators of norms at most 1) to the category of Hilbert spaces with distinguished unit vectors (the vector $1 \in \mathbb{C} \approx \mathcal{H}^{\otimes 0} \subset EXP(\mathcal{H})$) and contractions preserving the distinguished vectors.

7.3. The operators $\ell(\xi)$. Let $\mathcal{H}$ be a Hilbert space. For each vector $\xi \in \mathcal{H}$ and each integer $n \geq 0$, the linear map $\mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes(n+1)}$ defined by

$$\eta_1 \otimes \ldots \otimes \eta_n \longmapsto \xi \otimes \eta_1 \otimes \ldots \otimes \eta_n$$

extends to a bounded operator

$$\mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}^{\otimes(n+1)}$$

of norm $\|\xi\|$. As the later norm is independent of n, the direct sum over $n \geq 0$ of these operators is a bounded operator

$$\ell(\xi) : EXP(\mathcal{H}) \longrightarrow EXP(\mathcal{H})$$

of norm $\|\xi\|$. On has obviously

$$\langle \eta_0 \otimes \eta_1 \otimes \ldots \otimes \eta_n \mid \xi \otimes \eta_1' \otimes \ldots \otimes \eta_n' \rangle = \langle \langle \xi | \eta_0 \rangle\, \eta_1 \otimes \ldots \otimes \eta_n \mid \eta_1' \otimes \ldots \otimes \eta_n' \rangle$$

for each $n \geq 0$ and $\eta_0, ..., \eta_n, \eta'_1, ..., \eta'_n \in \mathcal{H}$, so that the adjoint $\ell(\xi)^*$ of $\ell(\xi)$ is given by

$$\ell(\xi)^* (\eta_0 \otimes ... \otimes \eta_n) = \langle \xi | \eta_0 \rangle \, \eta_1 \otimes ... \otimes \eta_n.$$

It is then equally obvious that

$$\ell(\eta)^* \ell(\xi) = \langle \eta | \xi \rangle \, id_{EXP(\mathcal{H})}$$

for all $\xi, \eta \in \mathcal{H}$.

7.4. Digression on the C*-algebra $C^*(\ell(\mathcal{H}))$. This is the C*-algebra of operators on $EXP(\mathcal{H})$ generated by the operators $\ell(\xi)$, for all $\xi \in \mathcal{H}$.

If $\mathcal{H}$ is infinite dimensional, $C^*(\ell(\mathcal{H}))$ is the so-called *Cuntz algebra* O_∞ of [Cun].

Suppose $\mathcal{H}$ is of finite dimension, say n, and let $e_1, ..., e_n$ be an orthonormal basis of $\mathcal{H}$. Then

$$p = \sum_{j=1}^{n} \ell(e_j) \ell(e_j)^*$$

is an orthogonal projection of $EXP(\mathcal{H})$. One may show that p is independent of the choice of the orthonormal basis. If $\langle 1-p \rangle$ denotes the principal two-sided ideal generated by $1-p$ in $C^*(\ell(\mathcal{H}))$, one has a short exact sequence

$$0 \longrightarrow \mathcal{K} \approx \langle 1-p \rangle \longrightarrow C^*(\ell(\mathcal{H})) \longrightarrow O_n \longrightarrow 0$$

where $\mathcal{K}$ denotes the algebra of compact operators on some separable infinite dimensional Hilbert space and where O_n is the n^{th} *Cuntz algebra*. (See [Cun, Proposition 3.1], as well as [VDN, Proposition 1.5.9].)

Let S_j denote the image of $\ell(\xi)$ in O_n. One has the relations

$$S_j^* S_k = \delta_{j,k} \qquad j,k \in \{1, ..., n\} \qquad\qquad \sum_{j=1}^{n} S_j S_j^* = 1$$

which show that O_n is generated by the isometries $S_1, ..., S_n$ and that the image projections of the S_j 's add up to 1.

7.B. The Fock space and the definition of the CAR C*-algebra.

7.5. The antisymmetric Fock space. Consider a Hilbert space $\mathcal{H}$, an integer $n \geq 1$ and the tensor product $\mathcal{H}^{\otimes n}$. There is a unitary representation $\sigma \mapsto u_\sigma$ of the symmetric group on n letters $\mathcal{S}_n$ on the space $\mathcal{H}^{\otimes n}$ defined by

$$u_\sigma (\xi_1 \otimes ... \otimes \xi_n) = \xi_{\sigma(1)} \otimes ... \otimes \xi_{\sigma(n)}$$

for $\xi_1, ..., \xi_n \in \mathcal{H}$.

We define $\bigwedge^n \mathcal{H}$ to be the subspace of $\mathcal{H}^{\otimes n}$ of vectors on which $\mathcal{S}_n$ acts by the signature, and we denote by

$$P_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma u_\sigma$$

the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\bigwedge^n \mathcal{H}$, where $(-1)^\sigma$ denotes the signature of the permutation σ. For $\xi_1, .., \xi_n \in \mathcal{H}$, we write

$$\xi_1 \wedge ... \wedge \xi_n = \sqrt{n!}\, P_n \, (\xi_1 \otimes ... \otimes \xi_n) \in \bigwedge^n \mathcal{H}.$$

Observe the factor $\sqrt{n!}$, which is crucial below. One has of course

$$\xi_{\sigma(1)} \wedge ... \wedge \xi_{\sigma(n)} = (-1)^\sigma \xi_1 \wedge ... \wedge \xi_n$$

for all $\sigma \in \mathcal{S}_n$.

The *antisymmetric Fock space* of $\mathcal{H}$ is the Hilbert space direct sum

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n \geq 0} \bigwedge^n \mathcal{H}$$

where $\bigwedge^0 \mathcal{H} = \mathbb{C}$ by convention. It is also called the *Fermi Fock space*, or *here* shortly the *Fock space* of $\mathcal{H}$. It is important to realize that $\bigwedge^n \mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ are defined as subspaces of $\mathcal{H}^{\otimes n}$ and $EXP(\mathcal{H})$ respectively; scalar products of vectors in these spaces do not have to be further *defined*, but the following proposition is important for *computations.*

7.6. Proposition. *Let $\xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in \mathcal{H}$. Then*

$$\langle \xi_1 \wedge ... \wedge \xi_n \mid \eta_1 \wedge ... \wedge \eta_n \rangle = \det \left(\langle \xi_j | \eta_k \rangle_{1 \leq j,k \leq n} \right).$$

Proof. One has

$$\begin{aligned}
&\langle \xi_1 \wedge ... \wedge \xi_n \mid \eta_1 \wedge ... \wedge \eta_n \rangle \\
&\qquad = \left\langle \sqrt{n!}\, P_n(\xi_1 \otimes ... \otimes \xi_n) \mid \sqrt{n!}\, P_n(\eta_1 \otimes ... \otimes \eta_n) \right\rangle \qquad \text{by definition} \\
&\qquad = n! \, \langle \xi_1 \otimes ... \otimes \xi_n \mid P_n(\eta_1 \otimes ... \otimes \eta_n) \rangle \qquad \text{because} \qquad P_n = P_n^* = P_n^2 \\
&\qquad = \sum_{\sigma \in \mathcal{S}_n} \prod_{1 \leq j \leq n} (-1)^\sigma \, \langle \xi_j | \eta_{\sigma(j)} \rangle \qquad \text{by definition of } P_n \\
&\qquad = \det \left(\langle \xi_j | \eta_k \rangle_{1 \leq j,k \leq n} \right)
\end{aligned}$$

where the last equality is one possible definition of the determinant (see Bourbaki, Algèbre, Chapitre III, page III.94). □

7.7. Remarks. (i) Here is a first consequence of Proposition 7.6: if $(e_\iota)_{\iota\in I}$ is an orthonormal basis of $\mathcal{H}$ indexed by a totally ordered set I, then vectors of the form

$$e_{\iota_1} \wedge ... \wedge e_{\iota_n}$$

with $\iota_1, ..., \iota_n \in I$ and $\iota_1 < ... < \iota_n$ constitute an orthonormal basis of $\bigwedge^n \mathcal{H}$.

(ii) Here is a second consequence: for $\xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in \mathcal{H}$, one has

$$\langle \xi_1 \wedge ... \wedge \xi_n \mid \eta_1 \wedge ... \wedge \eta_n \rangle \; = \; 0$$

if and only if the linear span of $\{\xi_1, ..., \xi_n\}$ contains a nonzero vector which is orthogonal to the linear span of $\{\eta_1, ..., \eta_n\}$.

(iii) If $n \geq 2$, it is important to note that the vector $e_{\iota_1} \otimes ... \otimes e_{\iota_n}$, of norm 1 in $\mathcal{H}^{\otimes n}$, is projected by P_n onto the vector $\frac{1}{\sqrt{n!}} e_{\iota_1} \wedge ... \wedge e_{\iota_n}$, not onto the unit vector $e_{\iota_1} \wedge ... \wedge e_{\iota_n}$!

7.8. Lemma. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces; for $n \geq 1$, let $L : \mathcal{H}_1^{\otimes n} \to \mathcal{H}_2$ be a bounded operator such that $Lu_\sigma = (-1)^\sigma L$ for all $\sigma \in \mathcal{S}_n$ (where u_σ is as in 7.5). Then*

$$L\,(\xi_1 \wedge ... \wedge \xi_n) = \sqrt{n!}\; L\,(\xi_1 \otimes ... \xi_n)$$

for all $\xi_1, ..., \xi_n \in \mathcal{H}$.

Proof. One has

$$\begin{aligned} L\,(\xi_1 \wedge ... \wedge \xi_n) \;&=\; \sqrt{n!}\; L\; P_n\; \; (\xi_1 \otimes ... \otimes \xi_n) \\ &=\; \frac{1}{\sqrt{n!}}\, L \left(\sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma u_\sigma\,(\xi_1 \otimes ... \otimes \xi_n) \right) \\ &=\; \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{S}_n} L\,(\xi_1 \otimes ... \otimes \xi_n) \\ &=\; \sqrt{n!}\; L\,(\xi_1 \otimes ... \otimes \xi_n) \end{aligned}$$

□

7.9. The operators $a_n(\xi)$. Consider a Hilbert space $\mathcal{H}$, a vector $\xi \in \mathcal{H}$ and an integer $n \geq 0$. Define an operator $A_n(\xi) : \mathcal{H}^{\otimes n} \to \bigwedge^{n+1} \mathcal{H}$ by

$$A_n(\xi)\big(\eta_1 \otimes ... \otimes \eta_n\big) \;=\; \frac{1}{\sqrt{n!}}\; \xi \wedge \eta_1 \wedge ... \wedge \eta_n \;=\; \sqrt{\frac{(n+1)!}{n!}}\; P_{n+1}\big(\xi \otimes \eta_1 \otimes ... \otimes \eta_n\big)$$

for all $\eta_1, ..., \eta_n \in \mathcal{H}$; it is clearly a bounded operator of norm at most $\sqrt{\frac{(n+1)!}{n!}}\, \|\xi\|$. By the previous lemma, one has

$$(*) \qquad A_n(\xi)\big(\eta_1 \wedge ... \wedge \eta_n\big) \;=\; \sqrt{(n+1)!}\; P_{n+1}\big(\xi \otimes \eta_1 \otimes ... \otimes \eta_n\big).$$

We denote the restriction of $A_n(\xi)$ to $\bigwedge^n \mathcal{H}$ by

$$a_n(\xi) \; : \; \begin{cases} \displaystyle\bigwedge^n \mathcal{H} & \longrightarrow \quad \displaystyle\bigwedge^{n+1} \mathcal{H} \\ \eta_1 \wedge \ldots \wedge \eta_n & \longmapsto \quad \xi \wedge \eta_1 \wedge \ldots \wedge \eta_n \end{cases}$$

and one has $\|a_n(\xi)\| \leq \sqrt{(n+1)!}\,\|\xi\|$ by $(*)$. But we shall see in Corollary 7.11 below that, notwithstanding what the above estimate may suggest, the norm of $a_n(\xi)$ is in fact *independent on n.*

7.10. Proposition. *For $\xi, \eta \in \mathcal{H}$, one has*

$$\begin{aligned} a_n^*(\xi)a_n(\eta) \; + \; a_{n-1}(\eta)a_{n-1}^*(\xi) \; &= \; \langle \xi|\eta\rangle \, id_{\bigwedge^n \mathcal{H}} \\ a_{n+1}(\xi)a_n(\eta) \; + \; a_{n+1}(\eta)a_n(\xi) \; &= \; 0 \end{aligned}$$

Proof. Let $\chi_1, ..., \chi_{n+1}, \psi_1, ..., \psi_n \in \mathcal{H}$. By Lemma 7.6, one has

$$\begin{aligned} &\langle \chi_1 \wedge ... \wedge \chi_{n+1} \mid \xi \wedge \psi_1 \wedge ... \wedge \psi_n \rangle \\ &\quad = \det \begin{pmatrix} \langle\chi_1|\xi\rangle & \langle\chi_1|\psi_1\rangle & \cdots & \langle\chi_1|\psi_n\rangle \\ \langle\chi_2|\xi\rangle & \langle\chi_2|\psi_1\rangle & \cdots & \langle\chi_2|\psi_n\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle\chi_{n+1}|\xi\rangle & \langle\chi_{n+1}|\psi_1\rangle & \cdots & \langle\chi_{n+1}|\psi_n\rangle \end{pmatrix} \\ &\quad = \sum_{j=1}^{n+1} (-1)^{j+1} \langle\chi_j|\xi\rangle \left\langle \chi_1 \wedge ...\hat{j}... \wedge \chi_{n+1} \mid \psi_1 \wedge ... \wedge \psi_n \right\rangle \\ &\quad = \left\langle \sum_{j=1}^{n+1} (-1)^{j+1} \langle\xi|\chi_j\rangle \chi_1 \wedge ...\hat{j}... \wedge \chi_{n+1} \;\middle|\; \psi_1 \wedge ... \wedge \psi_n \right\rangle \end{aligned}$$

by expansion of the determinant in terms of the first column. It follows that the adjoint of $a_n(\xi)$ is given by

$$a_n^*(\xi)\,(\chi_1 \wedge ... \wedge \chi_{n+1}) \; = \; \sum_{j=1}^{n+1} (-1)^{j+1} \langle\xi|\chi_j\rangle \chi_1 \wedge ...\hat{j}... \wedge \chi_{n+1}.$$

One has consequently

$$a_n^*(\xi)a_n(\eta)\,(\psi_1 \wedge ... \wedge \psi_n) \; = \; \langle\xi|\eta\rangle \; \psi_1 \wedge ... \wedge \psi_n \; + \; \sum_{j=1}^{n} (-1)^j \; \langle\xi|\psi_j\rangle \; \eta \wedge \psi_1 \wedge ...\hat{j}... \wedge \psi_n$$

$$a_{n-1}(\eta)a_{n-1}^*(\xi)\,(\psi_1 \wedge ... \wedge \psi_n) \; = \eta \wedge \sum_{j=1}^{n} (-1)^{j+1} \; \langle\xi|\psi_j\rangle \; \psi_1 \wedge ...\hat{j}... \wedge \psi_n$$

and the first relation follows. We leave it to the reader to check the second one. □

7.11. Corollary. *For each $\xi \in \mathcal{H}, \xi \neq 0$ and for all $n \geq 0$, the operator $\frac{1}{\|\xi\|^2} a_n^*(\xi) a_n(\xi)$ is a projection on $\bigwedge^n \mathcal{H}$. In particular*

$$\|a_n(\xi)\| \;=\; \|\xi\|$$

for all $n \geq 0$ and for all $\xi \in \mathcal{H}$.

