# POLYNOMIAL SYSTEMS WITH FEW REAL ZEROES

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Abstract. We study some systems of polynomials whose support lies in the convex hull of a circuit, giving a sharp upper bound for their numbers of real solutions. This upper bound is non-trivial in that it is smaller than either the Kouchnirenko or the Khovanskii bounds for these systems. When the support is exactly a circuit whose affine span is $\mathbb{Z}^n$, this bound is $2n+1$, while the Khovanskii bound is exponential in $n^2$. The bound $2n+1$ can be attained only for non-degenerate circuits. Our methods involve a mixture of combinatorics, geometry, and arithmetic.

## Introduction

The notion of degree for a multivariate Laurent polynomial $f$ is captured by its support, which is the set of exponent vectors of monomials in $f$. For example, Kouchnirenko [6] generalized the classical Bézout Theorem, showing that the number of solutions in the complex torus $(\mathbb{C}^*)^n$ to a generic system of $n$ polynomials in $n$ variables with common support $\mathcal{A}$ is the volume $v(\mathcal{A})$ of the convex hull of $\mathcal{A}$, normalized so that the unit cube $[0,1]^n$ has volume $n!$. This Kouchnirenko number is also a (trivial) bound on the number of real solutions to such a system of real polynomials. This bound is reached, for example, when the common support $\mathcal{A}$ admits a regular unimodular triangulation [8].

Khovanskii [5] gave a bound on the number of real solutions to a polynomial system that depends only upon the cardinality $|\mathcal{A}|$ of its support $\mathcal{A}$:

$$\text{Number of real solutions} \ \leq\ 2^n 2^{\binom{|\mathcal{A}|}{2}} \cdot (n+1)^{|\mathcal{A}|}\,.$$

This enormous fewnomial bound is non-trivial (smaller than the Kouchnirenko bound) only when the cardinality of $\mathcal{A}$ is very small when compared to the volume of its convex hull. It is widely believed that significantly smaller bounds should hold. For example, Li, Rojas, and Wang [7] showed that two trinomials in 2 variables have at most 20 common solutions, which is much less than the Khovanskii bound of 995,328. While significantly lower bounds are expected, we know of no reasonable conjectures about the nature of hypothetical lower bounds.

There are few other examples of non-trivial polynomial systems for which it is known that not all solutions can be real via a bound smaller than the Khovanskii bound. We describe a class of supports and prove an upper bound (which is often sharp) for the

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number of real solutions to a polynomial system with those supports that is non-trivial in that it is smaller than either the Kouchnirenko or the Khovanskii bound.

A finite subset $\mathcal{A}$ of $\mathbb{Z}^n$ that affinely spans $\mathbb{Z}^n$ is *primitive*. A (possibly degenerate) *circuit* is a collection $\mathcal{C} := \{0, w_0, w_1, \ldots, w_n\} \subset \mathbb{Z}^n$ of $n+2$ integer vectors which spans $\mathbb{R}^n$. Here is the simplest version of our main results, which are proven in Sections 4 and 5.

**Theorem.** *A polynomial system with support a primitive circuit has at most $2n+1$ real solutions. There exist systems with support a primitive circuit having $2n+1$ non-degenerate real solutions.*

This sharp bound for circuits suggests that one may optimistically expect similar dramatic improvements in the doubly exponential Khovanski bound for other sets $\mathcal{A}$ of supports.

Adding the vectors $2w_0, 3w_0, \ldots, kw_0$ to a circuit $\mathcal{C}$ produces a *near circuit* if $\mathbb{R}w_0 \cap \mathcal{C} = \{0, w_0\}$. Let $\nu$ be the cardinality of $\mathcal{D} \cap \{w_1, \ldots, w_n\}$, where $\mathcal{D} \subset \mathcal{C}$ is a minimal affinely dependent subset. Let $\ell$ be the largest integer so that $w_0/\ell$ is integral. We show that a polynomial system with support a primitive near circuit has at most $k(2\nu-1)+2$ real solutions if $\ell$ is odd, or at most $2k\nu+1$ real solutions if $\ell$ is even, and these bounds are tight among all such near circuits. These bounds coincide when $k=1$, that is, for primitive circuits. This is proven in Theorem 5.6.

For a given near circuit, we use its geometry and arithmetic to give tighter upper bounds and construct systems with many real zeroes.

An important step is to determine an eliminant for the system, which has the form

$$(*) \qquad x^N \prod_{i=1}^{p} (g_i(x^\ell))^{\lambda_i} \; - \prod_{i=p+1}^{\nu} (g_i(x^\ell))^{\lambda_i} \; ,$$

where the polynomials $g_i$ all have degree $k$, and $N$ and $\lambda_i$ are the coefficients of the minimal linear dependence relation among $\mathcal{D} \cup \{w_0/\ell\}$. This reduces the problem to studying the possible numbers of real zeroes of such a univariate polynomial. We adapt a method of Khovanskii to establish an upper bound for the number of real zeroes of such a polynomial. Our upper bound uses a variant of the Viro construction, which also allows us to construct polynomials $(*)$ with many real zeroes.

