A short introduction to topology of real algebraic varieties

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What shapes can be formed by the zeroes in $\mathbb{R}^n$ of a polynomial of degree d in n variables and with real coefficients ?

This is roughly what topology of real algebraic varieties is about. One classically prefers to work with compact hypersurfaces inside the compactification $\mathbb{RP}^n$ of $\mathbb{R}^n$, which raises the following definition for the objects we are interested in.

Definition 0.1. A real algebraic hypersurface A of degree m in $\mathbb{RP}^n$ is given by the datum of a homogeneous degree m polynomial P_A in $n+1$ variables with real coefficients up to multiplication by a non-zero real constant.
The hypersurface A has the real part $\mathbb{R}A := \{[x_0 : ... : x_n] \in \mathbb{RP}^n | P_A([x_0 : ... : x_n]) = 0\}$ and the complex part $\mathbb{C}A := \{[z_0 : ... : z_n] \in \mathbb{CP}^n | P_A([z_0 : ... : z_n]) = 0\}$.
The complex conjugation on $\mathbb{CP}^n$ induces an involution on $\mathbb{C}A$, whose fixed point set is $\mathbb{R}A$. If there does not exist any point $z \in \mathbb{CP}^n$ such that $P_A(z) = \frac{\partial P_A}{\partial z_0}(z) = ... = \frac{\partial P_A}{\partial z_n}(z) = 0$, then the hypersurface A is said to be non-singular. In this case, $\mathbb{C}A$ and $\mathbb{R}A$ are smooth manifolds.

The essential questions about a real algebraic hypersurface of a given degree d and dimension $n-1$ are the following ones.

- What could $\mathbb{R}A$ look like?
- How is $\mathbb{R}A$ situated inside $\mathbb{RP}^n$?
- How do the answers to the preceeding questions interact with one another?

To answer these questions, there are two main directions:

- finding restrictions on the topology of the considered objects;
- constructing real algebraic hypersurfaces with topological properties that are interesting in regard of the known restrictions.

1 Early restrictions

The essential questions about real algebraic hypersurfaces and the two main directions to answer those questions were already stated in the second half of the 19-th century in the case of curves in the fundamental works of A. Harnack and D. Hilbert. Harnack studied the topology of the real parts of the curves themselves (*i. e.* their number of connected components) while Hilbert focused on the possible mutual positions for the connected components of the real parts of the curves in $\mathbb{RP}^2$. Let us first establish some definitions.

Remark 1.1. A circle can be smoothly embedded in $\mathbb{RP}^2$ in two ways. Either the image of the embedding divides the real projective plane in two halves, one of which is homeomorphic to a disk, or it does not divide the real projective plane. If it bounds a disk, we call the image of the embedding an *oval*. The bounded disk is said to be *internal* to the oval. Otherwise, the image of the embedding is called a *pseudoline*. The real part of a smooth real algebraic curve C contains exactly one pseudoline if the degree of C is odd. If the degree is even, $\mathbb{R}C$ contains only ovals.

Definition 1.1. A *nest of depth* k is a union of N disjoint ovals O_1, ..., O_N in $\mathbb{RP}^2$ such that

- O_2, ..., O_N lie inside the disk internal to O_1;
- there exist $k-1$ integers i_2,...,i_k in $\{2, ..., N\}$ such that $\forall j \in \{2, ..., k-1\}$, $O_{i_{j+1}}$ lies in the disk internal to O_{i_j}.

We are now ready to state and prove Harnack's and Hilbert's main results regarding real algebraic curves.

Theorem 1.1. (Harnack's theorem) The real part of an irreducible non-singular real algebraic plane curve of degree d has at most $\frac{(d-1)(d-2)}{2}+1$ connected components. Furthermore, this bound is sharp for any degree d.

Definition 1.2. The curves of degree d whose real part has exactly $\frac{(d-1)(d-2)}{2}+1$ connected components are said to be *maximal*.

The two essential tools used by Harnack to prove this bound are the following classical results from algebraic/enumerative geometry.

Theorem 1.2. (Bézout) The number of intersection points (counted with multiplicity) of the complex parts of two (real) algebraic curves of respective degrees d_1 and d_2 is equal to $d_1 d_2$.

Corollary 1.1. (Real Bézout) Let C_1 and C_2 be two real algebraic curves of respective degree d_1 and d_2 in $\mathbb{RP}^2$. Then the number of intersection points (counted with multiplicity) of $\mathbb{R}C_1$ and $\mathbb{R}C_2$ is less than or equal to $d_1 d_2$.