Proof. Using the two relations of Proposition 7.10 (case $\xi = \eta$), one has

$$(a_n^*(\xi)a_n(\xi))^2 \;=\; a_n^*(\xi)\left(\|\xi\|^2\, id_{\wedge^{n+1}\mathcal{H}} \;-\; a_{n+1}^*(\xi)a_{n+1}(\xi)\right) a_n(\xi) \;=\; \|\xi\|^2\, a_n^*(\xi)a_n(\xi)$$

and the corollary follows. □

7.12. Definition. For each $\xi \in \mathcal{H}$, the corresponding *creation operator*

$$a(\xi) : \mathcal{F}(\mathcal{H}) \longrightarrow \mathcal{F}(\mathcal{H})$$

is the direct sum of the $a_n(\xi)$'s on the $\bigwedge^n \mathcal{H}$'s; it is a bounded operator of norm $\|\xi\|$ which depends linearly on ξ. Its adjoint is the *annihilation operator* $a^*(\xi)$ which depends antilinearly on ξ. If $\|\xi\| = 1$, then $a^*(\xi)a(\xi)$ and $a(\xi)a^*(\xi)$ are projections in $\mathcal{B}(\mathcal{F}(\mathcal{H}))$.

The *CAR algebra* is the C*-algebra $CAR(\mathcal{H})$ of operators on $\mathcal{F}(\mathcal{H})$ generated by the creation operators. It is a C*-algebra with unit. The map

$$\begin{cases} \mathcal{H} \longrightarrow CAR(\mathcal{H}) \\ \ \xi \longmapsto a(\xi) \end{cases}$$

is a linear isometry. One has the *CAR relations*

$$\begin{aligned} a^*(\xi)a(\eta) \;+\; a(\eta)a^*(\xi) \;&=\; \langle \xi | \eta \rangle\, 1 \\ a(\xi)a(\eta) \;+\; a(\eta)a(\xi) \;&=\; 0 \end{aligned}$$

for all $\xi, \eta \in \mathcal{H}$.

7.13. Remarks. (i) Let $(e_\iota)_{\iota\in I}$ be an orthonormal basis of $\mathcal{H}$, and set $x_\iota = a(e_\iota)$ for each $\iota \in I$. Then $(x_\iota)_{\iota\in I}$ generate $CAR(\mathcal{H})$ as a C*-algebra. This is because the map $a : \mathcal{H} \to CAR(\mathcal{H})$ is an isometry; thus, for any $\xi = \sum_{\iota\in I} \xi_\iota e_\iota \in \mathcal{H}$, the creation operator $a(\xi)$ is a limit in $CAR(\mathcal{H})$ of finite linear combinations of the x_ι 's.

(ii) We have chosen to denote by $\xi \mapsto a(\xi)$ the *linear* map giving *creation* operators, as in [PoS], [Sla]. In many references, our $a(\xi)$ is denoted by $a^*(\xi)$, so that $\xi \mapsto a(\xi)$ is an anti-linear map corresponding to annihilation operator .... [BrR], [Eva].

7.14. On physics and etymology. There are difficulties to build up consistent theories obeying both quantum requirements and relativistic requirements. One difficulty is that any description of one particle has to include a description of arbitrarily many particles. This motivates the introduction of a formalism which can describe an arbitrary number of particles. In case of fermions (e.g. of electrons), if one particle has states which can be described by vectors in a Hilbert space $\mathcal{H}$, then states with n particles are described by vectors in $\bigwedge^n \mathcal{H}$ and the operator

$$a(\xi) : \xi_1 \wedge \dots \wedge \xi_n \mapsto \xi \wedge \xi_1 \wedge \dots \wedge \xi_n$$

"creates" one more fermion in state ξ. (This "creation" being in fact an annihilation in case ξ lies in the linear space spanned by the ξ_j 's, in accordance with the Pauli exclusion principle.)

7.C. The CAR algebra as an UHF-algebra.

What we retain from the construction of 7.B is that, for any Hilbert space $\mathcal{H}$, there is a C*-algebra with unit $CAR(\mathcal{H})$ and a linear map $a : \mathcal{H} \to CAR(\mathcal{H})$ such that the CAR relations hold, namely such that

$$\begin{aligned} a^*(\xi)a(\eta) \ + \ a(\eta)a^*(\xi) \ &= \ \langle \xi | \eta \rangle \ 1 \\ a(\xi)a(\eta) \ + \ a(\eta)a(\xi) \ &= \ 0 \end{aligned}$$

for all $\xi, \eta \in \mathcal{H}$, and such that $a(\mathcal{H})$ generates $CAR(\mathcal{H})$ as a C*-algebra.

7.15. Lemma. *Let $\mathcal{H}$ be a Hilbert space of dimension 1, let $\xi \in \mathcal{H}$ be a unit vector and set $x = a(\xi)$.*

Then $CAR(\mathcal{H})$ is isomorphic to $M_2(\mathbb{C})$. More precisely the operators

$$\begin{aligned} e_{1,1} &= x^* x & \qquad e_{1,2} &= x^* \\ e_{2,1} &= x & \qquad e_{2,2} &= x x^* \end{aligned}$$

constitue a system of matrix units in $CAR(\mathcal{H})$ such that $e_{1,1} + e_{2,2} = 1$, and this system linearly generates $CAR(\mathcal{H})$.

Proof. The CAR relations read here

$$x^* x + x x^* = 1 \qquad \text{and} \qquad x^2 = 0.$$

Repeating the proof of Corollary 7.11

$$(x^* x)^2 \ = \ x^*(1 - x^* x) x \ = \ x^* x$$

we see that $e_{1,1}$ is a projection. Other relations, such as $e_{1,1} e_{1,2} = e_{1,2}$, are equally straightforward to check. □

7.16. Lemma. *Let A be a C^*-algebra with unit, and let $A_1, ..., A_n$ be pairwise commuting sub-C^*-algebras of A which contain 1. Assume that A_j is isomorphic to a matrix algebra $M_{k_j}(\mathbb{C})$ for each $j \in \{1, ..., n\}$, and set $k = \prod_{j=1}^n k_j$. Then the sub-C^*-algebra of A generated by $A_1 \cup ... \cup A_n$ is isomorphic to $M_k(\mathbb{C})$.*

Proof. The multilinear map

$$\begin{cases} A_1 \times ... \times A_n & \longrightarrow & A \\ (a_1, ..., a_n) & \longmapsto & a_1 a_2 ... a_n \end{cases}$$

defines a linear map $\phi : A_1 \otimes ... \otimes A_n \to A$. The commutation hypothesis implies that ϕ is a morphism of C*-algebras. The domain of ϕ is isomorphic to the matrix algebra

$$M_{k_1}(\mathbb{C}) \otimes ... \otimes M_{k_n}(\mathbb{C}) \approx M_k(\mathbb{C}).$$

As this algebra is simple and as ϕ is obviously non zero, ϕ is one-to-one and its image is precisely the sub-C*-algebra generated by $A_1 \cup ... \cup A_n$. $\square$

7.17. Lemma. *Let $\mathcal{H}$ be a Hilbert space and let $V \subset \mathcal{H}$ be a subspace of dimension $n < \infty$. Then the sub-C^*-algebra $C^*(a(V))$ of $CAR(\mathcal{H})$ generated by $a(V)$ is isomorphic to*

$$M_{2^n}(\mathbb{C}).$$

Proof. Choose an orthonormal basis $\{v_1, ..., v_n\}$ of V and set $x_j = a(v_j)$ for all $j \in \{1, .., n\}$. The CAR relations read

$$x_j^* x_k + x_k x_j^* = \delta_{j,k} \ ,$$
$$x_j x_k + x_k x_j = 0$$

$(1 \leq j, k \leq n)$. For each $l \in \{0, 1..., n\}$, set

$$u_l = \prod_{j=1}^{l} \ (1 - 2x_j x_j^*)$$

with $u_0 = 1$. For each $j \in \{1, ..., n\}$, we know from Lemma 7.15 that $x_j x_j^*$ is a projection; thus $1 - 2x_j x_j^*$ is a selfadjoint unitary (namely a unitary of square 1). From the CAR relations we know also that the $x_j x_j^*$'s commute pairwise; thus u_l is a self-adjoint unitary for all $l \in \{1, ..., n\}$.

Set also

$$\begin{array}{lllll} e_{1,1}^{(l)} & = & u_{l-1} x_l^* x_l u_{l-1} & = & x_l^* x_l \qquad\qquad e_{1,2}^{(l)} = u_{l-1} x_l^* \\ e_{2,1}^{(l)} & = & x_l u_{l-1} & & \qquad\qquad\qquad e_{2,2}^{(l)} = x_l x_l^* \end{array}$$

where the equality $u_{l-1} x_l^* x^* u_{l-1} = x_l^* x_l$ holds because $x_l^* x_l$ commutes with u_{l-1} (observe that $x_l^* x_l$ and $x_j x_j^*$ commute for all $l, j \in \{1, ..., n\}$).

We claim firstly that $e^{(l)}_{j,k}$ and $e^{(m)}_{j',k'}$ commute if $l \neq m$. Indeed, one computes successively

$$\begin{aligned} \left(1-2x_jx_j^*\right) x_j \left(1-2x_jx_j^*\right) &= x_j \left(1-2(1-x_j^*x_j)\right) = -x_j \\ \left(1-2x_jx_j^*\right) x_l \left(1-2x_jx_j^*\right) &= x_l \qquad \text{if} \qquad j \neq l \end{aligned}$$

and

$$u_m x_l u_m = \begin{cases} -x_l & \text{if} \quad l \leq m \\ \ \ x_l & \text{if} \quad l > m \end{cases}$$

One has then, say if $l < m$,

$$\begin{aligned} e^{(l)}_{2,1} e^{(m)}_{2,1} &= x_l u_{l-1} x_m u_{m-1} \\ &= +x_l x_m u_{l-1} u_{m-1} && \text{by the computation above } (m > l-1) \\ &= -x_m x_l u_{m-1} u_{l-1} && \text{because } u_{l-1}u_{m-1} = u_{m-1}u_{l-1} \text{ and } x_lx_m + x_mx_l = 0 \\ &= +x_m u_{m-1} x_l u_{l-1} && \text{by the computation above } (l \leq m-1) \\ &= e^{(m)}_{2,1} e^{(l)}_{2,1} \end{aligned}$$

and more generally $e^{(l)}_{j,k} e^{(m)}_{j',k'} = e^{(m)}_{j',k'} e^{(l)}_{j,k}$ for all $j,k,j',k' \in \{1,2\}$ by similar computations.

We claim secondly that $\left(e^{(l)}_{j,k}\right)_{1 \leq j,k \leq 2}$ is a system of matrix units of order 2 for each $l \in \{1,...,n\}$. This follows from arguments as in the proof of Lemma 7.15.

Thus, for each $l \in \{1,...,n\}$, the system $\left(e^{(l)}_{j,k}\right)_{1 \leq j,k \leq 2}$ generates a sub-C*-algebra of $C^*(a(V))$ isomorphic to $M_2(\mathbb{C})$, and these sub-C*-algebras pairwise commute. Lemma 7.17 follows now from Lemma 7.16.

The trick of introducing the u_l 's to obtain commuting systems of matrix units is taken from [PoS]. □

7.18. Theorem. *Let $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{H}$ is of finite dimension n, then*

$$CAR(\mathcal{H}) \approx M_{2^n}(\mathbb{C}).$$

If $\mathcal{H}$ is infinite dimensional, then

$$CAR(\mathcal{H}) \approx \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})_j$$

is a UHF-algebra as in example 5.10.

Proof. In case $dim_{\mathbb{C}}\mathcal{H} < \infty$, the claim is contained in the previous lemma.

If $dim_{\mathbb{C}}\mathcal{H} = \infty$, choose an orthonormal basis $(e_n)_{n \geq 1}$ of $\mathcal{H}$. For each $n \geq 1$, let V_n denote the subspace of $\mathcal{H}$ generated by $\{e_1, ..., e_n\}$ and set $A_n = C^*(a(V_n))$. Then one has a tower

$$A_1 \approx M_2(\mathbb{C}) \ \subset \ ... \ \subset \ A_n \approx M_{2^n}(\mathbb{C}) \ \subset ...$$

of subalgebras of $CAR(\mathcal{H})$. Their union is dense in $CAR(\mathcal{H})$, by the argument of Remark 7.13. The claim follows. □

7.19. Corollary. *The action of the CAR algebra $CAR(\mathcal{H})$ in the Fock space $\mathcal{F}(\mathcal{H})$ is irreducible*

Proof. If $\mathcal{H}$ is of finite dimension, say $n < \infty$, then $\mathcal{F}(\mathcal{H})$ is of dimension 2^n and, by comparison of dimensions, one has $CAR(\mathcal{H}) = \mathcal{B}(\mathcal{F}(\mathcal{H}))$.

If $\mathcal{H}$ is infinite dimensional, let $(V_n)_{n\geq 1}$ and $(A_n)_{n\geq 1}$ be as in the proof of Theorem 7.18. If "primes" denote commutants in $\mathcal{B}(\mathcal{F}(\mathcal{H}))$, one has

$$CAR(\mathcal{H})' = \bigcap_{n\geq 1} A_n' \ .$$

For each $n \geq 1$, the commutant A_n' consists of operators of the form $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & * \end{pmatrix}$ with respect to the decomposition $\mathcal{F}(\mathcal{H}) = \mathcal{F}(V_n) \oplus \mathcal{F}(V_n)^{\perp}$. As $\bigcup_{n\geq 1} \mathcal{F}(V_n)$ is dense in $\mathcal{F}(\mathcal{H})$, it follows that $\bigcap_{n\geq 1} A_n' = \mathbb{C}$ and this ends the proof.

Alternatively, one may also observe that $1 \in \mathbb{C} \approx \bigwedge^0 \mathcal{H} \subset \mathcal{F}(\mathcal{H})$ is a cyclic vector for the representation of $CAR(\mathcal{H})$ on $\mathcal{F}(\mathcal{H})$, and then observe that the corresponding state on $CAR(\mathcal{H})$ is pure by Proposition 6.29.

We present finally a third proof using Schur's lemma. Let T be a bounded operator on $\mathcal{F}(\mathcal{H})$ which commutes with $a(\xi)$ and $a^*(\xi)$ for all $\xi \in \mathcal{H}$; let us show that T is a scalar multiple of the identity. Denote by Ω the vector $1 \in \mathbb{C} \approx \bigwedge^0 \mathcal{H}$ and let $\xi_1, ..., \xi_m, \eta_1, ..., \eta_n$ be vectors in $\mathcal{H}$.

The scalar product

$$(*) \qquad \langle\ \xi_1 \wedge ... \wedge \xi_m \mid T\,(\eta_1 \wedge ... \wedge \eta_n)\ \rangle$$

is equal to $\langle\ a(\xi_1)...a(\xi_m)\Omega \mid a(\eta_1)...a(\eta_n)T\Omega\ \rangle$, and is thus zero if $n > m$; it is also equal to $\langle\ a(\xi_1)...a(\xi_m)T^*\Omega \mid a(\eta_1)...a(\eta_n)\Omega\ \rangle$, and is thus zero if $m > n$.