In Section 1, we establish some basics on sparse polynomial systems, and then devote Section 2 to an example of a family of near circuits for which it is easy to establish sharp upper bounds, as simple linear algebra suffices for the elimination, Descartes's rule of signs gives the bounds, and the Viro construction establishes their sharpness. In Section 3, we compute an eliminant of the form $(*)$ for a system supported on a near circuit. Section 4 is devoted to proving an upper bound for the number of real solutions to a polynomial of the form $(*)$, while in Section 5 we construct such polynomials with many real zeroes, and describe some cases in which our bounds are sharp, including when $\mathcal{A}$ is a circuit.

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## 1. Basics on sparse polynomial systems

We emphasize that we look for solutions to polynomial systems which have only non-zero coordinates and thus lie in $(\mathbb{C}^*)^n$. Write $x^w$ for the monomial with exponent vector $w \in \mathbb{Z}^n$.

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set which does not lie in an affine hyperplane. A polynomial $f$ has *support* $\mathcal{A}$ if the exponent vectors of its monomials lie in $\mathcal{A}$. A polynomial system with support $\mathcal{A}$ is a system

$$f_1(x_1,\ldots,x_n) \;=\; f_2(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,, \tag{1.1}$$

where each polynomial $f_i$ has support $\mathcal{A}$. Multiplying a polynomial $f$ by a monomial $x^w$ does not change its set of zeroes in $(\mathbb{C}^*)^n$, but does translate its support by the vector $w$. Thus it is no loss to assume that $0 \in \mathcal{A}$. A system (1.1) is *generic* if its number of solutions in $(\mathbb{C}^*)^n$ equals $v(\mathcal{A})$, the volume of the convex hull of $\mathcal{A}$ normalized so that the unit cube $[0,1]^n$ has volume $n!$, which is the Kouchnirenko bound [6]. This condition forces each solution to be simple. We will always assume that our systems are generic in this sense.

Let $\mathbb{Z}\mathcal{A} \subset \mathbb{Z}^n$ be the full rank sublattice generated by the vectors in $\mathcal{A}$. (This is the affine span of $\mathcal{A}$ as $0 \in \mathcal{A}$.) If $\mathbb{Z}\mathcal{A} = \mathbb{Z}^n$, then $\mathcal{A}$ is *primitive*. The *index* of $\mathcal{A}$ is the index of $\mathbb{Z}\mathcal{A}$ in $\mathbb{Z}^n$. The fundamental theorem of abelian groups implies that

$$\frac{\mathbb{Z}^n}{\mathbb{Z}\mathcal{A}} \;\simeq\; \frac{\mathbb{Z}}{a_1\mathbb{Z}} \oplus \frac{\mathbb{Z}}{a_2\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{a_n\mathbb{Z}}\,,$$

where $a_i$ divides $a_{i+1}$, for $i = 1,\ldots,n-1$. These numbers $a_1,\ldots,a_n$ are the *invariant factors of* $\mathcal{A}$. When $\mathcal{A}$ is a simplex—so there are $n$ non-zero vectors in $\mathcal{A}$—then the numbers $a_i$ are the invariant factors of the matrix whose columns are these vectors. The index of $\mathcal{A}$ is the product of its invariant factors.

1.1. **Polynomial systems with support a simplex.** Let $e(\mathcal{A})$ be the number of even invariant factors of $\mathcal{A}$. The following result can be found in Section 3 of [8].

**Proposition 1.1.** *Suppose that $\mathcal{A}$ is the set of vertices of a simplex. Then the number of real solutions to a generic system with support $\mathcal{A}$ is*

(i) $0$ *or* $2^{e(\mathcal{A})}$ *if* $v(\mathcal{A})$ *is even.*
(ii) $1$ *if* $v(\mathcal{A})$ *is odd.*

*Proof.* Suppose that $0 \in \mathcal{A}$. Given a polynomial system (1.1) with support $\mathcal{A}$ whose coefficients are generic, we may perform Gaussian elimination on the matrix of its coefficients and convert it into a system of the form

$$x^{w_i} \;=\; \beta_i\,, \quad \text{for} \quad i = 1,\ldots,n\,, \tag{1.2}$$

where $\beta_i \neq 0$ and $w_1,\ldots,w_n$ are the non-zero elements of $\mathcal{A}$. Solutions to this system have the form $\varphi_{\mathcal{A}}^{-1}(\beta)$, where $\beta = (\beta_1,\ldots,\beta_n) \in (\mathbb{C}^*)^n$ and $\varphi_{\mathcal{A}}$ is the homomorphism

$$(\mathbb{C}^*)^n \;\ni\; (x_1,\ldots,x_n) \;\longmapsto\; (x^{w_1},\ldots,x^{w_n}) \;\in\; (\mathbb{C}^*)^n\,.$$

Real solutions to (1.2) are $\psi^{-1}(\beta)$, where $\beta \in (\mathbb{R}^*)^n$ and $\psi\colon (\mathbb{R}^*)^n \to (\mathbb{R}^*)^n$ is the restriction of $\varphi_{\mathcal{A}}$ to $(\mathbb{R}^*)^n$. The kernel of $\psi$ consists of those points $x \in \{\pm 1\}^n$ that satisfy

$$x^{w_i} \;=\; 1 \quad \text{for } i = 1,\ldots,n\,.$$

Let $A$ be a matrix whose columns are the non-zero elements of $\mathcal{A}$. Identifying $\{\pm 1\}$ with $\mathbb{Z}/2\mathbb{Z}$ identifies the kernel of $\psi$ with the kernel of the reduction of $A$ modulo 2, which has dimension $e(\mathcal{A})$. The result follows as $\psi$ is surjective if (and only if) $e(A) = 0$, which is equivalent to $v(\mathcal{A})$ being odd.