Theorem 1.3. (Cramer) Through $\frac{d(d+3)}{2}$ points in generic position in $\mathbb{CP}^2$, there passes a unique curve of degree d.

Proof. **(Harnack's inequality)** Assume that $d > 2$. Suppose that C is an irreducible non-singular real algebraic plane curve of degree d such that $\mathbb{R}C$ has more than $\frac{(d-1)(d-2)}{2}+1$ connected components. Then $\mathbb{R}C$ has at least $\frac{(d-1)(d-2)}{2}+1$ ovals $\mathcal{O}_1$,...,$\mathcal{O}_p$ and one other connected component. We choose $\frac{(d-2)(d+1)}{2}$ points on $\mathbb{R}P^2$; one on each oval $\mathcal{O}_i$ and all the others on a different connected component. Through these points there passes a degree $(d-2)$ curve D. The curves C and D have no irreducible component in common. Up to some transversality hypotheses (or up to counting the points in $C \bigcap D$ with multiplicities), D intersects each oval $\mathcal{O}_i$ in a at least 2 points and we have $\#(C \bigcap D) \geq \frac{(d-2)(d+1)}{2} + \frac{(d-1)(d-2)}{2} + 1 = (d-1)^2$. But by Bézout's theorem, $\#(C \bigcap D) \leq d(d-2)$. We reach a contradiction. $\square$

Bézout's theorem also give some information about how the connected components of the real part of a curve of given degree can be positionned in $\mathbb{RP}^2$.

Theorem 1.4. (Hilbert's theorem on nests) The maximal depth for a nest in the real part of a non-singular real algebraic curve of degree d is $\lfloor \frac{d}{2} \rfloor$. Likewise, the sum of the depths of two disjoint nests of the real part of a non-singular real algebraic curve of degree d is $\lfloor \frac{d}{2} \rfloor$.

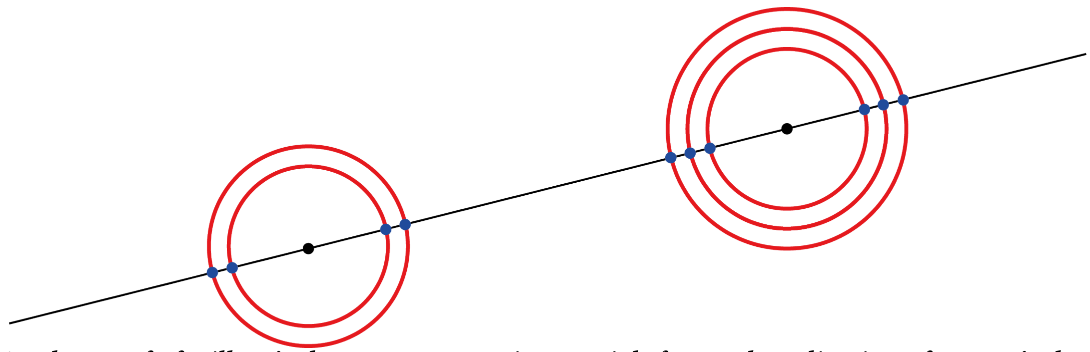

Fig. 1: The proof of Hilbert's theorem on nests is a straightforward application of Bezout's theorem. Pick a point inside two disjoint nests and trace a line through these two points. How many times does the line intersect the curve at least ?

Hilbert's interest in topology of real algebraic plane curves led him to formulate the following question.

First part of Hilbert's sixteenth problem: What are the possible arrangements in $\mathbb{RP}^2$ for the 11 connected components of the real part of a maximal curve of degree 6?

For decades, the existence of the central arrangement was unknown. It was even thought to be impossible, until it was realized in 1974 by Gudkov.

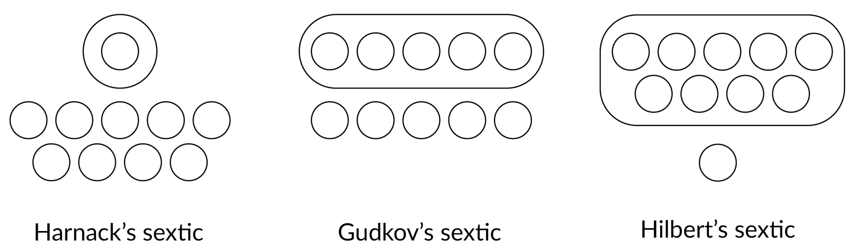

Fig. 2: The three possible arrangements for the 11 connected components of a real plane sextic.