In case $m = n$, observe firstly that there exists some $\lambda \in \mathbb{C}$ such that

$$a^*(\xi_n)...a^*(\xi_1)a(\eta_1)...a(\eta_n)\Omega = \lambda\Omega$$

because there are as many annihilator operators $a^*(\xi_j)$'s as creation operators $a(\eta_k)$'s. Observe then that λ is given by

$$\begin{aligned} \lambda = \langle\ \Omega \mid \lambda\Omega\ \rangle &= \langle\ a(\xi_1)...a(\xi_n)\Omega \mid a(\eta_1)...a(\eta_n)\Omega\ \rangle \\ &= \langle \xi_1 \wedge ... \wedge \xi_n \mid \eta_1 \wedge ... \wedge \eta_n\ \rangle\ . \end{aligned}$$

One has finally

$$\begin{aligned} \langle \xi_1 \wedge ... \wedge \xi_n \mid T\,(\eta_1 \wedge ... \wedge \eta_n)\ \rangle &= \langle\ T^*\Omega \mid a^*(\xi_n)...a^*(\xi_1)a(\eta_1)...a(\eta_n)\Omega\ \rangle \\ = \langle\ T^*\Omega \mid \lambda\Omega \rangle &= \langle\ \Omega \mid T\Omega\ \rangle \langle \xi_1 \wedge ... \wedge \xi_n \mid \eta_1 \wedge ... \wedge \eta_n\ \rangle \end{aligned}$$

and it follows that $T = \langle \Omega | T\Omega \rangle\, id_{\mathcal{F}(\mathcal{H})}$. Thus $CAR(\mathcal{H})$ acts irreducibly on $\mathcal{F}(\mathcal{H})$. $\square$

7.20. Remark. Let again Ω denote the unit vector $1 \in \mathbb{C} \approx \bigwedge^0 \mathcal{H} \subset \mathcal{F}(\mathcal{H})$. Let $l \in \{1,2,...\}$ and $j,k \in \{1,2\}$; with the notations of the proof of Lemma 7.17, one has $x_l^*(\Omega) = 0$ and $x_l^* x_l(\Omega) = \Omega$, so that

$$\left\langle \, \Omega \mid e_{j,k}^{(l)} \Omega \, \right\rangle \; = \; \begin{cases} 1 & \text{if} \quad (j,k) \; = \; (1,1) \\ 0 & \text{if} \quad (j,k) \; \in \; \{(1,2),(2,1),(2,2)\} \; . \end{cases}$$

It follows that the vector state $\omega_\Omega(x) = \langle \Omega | x\Omega \rangle$ coïncides with the state ϕ_Λ of Exercice 6.33 corresponding to the constant sequence $\Lambda = (1,1,1,...)$.

7.21. Theorem. *Let $\mathcal{H}$ be a Hilbert space. There exists a C^*algebra $CAR(\mathcal{H})$ with unit and an linear map $a : \mathcal{H} \to CAR(\mathcal{H})$ such that*

(i) as a C^*algebra, $CAR(\mathcal{H})$ is generated by $a(\mathcal{H})$,
(ii) one has the CAR relations

$$\begin{aligned} a^*(\xi)a(\eta) \; + \; a(\eta)a^*(\xi) \; &= \; \langle \xi | \eta \rangle \, 1 \\ a(\xi)a(\eta) \; + \; a(\eta)a(\xi) \; &= \; 0. \end{aligned}$$

Moreover the pair $\big(a, CAR(\mathcal{H})\big)$ is unique in the following sense:
for any pair $\big(a', CAR'(\mathcal{H})\big)$ satisfying properties (i) and (ii),
there exists an isomorphism $\phi : CAR(\mathcal{H}) \to CAR'(\mathcal{H})$ such that $a' = \phi \circ a$.

Proof. The existence part of the theorem has been proved in Section 7.B.

The unicity part is a straightforward consequence of the analysis of the present Section 7.C. (As already stated early in 7.C, we have only used the CAR relations of (ii), not the actual construction of 7.B.)

The previous arguments carries over with minor adjustments to the case of a non separable Hilbert space $\mathcal{H}$. $\square$

7.22. Corollary. *For any unitary operator u on $\mathcal{H}$, there is a unique automorphism $Bog(u)$ of $CAR(\mathcal{H})$ such that*

$$Bog(u)\ (a(\xi)) = a(u(\xi))$$

for all $\xi \in \mathcal{H}$. Moreover, the resulting "representation"

$$Bog : \mathcal{U}(\mathcal{H}) \longrightarrow Aut\big(CAR(\mathcal{H})\big)$$

is continuous for the strong topology on $\mathcal{U}(\mathcal{H})$ and the topology of pointwise norm convergence on $Aut\big(CAR(\mathcal{H})\big)$.

Proof. Observe that the map

$$a_u \;:\; \begin{cases} \mathcal{H} & \longrightarrow \; CAR(\mathcal{H}) \\ \xi & \longmapsto \; a(u(\xi)) \end{cases}$$

satisfies the CAR relations. The first claim of the Corollary follows, by the unicity part of the previous previous theorem.

The topology of pointwise norm convergence is the topology on $Aut\big(CAR(\mathcal{H})\big)$ for which a basis of neighbourhoods of the identity is the family of finite intersections of sets of the form

$$\big\{\phi \in Aut\big(CAR(\mathcal{H})\big) \mid \|\phi(x) - x\| < 1\big\}$$

with $x \in CAR(\mathcal{H})$. We leave it to the reader to check that this topology makes $Aut\big(CAR(\mathcal{H})\big)$ a topological group. We recall from Proposition 2.16 that $\mathcal{U}(\mathcal{H})$ with the strong topology is also a topological group. (Indeed these are two Polish groups if $\mathcal{H}$ is separable.)

As one has

$$Bog(u)\ \big(a(\xi)\big) - a(\xi) = a\big(u(\xi) - \xi\big)$$

for all $u \in \mathcal{U}(\mathcal{H})$ and $\xi \in \mathcal{H}$, the continuity of Bog follows from the fact that $a(\mathcal{H})$ generates $CAR(\mathcal{H})$ and from the fact that $a : \mathcal{H} \to CAR(\mathcal{H})$ is an isometric inclusion. $\square$

7.23. Remarks (i) The automorphisms $Bog(u)$ are called *Bogoliubov automorphisms.*

(ii) The Bogoliubov action of the "continuous group" $\mathcal{U}(\mathcal{H})$ on the C*-algebra $\bigotimes_{j=1}^{\infty} M_2(\mathbb{C})$ is quite remarkable. Indeed, this algebra has a "commutative analogue" $\bigotimes_{j=1}^{\infty} (\mathbb{C} \oplus \mathbb{C})$ which is the C*-algebra of continuous functions on the standard Cantor set, and no continuous group may act nontrivially on a totally disconnected space.

7.24. Example. There is a canonical action of the group $SO(2)$ of complex numbers of modulus 1 on $\mathcal{H}$, given by $(e^{i\theta}, \xi) \mapsto e^{i\theta}\xi$. Thus, there is a family $(\alpha_\theta)_{\theta \in SO(2)}$ of automorphisms of $CAR(\mathcal{H})$ such that

$$\alpha_\theta\left(a(\xi)\right) \;=\; a\left(e^{i\theta}\xi\right)$$

for all $\theta \in [0, 2\pi]$ and for all $\xi \in \mathcal{H}$. In the tensor product picture (see Theorem 7.18), it can be checked that

$$\begin{aligned}
&\alpha_\theta\left(\bigotimes_{j=1}^{\infty}\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}\right) \\
&\qquad = \left(\bigotimes_{j=1}^{\infty}\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\right)\left(\bigotimes_{j=1}^{\infty}\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}\right)\left(\bigotimes_{j=1}^{\infty}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}\right) \\
&\qquad = \left(\bigotimes_{j=1}^{\infty}\begin{pmatrix} a_j & e^{i\theta}b_j \\ e^{-i\theta}c_j & d_j \end{pmatrix}\right)
\end{aligned}$$

for all

$$\bigotimes_{j=1}^{\infty}\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \;\in\; \bigotimes_{j=1}^{\infty} M_2(\mathbb{C})_j \;\approx\; CAR(\mathcal{H})\ .$$

More generally, for any locally compact group G and any unitary representation of G on a Hilbert space $\mathcal{H}$, there is an associated action of G on $CAR(\mathcal{H})$.

7.25. Exercise. Let A be a C*-algebra, let $x_1, ..., x_n \in A$ and set

$$u_{2j-1} = x_j + x_j^* \qquad u_{2j} = \frac{1}{i}\left(x_j - x_j^*\right)$$

for all $j \in \{1, ..., n\}$. Show that one has the CAR relations

$$x_j^* x_k + x_k x_j^* = \delta_{j,k} \qquad \text{and} \qquad x_j x_k + x_k x_j = 0$$

for all $j, k \in \{1, ..., n\}$ if and only if the following holds:

$$\begin{aligned} u_1, ..., u_{2n} & \qquad \text{are self-adjoint unitaries} \\ u_l u_m = -u_m u_l & \qquad \text{for all} \;\; l, m \in \{1, ..., 2n\} \;\; \text{with} \;\; l \neq m. \end{aligned}$$

(It follows that representations of a CAR-algebra $CAR(\mathcal{H})$ can be interpreted in terms of representations of an appropriate group. More on this in [Gui, Proposition 3.4].)

7.26. Exercice. An *antilinear operator* on $\mathcal{H}$ is a $\mathbb{R}$-linear map $v : \mathcal{H} \to \mathcal{H}$ such that $v(i\xi) = -iv(\xi)$ for all $\xi \in \mathcal{H}$ and $\|v\| = \sup\{\ \|v(\xi)\| \mid \xi \in \mathcal{H}\ ,\ \|\xi\| \leq 1\ \} < \infty$. Such an operator has an *adjoint* v^* defined by

$$\overline{\langle\ v^*\eta \mid \xi\ \rangle} \;=\; \langle\ \eta \mid v\xi\ \rangle$$

for all $\xi, \eta \in \mathcal{H}$.

Let u be a linear operator on $\mathcal{H}$ and let v be an antilinear operator on $\mathcal{H}$ such that

$$\begin{aligned} u^*u + v^*v \;&=\; uu^* + vv^* \;=\; 1 \\ u^*v + v^*u \;&=\; uv^* + vu^* \;=\; 0\ , \end{aligned}$$

namely such that

$$\begin{pmatrix} u^* & v^* \\ v^* & u^* \end{pmatrix} \begin{pmatrix} u & v \\ v & u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} u^* & v^* \\ v^* & u^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that there is a unique automorphism ϕ of $CAR(\mathcal{H})$ such that

$$\phi(a(\xi)) \;=\; a(u(\xi)) + a^*(v(\xi))$$

for all $\xi \in \mathcal{H}$.

7.27. Remark on the CCR-algebra. For physical reasons, it is equally important to study algebras of *Canonical Commutation Relations,* which are C*-algebras acting on the symmetric parts of spaces of the form $EXP(\mathcal{H})$. A nice introduction to these is the volume of Petz' lectures [Pet].

7.28. Remark on Clifford algebras. Let V be a vector space (say here over the reals) and let $q : V \to \mathbb{R}$ be a quadratic form. The *Clifford algebra* $Cliff(V, q)$ is the quotient of the tensor algebra of V by the two-sided ideal generated by $(v \otimes v - q(v)1)_{v \in V}$. There is a canonical inclusion $\iota : V \to Cliff(V,q)$, and the pair $(Cliff(V,q), \iota)$ has the following universal property: maps f from V to a real algebra A with unit such that $f(v)^2 = q(v)1_A$ for all $v \in V$ are in natural bijection with homomorphisms of algebras $F : Cliff(V,q) \to A$, the correspondence being such that $F(\iota(v)) = f(v)$ for all $v \in V$. For the theory of Clifford algebras, see an introduction in Chapter III of [Ch2], or the book [Ch1].

Let $\mathcal{H}_{\mathbb{R}}$ be a *real* Hilbert space. We denote by $Cliff_0(\mathcal{H}_{\mathbb{R}})$ the Clifford algebra defined as above for the quadratic form $q : \xi \mapsto \|\xi\|^2$. Let $\mathcal{H}$ be a *complex* Hilbert space, and let $\mathcal{H}_{\mathbb{R}}$ denote the underlying real Hilbert space. The $\mathbb{R}$-linear map

$$f \ : \ \begin{cases} \mathcal{H}_{\mathbb{R}} & \longrightarrow \quad CAR(\mathcal{H}) \\ \quad \xi & \longmapsto \quad a(\xi) + a(\xi)^* \end{cases}$$

satisfies $f(\xi)^2 = \|\xi\|^2$ for all $\xi \in \mathcal{H}_{\mathbb{R}}$ (a straightforward consequence of the CAR relations). Hence, using the universal property of Clifford algebras, one has a morphism of algebras

$$Cliff_0\,(\mathcal{H}_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} \ \longrightarrow \ CAR(\mathcal{H})$$

which can be shown to be an *injection with dense image.*

One may then approach the theory of the CAR algebra by viewing it firstly as a Clifford algebra [PlR]. For example, as Clifford algebras are naturally $\mathbb{Z}/2\mathbb{Z}$-graded, one may use graded tensor products to find out the structure of $CAR(\mathcal{H})$ when $\mathcal{H}$ is finite dimensional, as in [ABS]; the introduction of the u_l 's in the proof of Lemma 7.17 makes it possible to forget the grading at this point.

P. de la Harpe and V. Jones, July 1995.

CHAPTER 8. QUASI-FREE STATES ON THE CAR ALGEBRA

In this chapter, $\mathcal{H}$ denotes a Hilbert space, $CAR(\mathcal{H})$ the corresponding CAR algebra and $a : \mathcal{H} \to CAR(\mathcal{H})$ the canonical linear isometric immersion, as in the previous chapter. We start here the study of states on $CAR(\mathcal{H})$ which are sometimes called "free", but more traditionally (and with no particular reason) "quasi-free".

8.A. Definition of the quasi-free states.

8.1. Two-point function of a state on the CAR-algebra. Let ω be a state on the CAR-algebra $CAR(\mathcal{H})$. For $\xi, \eta \in \mathcal{H}$, set

$$\langle \xi | \eta \rangle_\omega \; = \; \omega\left(a^*(\xi) a(\eta)\right).$$

Then $\langle \bullet | \bullet \rangle_\omega$ is a sesqui-linear form on $\mathcal{H}$, and

$$|\langle \xi | \eta \rangle_\omega| \;\le\; \|a^*(\xi)a(\eta)\| \;\le\; \|\xi\| \, \|\eta\|$$

for all $\xi, \eta \in \mathcal{H}$ because $\|\omega\| \le 1$. Hence there exists a well defined operator b on $\mathcal{H}$ such that

$$\omega(a^*(\xi)a(\eta)) \; = \; \langle \xi | b\eta \rangle$$

for all $\xi, \eta \in \mathcal{H}$, and one has $\|b\| \le 1$. As ω is positive, one has $\langle \xi | b\xi \rangle = \omega(a^*(\xi)a(\xi)) \ge 0$ for all $\xi \in \mathcal{H}$, namely $0 \le b \le 1$.