**Remark 1.2.** The upper bound of Proposition 1.1 for a system with support $\mathcal{A}$ is the volume of $\mathcal{A}$ if and only if no invariant factor of $\mathcal{A}$ exceeds 2.

1.2. **Generic systems.** The system (1.2) is generic (has $v(\mathcal{A})$ simple roots in $(\mathbb{C}^*)^n$) when the numbers $\beta_i$ are non-zero. We give a proof of the following elementary result, as we will use the proof later.

**Proposition 1.3.** *Suppose that $\mathcal{A}$ does not lie in an affine hyperplane. Then there is a non-empty Zariski open subset in the space of coefficients of monomials appearing in a system with support $\mathcal{A}$ such that the system has $v(\mathcal{A})$ simple solutions in $(\mathbb{C}^*)^n$.*

*Proof.* Suppose that $0 \in \mathcal{A}$. Then the affine span $\mathbb{Z}\mathcal{A}$ is a full rank sublattice of $\mathbb{Z}^n$. Let $a := |\mathcal{A}| - 1$ and consider the map $\varphi_{\mathcal{A}}$ defined by

$$\varphi_{\mathcal{A}} \,:\, (\mathbb{C}^*)^n \ni (x_1, \ldots, x_n) \;\longmapsto\; [x^w \mid w \in \mathcal{A}] \;\in\; \mathbb{P}^a\,. \tag{1.3}$$

Its image is a subgroup of the dense torus in $\mathbb{P}^a$, and it is a homomorphism of algebraic groups. Then a polynomial system (1.1) with support $\mathcal{A}$ is the pullback of $n$ linear forms (given by the coefficients of the $f_i$) along the map $\varphi_{\mathcal{A}}$. These $n$ linear forms determine a linear section of the closure $X_{\mathcal{A}}$ of the image of $\varphi_{\mathcal{A}}$, a (not necessarily normal) projective toric variety [9, §4,13]. Bertini's Theorem [4, p. 179] asserts that a general linear section of $X_{\mathcal{A}}$ consists of $\deg(X_{\mathcal{A}})$ simple points, all lying in the image of $\varphi_{\mathcal{A}}$. Each of these pull back along $\varphi_{\mathcal{A}}$ to $|\ker(\varphi_{\mathcal{A}})|$ distinct solutions to the original system.

The kernel of $\varphi_{\mathcal{A}}$ is the abelian group dual to the factor group $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$, which has order equal to the index of $\mathcal{A}$. Furthermore, the toric variety $X_{\mathcal{A}}$ has degree equal to the volume of the convex hull of $\mathcal{A}$, normalized so that a unit parallelepiped of $\mathbb{Z}\mathcal{A}$ has volume $n!$. The product of this volume with the index of $\mathcal{A}$ is the usual volume of $\mathcal{A}$. Thus a system with support $\mathcal{A}$ and generic coefficients has $v(\mathcal{A})$ simple solutions.

1.3. **Congruences on the number of real solutions.** The proof of Proposition 1.3 shows that the solutions to the system (1.1) are the fibres $\varphi_{\mathcal{A}}^{-1}(\beta)$ for the points $\beta$ in a linear section of $X_{\mathcal{A}}$. It follows that Proposition 1.1 gives some restrictions on the possible numbers of real solutions. The index of $\mathcal{A}$ factors as $2^{e(\mathcal{A})} \cdot N$.

**Corollary 1.4.** *The number of real solutions to (1.1) is at most $v(\mathcal{A})/N$ and is congruent to this number modulo $\max\{2, 2^{e(\mathcal{A})}\}$.*

*Proof.* Let $\varphi_{\mathcal{A}}$ be the map (1.3). The image of any real solution to (1.1) under $\varphi_{\mathcal{A}}$ is a real point, and there are $2^{e(\mathcal{A})}$ real solutions with the same image under $\varphi_{\mathcal{A}}$. Thus the maximum number of real solutions to (1.1) is $2^{e(\mathcal{A})} \cdot \deg X_{\mathcal{A}}$, which is $v(\mathcal{A})/N$. The congruence follows as the restriction of the map $\varphi_{\mathcal{A}}$ to the real subtorus is surjective on the real subtorus of $X_{\mathcal{A}}$ if and only if $e(\mathcal{A}) = 0$.

**Remark 1.5.** Any bound or construction for the number of real roots of polynomial systems associated to primitive vector configurations gives the same bounds and constructions for the number of real roots for configurations with odd index. Indeed, let $\mathcal{A} \subset \mathbb{Z}^n$ be finite and $\mathcal{B} \subset \mathbb{Z}^n$ be a basis for $\mathbb{Z}\mathcal{A}$ so that linear combinations of vectors in $\mathcal{B}$ identify $\mathbb{Z}\mathcal{A}$ with $\mathbb{Z}^n$. This identification maps $\mathcal{A}$ to a primitive vector configuration $\mathcal{A}'$, which has the same geometry and arithmetic as $\mathcal{A}$. The map $\varphi_{\mathcal{A}}$ (1.3) factors

$$(\mathbb{C}^*)^n \xrightarrow{\varphi_{\mathcal{B}\cup\{0\}}} (\mathbb{C}^*)^n \xrightarrow{\varphi_{\mathcal{A}'}} \mathbb{P}^a .$$

Since $\mathcal{A}$ and $\mathcal{B}\cup\{0\}$ have the same (odd) index, $\varphi_{\mathcal{B}\cup\{0\}}$ is bijective on $(\mathbb{R}^*)^n$. Thus $\varphi_{\mathcal{B}\cup\{0\}}$ gives a bijection between real solutions to systems with support the primitive vector configuration $\mathcal{A}'$ and real solutions to systems with support $\mathcal{A}$.