2 Early constructions

The second part of Harnack's work consisted in a construction of maximal curves of arbitrary degree. At the time, the only available construction tool was the method of *small perturbation* of curves with isolated doubles points.

Let P, Q be homogeneous polynomials with real coefficients in three variables of respective degree d_P, d_Q. Suppose that the real algebraic curves C_P and C_Q defined by P and Q are non-singular and intersect transversely in a finite number of real ordinary double points. Choose a polynomial R of degree $d_P + d_Q$ such that the curve C_R defined by R intersects $C_P \bigcup C_Q$ transversaly and away from the double points of $C_P \bigcup C_Q$. For a sufficiently small ϵ, the real part of the curve defined by $PQ + \epsilon R$ is non-singular, lies in a small tubular neighbourhood of $\mathbb{R}C_P \bigcup \mathbb{R}C_Q$ and its topology can be easly understood from the way $\mathbb{R}C_P$, $\mathbb{R}C_Q$ and $\mathbb{R}C_R$ are positionned with respect to one another.

Below is a representation of the first few steps of Harnack's recursive construction of maximal real algebraic plane curves. The curve to perturb is the union of a red curve (generally obtained by the previous step of the construction) and a line (in blue). The perturbator curve appears in green and is the union a several vertical lines.

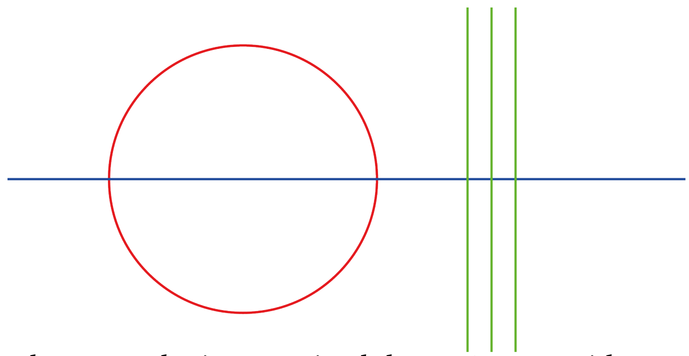

Fig. 3: The necessary datum to obtain a maximal degree 3 curve with Harnack's construction: a conic (in red), a line (in blue), and the perturbator curve in green.

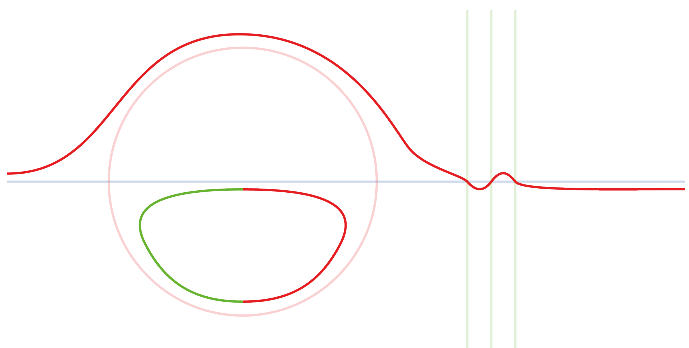

Fig. 4: The Harnack cubic obtained by a small perturbation using the datum in Fig. 3.

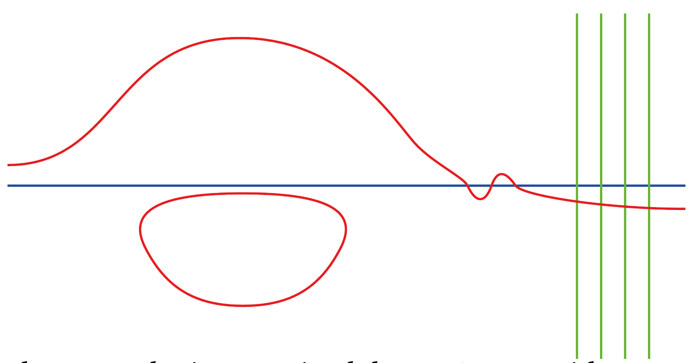

Fig. 5: The necessary datum to obtain a maximal degree 4 curve with Harnack's construction: the Harnack cubic (in red) constructed at the previous step, a line (in blue, it is the same line as in the previous step, which allows to control the number and position of its intersection points with the Harnack cubic), and the perturbator curve in green.