8.2. Theorem. *Let $b \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator such that $0 \le b \le 1$. Then there exists a unique state ϕ_b on $CAR(\mathcal{H})$ such that*

$$\phi_b\Big(a^*(\xi_m)a^*(\xi_{m-1})...a^*(\xi_1) \; a(\eta_1)a(\eta_2)...a(\eta_n) \Big) \; = \; \delta_{m,n} \; \det \Big(\langle \xi_j \mid b\eta_k \rangle_{1 \le j \le m, 1 \le k \le n} \Big)$$

for all $\xi_1, ..., \xi_m, \eta_1, ..., \eta_n \in \mathcal{H}$.

8.3. Definition. The state ϕ_b on $CAR(\mathcal{H})$ is called the *quasi-free state of covariance b*.

A direct verification that the formula of Theorem 8.2 defines a state would be cumbersome (especially for the proof of positivity). Our strategy will be to prove Theorem 8.2 in the special case of a projection, and then to use the well-known "two-by-two-matrix-trick" (compare with the proof of Kaplansky Density Theorem 6.18).

The unicity part of the theorem is straightforward, because linear combinations of elements of the form $a^*(\xi_m)...a^*(\xi_1)a(\eta_1)...a(\eta_n)$ are dense in $CAR(\mathcal{H})$; see Remark 7.13.

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8.4. Lemma. *Theorem 8.2 holds in case $\mathcal{H}$ is finite dimensional and b is a projection, written p below.*

In this case, ϕ_p is the vector state defined by the Fock representation and by a unit vector $v \in \mathcal{F}(\mathcal{H})$ corresponding to the subspace $Im(p)$ of $\mathcal{H}$.

Comment. Let $Grass_r$ denote the Grassmannian of subspaces of $\mathcal{H}$ of some dimension r. Recall that there is a classical embedding

$$pl \ : \ Grass_r \ \longrightarrow \ \mathbb{P}\left(\bigwedge^r \mathcal{H}\right) \ \subset \ \mathbb{P}\Big(\mathcal{F}(\mathcal{H})\Big)$$

(where $\mathbb{P}(\mathcal{K})$ denotes the projective space of a space $\mathcal{K}$ and where pl refers to Plücker) defined as follows : for an element V of $Grass_r$, consider a basis $v_1, ..., v_r$ of V and let $pl(V)$ be the line defined by $v_1 \wedge ... \wedge v_r$ (see e.g. [Di3], § 3, n^0 5). Observe that, if $v_1, ..., v_r$ is an orthonormal basis of V, the unit vector $v_1 \wedge ... \wedge v_r \in \wedge^r \mathcal{H}$ is well defined by V up to multiplication by a complex number of modulus 1; a fortiori, the corresponding state on $CAR(\mathcal{H})$ depends only on V.

Proof. Let r be the codimension of the image of p and let s be its dimension. Choose orthonormal basis

$$\begin{array}{ccc} v_1, ..., v_r & \text{of} & (1-p)(\mathcal{H}) \\ w_1, ..., w_s & \text{of} & p(\mathcal{H}). \end{array}$$

Consider the vector $v = v_1 \wedge ... \wedge v_r \in \mathcal{F}(\mathcal{H})$ and the vector state ω_v on $CAR(\mathcal{H})$, such that

$$\omega_v\left(a^*(\xi_m)...a^*(\xi_1)a(\eta_1)...a(\eta_n)\right) \ = \ \langle v \mid a^*(\xi_m)...a^*(\xi_1)a(\eta_1)...a(\eta_n)v\rangle$$

for all $\xi_1, ..., \xi_m, \eta_1, ..., \eta_n \in \mathcal{H}$.

We claim that the formula of Theorem 8.2 holds with $\phi_p = \omega_v$, namely that

$$(*) \qquad \omega_v\left(a^*(\xi_m)...a^*(\xi_1)a(\eta_1)...a(\eta_n)\right) \ = \ \delta_{m,n} \ \det\Big(\langle \xi_j \mid p\eta_k\rangle_{1\le j\le m, 1\le k\le n}\Big)$$

for all $\xi_1, ..., \xi_m, \eta_1, ..., \eta_n \in \mathcal{H}$.

As both terms of $(*)$ are anti-multilinear in the ξ_j 's and multilinear in the η_k 's, it suffices to check $(*)$ in case $\xi_1, ..., \xi_m, \eta_1, ..., \eta_n$ are vectors in some basis of $\mathcal{H}$, and indeed in the basis $\{v_1, ..., v_r, w_1, ..., w_s\}$ of $\mathcal{H}$.

Now

$$\omega_v\left(a^*(\xi_m)...a^*(\xi_1)a(\eta_1)...a(\eta_n)\right) \ = \ \langle \xi_1 \wedge ... \wedge \xi_m \wedge v \mid \eta_1 \wedge ... \wedge \eta_n \wedge v\rangle$$

is clearly 0 if at least one of the ξ_j 's or the η_k 's is in $\{v_1, ..., v_r\}$, or if $m \neq n$. Similarly the right-hand term of $(*)$ is 0 if one of these hold (because $pv_i = 0$ for $i \in \{1, ..., r\}$). So it suffices to consider the case in which $m = n$ and in which all the ξ_j 's and η_k 's are in $\{w_1, ..., w_s\}$. In this case, using Proposition 7.6, one has

$$\begin{aligned} \omega_v\left(a^*(\xi_n)...a^*(\xi_1)a(\eta_1)...a(\eta_n)\right) \ &= \ \langle \xi_1 \wedge ... \wedge \xi_n \wedge v_1 ... \wedge v_r \mid \eta_1 \wedge ... \wedge \eta_n \wedge v_1 ... \wedge v_r\rangle \\ &= \ \det\begin{pmatrix} \langle \xi_j | \eta_k\rangle_{1\le j,k\le n} & 0 \\ 0 & (\delta_{j,k})_{1\le j,k\le r} \end{pmatrix} \\ &= \ \det\Big(\langle \xi_j | \eta_k\rangle_{1\le j,k\le n}\Big) \end{aligned}$$

and this proves Formula $(*)$. $\square$

8.5. Lemma. *Theorem 8.2 holds in case b is a projection, written p below.*

Proof. One may choose a tower

$$V_1 \subset V_2 \subset \ldots \subset V_n \subset V_{n+1} \subset \ldots$$

of finite dimensional subspaces of $\mathcal{H}$ such that

$$V_n = p(V_n) \oplus (1-p)(V_n)$$

for all $n \geq 1$ and such that $\bigcup_{n\geq 1} V_n$ is dense in $\mathcal{H}$. For each $n \geq 1$, let ϕ_n denote the state on the sub-C*-algebra $CAR(V_n)$ of $CAR(\mathcal{H})$ defined as in the proof of Lemma 8.4. It is clear that ϕ_n coincides with the restriction of ϕ_{n+1} from $CAR(V_{n+1})$ to $CAR(V_n)$. Thus, in the formula of Theorem 8.2, one may take for ϕ_p the limit state defined by the ϕ_n 's (see Number 6.28). $\square$

8.6. Lemma. *Let $\mathcal{H}$ be a Hilbert space and let $b \in \mathcal{B}(\mathcal{H})$ be such that $0 \leq b \leq 1$. Then*

$$p = \begin{pmatrix} b & \sqrt{b(1-b)} \\ \sqrt{b(1-b)} & 1-b \end{pmatrix}$$

is a projection on $\mathcal{H} \oplus \mathcal{H}$ and

$$p\begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} b\xi \\ * \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}.$$

Proof: straightforward. $\square$

8.7. End of proof of Theorem 8.2. Let b be as in Theorem 8.2, let p be as in Lemma 8.6, and let

$$\phi_p : CAR(\mathcal{H} \oplus \mathcal{H}) \longrightarrow \mathbb{C}$$

be the state defined as in the proof of Lemma 8.5. Let ϕ_b denote the restriction of ϕ_p to the sub-C*-algebra $CAR(\mathcal{H})$ of $CAR(\mathcal{H} \oplus \mathcal{H})$ corresponding to the first factor of the direct sum. Then the formula of Theorem 8.2 holds for ϕ_b. $\square$

8.8. Proposition. *Let $p \in \mathcal{B}(\mathcal{H})$ be a projection. Then the quasi-free state ϕ_p of covariance p is pure.*

Proof. In case $\mathcal{H}$ is finite dimensional, $CAR(\mathcal{H})$ is the space of all operators on $\mathcal{F}(\mathcal{H})$, and in particular is irreducible on $\mathcal{F}(\mathcal{H})$. As the proof of Lemma 8.4 shows that ϕ_p is a vector state on $CAR(\mathcal{H})$, it is indeed a pure state (see Proposition 6.16).

In case $\mathcal{H}$ is infinite dimensional, the argument of Lemma 8.5 shows that ϕ_p is a limit state of states $\phi_n : CAR(V_n) \to \mathbb{C}$, so that ϕ_p is again pure by Proposition 6.29. $\square$

8.9. Remark. Conversely, it is easy to show that a quasi-free state ϕ_b of covariance b is pure *if and only if* b is a projection.

It is also known that the states ϕ_b are always factorial: see [PoS, Lemma 1.3] for the easy case of an operator b with pure point spectrum, and [PoS, Theorem 5.1] for the general case.

8.10. On states with two-point functions defined by projections. For each integer $n \geq 0$, let $F_n = F_n CAR(\mathcal{H})$ denote the subspace of $CAR(\mathcal{H})$ linearly generated by products of the form $c_1(\zeta_1)c_2(\zeta_2)...c_k(\zeta_k)$ where $k \leq n$, where each c_j holds for either a or a^* and where $\zeta_1, ..., \zeta_k$ are vectors in $\mathcal{H}$. The filtration $(F_n)_{n \geq 0}$ defined this way has the following property : if σ is any permutation of $\{1, ..., n\}$, then

$$c_{\sigma(1)}\left(\zeta_{\sigma(1)}\right)) \dots c_{\sigma(n)}\left(\zeta_{\sigma(n)}\right) \equiv c_1(\zeta_1) \dots c_n(\zeta_n) \mod F_{n-2}$$

for all $c_1, ..., c_n \in \{a, a^*\}$ and for $\zeta_1, ..., \zeta_n \in \mathcal{H}$ (this is a straightforward consequence of the CAR relations).

Let ω be a state on $CAR(\mathcal{H})$ with two-point function defined by a projection p on $\mathcal{H}$, namely such that

$$\omega\left(a^*(\zeta_1)a(\zeta_2)\right) = \langle\zeta_1|p\zeta_2\rangle$$

for all $\zeta_1, \zeta_2 \in \mathcal{H}$. (We do *not* assume a priori that ω is a quasi-free state, even though it will follow from the analysis below.) Denote by $\pi_\omega : CAR(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_\omega)$ the representation and by Ω_ω the cyclic vector obtained by the GNS construction.

For $\xi \in Im(1-p)$ one has

$$\|\pi_\omega\left(a(\xi)\right)\Omega_\omega\|^2 = \omega\left(a^*(\xi)a(\xi)\right) = \langle\xi|p\xi\rangle = 0$$

and consequently

$$\pi_\omega\left(a(\xi)\right)\Omega_\omega = 0.$$

For $\eta \in p(\mathcal{H})$ one has

$$\|\pi_\omega\left(a^*(\eta)\right)\Omega_\omega\|^2 = \omega\left(\|\eta\|^2 - a^*(\eta)a(\eta)\right) = \|\eta\|^2 - \langle\eta|p\eta\rangle = 0$$

and consequently

$$\pi_\omega\left(a^*(\eta)\right)\Omega_\omega = 0.$$

Let $n \geq 2$. For $x \in F_n$ of the form $c_1(\zeta_1) \dots c_n(\zeta_n)$ as above, the n-point function

$$\omega\Big(c_1(\zeta_1) \dots c_n(\zeta_n)\Big) = \langle\ \Omega_\omega \mid \pi_\omega(c_1(\zeta_1) \dots \pi_\omega(c_n(\zeta_n))\Omega_\omega\ \rangle$$

is zero as soon as one at least of

$$\begin{array}{rcl} c_1 = a^* & \text{and} & \zeta_1 \in Im(1-p) \\ c_1 = a & \text{and} & \zeta_1 \in Im(p) \\ c_n = a & \text{and} & \zeta_n \in Im(1-p) \\ c_n = a^* & \text{and} & \zeta_n \in Im(p) \end{array}$$

holds. It follows that one has :

$$\text{for any } \ x \in F_n \ \text{ there exists } \ y \in F_{n-2} \ \text{ such that } \ \omega(y) = \omega(x) \ .$$

We have proved the following :

a state on $CAR(\mathcal{H})$ with 2-point functions of the form
$\omega\big(a^*(\zeta_1)a(\zeta_2)\big) \ = \ \langle\zeta_1|p\zeta_2\rangle$ *for some projection p on $\mathcal{H}$*
is necessarily the quasi-free state of covariance p.

8.11. Quasi-free states and the GNS-construction. Given a complex Hilbert space $\mathcal{K}$, denote by $\overline{\mathcal{K}}$ the *conjugate space.* Recall that there is a $\mathbb{R}$-linear bijection

$$\begin{cases} \mathcal{K} & \longrightarrow \ \overline{\mathcal{K}} \\ \kappa & \longmapsto \ \overline{\kappa} \end{cases}$$

such that $\overline{(z\kappa)} = \overline{z}\ \overline{\kappa}$ and $\left\langle \overline{\kappa}|\overline{\kappa'}\right\rangle_{\overline{\mathcal{K}}} \ = \ \left\langle \overline{\kappa'}|\overline{\kappa}\right\rangle_{\mathcal{K}}$ for all $z \in \mathbb{C}$ and $\kappa, \kappa' \in \mathcal{K}$. One has clearly a canonical isomorphism

$$\mathcal{F}\left(\overline{\mathcal{K}}\right) \ \approx \ \overline{\mathcal{F}(\mathcal{K})}$$

at the level of Fock spaces.