## 2. A family of systems with a sharp bound

We describe a family of supports $\Delta \subset \mathbb{Z}^n$ and prove a non-trivial sharp upper bound on the number of real solutions to polynomial systems with support $\Delta$. That is, the points in $\Delta$ affinely span $\mathbb{Z}^n$, but there are fewer than $v(\Delta)$ real solutions to polynomial systems with support $\Delta$. These sets $\Delta$ also have the property that they consist of all the integer points in their convex hull.

Let $l > k > 0$ and $n \geq 3$ be integers and $\epsilon = (\epsilon_1, \ldots, \epsilon_{n-1}) \in \{0,1\}^{n-1}$ be non-zero. Then $\Delta^{\epsilon}_{k,l} \subset \mathbb{R}^n$ consists of the points

$$(0,\ldots,0),\ (1,0,\ldots,0),\ldots,(0,\ldots,0,1,0),\ (0,\ldots,0,k),\ (\epsilon_1,\ldots,\epsilon_{n-1},l)\,.$$

together with the points along the last axis

$$(0,\ldots,0,1),\ (0,\ldots,0,2),\ \ldots,\ (0,\ldots,0,k{-}1)\,.$$

Since these include the standard basis and the origin, $\Delta^{\epsilon}_{k,l}$ is primitive.

Set $|\epsilon| := \sum_i \epsilon_i$. Then the volume of $\Delta^{\epsilon}_{k,l}$ is $l + k|\epsilon|$. Indeed, the configuration $\Delta^{\epsilon}_{k,l}$ can be triangulated into two simplices $\Delta^{\epsilon}_{k,l} \setminus \{(\epsilon_1,\ldots,\epsilon_{n-1},l)\}$ and $\Delta^{\epsilon}_{k,l} \setminus \{0\}$ with volumes $k$ and $l - k + k|\epsilon|$, respectively. One way to see this is to apply the affine transformation

$$(x_1,\ldots,x_n) \longmapsto (x_1,\ldots,x_{n-1},x_n - k + k\sum_{i=1}^{n-1} x_i)\,.$$

**Theorem 2.1.** *The number, $r$, of real solutions to a generic system of $n$ real polynomials with support $\Delta^{\epsilon}_{k,l}$ lies in the interval*

$$0 \;\leq\; r \;\leq\; k + k|\epsilon| + 2\,,$$

*and every number in this interval with the same parity as $l + k|\epsilon|$ occurs.*

This upper bound does not depend on $l$ and, since $k < l$, it is smaller than or equal to the number $l + k|\epsilon|$ of complex solutions. We use elimination to prove this result.

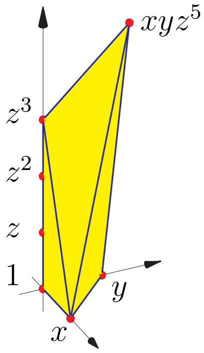
**Example 2.2.** Suppose that $n = k = 3$, $l = 5$, and $\epsilon = (1, 1)$.

Then the system

$$\begin{aligned} x + y + xyz^5 + 1 + z + z^2 + z^3 &= 0 \\ x + 2y + 3xyz^5 + 5 + 7z + 11z^2 + 13z^3 &= 0 \\ 2x + 2y + xyz^5 + 4 + 8z + 16z^2 + 32z^3 &= 0 \end{aligned}$$

is equivalent to

$$\begin{aligned} x - (5 + 11z + 23z^2 + 41z^3) &= 0 \\ y + (8 + 18z + 38z^2 + 72z^3) &= 0 \\ xyz^5 - (2 + 6z + 14z^2 + 30z^3) &= 0 \end{aligned}$$




And thus its number of real roots equals the number of real roots of

$$z^5(5 + 11z + 23z^2 + 41z^3)(8 + 18z + 38z^2 + 72z^3) - (2 + 6z + 14z^2 + 30z^3)\,,$$

which, as we invite the reader to check, is 3.

*Proof.* A generic real polynomial system with support $\Delta^{\epsilon}_{k,l}$ has the form

$$\sum_{j=1}^{n-1} c_{ij}x_j \ + c_{in}x^{\epsilon}x_n^l \ + \ f_i(x_n) \ = \ 0 \qquad \text{for } i = 1, \ldots, n\,,$$

where each polynomial $f_i$ has degree $k$ and $x^{\epsilon}$ is the monomial $x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}}$.