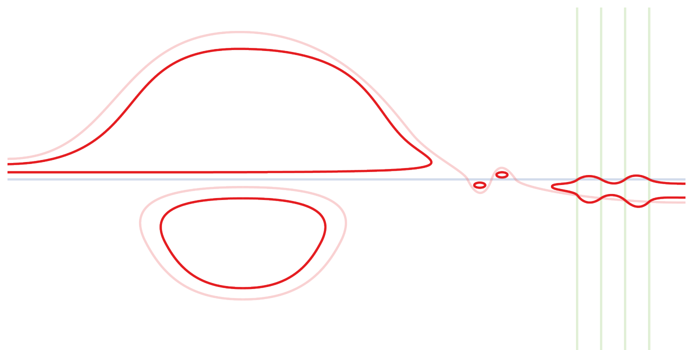

Fig. 6: The Harnack quartic. By perturbing its union with the same line as before, one can obtain a maximal curve of degree 5, then of degree 6, etc...

3 Modern restriction tools

Already in the case of surfaces, Bézout's theorem fails to provide an analogue of Harnack's inequality. The more recent restrictions, proven in the second half of the 20-th century, exploit diverse techniques from algebraic topology together with the fact that the real part of a real algebraic hypersurface is naturally the fixed point set of an antiholomorphic involution of the complex part.

Theorem 3.1. (Smith-Thom inequality and congruence) Let $conj$ be a continuous involution on a simplicial complex Z of dimension $2n$ and let Z^{conj} be the fixed points set of $conj$. Then

$$\sum_{i=0}^{n} dim_{\mathbb{Z}_2} H_i(Z^{conj};\mathbb{Z}_2) \leq \sum_{i=0}^{2n} dim_{\mathbb{Z}_2} H_i(Z;\mathbb{Z}_2).$$

Furthermore, $\sum_{i=0}^{n} dim_{\mathbb{Z}_2} H_i(Z^{conj};\mathbb{Z}_2) \equiv \sum_{i=0}^{2n} dim_{\mathbb{Z}_2} H_i(Z;\mathbb{Z}_2)$ (mod 2).

Theorem 3.1 can be applied to the complex part of a real algebraic variety A endowed with the natural involution induced by the complex conjugation. We obtain

$$\sum_{i=0}^{n} dim_{\mathbb{Z}_2} H_i(\mathbb{R}A;\mathbb{Z}_2) \leq \sum_{i=0}^{2n} dim_{\mathbb{Z}_2} H_i(\mathbb{C}A;\mathbb{Z}_2);$$
$$\sum_{i=0}^{n} dim_{\mathbb{Z}_2} H_i(\mathbb{R}A;\mathbb{Z}_2) \equiv \sum_{i=0}^{2n} dim_{\mathbb{Z}_2} H_i(\mathbb{C}A;\mathbb{Z}_2) \text{ (mod 2)}.$$

Definition 3.1. The real algebraic variety A is said to be an $(M-d)$-*variety* if $\sum_{i=0}^{n} dim_{\mathbb{Z}_2} H_i(\mathbb{R}A;\mathbb{Z}_2) = \sum_{i=0}^{2n} dim_{\mathbb{Z}_2} H_i(\mathbb{C}A;\mathbb{Z}_2) - 2d$. When $d=0$, A is said to be an M-*variety*, or to be *maximal*.

Maximal varieties have exceptional topological properties, the main example being Rokhlin's congruence.

Theorem 3.2. (Rokhlin's congruence) Let A be a non-singular maximal real algebraic variety of even dimension. Then $\chi(\mathbb{R}A) \equiv \sigma(\mathbb{C}A)$ mod 16, where $\chi(\mathbb{R}A)$ is the Euler characteristic of $\mathbb{R}A$ and $\sigma(\mathbb{C}A)$ is the signature of $\mathbb{C}A$.

Theorem 3.3. (Petrovskii-Oleinik-Kharlamov inequality) Let $\mathbb{C}A$ be a compact connected Kähler manifold of even complex dimension $2n$. Let $conj$ be an antiholomorphic involution on $\mathbb{C}A$ and let $\mathbb{R}A$ be the fixed point set of $conj$. Then, one has

$$|\chi(\mathbb{R}A) - 1| \leq h^{n,n}(\mathbb{C}A) - 1.$$

These results can be applied directly to real algebraic hypersurfaces to obtain restrictions on the topology of the real part, but they can also be used to understand how the real part of the hypersurface is positioned inside its ambient space. The trick is to consider a double covering of the complex part of the ambiant space ramified along the complex part of the hypersurface.