Consider a projection p on $\mathcal{H}$. We define the Hilbert space

$$\Gamma_p(\mathcal{K}) \ = \ \mathcal{F}\left(\overline{(1-p)(\mathcal{H})}\right) \bigotimes \mathcal{F}(p(\mathcal{H})).$$

We denote by the same symbol 1 the unit vector in $\mathbb{C} = \wedge^0\overline{(1-p)(\mathcal{H})}$ and the unit vector in $\mathbb{C} = \wedge^0 p(\mathcal{H})$, and we set

$$\Omega_p \ = \ 1 \otimes 1 \ \in \ \Gamma_p(\mathcal{H}).$$

Let D be the *parity operator* in $\mathcal{F}\left(\overline{(1-p)(\mathcal{H})}\right)$, defined by

$$Dx \ = \ \begin{cases} x & \text{for all} \quad x \ \in \ \displaystyle\bigoplus_{n\geq 0} \bigwedge^{2n} \left(\overline{(1-p)(\mathcal{H})}\right) \ \subset \ \mathcal{F}\left(\overline{(1-p)(\mathcal{H})}\right) \\ -x & \text{for all} \quad x \ \in \ \displaystyle\bigoplus_{n\geq 0} \bigwedge^{2n+1} \left(\overline{(1-p)(\mathcal{H})}\right) \ \subset \ \mathcal{F}\left(\overline{(1-p)(\mathcal{H})}\right). \end{cases}$$

For

$$\zeta \ = \ \xi \oplus \eta \ \in \ (1-p)(\mathcal{H}) \oplus p(\mathcal{H}) \ = \ \mathcal{H}$$

we define

$$A_p(\zeta) \ = \ A_p(\xi \oplus \eta) \ = \ a^*(\overline{\xi}) \otimes 1 \ + \ D \otimes a(\eta) \ \in \ \mathcal{B}\left(\Gamma_p(\mathcal{H})\right).$$

We leave it as an exercice for the reader to check that the CAR relations

$$\begin{aligned} A_p^*(\xi \oplus \eta)A_p(\xi' \oplus \eta') \ + \ A_p(\xi' \oplus \eta')A_p^*(\xi \oplus \eta) \ &= \ \langle \xi \oplus \eta \mid \xi' + \eta' \rangle \\ A_p(\xi \oplus \eta)A_p(\xi' \oplus \eta') \ + \ A_p(\xi' \oplus \eta')A_p(\xi \oplus \eta) \ &= \ 0 \end{aligned}$$

hold for all $\xi \oplus \eta, \xi' \oplus \eta' \in \mathcal{H}$ (with $p\xi = 0$, $p\eta = \eta$, $p\xi' = 0$, $p\eta' = \eta'$). Thus the assignment

$$\zeta = \xi + \eta \in \mathcal{H} \ \longmapsto \ A_p(\zeta) = a^*(\overline{\xi}) \otimes 1 \ + \ D \otimes a(\eta) \ \in \ \mathcal{B}(\Gamma_p(\mathcal{H}))$$

extends to a representation

$$\pi_p \ : \ CAR(\mathcal{H}) \ \longrightarrow \ \mathcal{B}(\Gamma_p(\mathcal{H})) \, .$$

The vector state ω_p defined by the representation π_p and by the (clearly cyclic) vector Ω_p satisfies

$$\begin{aligned} \omega_p\Big(\pi_p\big(a(\xi \oplus \eta)\big)^* \pi_p\big(a(\xi' \oplus \eta')\big)\Big) \ &= \ \langle A_p(\xi \oplus \eta)\Omega_p | A_p(\xi' \oplus \eta')\Omega_p \rangle \ = \ \langle \eta | \eta' \rangle \\ &= \ \langle \xi \oplus \eta | p(\xi' + \eta') \rangle \end{aligned}$$

for all $\xi \oplus \eta$, $\xi' \oplus \eta' \in (1-p)(\mathcal{H} = \oplus p(\mathcal{H}) = \mathcal{H}$. It follows from 8.2 and 8.10 that ω_p is the quasi-free state of covariance p.

8.12. On Dirac holes.
See the discussion in [KaR], Section 4.2.

8.13. Example. On the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$, consider the (unbounded !) self-adjoint operator $H = -i\frac{d}{d\theta}$ and the corresponding one parameter unitary group $(U_t)_{t \in \mathbb{T}}$ defined by

$$U_t \ = \ e^{it\left(-i\frac{d}{d\theta}\right)}$$

or

$$(U_t f)(\theta) \ = \ f(\theta + t)$$

for all $f \in L^2(\mathbb{T})$ and $\theta \in \mathbb{T}$. Let $(e_n)_{n \in \mathbb{Z}}$ be the usual basis of $L^2(\mathbb{T})$, defined by $e_n(\theta) = e^{in\theta}$. As $-i\frac{d}{d\theta}$ is diagonal with respect to this basis, it is straightforward to compute the spectrum of H which is

$$\sigma(H) \ = \ \mathbb{Z}.$$

If H is to be a model of a Hamiltonian for a physical system, there is a problem because $\sigma(H)$ is not bounded below whereas the energy should be bounded below.

Let $H^2(\mathbb{T}) = \text{span}\,\{(e_n)_{n \geq 0}\}$ be the Hardy space and let $p \in \mathcal{B}(L^2(\mathbb{T}))$ be the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$, as in Example 3.23. For all $t \in \mathbb{T}$ the unitary operator U_t commutes with p. It follows that the automorphism α_t of $CAR(\mathcal{H})$ induced by U_t (via the usual formula $\alpha_t(a(\xi)) = a(U_t(\xi))$, see Corollary 7.32) and the quasi-free state ϕ_p satisfy

$$\phi_p \ \alpha_t \ = \ \phi_p$$

for all $t \in \mathbb{T}$. Thus there exists a unitary operator u_t on $\Gamma_p(\mathcal{H})$ such that

$$\pi_p(\alpha_t(x)) \;=\; u_t \pi_p(x) u_t^*$$

for all $t \in \mathbb{T}$ and for all $x \in CAR(\mathcal{H})$; see 6.11.ii.

The Hilbert space $\Gamma_p(\mathcal{H})$ has an orthonormal basis of vectors of the form

$$X \;=\; \pi_p\Big(a^*(\xi_1)...a^*(\xi_k)a(\eta_1)...a(\eta_l)\Big)\Omega_p^{GNS}$$

where

$$\begin{array}{ll} \xi_i = e_{-m_i} \;\in\; \overline{(1-p)(\mathcal{H})} \;\;,\;\; m_i > 0 \quad \text{for} \quad 1 \le i \le k \\ \eta_j = e_{n_j} \;\in\; p(\mathcal{H}) \;\;,\;\; n_j \ge 0 \quad \text{for} \quad 1 \le j \le l. \end{array}$$

As $U_t e_n = exp(\sqrt{-1}nt)e_n$ for all $n \in \mathbb{Z}$, one has

$$\begin{array}{l} u_t \pi_p\Big(a^*(\xi_1)...a^*(\xi_k)a(\eta_1)...a(\eta_l)\Big)\Omega_p^{GNS} \\ \quad \pi_p\Big(a^*(U_t\xi_1)...a^*(U_t\xi_k)a(U_t\eta_1)...a(U_t\eta_l)\Big)\Omega_p^{GNS} \\ \quad exp\Big(\sqrt{-1}(m_1 + ... + m_k + n_1 ... + n_l)\Big)\pi_p\left(a^*(\xi_1)...a^*(\xi_k)a(\eta_1)...a(\eta_l)\right)\Omega_p^{GNS} \end{array}$$

and U_t acts on X by multiplication by

$$exp\Big(\sqrt{-1}(m_1 + ... + m_k + n_1 ... + n_l)\Big).$$

In particular, the infinitesimal generator of $(U_t)_{t\in\mathbb{T}}$ has positive spectrum, contained inside $\mathbb{N}$.

The crucial point is that the process of "second quantification", namely of replacing $\mathcal{H}$ by $\Gamma_p(\mathcal{H})$, makes the spectrum of the "Hamiltonian" positive. It is the conjugation in $\overline{(1-p)(\mathcal{H})}$ which restores positivity of the spectrum.

8.B. Equivalence of quasi-free states.

We show in Theorem 8.24 below a sufficent condition for the equivalence of two quasi-free states $CAR(\mathcal{H})$.

8.14. Theorem. *Let $\mathcal{H}$ be a Hilbert space, let p, q be two projections on $\mathcal{H}$ and let ϕ_p, ϕ_q be the corresponding quasi-free states on $CAR(\mathcal{H})$. Then*

$$\|\phi_p - \phi_q\| \;\le\; 2\,\|p - q\|_2$$

where $\|\bullet\|_2$ denotes a Hilbert-Schmidt norm.

8.15. Lemma. *Let p, q be two projections on a finite dimensional Hilbert space $\mathcal{H}$. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of the operator $c = (1-p)(1-q)(1-p)$ acting on the space $(1-p)(\mathcal{H})$, where $n = dim_{\mathbb{C}}((1-p)(\mathcal{H}))$.*

(i) If $dim_{\mathbb{C}}(p(\mathcal{H})) \neq dim_{\mathbb{C}}(q(\mathcal{H}))$ one has $\|p-q\|_2 \geq 1$.

(ii) If $dim_{\mathbb{C}}(p(\mathcal{H})) = dim_{\mathbb{C}}(q(\mathcal{H}))$ one has $\|p-q\|_2^2 = 2\sum_{j=1}^n (1-\lambda_j)$.

Proof. Let $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ denote the usual trace. By definition of the Hilbert-Schmidt norm (Definition 3.18), one has

$$\|p-q\|_2^2 = tr((p-q)^*(p-q)) = tr(p+q-pq-qp) = tr(p)+tr(q)-2tr(qpq)$$

because $tr(pq) = tr(p^2q) = tr(pqp) = tr(qp)$.

Assume firstly that $dim_{\mathbb{C}}(p(\mathcal{H})) > dim_{\mathbb{C}}(q(\mathcal{H}))$, so that $tr(p) - tr(q) \geq 1$. As $tr(q) \geq tr(qpq)$, one has

$$\|p-q\|_2^2 = tr(p) - tr(q) + 2\big(tr(q) - tr(qpq)\big) \geq 1$$

and (i) follows.

Assume secondly that $dim_{\mathbb{C}}(p(\mathcal{H})) = dim_{\mathbb{C}}(q(\mathcal{H}))$, so that $tr(p) = tr(q)$. One has

$$\begin{aligned} \|p-q\|_2^2 = \|(1-p)-(1-q)\|_2^2 &= 2\Big[tr(1-p) - tr\big((1-p)(1-q)(1-p)\big)\Big] \\ &= 2\Big[tr'(1-p) - tr'\big((1-p)(1-q)(1-p)\big)\Big] \\ &= 2\sum_{j=1}^{n}(1-\lambda_j) \end{aligned}$$

where tr' denotes the usual trace on $\mathcal{B}\big((1-p)(\mathcal{H})\big)$, which is also the natural restriction to this algebra of operators of the trace tr on $\mathcal{B}(\mathcal{H})$. $\square$

8.16. Lemma. *Theorem 8.14 holds if $dim(\mathcal{H}) < \infty$.*

Proof. If $dim_{\mathbb{C}}(p(\mathcal{H})) \neq dim_{\mathbb{C}}(q(\mathcal{H}))$, one has $2\|p-q\|_2 \geq 2$ by the previous lemma and there is nothing to prove (recall that states have norm 1). We assume from now on that $dim_{\mathbb{C}}(p(\mathcal{H})) = dim_{\mathbb{C}}(q(\mathcal{H}))$ and we choose

$$\begin{aligned} &\text{an orthonormal basis } v_1, ..., v_n \text{ of } (1-p)(\mathcal{H}), \\ &\text{an orthonormal basis } w_1, ..., w_n \text{ of } (1-q)(\mathcal{H}). \end{aligned}$$

Set $v = v_1 \wedge ... \wedge v_n \in \mathcal{F}\big((1-p)(\mathcal{H})\big)$ and $w = w_1 \wedge ... \wedge w_n \in \mathcal{F}\big((1-q)(\mathcal{H})\big)$. We know from the proof of Lemma 8.4 that ϕ_p coïncides with the vector state ω_v on $CAR(\mathcal{H})$, and similarly that $\phi_q = \omega_w$. From Propositions 6.27.i and 7.6 one has

$$\begin{aligned} \|\phi_p - \phi_q\| = \|\omega_v - \omega_w\| &\leq 2\sqrt{1 - |\langle\, v \mid w\,\rangle|^2} \\ &= 2\sqrt{1 - \Big|det\Big(\langle\, v_j \mid w_k\,\rangle_{1\leq j,k\leq n}\Big)\Big|^2}\,. \end{aligned}$$

Let $b = \big(b_{j,k}\big)_{1\leq j,k\leq n}$ denote the matrix $\big(\langle\, v_j|w_k\,\rangle\big)_{1\leq j,k\leq n}$ and set $c = bb^*$. On one hand we have

$$\big|det(b)\big|^2 \;=\; det(b)det(b^*) \;=\; det(c).$$

On the other hand c is the matrix of the linear map

$$(1-p)(1-q)(1-p) \;:\; (1-p)(\mathcal{H}) \;\longrightarrow\; (1-p)(\mathcal{H})$$

with respect to the basis $\{v_1, ..., v_n\}$; indeed :

$$\begin{aligned}
(1-p)(1-q)(1-p)v_i \;&=\; (1-p)(1-q)v_i \\
&=\; (1-p)\left(\sum_{j=1}^{n} \langle w_j|v_i\rangle\, w_j\right) \\
&=\; \sum_{j,k=1}^{n} \langle w_j|v_i\rangle\,\langle v_k|w_j\rangle\, v_k \\
&=\; \sum_{j,k=1}^{n} (b^*)_{j,i} b_{k,j} v_k \\
&=\; \sum_{k=1}^{n} c_{k,i} v_k
\end{aligned}$$

for all $i \in \{1, ..., n\}$. If $\lambda_1, ..., \lambda_n \in [0, 1]$ denote the eigenvalues of c, one has

$$\|\phi_p - \phi_q\|^2 \;\leq\; 4\big(1 - det(c)\big) \;=\; 4\left(1 \;-\; \prod_{j=1}^{n} \lambda_j\right).$$

Because of Claim (ii) of the previous lemma, it suffices to show that

$$(*) \qquad 1 \;-\; \prod_{j=1}^{n} \lambda_j \;\leq\; \sum_{j=1}^{n} (1-\lambda_j).$$

Assume inductively that $1 - \prod_{j=1}^{k} \lambda_j \;\leq\; \sum_{j=1}^{k}(1-\lambda_j)$ for some $k \in \{1, ..., n-1\}$ (this is tautological for $k = 1$). As $\lambda_{k+1} \leq 1$ one has $\lambda_{k+1} - \prod_{j=1}^{k+1} \lambda_j \;\leq\; \sum_{j=1}^{k}(1-\lambda_j)$. Adding $1 - \lambda_{k+1}$ on both sides, one obtains

$$1 \;-\; \prod_{j=1}^{k+1} \lambda_j \;\leq\; \sum_{j=1}^{k+1} (1-\lambda_j).$$

This shows $(*)$, and ends the proof. $\square$

8.17. Lemma. *Theorem 8.14 holds if $dim(\mathcal{H}) = \infty$.*

Proof. If $\|p-q\|_2 = \infty$, there is nothing to show. We assume from now on that $\|p-q\|_2 < \infty$. In particular $(p-q)^2$ is a positive compact operator on $\mathcal{H}$. Observe that $Ker\left((p-q)^2\right) = Ker(p-q)$. By the spectral theorem for compact self-adjoint operators, there exists an orthogonal decomposition

$$\mathcal{H} \;=\; \left(\bigoplus_{i\in I} W_i\right) \bigoplus Ker(p-q)$$

and a family $(\mu_i)_{i\in I}$ of strictly positive numbers, pairwise distinct, such that

$$\begin{array}{ll} (a) & \text{each } W_i \text{ is a finite dimensional subspace of } \;\; \mathcal{H} \\ (b) & (p-q)^2 = \mu_i \quad \text{on} \quad W_i \end{array}$$

(I may be a finite set or an infinite set). Moreover, as $(p-q)^2$ commutes with p and q,

$$(c)\;\; Ker(p-q) \;\text{ and each }\; W_i \text{ is invariant by } p \text{ and by } q.$$

We claim that there exists a nested sequence $V_1 \subset V_2 \subset \ldots \subset V_n \subset V_{n+1} \subset \ldots$ of subspaces of $\mathcal{H}$ such that

$$\begin{array}{ll} (d) & \text{each } \;V_n\; \text{ is finite dimensional, and } \;\; \bigcup_{n\geq 1} V_n \;\; \text{ is dense in } \;\; \mathcal{H} \\ (e) & \text{each } \;V_n\; \text{ is invariant by } p \text{ and by } q\;. \end{array}$$

To check this, consider firstly the case where $dim_{\mathbb{C}}\left(Ker(p-q)\right) < \infty$, so that I is infinite, say $I = \{1, 2, \ldots\}$. It suffices to set

$$V_n \;=\; \left(\bigoplus_{i=1}^{n} W_i\right) \bigoplus Ker(p-q)\;.$$

In the general case, observe that

$$Ker(p-q) \;=\; \left(Ker(p-q) \cap Ker(p)\right) \bigoplus \left(Ker(p-q) \cap Im(p)\right)$$

(one may of course write $Ker(q)$ for $Ker(p)$ and $Im(q)$ for $Im(p)$). Let $(\xi_j)_{j\in J}$ be an orthonormal basis of $Ker(p-q)\cap Ker(p)$ and let $(\eta_k)_{k\in K}$ be a basis of $Ker(p-q)\cap Im(p)$. If all of I, J, K are infinite sets, say $\{1, 2, \ldots\}$, set

$$V_n \;=\; \left(\bigoplus_{i=1}^{n} W_i\right) \bigoplus \left(\bigoplus_{j=1}^{n} \mathbb{C}\xi_j\right) \bigoplus \left(\bigoplus_{k=1}^{n} \mathbb{C}\eta_k\right)\;.$$

If some of I, J, K are finite sets, proceed similarly with truncated sums for n large.