Since all solutions to our system are simple, we may perturb the coefficient matrix $(c_{ij})_{i,j=1}^n$ if necessary and then use Gaussian elimination to obtain an equivalent system

$$x_1 - g_1(x_n) \ = \ \cdots \ = \ x_{n-1} - g_{n-1}(x_n) \ = \ x^{\epsilon}{x_n}^l - g_n(x_n) \ = \ 0\,, \tag{2.1}$$

where each polynomial $g_i$ has degree $k$. Using the first $n-1$ polynomials to eliminate the variables $x_1, \ldots, x_{n-1}$ gives the univariate polynomial

$${x_n}^l \cdot g_1(x_n)^{\epsilon_1} \cdots g_{n-1}(x_n)^{\epsilon_{n-1}} \ - \ g_n(x_n)\,, \tag{2.2}$$

which has degree $l + k|\epsilon| = v(\Delta^{\epsilon}_{k,l})$. Any zero of this polynomial leads to a solution of the original system (2.1) by back substitution. This implies that the number of real roots of the polynomial (2.2) is equal to the number of real solutions to our original system (2.1).

The eliminant (2.2) has no terms of degree $m$ for $k < m < l$, and so it has at most $k + k|\epsilon| + 2$ non-zero real roots, by Descartes's rule of signs ([3, 1], see also Remark 2.3). This proves the upper bound. We complete the proof by constructing a polynomial of the form (2.2) having $r$ roots, for every number $r$ in the interval between 0 and $k + k|\epsilon| + 2$ with the same parity as $l + k|\epsilon|$.

Choose real polynomials $f_1, \ldots, f_n$ of degree $k$ having simple roots and non-zero constant terms. We further assume that the roots of $f_1, \ldots, f_{n-1}$ are distinct. Put $f^{\epsilon} = f_1^{\epsilon_1} \cdots f_{n-1}^{\epsilon_{n-1}}$ and let $\alpha$ be the ratio of the leading term of $f_n$ to the constant term of $f^{\epsilon}$. Choose any piecewise-linear convex function $\nu : [0, k|\epsilon| + l] \to \mathbb{R}$ which is identically 0 on $[l, k|\epsilon| + l]$ and whose maximal domains of linearity are $[0, k]$, $[k, l]$, and $[l, k|\epsilon| + l]$.

Let $f(z) := f_n \pm \alpha z^l f^\epsilon$ (the sign $\pm$ will be determined later). The *Viro polynomial* $f_t$ associated to $f$ and $\nu$ is obtained by multiplying the monomial $a_p z^p$ in $f(z)$ by $t^{\nu(p)}$ ([10, 11, 2]). By the definition of $\nu$, $f_t = f_{n,t} + \alpha z^l f_1^{\epsilon_1} \cdots f_{n-1}^{\epsilon_{n-1}}$, where $f_{n,t}$ is the Viro polynomial obtained from $f_n$ and the restriction of $\nu$ to $[0, k]$. By Viro's Theorem ([10, 11, 2], see also Proposition 4.1), there exists a sufficiently small $t_0 > 0$, such that if $t_0 > t > 0$, then the polynomial $f_t$ will have $r = r_1 + r_2 + r_3$ simple real roots, where $r_1$ is the number of real roots of $f^\epsilon$, $r_2$ the number of real roots of $f_n$, and $r_3$ is the number of (non-zero) real roots of the binomial obtained as the truncation of $f$ to the interval $[k, l]$. By our choice of $\alpha$, this binomial is a constant multiple of $z^k \pm z^l$. Thus $r_3$ is 1 if $l - k$ is odd, and either 0 or 2 (depending on the sign $\pm$) if $l - k$ is even. If $k$ is even, then every possible value of $r$ between 0 and $k + k|\epsilon| + 2$ with the same parity as $l + k|\epsilon|$ can be obtained in this way. If $k$ is odd then this construction gives all admissible values of $r$ in the interval $[|\epsilon| + 2, k + k|\epsilon| + 2]$ but no values of $r$ less than $|\epsilon| + 2$.

Suppose now that $k$ is odd. Take one of the Viro polynomials $f_t$ with $r' \geq |\epsilon| + 2$ real roots. For a generic $t$, the crital values of $f_t$ are all different. Choose one such $t$ and suppose without loss of generality that the leading coefficient of $f_t$ is positive. For each $\lambda$, consider the polynomial $h_\lambda = -\lambda - f_t$. If $\lambda$ is larger than every critical value of $f_t$, then $h_\lambda$ has either 0 or 1 real roots, depending upon the parity of $k|\epsilon| + l$. Since the number of real roots of $h_\lambda$ changes by 2 when $\lambda$ passes through critical values of $f_t$, and $h_0$ has at least $|\epsilon| + 2$ real roots, every possible number of real roots between 0 and $|\epsilon| + 2$ having the same parity as $l + k|\epsilon|$ occurs for some $h_\lambda$.

**Remark 2.3.** [Descartes's bound] Let $f(x) = \sum_{i=1}^{d} a_i x^{p_i}$ be a univariate polynomial with exponents $p_1 < \cdots < p_d$ and $a_1, \ldots, a_d$ are non-zero real numbers. Descartes's rule of signs asserts that the number of positive roots of $f$ is no more than the number of $i \in \{1, \ldots, d-1\}$ with $a_i \cdot a_{i+1} < 0$. Applying this to $f(x)$ and $f(-x)$ shows that the number of non-zero real roots of $f$ is no more than $\sum_{i=1}^{d-1} \overline{(p_{i+1} - p_i)}$, where $\overline{a} = 1$ or 2 according as $a$ is odd or even, respectively.

## 3. Elimination for near circuits

We first consider (possibly degenerate) circuits, which are collections of $n+2$ integer vectors that affinely span $\mathbb{R}^n$.