Definition 3.2. Let A be a non-singular real algebraic hypersurface of even degree d in $\mathbb{P}^n$. A real ramified double covering B of $\mathbb{P}^n$ associated to A is a double covering $\mathbb{C}B$ of $\mathbb{C}\mathbb{P}^n$ ramified along $\mathbb{C}A$ together with an antiholomorphic involution that lifts the complex conjugation on $\mathbb{C}\mathbb{P}^n$ to $\mathbb{C}B$.

Remark 3.1. The complex variety $\mathbb{C}B$ exists and is unique up to isomorphisms of ramified double coverings.

There are two different ways of lifting the complex conjugation of $\mathbb{C}\mathbb{P}^n$ to $\mathbb{C}B$. Hence, for every hypersurface A we get two different real algebraic varieties B^+ and B^- with two different real parts $\mathbb{R}B^+$ and $\mathbb{R}B^-$. Each of these real parts lies over one of the two halves cut out in $\mathbb{R}\mathbb{P}^n$ by $\mathbb{R}A$. So applying the usual restrictions to B^+ and B^-, we obtain some information on the topology of $\mathbb{R}\mathbb{P}^n \setminus \mathbb{R}A$.

4 A modern construction tool : combinatorial patchworking

Combinatorial patchworking is a construction method introduced in in the 1980s by O. Viro. It has proven to be very fruitful and to provide a bridge between algebraic, tropical and combinatorial geometry. Here is, roughly, how it works.

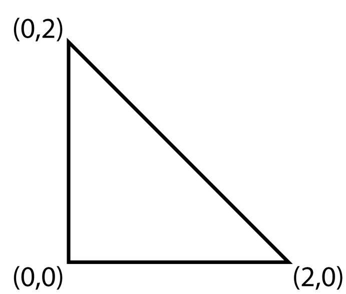

Fig. 7: Choose a convex polytope Δ in $\mathbb{R}^n_{\geq 0}$ with integer vertices. It is the Newton polytope of the to-be-constructed hypersurface. In this example, we will construct a conic in the plane.

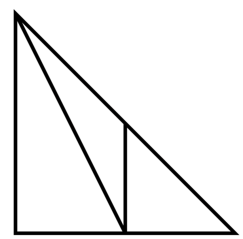

Fig. 8: Choose a convex triangulation of Δ with integer vertices.

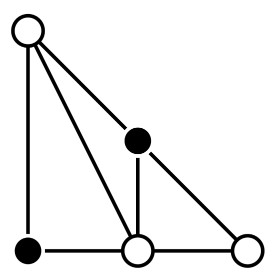

Fig. 9: Assign a sign (+ or −) to each vertex of the triangulation.

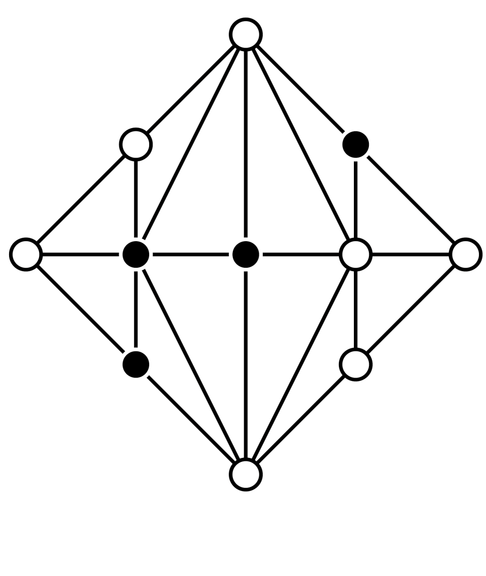

Fig. 10: Extend the triangulation to the copies of Δ obtained by taking symmetries with respect to coordinate hyperplanes. Extend the sign distribution as well using the following rule: two vertices that are image of each other under a symmetry with respect to a coordinate hyperplane bear the same sign if their distance to the hyperplane is even, and opposite signs otherwise. Notice that the signs associated to a copy of a certain point behave like the sign of the corresponding monomial. For example, compare the sign of the monomial x^1y^1 in the four quadrants of $\mathbb{R}^2$ and the signs associated to the copies of the integer point $(1,1)$ in the figure.