Let $p_n \in \mathcal{B}(V_n)$ denotes the restriction of p to V_n; one has

$$\phi_p | CAR(V_n) \;=\; \phi_{p_n}$$

because these two states on $CAR(V_n)$ are given by the same formula (see Theorem 8.2). As $\bigcup_{n\geq 1} CAR(V_n)$ is dense in $CAR(\mathcal{H})$, one has

$$\|\phi_p - \phi_q\| \;=\; \lim_{n\to\infty} \|\phi_{p_n} - \phi_{q_n}\| \,.$$

One has also

$$\|p-q\|_2 \;=\; \lim_{n\to\infty} \|p_n - q_n\|_2 \;.$$

Finally one has $\|\phi_{p_n} - \phi_{q_n}\| \leq 2\,\|p_n - q_n\|_2$ for all $n \geq 1$ by the previous lemma, so that

$$\|\phi_p - \phi_q\| \;\leq\; 2\,\|p-q\|_2$$

as was to be proved. □

8.18. Remarks. It follows from Theorem 8.14 and 6.26 that, if $\|p-q\|_2 \;<\; 1$, then ϕ_p and ϕ_q are equivalent. But see the stronger statement of Theorem 8.24 below.

7.8.bis. Lemma. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be three Hilbert spaces. For $m \geq 1, n \geq 1$, let $L : \mathcal{H}_1^{\otimes m} \otimes \mathcal{H}_2^{\otimes n} \to \mathcal{H}_3$ be a bounded operator such that $Lu_\sigma = (-1)^\sigma L$ for all $\sigma \in \mathcal{S}_m$ and $Lu_\tau = (-1)^\tau L$ for all $\tau \in \mathcal{S}_n$. Then*

$$L\big((\xi_1 \wedge ... \wedge \xi_m) \otimes (\eta_1 \wedge ... \wedge \eta_n)\big) \;=\; \sqrt{m!n!}\; L\big(\xi_1 \otimes ... \otimes \xi_m \otimes \eta_1 \otimes ... \otimes \eta_n\big)$$

for all $\xi_1, ..., \xi_m \in \mathcal{H}_1$ and $\eta_1, ..., \eta_n \in \mathcal{H}_2$.

Proof. Let $P_m : \mathcal{H}_1^{\otimes m} \to \bigwedge^m \mathcal{H}_1$ and $P_n : \mathcal{H}_2^{\otimes n} \to \bigwedge^n \mathcal{H}_2$ be defined as in 7.5. One has

$$\begin{aligned}
&L\Big((\xi_1 \wedge ... \wedge \xi_m) \otimes (\eta_1 \wedge ... \wedge \eta_n)\Big) \\
&\qquad = \sqrt{m!n!}\; L\big(P_m(\xi_1 \otimes ... \otimes \xi_m) \otimes P_n(\eta_1 \otimes ... \otimes \eta_n)\big) \\
&\qquad = \frac{1}{\sqrt{m!n!}} L\left(\sum_{\substack{\sigma\in\mathcal{S}_m\\ \tau\in\mathcal{S}_n}} (-1)^\sigma u_\sigma(\xi_1 \otimes ... \otimes \xi_m) \otimes (-1)^\tau u_\tau(\eta_1 \otimes ... \otimes \eta_n)\right) \\
&\qquad = \frac{1}{\sqrt{m!n!}} L\left(\sum_{\substack{\sigma\in\mathcal{S}_m\\ \tau\in\mathcal{S}_n}} \xi_1 \otimes ... \otimes \xi_m \otimes \eta_1 \otimes ... \otimes \eta_n\right) \\
&\qquad = \sqrt{m!n!} L\left(\xi_1 \otimes ... \otimes \xi_m \otimes \eta_1 \otimes ... \otimes \eta_n\right).
\end{aligned}$$

□

8.19. Lemma. *Let V be a closed subspace of $\mathcal{H}$. The isomorphism*

$$V \oplus V^{\perp} \xrightarrow{\approx} \mathcal{H}$$

induces an isometric isomorphism of Hilbert spaces

$$\mathcal{F}(V) \otimes \mathcal{F}(V^{\perp}) \xrightarrow{\approx} \mathcal{F}(\mathcal{H}).$$

Proof. Given two integers $m \geq 0$ and $n \geq 0$, one has a linear map

$$\mathcal{T} : \begin{cases} (\otimes^m V) \otimes (\otimes^n V^{\perp}) & \longrightarrow \quad \bigwedge^{m+n} \mathcal{H} \\ (\xi_1 \otimes \dots \otimes \xi_m) \otimes (\eta_1 \otimes \dots \otimes \eta_n) & \longmapsto \sqrt{\dfrac{(m+n)!}{m!n!}} P_{m+n} (\xi_1 \otimes \dots \otimes \xi_m \otimes \eta_1 \otimes \dots \otimes \eta_n) . \end{cases}$$

By the previous lemma, $\mathcal{T}$ restricts to a linear map

$$\begin{cases} (\wedge^m V) \otimes (\wedge^n V^{\perp}) & \longrightarrow \quad \bigwedge^{m+n} \mathcal{H} \\ (\xi_1 \wedge \dots \wedge \xi_m) \otimes (\eta_1 \wedge \dots \wedge \eta_n) & \longmapsto \quad \xi_1 \wedge \dots \wedge \xi_m \wedge \eta_1 \wedge \dots \wedge \eta_n . \end{cases}$$

The sum of the latter maps on $m \geq 0$ and $n \geq 0$ provide the isomorphism $\mathcal{F}(V) \otimes \mathcal{F}(V^{\perp}) \approx \mathcal{F}(\mathcal{H})$. □

8.20. Remark. One has obvious inclusions

$$i_V \ : \ CAR(V) \hookrightarrow CAR(\mathcal{H}) \qquad \text{and} \qquad i_{V^{\perp}} \ : \ CAR(V^{\perp}) \hookrightarrow CAR(\mathcal{H}),$$

of which the first has already appeared in Lemma 7.17. These could be used to define an isomorphism from the appropriate *graded* tensor product of $CAR(V)$ and $CAR(V^{\perp})$ onto $CAR(\mathcal{H})$. As we have not introduced any graded tensor product, we have to make use of a unitary operator analogous to the u_l 's of the proof of Lemma 7.17.

8.21. Lemma. *Let V be a finite dimensional Hilbert space. There exists a unitary element $u \in CAR(V)$ such that $u^2 = 1$ and*

$$ua(v)u \ = \ a(-v)$$

for all $v \in V$.

Proof. Let $(v_1, \dots, v_n)$ be an orthonormal basis of V and set

$$u \ = \ \prod_{j=1}^{n} \Big(1 - 2a(v_j)a(v_j)^*\Big).$$

The proof of Lemma 7.17 shows that u has the required properties.

(Here is another argument to show the existence of u. Let α be the Bogoliubov automorphism of $CAR(\mathcal{H})$ defined by $\alpha(a(v)) = a(-v)$ for all $v \in V$; as $CAR(V)$ is a matrix alegbra, α is inner, so that there exists a unitary $u \in CAL(V)$ with the desired properties.)
□

8.22. Proposition. *Let V be a finite dimensional subspace of a Hilbert space $\mathcal{H}$, let $b \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator such that $0 \leq b \leq 1$ and $b(V) \subset V$, as in Theorem 8.2, and let $u \in CAR(V)$ be a self-adjoint unitary element, as in Lemma 8.21. Then theere exists a unique morphism of C*-algebras*

$$\psi \ : \ CAR(V) \otimes CAR(V^\perp) \ \longrightarrow \ CAR(\mathcal{H})$$

such that

$$\begin{aligned} a(\xi) \otimes 1 \ &\longmapsto \ a(\xi) \\ 1 \otimes a(\eta) \ &\longmapsto \ ua(\eta) \end{aligned}$$

for all $\xi \in V$ and $\eta \in V^\perp$. Moreover ψ is an isomorphism and one has

$$\phi_b \circ \psi \ = \ \phi_{(b|V)} \otimes \phi_{(b|V^\perp)}.$$

Proof. The inclusion $V \subset \mathcal{H}$ provides an inclusion

$$\psi_1 \ : \ CAR(V) \longrightarrow CAR(\mathcal{H})$$

such that $\psi_1(a(\xi)) = a(\xi)$ for all $\xi \in V$.

For all $\eta \in V^\perp$, one has $ua(\eta) = a(\eta)u$. Consequently, for $\eta_1, \eta_2 \in V^\perp$, one has

$$\Big(ua(\eta_1)\Big)^* ua(\eta_2) + ua(\eta_2)\Big(ua(\eta_1)\Big)^* \ = \ a(\eta_1)^* a(\eta_2) + a(\eta_2)a(\eta_1)^* \ = \ \langle \eta_1 | \eta_2 \rangle$$

and similarly

$$ua(\eta_1)ua(\eta_2) + ua(\eta_2)ua(\eta_1) \ = \ 0.$$

It follows from Theorem 7.21 that there exists a morphism

$$\psi_2 \ : \ Car(V^\perp) \longrightarrow CAR(\mathcal{H})$$

such that $\psi_2(a(\eta)) = ua(\eta)$ for all $\eta \in V^\perp$.

For $\xi \in V$ and $\eta \in V^\perp$, one has

$$\psi_1\Big(a(\xi)\Big)\psi_2\Big(a(\eta)\Big) \ = \ uua(\xi)ua(\eta) \ = \ ua(-\xi)a(\eta) \ = \ ua(\eta)a(\xi) \ = \ \psi_2\Big(a(\eta)\Big)\psi_1\Big(a(\xi)\Big).$$

Hence the images of ψ_1 and ψ_2 commute and the application

$$\psi \ : \ \begin{cases} CAR(V) \otimes CAR(V^\perp) \ \longrightarrow \ CAR(\mathcal{H}) \\ \qquad\qquad m \otimes x \ \longmapsto \ mx \end{cases}$$

is a morphism of C*-algebras.

The image of ψ contains $a(\xi)$ for all $\xi \in \mathcal{H}$, so that ψ is onto. As $CAR(V)$ and $CAR(V^\perp)$ are simple, so is their tensor product, and ψ is an injection. This proves the proposition. $\square$

8.23. Theorem. *Let $p, q \in \mathcal{B}(\mathcal{H})$ be two projections which differ by a Hilbert-Schmidt operator. Then the states ϕ_p and ϕ_q on $CAR(\mathcal{H})$ are equivalent.*

Proof. The proof of Lemma 8.17 shows that there exists a finite dimensional subspace V of $\mathcal{H}$ which is invariant by p and q, and such that

$$\left\|(p-q)|V^{\perp}\right\|_2 \;<\; 1.$$

One has also

$$\phi_p \;=\; \phi_{(p|V)} \otimes \phi_{(p|V^{\perp})} \qquad \text{and} \qquad \phi_q \;=\; \phi_{(q|V)} \otimes \phi_{(q|V^{\perp})}$$

by the previous proposition.

As $\phi_{(p|V)}$ and $\phi_{(q|V)}$ are two pure states on the matrix algebra $CAR(V)$, they are necessarily equivalent. As $\left\|(p-q)|V^{\perp}\right\|_2 \;<\; 1$, the states $\phi_{(p|V^{\perp})}$ and $\phi_{(q|V^{\perp})}$ are equivalent by Remark 8.18.i. It follows that ϕ_p and ϕ_q are equivalent. □

8.24. Remarks. Let $b, c \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators such that $0 \leq b, c \leq 1$ (not necessarily projections) and let ϕ_b, ϕ_c be as in Theorem 8.2. One may show more generally that ϕ_b and ϕ_c are equivalent if and only if both $b^{\frac{1}{2}} - c^{\frac{1}{2}}$ and $(1-b)^{\frac{1}{2}} - (1-c)^{\frac{1}{2}}$ are Hilbert-Schmidt operators. (this is the main theorem in [PoW]).

8.c. The projective representation of the restricted unitary group

8.25. Lemma. *Let A be a C^*-algebra with unit, let α be an automorphism of A, and let ϕ be a state on A. Let π_ϕ and $\pi_{\phi\alpha}$ be the GNS-representations defined by the states ϕ and $\phi \circ \alpha$. Then there exists a unitary operator $u : \mathcal{H}_\phi \to \mathcal{H}_{\phi\alpha}$ such that*

$$u\pi_\phi\big(\alpha(a)\big) \;=\; \pi_{\phi\alpha}(a)u$$

for all $a \in A$.

Proof. We use the notations of Theorem 6.11. The application

$$\begin{cases} A & \longrightarrow & A \\ a & \longmapsto & \alpha^{-1}(a) \end{cases}$$

maps V_ϕ isomorphically onto $V_{\phi\alpha}$. It induces a linear map $A/V_\phi \to A/V_{\phi\alpha}$ which is unitary because

$$\langle b + V_\phi | a + V_\phi \rangle_\phi \;=\; \phi(b^*a) \;=\; \phi\alpha\left(\alpha^{-1}(b^*)\alpha^{-1}(a)\right) \;=\; \left\langle \alpha^{-1}(b) + V_{\phi\alpha} | \alpha^{-1}(a) + V_{\phi\alpha} \right\rangle_{\phi\alpha}$$

for all $a, b \in A$, and thus an isometric isomorphism $u : \mathcal{H}_\phi \to \mathcal{H}_{\phi\alpha}$ such that $u\pi_\phi(a) = \pi_{\phi\alpha}\left(\alpha^{-1}(a)\right) u$ for all $a \in A$. □

8.26 Lemma. *Let A be a C^*-algebra with unit, let α be an automorphism of A, and let ϕ be a state on A such that ϕ and $\phi \circ \alpha$ are equivalent. Let π_ϕ and $\pi_{\phi\alpha}$ be the GNS-representations defined by the states ϕ and $\phi \circ \alpha$. Then there exists a unitary operator v on $\mathcal{H}_\phi^{GNS}$ such that*

$$v\pi_\phi\big(\alpha(a)\big)v^* \;=\; \pi_\phi(a)$$

for all $a \in A$.