3.1. **Arithmetic of circuits.** Suppose that $\mathcal{C} := \{w_{-1}, w_0, w_1, \ldots, w_n\} \subset \mathbb{Z}^n$ affinely spans $\mathbb{R}^n$. For each $i = -1, 0, \ldots, n$, let $\mathcal{A}_i$ be the (possibly degenerate) simplex with vertices $\mathcal{C} \setminus \{w_i\}$. For any $j \in \{-1, 0, \ldots, n\}$, Cramer's rule implies that

$$\sum_{i=-1}^{j-1} (-1)^i \det(W_i)(w_i - w_j) \quad = \quad \sum_{i=j+1}^{n} (-1)^i \det(W_i)(w_i - w_j)\,,$$

where $W_i$ is the matrix whose columns are the vectors $w_{-1} - w_j, \ldots, w_n - w_j$, with $w_i - w_j$ and $w_j - w_j$ omitted. Thus if $i \neq j$, $|\det W_i|$ is the volume $v(\mathcal{A}_i)$ of $\mathcal{A}_i$.

**Lemma 3.1.** *Suppose that* $\{0, v_0, \ldots, v_n\} \subset \mathbb{Z}^n$ *is primitive with primitive relation*

$$\sum_{i=0}^{n} \alpha_i v_i \;=\; 0\,.$$

*If* $\mathcal{A}_q := \{0, v_0, \ldots, \widehat{v_q}, \ldots, v_n\}$*, then* $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}_q \simeq \mathbb{Z}/\alpha_q\mathbb{Z}$*.*

*Proof.* We can assume that $q = 0$. Since $\{0, v_0, \ldots, v_n\}$ is primitive, $\mathbb{Z}^n = \mathbb{Z}\{v_0, v_1, \ldots, v_n\}$. Then the image of $v_0$ generates the factor group $\mathbb{Z}^n/\mathbb{Z}\{v_1, \ldots, v_n\}$, and so this factor group is cyclic. Since the relation is primitive, $|\alpha_0|$ is the least positive multiple of $v_0$ lying in the lattice $\mathbb{Z}\{v_1, \ldots, v_n\}$, which implies that $\mathbb{Z}^n/\mathbb{Z}\{v_1, \ldots, v_n\} \simeq \mathbb{Z}/\alpha_0\mathbb{Z}$.

When $\mathcal{C} = \{w_{-1}, w_0, w_1, \ldots, w_n\}$ is primitive, this proof shows that the invariant factors of $\mathcal{A}_q$ are $1, \ldots, 1, |\alpha_q|$, and $v(\mathcal{A}_q) = |\alpha_q|$. In general, the index $a$ of $\mathcal{C}$ is the greatest common divisor of the volumes $v(\mathcal{A}_i)$, and the primitive affine relation on $\mathcal{C}$ has the form

$$\sum_{i=-1}^{n} \alpha_i w_i \;=\; 0 \qquad \text{with} \qquad \sum_{i=-1}^{n} \alpha_i \;=\; 0\,,$$

where $a|\alpha_i| = v(\mathcal{A}_i)$. Note that the configuration $\mathcal{C}$ admits two triangulations. One is given by those $\mathcal{A}_i$ with $\alpha_i > 0$ and the other by those $\mathcal{A}_i$ with $\alpha_i < 0$.

From now on, we make the following assumptions. First, assume $w_{-1} = 0$ and choose signs so that $\alpha_0 \geq 0$. Furthermore, assume the vectors $w_1, \ldots, w_n$ are ordered so that $\alpha_1, \ldots, \alpha_p > 0$, $\alpha_{p+1}, \ldots, \alpha_\nu < 0$, and $\alpha_{\nu+1}, \ldots, \alpha_n = 0$, for some integers $0 \leq p \leq \nu \leq n$. If $a$ is the index of $\mathcal{C}$ and we write $\lambda_i = |\alpha_i| = v(\mathcal{A}_i)/a$, then the primitive relation on $\mathcal{C}$ is

$$\sum_{i=0}^{p} \lambda_i w_i \;=\; \sum_{i=p+1}^{\nu} \lambda_i w_i\ ,$$

and we have

$$v(\mathcal{C}) \;=\; a \cdot \max\left\{ \sum_{i=0}^{p} \lambda_i,\ \sum_{i=p+1}^{\nu} \lambda_i \right\}\,, \tag{3.1}$$

as at least one of $\{\mathcal{A}_0, \ldots, \mathcal{A}_p\}$ or $\{\mathcal{A}_{p+1}, \ldots, \mathcal{A}_\nu\}$ is a triangulation of $\mathcal{C}$. If $\mathcal{A}_{-1}$ is non-degenerate, then adding it to the non-triangulation produces a triangulation. Thus $v(\mathcal{A}_{-1})$ is the difference of the two numbers in (3.1).

3.2. **Systems with support a near circuit.** Let $\{0, w_0, w_1, \ldots, w_n\} \subset \mathbb{Z}^n$ span $\mathbb{R}^n$ and suppose that $w_0 = \ell e_n$, where $e_n$ is the $n$th standard basis vector and $\ell$ is a positive integer. Let $k > 0$ and consider a generic polynomial system with support

$$\mathcal{C} \;:=\; \{0, w_0, 2w_0, \ldots, kw_0,\ w_1, \ldots, w_n\}\,. \tag{3.2}$$

We will call such a set of vectors a *near circuit* if no $w_i$ for $i > 0$ lies in $\mathbb{R}w_0$. We will assume that $\mathcal{C}$ is primitive, which implies that $\{0, e_n, w_1, \ldots, w_n\}$ is primitive. By Remark 1.5, this will enable us to deduce results for near circuits having odd index.