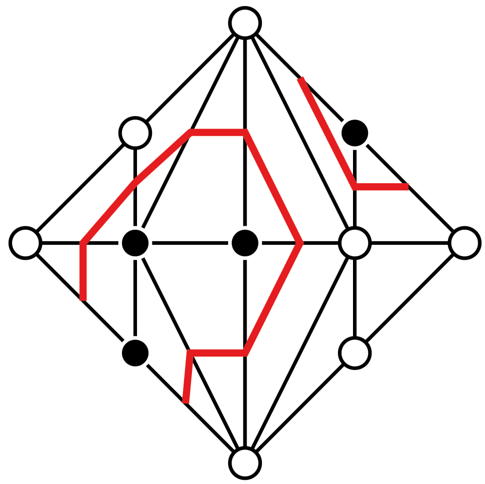

Fig. 11: Separate the vertices bearing the sign $+$ from the vertices bearing the sign $-$ using a piecewise linear hypersurface.

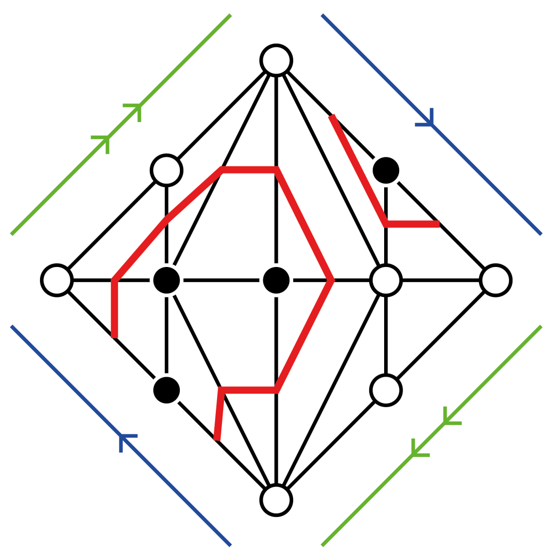

Fig. 12: Glue the faces of the symmetric copies of the faces of the polytope Δ in a prescribed way, as in the classical construction of the real part of the toric variety associated to Δ from copies of Δ.

At the end of the process, one obtains a piecewise linear hypersurface $\tilde{\Gamma}$ inside a piecewise linear space $\tilde{\Delta}$ obtained by gluing multiple copies of Δ.

Theorem 4.1. (Viro's combinatorial patchworking theorem) There exists a polynomial P with Newton polytope Δ that defines a Newton-non-degenerate real algebraic hypersurface A_P inside the toric variety X_Δ associated to Δ such that the pair $(\mathbb{R}X_\Delta, \mathbb{R}A_P)$ is homeomorphic to the pair $(\tilde{\Delta}, \tilde{\Gamma})$.

Remark 4.1. This is a simplified version of the theorem. Viro's work allows one to recover an explicit expression of the polynomial P.

Remark 4.2. In this statement, Newton-non-degenerate roughly means "as smooth as possible", in the sense that A_P is smooth if X_Δ is.

Combinatorial patchworking has proven to be a very powerful tool. For example, it allowed I. Itenberg and Viro to prove the following result.

Theorem 4.2. (Itenberg-Viro) For any integers n and d, there exists a maximal smooth real algebraic hypersurface of degree d in $\mathbb{P}^n$.

But the hypersurfaces obtained by combinatorial patchworking have very specific topological properties. For example, if one restricts to primitive triangulations (*i.e.* triangulations where each top-dimensional simplex has the smallest possible euclidean volume), then the following result holds.

Theorem 4.3. (B. Bertrand) A smooth real algebraic hypersuface A obtained by combinatorial patchworking starting with a primitive triangulation of a polytope Δ satisfies $\chi(\mathbb{R}A) = \sigma(\mathbb{C}A)$.

Remark 4.3. This theorem is to compare with Rokhlin's congruence. It is much stronger!

My current goal at the Section de Mathématiques de l'Université de Genève is to try to better understand what can or cannot be constructed using combinatorial patchworking.

5 What to read if you want to know more ?

- Viro's professional webpage has a very nice and accessible section about topology of real algebraic curves.
- A good place to start if you want to learn some details about the modern tools used to study real algebraic varieties is G. Wilson's survey *Hilbert's sixteenth problem*.
- If you are interested in combinatorial patchworking, maybe reading Itenberg's paper *Counter-examples to Ragsdale conjecture and T-curves* is a good idea.