If $\tilde{v}$ is another unitary operator such that $\tilde{v}\pi_\phi\big(\alpha(a)\big)\tilde{v}^* \;=\; \pi_\phi(a)$ for all $a \in A$, then there exists a complex number z of modulus 1 such that $\tilde{v} = zv$.

Proof. By the previous lemma, there exists a unitary operator $u : \mathcal{H}_\phi \to \mathcal{H}_{\phi\alpha}$ such that

$$u\pi_\phi\big(\alpha(a)\big) \;=\; \pi_{\phi\alpha}(a)u$$

for all $a \in A$. By the hypothesis of equivalence, there exists a unitary operator $u' : \mathcal{H}_{\phi\alpha} \to \mathcal{H}_\phi$ such that

$$u'\pi_{\phi\alpha} \;=\; \pi_\phi(a)u'$$

for all $a \in A$. It is enough to set $v = u'u$.

The last statement is a consequence of Schur's lemma. $\square$

8.27. Definition. Let $p \in \mathcal{B}(\mathcal{H})$ be a projection. We define the *restricted unitary group*

$$U_{res,p}(\mathcal{H}) \;=\; \{\ u \in \mathcal{U}(\mathcal{H}) \mid \|pu - up\|_2 < \infty\ \}$$

of the unitary group $\mathcal{U}(\mathcal{H})$. It is a proper subgroup in case $dim_{\mathbb{C}}(p(\mathcal{H}))$ and $dim_{\mathbb{C}}((1-p)(\mathcal{H}))$ are both infinite.

Denote by $\mathcal{B}(\mathcal{H})_s$ the space of all operators on $\mathcal{H}$ with the strong topology, and by $HS(\mathcal{H})$ the space of all Hilbert-Schmidt operators on $\mathcal{H}$ with the topology of the Hilbert-Schmidt norm. The natural topology on $U_{res,p}(\mathcal{H})$ for what follows is that induced by the inclusion

$$\begin{cases} U_{res,p}(\mathcal{H}) & \longrightarrow & \mathcal{B}(\mathcal{H})_s \oplus HS(\mathcal{H}) \\ u & \longmapsto & \Big(pup + (1-p)u(1-p)\ \ ,\ \ pu - up\Big). \end{cases}$$

It makes it a topological group.

8.28. Proposition. *Let $p \in \mathcal{B}(\mathcal{H})$ be a projection and let $u \in \mathcal{U}_{res,p}(\mathcal{H})$. Then there exists a unitary operator $\tilde{\Gamma}_p(u)$ on $\Gamma_p(\mathcal{H})$ such that*

$$\tilde{\Gamma}_p(u)a(\xi)\tilde{\Gamma}_p(u)^* \;=\; a(u(\xi))$$

for all $\xi \in \mathcal{H}$.

Proof. For $u \in \mathcal{U}(\mathcal{H})$, let α_u denote the automorphism of $CAR(\mathcal{H})$ defined by

$$\alpha_u(a(\xi)) \;=\; a(u(\xi))$$

for all $\xi \in \mathcal{H}$. If ϕ_p is the quasi-free state of covariance p, one has

$$\Big(\phi_p \circ \alpha_u\Big)(a^*(\xi)a(\eta)) \; = \; \phi_p(a^*(u(\xi))a(u(\eta))) \; = \; \langle u\xi | pu\eta\rangle \; = \; \phi_{u^*pu}(a^*(\xi)a(\eta))$$

for all $\xi, \eta \in \mathcal{H}$, so that

$$\phi_p \circ \alpha_u \; = \; \phi_{u^*pu}.$$

If moreover $u \in \mathcal{U}_{res,p}(\mathcal{H})$, one has

$$\|u^*pu - p\|_2 \; = \; \|pu - up\|_2 \; < \; \infty$$

and $\phi_p \circ \alpha_u, \phi_p$ are equivalent.

The proposition follows from the previous lemma. □

8.29. What next. Observe that $\tilde{\Gamma}_p(u)$ is well defined up to a complex number of modulus one. Thus one has a well defined elememt $\Gamma_p(u) \in P\mathcal{U}\big(\Gamma_p(\mathcal{H})\big)$ and it is straightforward to check that the resulting map

$$\begin{cases} \mathcal{U}_{res,p}(\mathcal{H}) & \longrightarrow & P\mathcal{U}\big(\Gamma_p(\mathcal{H})\big) \\ u & \longmapsto & \Gamma_p(u) \end{cases}$$

is a **projective representation** of the **restricted unitary group** (more on this in Chapter 9).

It can be shown to be a *continuous* projective representation.

In case $\mathcal{H}$ is a Hilbert space of the form

$$L^2\Big(\mathbb{S}^1, \mathbb{C}^n\Big)$$

and p is the projection on the corresponding Hardy space, loop groups of the form

$$\mathcal{C}^\infty\Big(\mathbb{S}^1, SU(n)\Big)$$

are naturally subgroups of $\mathcal{U}_{res,p}(\mathcal{H})$, and one obtains by restriction projective representations of these loop groups.

For more on this, the canonical reference is the book by A. Priestley and G. Segal [PrS].

P. de la Harpe and V. Jones, July 1995.

CHAPTER 9. UNITARY PROJECTIVE REPRESENTATIONS OF GROUPS.

9.A. Generalities on unitary projective representations

9.1. The projective unitary group of a Hilbert space Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{U}(\mathcal{H})$ denote the unitary group of $\mathcal{H}$. We denote by $\mathbb{T}$ the group of complex numbers of modulus 1 and we identify it with the subgroup of unitary homotheties of $\mathcal{H}$. We denote by

$$P\mathcal{U}(\mathcal{H}) \; = \; \mathcal{U}(\mathcal{H})/\mathbb{T}$$

the quotient group and by

$$p : \mathcal{U}(\mathcal{H}) \to P\mathcal{U}(\mathcal{H})$$

the canonical projection.

We consider $\mathcal{U}(\mathcal{H})$ endowed with the strong topology of operators (or equivalently with the weak topology, see Proposition 2.16.iv) and $P\mathcal{U}(\mathcal{H})$ with the quotient topology. In case $\mathcal{H}$ is of finite dimension n, the group $P\mathcal{U}(\mathcal{H})$ is the Lie group $U(n)/\mathbb{T} \approx SU(n)/C_n$ (where C_n denotes here the group of n^{th} roots of unity).

9.2. A digression on symmetry operations and Wigner's theorem. Let $P\mathcal{H}$ denote the projective space of lines in $\mathcal{H}$. There is an application

$$t \; : \; \begin{cases} P\mathcal{H} \times P\mathcal{H} & \longrightarrow \; [0,1] \\ \qquad (\underline{\xi}, \underline{\eta}) & \longmapsto \; |\langle \xi | \eta \rangle|^2 \end{cases}$$

where $\xi, \eta \in \mathcal{H}$ are unit vectors which represent the lines $\underline{\xi}, \underline{\eta}$ respectively. An *automorphism* of $P\mathcal{H}$ is a bijection $\alpha : P\mathcal{H} \to P\mathcal{H}$ such that

$$t(\alpha\underline{\xi}, \alpha\underline{\eta}) \; = \; t(\underline{\xi}, \underline{\eta})$$

for all $\underline{\xi}, \underline{\eta} \in P\mathcal{H}$. Every unitary operator u on $\mathcal{H}$ provides obviously an automorphism of $P\mathcal{H}$, which depends only on the class of u in $P\mathcal{U}(\mathcal{H})$. In the formalism of quantum mechanics, it is a fundamental result of E. Wigner ([Wig], [Bar]) that $P\mathcal{U}(\mathcal{H})$ is a subgroup of index 2 in the group of all automorphisms of $P\mathcal{H}$ (the other automorphisms are classes modulo $\mathbb{T}$ of anti-unitary operators on $\mathcal{H}$).

9.3. Definition. Let G be a locally compact group and let $\mathcal{H}$ be a Hilbert space. A *unitary projective representation* of G on $\mathcal{H}$ is a continuous group homomorphism

$$\pi \; : \; G \longrightarrow P\mathcal{U}(\mathcal{H}).$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Such a π *admits a lifting* if there exists a continuous unitary representation $\tilde{\pi} : G \to \mathcal{U}(\mathcal{H})$ such that $\pi = p \circ \tilde{\pi}$.

9.4. Examples. (i) Any continuous unitary representation $\tilde{\pi} : G \to \mathcal{U}(\mathcal{H})$ provides a continuous unitary projective representation $\pi : G \to P\mathcal{U}(\mathcal{H})$ by composition with the canonical projection $p : \mathcal{U}(\mathcal{H}) \to P\mathcal{U}(\mathcal{H})$.

(ii) The unit 2-by-2 matrix and the *Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

provide a unitary projective representation of the Klein group $\mathbb{V} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^2$. Observe that

$$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z \qquad \sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x \qquad \sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y$$

so that, in particular, this $\pi : \mathbb{V} \to PU(2)$ *does not* admit any lifting.

(iii) Any unitary representation $\pi : SU(2) \to U(N)$ provides a projective representation $\underline{\pi}$ of the rotation group $SO(3)$ such that the diagram

$$\begin{array}{ccc} SU(2) & \xrightarrow{\pi} & U(N) \\ \downarrow & & \downarrow \\ SO(3) & \xrightarrow{\underline{\pi}} & PU(N) \end{array}$$

commutes. In case π is "the" irreducible representation of $SU(2)$ of dimension N (which is well defined up to equivalence), it is known that $\underline{\pi}$ admits a lifting if and only if $\pi\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the identity on $\mathbb{C}^N$, namely if and only if N is odd.

9.5. Cocycles and coboundaries. Let G be a group. A $\mathbb{T}$*-valued 2-cocycle* on G is a map

$$\gamma \ : \ G \times G \longrightarrow \mathbb{T}$$

such that

$$\gamma(g,h)\gamma(gh,k) \ = \ \gamma(g,hk)\gamma(h,k)$$

for all $g, h, k \in G$. The set of all $\mathbb{T}$-valued 2-cocycles on G is denoted by

$$Z^2(G, \mathbb{T})$$

and is an abelian group, for the multiplication defined by $\gamma_1\gamma_2 : (g,h) \mapsto \gamma_1(g,h)\gamma_2(g,h)$.

For any map $\mu : G \to \mathbb{T}$, the map

$$\delta\mu \ : \ \begin{cases} G \times G \longrightarrow & \mathbb{T} \\ (g,h) \longmapsto & \mu(g)\mu(h)\mu(gh)^{-1} \end{cases}$$

is a 2-cocycle. Cocycles of this form are called $\mathbb{T}$-*valued 2-coboundaries* on G and the set of all these is a subgroup

$$B^2(G, \mathbb{T})$$

of $Z^2(G, \mathbb{T})$.

The *second cohomology group* of G in $\mathbb{T}$ is the quotient group

$$H^2(G, \mathbb{T}) \;=\; Z^2(G, \mathbb{T})/B^2(G, \mathbb{T}).$$

9.6. The cohomology class of a unitary projective representation. Let G be a group and let $\pi : G \to P\mathcal{U}(\mathcal{H})$ be a unitary projective representation of G on a Hilbert space $\mathcal{H}$.

For each $g \in G$, choose $u_g \in \mathcal{U}(\mathcal{H})$ such that $\pi(g) = p(u_g)$. For each pair $(g, h) \in G \times G$, let $\gamma(g, h)$ denote the number in $\mathbb{T}$ such that

$$u_g u_h \;=\; \gamma(g, h) u_{gh}.$$

Using associativity in $\mathcal{U}(\mathcal{H})$, one computes for all $g, h, k \in G$

$$\begin{aligned} (u_g u_h)\, u_k \;&=\; \gamma(g, h) u_{gh} u_k \;=\; \gamma(g, h)\gamma(gh, k) u_{ghk} \\ = u_g\,(u_h u_k) \;&=\; u_g \gamma(h, k) u_{hk} \;=\; \gamma(g, hk)\gamma(h, k) u_{ghk} \end{aligned}$$

so that $(g, h) \mapsto \gamma(g, h)$ is a cocycle in $Z^2(G, \mathbb{T})$.

Let $g \mapsto u'_g$ denote another choice of representatives, such that $\pi(g) = p(u'_g)$. For each $g \in G$, let $\mu(g)$ denote the number in $\mathbb{T}$ such that $u'_g = \mu(g) u_g$. Define as before $\gamma' : G \times G \to \mathbb{T}$ by $u'_g u'_h = \gamma'(g, h) u'_{gh}$. Then

$$\gamma'(g, h) \;=\; \mu(g)\mu(h)\mu(gh)^{-1}\gamma(g, h)$$

so that γ and γ' define the same class $[\gamma] \in H^2(G, \mathbb{T})$. This class depends only on π.

9.7. Proposition. *Let G be a group, let $\pi : G \to P\mathcal{U}(\mathcal{H})$ be a unitary projective representation of G on a Hilbert space $\mathcal{H}$ and let $[\gamma] \in H^2(G, \mathbb{T})$ be the associated cohomology class. Then π admits a lifting $\tilde{\pi} : G \to \mathcal{U}(\mathcal{H})$ if and only if $[\gamma] = 1$.*

Proof. See e.g. [Kir]. □

9.8. Remarks. (i) The previous proposition carries over to topological groups and continuous representations, but there are subtle points about various classes of coycles (Borel-measurable, continuous, ...). See [Mac] and quote perhaps C.C. Moore, Trans. AMS **113** (1964) 40-63 and 64-86, and **221** (19??) 1-33 and 35-58.

(ii) It can be checked that the cohomology class of Example 4.ii is not 1, and that the cohomology class of Example 4.iii is 1 if and only if N is odd.

9.B. Some projective representations of symplectic groups

The purpose of this Section is to expose a family of standard examples of projective representations, involving finite symplectic groups.

9.9. Symplectic spaces. Let $\mathbb{F}$ be a field and let V be a finite dimensional vector space over $\mathbb{F}$. A *symplectic form* on V is a bilinear form

$$\omega \ : \ V \times V \longrightarrow \mathbb{F}$$

which is non degenerate (i.e. if $v \in V$ is such that either $\omega(v, V) = 0$ or $\omega(V, v) = 0$, then $v = 0$) and alternating (i.e. $\omega(v, v) = 0$ for all $v \in V$).

Let ω be a symplectic form on V. It is a classical result that the dimension of V is even, say $dim_{\mathbb{F}}(V) = 2n$, and that there exists a *symplectic basis*, namely a basis $\{\epsilon_1, ..., \epsilon_{2n}\}$ such that the matrix $\Big((\omega(\epsilon_j, \epsilon_k)_{1 \leq j,k \leq 2n}\Big)$ has the form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I [respectively 0] denote the n-by-n unit matrix [resp. null matrix]. See e.g. [Art, Theorem 3.7] or [MiH, § I.3].