Write each vector $w_i = v_i + l_i \cdot e_n$, where $0 \neq v_i \in \mathbb{Z}^{n-1}$. Then $\{0, v_1, \ldots, v_n\}$ is primitive. Let $\lambda_1, \ldots, \lambda_\nu$ be the positive integral coefficients in the primitive relation on $\{0, v_1, \ldots, v_n\}$,

$$\sum_{i=1}^{p} \lambda_i v_i \;=\; \sum_{i=p+1}^{\nu} \lambda_i v_i \;.$$

Here, we could have $p = 0$ or $p = \nu$, so that one of the two sums collapses to 0.

Assume that the vectors are ordered so that

$$Ne_n \;+\; \sum_{i=1}^{p} \lambda_i w_i \;-\; \sum_{i=p+1}^{\nu} \lambda_i w_i \;=\; 0 \tag{3.3}$$

is the primitive relation on $\{0, e_n, w_1, \ldots, w_n\}$, where

$$N \;:=\; \sum_{i=p+1}^{\nu} \lambda_i l_i \;-\; \sum_{i=1}^{p} \lambda_i l_i \;\geq\; 0\,.$$

Consider a generic real polynomial system with support the near circuit $\mathcal{C}$. Perturbing the matrix of coefficients of the monomials $x^{w_i}$ for $i = 1, \ldots, n$ and applying Gaussian elimination gives a system with the same number of real solutions, but of the form

$$x^{w_i} \;=\; g_i(x_n^\ell) \qquad \text{for } i = 1, \ldots, n\,, \tag{3.4}$$

where each $g_i$ is a generic polynomial of degree $k$.

Define a polynomial $f \in \mathbb{R}[x_n]$ by

$$f(x_n) \;:=\; x_n^N \prod_{i=1}^{p} (g_i(x_n^\ell))^{\lambda_i} \;-\; \prod_{i=p+1}^{\nu} (g_i(x_n^\ell))^{\lambda_i}\;. \tag{3.5}$$

Here, empty products are equal to 1. By (3.1), the degree of $f$ is equal to the volume of the near circuit $\mathcal{C}$, as $N$ is the volume of $\mathcal{A}_0$ and $k\ell\lambda_i$ is the volume of the simplex $\mathcal{A}_i$, which is the convex hull of $\{kw_0 = k\ell e_n, 0, w_1, \ldots, \widehat{w_i}, \ldots, w_n\}$. Lastly, the absolute value of the difference

$$\delta \;:=\; N + \sum_{i=1}^{p} k\ell\lambda_i \;-\; \sum_{i=p+1}^{\nu} k\ell\lambda_i \tag{3.6}$$

in the degrees of the two terms of $f$ is the volume of the convex hull of $\{k\ell e_n, w_1, \ldots, w_n\}$, which may be zero.

**Theorem 3.2.** *Assume that $g_1, \ldots, g_n$ are generic polynomials of degree $k$. Then, the association of a solution $x$ of* (3.4) *to its $n$th coordinate $x_n$ gives a one-to-one correspondence between solutions of* (3.4) *and roots of the univariate polynomial $f$* (3.5) *which restricts to a bijection between real solutions to* (3.4) *and real roots of $f$.*

In particular, bounds on the number of real roots of polynomials of the form (3.5) give bounds on the number of real solutions to a generic polynomial system with support $\mathcal{C}$. The polynomial $f$ is the eliminant of the system (3.4), but our proof is not as direct as the corresponding proof in Section 2.

**Lemma 3.3.** *For each $q = 1, \ldots, \nu$, the system*

$$I_q \; : \; \left\{ \begin{array}{rcll} x^{w_i} & = & g_i(x_n^\ell), & \textit{for} \;\; i = 1, \ldots, n, i \neq q \\ f(x_n) & = & 0 & \end{array} \right.$$

*is equivalent to the system*

$$J_q \; : \; \left\{ \begin{array}{rcll} x^{w_i} & = & g_i(x_n^\ell), & \textit{for} \;\; i = 1, \ldots, n, i \neq q \\ \left(x^{w_q}\right)^{\lambda_q} & = & \left(g_q(x_n^\ell)\right)^{\lambda_q} & \end{array} \right.$$

*Proof.* This follows from (3.3) and the form (3.5) of $f$. ∎

**Remark 3.4.** Observe that the last polynomial in the system $J_q$ factors

$$(x^{w_q})^{\lambda_q} \; - \; (g_q(x_n^\ell))^{\lambda_q} \; = \; \prod_{\zeta \in Z_{\lambda_q}} \left( \zeta x^{w_q} \; - \; g_q(x_n^\ell) \right),$$

where $Z_{\lambda_q}$ is the set of roots of $z^{\lambda_q} - 1$. For a solution $x$ to $J_q$, the number $\zeta$ is $x^{-w_q} g_q(x_n^\ell)$. This factorization reveals that the system $J_q$ is a disjunction of $\lambda_q$ systems with support $\mathcal{C}$. The subsystem with $\zeta = 1$ is our original system (3.4).