9.10. Symplectic groups. Given a symplectic form ω on a space V as above, the corresponding *symplectic group* is the group

$$Sp(V, \omega) \ = \ \Big\{ \ g \in GL(V) \ \Big| \ \omega(gv, gw) = \omega(v, w) \ \text{ for all } \ v, w \in V \ \Big\}.$$

The result quoted above on symplectic bases shows that this group depends only on the field $\mathbb{F}$ and the dimension $2n$ of V, and it is denoted by

$$Sp(2n, \mathbb{F}).$$

We collect now some classical facts about these groups.

(i) The centre of $Sp(2n, \mathbb{F})$ is reduced to $\{\pm id_V\}$. It is of order 2 if the characteristic of $\mathbb{F}$ is not 2, and of order 1 if this characteristic is 2 [Art, Theorem 3.26].

The quotient of $Sp(2n, \mathbb{F})$ by its center is the *projective symplectic group* denoted by $PSp(2n, \mathbb{F})$. In the ATLAS, one writes also $S_{2n}(q)$ for $PSp(2n, \mathbb{F}_q)$.

(ii) If $\mathbb{F}_q$ denotes the finite field with q elements,

$$|Sp(2n, \mathbb{F}_q)| \ = \ q^{n^2} \prod_{j=1}^{n} \Big(q^{2j} - 1\Big)$$

[Art, § III.6]. In particular

$$|Sp(2, \mathbb{F}_2)| = 6 \qquad |Sp(2, \mathbb{F}_3)| = 24 \qquad |Sp(4, \mathbb{F}_2)| = 720.$$

It can be shown that

$PSp(2, \mathbb{F}_2)$	is isomorphic to the symmetric group	$\mathcal{S}_3$,
$PSp(2, \mathbb{F}_3)$	is isomorphic to the alternating group	$\mathcal{A}_4$,
$PSp(4, \mathbb{F}_2)$	is isomorphic to the symmetric group	$\mathcal{S}_6$,
$PSp(4, \mathbb{F}_3)$	is the group of order $25'920 = 2^6 3^4 5$ which turns up as the group of the 27 straight lines of a cubic surface,	
$PSp(6, \mathbb{F}_2)$	is the group of order $1'451'520 = 4(9!)$ which turns up as the group of the 28 double tangents to a plane curve of degree four.	

(iii) With the exceptions of $Sp(2, \mathbb{F}_2)$, $Sp(2, \mathbb{F}_3)$ and $Sp(4, \mathbb{F}_2)$, the group $Sp(2n, \mathbb{F})$ does not contain any normal subgroup which is proper and not contained in its center [Art, Theorem 5.1].

It follows that the corresponding group $PSp(2n, \mathbb{F})$ is simple, with the three exceptions $PSp(2, \mathbb{F}_2)$, $PSp(2, \mathbb{F}_3)$ and $PSp(4, \mathbb{F}_2)$.

9.11. Remark. Let $\chi : \mathbb{F} \to \mathbb{T}$ be an additive character of the field $\mathbb{F}$. Then the map

$$\begin{cases} V \times V & \longrightarrow & \mathbb{T} \\ (v, w) & \longmapsto & \chi\Big(\omega(v, w)\Big) \end{cases}$$

is a cocycle in $Z^2(V_{add}, \mathbb{T})$, where V_{add} denotes the abelian group underlying the vector space V.

9.12. Metapectic algebras. Let $\mathbb{F}$ be a finite field, let $\chi : \mathbb{F} \to \mathbb{T}$ be an additive character of $\mathbb{F}$ and let (V, ω) be a finite dimensional symplectic space of dimension $2n$ over $\mathbb{F}$. The corresponding *metaplectic algebra* $Met_\chi(2n, \mathbb{F})$ is the *complex* involutive algebra with basis $(e_v)_{v \in V}$, with the multiplication defined by

$$e_v e_w = \chi(\omega(v, w)) e_{v+w}$$

for all $v, w \in V$ and with the involution defined by

$$(e_v)^* = e_{-v}$$

for all $v \in V$. It is straightforward to check that these make $Met_\chi(2n, \mathbb{F})$ an associative algebra with involution, with unit e_0; its complex dimension is q^{2n}, where q is the cardinality of $\mathbb{F}$. For each $v \in V$, the basis element e_v is invertible with inverse e_{-v}; moreover one has

$$e_v e_w (e_v)^{-1} = \chi(2\omega(v, w)) e_w$$

for all $v, w \in V$.

9.13. A representation of the metaplectic algebra. For each $v \in V$, let u_v be the operator on the q^{2n}-dimensional Hilbert space $\ell^2(V)$ defined by

$$(u_v \xi)(x) \; = \; \chi(\omega(v,x))\xi(v-x)$$

for all $\xi \in \ell^2(V)$ and $x \in V$. It is straightforward to check that u_v is unitary for each $v \in V$ and that

$$\begin{aligned} u_v u_w &= \chi(\omega(v,w))u_{v+w} \\ (u_v)^* &= u_{-v} \end{aligned}$$

for all $v, w \in V$. Consequently the map $e_v \mapsto u_v$ defines a $*$-representation

$$\pi \; : \; Met_\chi(2n, \mathbb{F}) \;\longrightarrow\; \mathcal{B}(\ell^2(V))$$

of the metaplectic algebra.

This representation is faithful. Indeed, let $(c_v)_{v \in V}$ be a family of complex numbers such that $\pi\left(\sum_{v\in V} c_v e_v\right) = 0$. Then $\left(\pi\left(\sum_{v\in V} c_v e_v\right)\xi\right)(0) = \sum_{v\in V} c_v \chi(\omega(v,x))\xi(-v) = 0$. This implies $c_v = 0$ for all $v \in V$.

At this point, we know that $Met_\chi(2n, \mathbb{F})$ is a C*-algebra of dimension q^{2n}. Let us assume moreover that

(a) $\mathbb{F}$ is the prime field $\mathbb{F}_p$ of characteristic p,

(b) $p \neq 2$,

(c) χ is not the unit character of $\mathbb{F}_p$.

It follows that $\chi^2 : \mathbb{F}_p \to \mathbb{T}$ is faithful. We claim now that the center of the metaplectic algebra is reduced to the scalar multiples of the identity, so that the C*-algebra $Met_\chi(2n, \mathbb{F}_p)$ is isomorphic to the full matrix algebra $M_{p^n}(\mathbb{C})$. Let $x = \sum_{v\in V} c_v e_v \in Met_\chi(2n, \mathbb{F}_p)$ be a central element. Then

$$u_w \pi(x) \; = \; \sum_{v\in V} c_v \chi(\omega(w,v)) u_{v+w} \; = \; \sum_{v \in V} c_v \chi(\omega(v,w)) u_{v+w} \; = \; \pi(x) u_w$$

for all $w \in V$, namely

$$c_v \chi(2\omega(w,v)) \; = \; c_v$$

for all $v, w \in V$. If $c_v \neq 0$ then $\omega(V, v) = 0$ by hypothesis (a) to (c), so that $v = 0$ by the non-degeneracy of ω. This proves the claim.

We can sum up these results as follows.

9.14. Proposition. *Let $\mathbb{F}_p$ be the prime field of characteristic $p \neq 2$ and let $\chi : \mathbb{F}_p \to \mathbb{T}$ be an additive character distinct from the unit character. Then one has*

$$Met_\chi(2n, \mathbb{F}_p) \; \approx \; M_{p^n}(\mathbb{C})$$

(isomorphism of C^*-algebras).

9.15. Exercise. The hypothesis being as in the previous proposition, let L be a Lagrangian subspace of V, namely a n-dimensional subspace of V on which the restriction of ω is identically zero. Show that the linear span of $(e_l)_{l\in L}$ is a maximal self-adjoint algebra in $Met_\chi(2n,\mathbb{F})$.

9.16. Exercise. Let L be a Lagrangian subspace of $V \approx (\mathbb{F}_p)^{2n}$. Check that

$$f_L \;=\; p^{-n}\sum_{l\in L} e_l$$

is an *idempotent* of $Met_\chi(2n,\mathbb{F}_p)$.

Consider the linear map $\tau : Met_\chi(2n,\mathbb{F}_p) \to \mathbb{C}$ defined by $\tau(e_0)=1$ and $\tau(e_v)=0$ if $v\in V, v\neq 0$. Check that τ is the normalised trace on $Met_\chi(2n,\mathbb{F}_p)\approx M_{p^n}(\mathbb{C})$. Compute $\tau(f_L)$ and deduce that f_L is a *minimal* idempotent of $Met_\chi(2n,\mathbb{F}_p)$.

More generally, for any additive character $\lambda : L\to\mathbb{T}$, check that

$$f_L \;=\; p^{-n}\sum_{l\in L} \lambda(l)e_l$$

is a minimal idempotent of $Met_\chi(2n,\mathbb{F}_p)$.

9.17. Exercise. Show that there exists a basis $s_1,...,s_{2n}$ of V such that

$$\omega(s_j,s_k) \;=\; \begin{cases} 1 & \text{if} \quad k=j+1 \\ -1 & \text{if} \quad k=j-1 \\ 0 & \text{if} \quad |k-j|\neq 1. \end{cases}$$

for $j,k\in\{1,...,2n\}$. Set $\eta=\chi(1)$, which is a primitive p^{th} root of unity, and set $u_j=e_{s_j}$ for $1\le j\le 2n$. Check that

$$\begin{aligned} u_j^p &= 1 \qquad \text{for all} j\in\{1,...,2n\}, \\ u_ju_{j+1} &= \eta^2 u_{j+1}u_j \qquad \text{if} \qquad j\in\{1,...,2n-1\}, \\ u_ju_k &= u_ju_k \qquad \text{if} \qquad |j-k|\ge 2. \end{aligned}$$

Define then

$$e_j \;=\; \frac{1}{p}\sum_{k=0}^{p-1}(u_j)^k$$

and check that one has

$$\begin{aligned} e_j^2 &= e_j \qquad (1\le j\le 2n), \\ e_je_{j+1}e_j \;-\; \frac{1}{p}e_j &= e_{j+1}e_je_{j+1} \;-\; \frac{1}{p}e_{j+1} \;=\; 0 \qquad (1\le j\le 2n-1), \\ e_je_k &= e_ke_i \qquad (1\le j,k\le 2n\;,\;|k-j|\ge 2). \end{aligned}$$

Define also

$$\sigma_j' \;=\; \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \alpha^{r^2} u_j^r$$

and check that one has

$$\begin{aligned} \sigma_j' \quad & \text{is unitary} \qquad (1 \le j \le 2n), \\ \sigma_j'\sigma_{j+1}'\sigma_j' \;&=\; \sigma_{j+1}'\sigma_j'\sigma_{j+1}' \qquad (1 \le j \le 2n-1), \\ \sigma_j'\sigma_k' \;&=\; \sigma_k'\sigma_j' \qquad (1 \le j,k \le 2n \;,\; |k-j| \ge 2). \end{aligned}$$

If B_{2n+1} denotes the group of braids on $2n+1$ strings with the usual Artin's generators $\sigma_1, ..., \sigma_{2n}$, show that

$$\sigma_j \;\longmapsto\; \left(\frac{p}{2} - 1 + \frac{\sqrt{p^2-4p}}{2}\right) e_j \;-\; (1-e_j)$$

defines a representation of B_{2n+1}, which is unitary if and only if $p = 3$. (See [Jon].)

Similarly

$$\sigma_j \;\longmapsto\; \sigma_j'$$

defines a unitary representation of B_{2n+1} for all odd prime p. (See [GoJ].)

9.18. The metaplectic projective representation of $Sp(2n, \mathbb{F}_p)$. There is an obvious action of the symplectic group by $*$-automorphisms of the metaplectic algebra

$$\begin{cases} Sp(2n,\mathbb{F}) \longrightarrow Aut\big(Met_\chi(2n,\mathbb{F})\big) \\ \qquad g \qquad \longmapsto \qquad \alpha_g \end{cases}$$

given by $\alpha_g(e_v) = e_{gv}$ for all $g \in Sp(2n,\mathbb{F})$ and $v \in V$. With the assumptions (a) to (c) of 9.13, the algebra $Met_\chi(2n,\mathbb{F}_p)$ is a full matrix algebra by Proposition 9.14; thus all its $*$-automorphisms are inner, of the form $x \mapsto uxu^*$ for some unitary element $u \in Met_\chi(2n,\mathbb{F}_p)$; we write abusively $u \in \mathcal{U}(p^{2n})$. Observe that u is not uniquely defined by the automorphism, but that is class $\underline{u} \in P\mathcal{U}(p^{2n})$ is well defined. Thus, one has a projective representation

$$\begin{cases} Sp(2n,\mathbb{F}_p) & \longrightarrow & P\mathcal{U}(p^{2n}) \\ \qquad g & \longmapsto & \underline{u}_g \end{cases}$$

such that $\alpha_g(x) = u_g x u_g^*$ for all $g \in Sp(2n,\mathbb{F}_p)$ and $x \in Met_\chi(2n,\mathbb{F}_p)$, where $u_g \in \mathcal{U}(p^{2n})$ is some representant of $\underline{u}_g$.

9.19. Proposition. *(i) The above projective representation lifts to a representation*

$$Sp(2n, \mathbb{F}) \to \mathcal{U}(p^{2n}).$$

(ii) The latter representation is a direct sum of p^n equivalent representations

$$\pi_j : Sp(2n, \mathbb{F}_p) \to \mathcal{U}(p^n).$$

(iii) Each π_j splits as the direct sum of two irreducible representations, of dimensions $\frac{p^n+1}{2}$ and $\frac{p^n-1}{2}$.

Proof. Find a good reference !!! or give the argument ???

9.20. Comment. Let $H_L \subset Met_\chi(2n, \mathbb{F}_p)$ be a minimal left-ideal of the form

$$\Big(Met_\chi(2n, \mathbb{F}_p)\Big) f_L,$$

where f_L is as in Exercise 9.16. One has the following correspondence with the material of Chapter 8 (and of further chapters).

$$\begin{array}{rcl}
Met_\chi(2n, \mathbb{F}_p) & \leftrightsquigarrow & CAR(\mathcal{H}) \\
Sp(2n, \mathbb{F}_p) & \leftrightsquigarrow & U_{res}(\mathcal{H}) \\
H_L & \leftrightsquigarrow & \Gamma_p(\mathcal{H}) \\
Sp(2n, \mathbb{F}_p) \ \text{ acts on } \ Met_\chi(2n, \mathbb{F}_p) & \leftrightsquigarrow & U_{res}(\mathcal{H}) \ \text{ acts on } \ CAR(\mathcal{H}) \\
Met_\chi(2n, \mathbb{F}_p) \ \text{ irreducible on } \ H_L & \leftrightsquigarrow & CAR(\mathcal{H}) \ \text{ irreducible on } \ \Gamma_p(\mathcal{H}) \\
Sp(2n, \mathbb{F}_p) \ \text{ projective on } \ H_L & \leftrightsquigarrow & U_{res}(\mathcal{H}) \ \text{ projective on } \ \Gamma_p(\mathcal{H}) \\
Sp(2n, \mathbb{F}_p) \ \text{ lifts} & !!!!!!! & U_{res}(\mathcal{H}) \ \text{ does not lift .}
\end{array}$$