*Proof of Theorem* 3.2. Since the integers $\lambda_1, \ldots, \lambda_\nu$ are coprime, at least one $\lambda_i$ is odd. We restrict ourselves to the case where $\lambda_1$ is odd since the proof with any other $\lambda_i$ odd is similar. Let $q = 1$ in Lemma 3.3 and let $x_n$ be a root of $f$. We show that $x_n$ extends to a unique solution of (3.4), and that the solution is real if and only if $x_n$ is real.

We first prolong $x_n$ to solutions to the system $I_1$ by solving the system for $y \in (\mathbb{C}^*)^{n-1}$

$$y^{v_i} \; = \; x_n^{-l_i} g_i(x_n^\ell) \qquad \text{for } i = 2, \ldots, n\,. \tag{3.7}$$

The numbers $\beta_i := x_n^{-l_i} g_i(x_n^\ell)$ are well-defined and non-zero since the $g_i$ are generic polynomials of degree $k$. Hence (3.7) is a system associated to the simplex $\mathcal{B} = \{0, v_2, \ldots, v_n\} \subset \mathbb{Z}^{n-1}$ as studied in Proposition 1.1. Its set of solutions is $\varphi_{\mathcal{B}}^{-1}(\beta)$, where $\varphi_{\mathcal{B}} \colon (\mathbb{C}^*)^{n-1} \to (\mathbb{C}^*)^{n-1}$ is the homomorphism

$$\varphi_{\mathcal{B}} \; : \; t \; \longmapsto \; (t^{v_2}, \ldots, t^{v_n}) \; \in \; (\mathbb{C}^*)^{n-1}\,.$$

Let $y \in \varphi_{\mathcal{B}}^{-1}(\beta)$. Then the fibre $\varphi_{\mathcal{B}}^{-1}(\beta)$ is the set $\{ty \mid t \in \ker(\varphi_{\mathcal{B}})\}$. We determine which of these, if any, is a solution to (3.4) by computing the number $\zeta$ of Remark 3.4. For $ty \in \varphi_{\mathcal{B}}^{-1}(\beta)$, this number is $t^{-v_1} y^{-v_1} x_n^{-l_1} g_1(x_n^\ell)$. Consider the map

$$\psi \colon \ker(\varphi_{\mathcal{B}}) \ni t \; \longmapsto \; t^{v_1} \in Z_{\lambda_1}.$$

Since $\mathcal{D} = \{0, v_1, \ldots, v_n\} \subset \mathbb{Z}^{n-1}$ is primitive, $\psi$ is well-defined by Lemma 3.1. Similarly, $\varphi_{\mathcal{D}}$ is injective (see Proposition 1.3), and thus so is $\psi$. Note that $\ker(\varphi_{\mathcal{B}})$ is isomorphic to $Z_{\lambda_1}$, thus $\psi$ is an isomorphism. Thus exactly one of these prolongations of $x_n$ given by $\varphi_{\mathcal{B}}^{-1}(\beta)$ is a solution to (3.4).

Suppose now that $x_n$ is real. Then $\beta$ is real. Since $\lambda_1$ is odd, there is a unique real solution $y$ to (3.7), by Proposition 1.1. But then $y^{-v_1} x_n^{-l_1} g_1(x_n^\ell)$ is real. As $\lambda_1$ is odd, there is eactly one real $\lambda_1$-th root of unity, namely 1, which proves that the real solution $(y, x_n)$ is a real solution to (3.4). This completes the proof of the theorem. ∎

**Remark 3.5.** The primitivity of $\mathcal{C}$ implies that $N$ and $\ell$ are coprime if $N \neq 0$, or $\ell = 1$ if $N = 0$. Indeed, the affine span of $\mathcal{C}$ is equal to that of $\{\ell e_n, w_1, \ldots, w_n\}$ and the Cramer relation on this set is obtained by multiplying both sides of (3.3) by $\ell$. Hence, the index of $\mathcal{C}$ is the greatest common divisor of $N, \ell\lambda_1, \ldots, \ell\lambda_\nu$, and the result follows as the $\lambda_i$ are coprime. In particular, either $\ell$ is odd, or $\ell$ is even and $N$ and $\delta$ are odd.

**Example 3.6.** We show that any positive integers $p$, $\ell$, $N$, with $N, \ell$ coprime if $N \neq 0$ or $\ell = 1$ if $N = 0$, and any positive coprime integers $\lambda_1, \ldots, \lambda_\nu$ with $\nu \leq n$ correspond to a primitive near circuit, when one $\lambda_i = 1$. Thus any polynomial of the form (3.5) is the eliminant of a system with support a primitive near circuit, when one exponent $\lambda_i = 1$.

Assume without loss of generality that $\lambda_\nu = 1$. Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. Let $v_i := e_i$ for $i = 1, \ldots, \nu - 1$. Set

$$v_\nu \;:=\; \sum_{i=1}^{p} \lambda_i v_i \;-\; \sum_{i=p+1}^{\nu-1} \lambda_i v_i \,.$$

Since $\lambda_1, \ldots, \lambda_\nu$ are coprime, there exist integers $l_1, \ldots, l_\nu$ such that $N = \sum_{i=p+1}^{\nu} \lambda_i l_i - \sum_{i=1}^{p} \lambda_i l_i$. If we set $w_i := v_i + l_i e_n$ for $i = 1, \ldots, \nu$ and $w_i := e_{i-1}$ for $i = \nu+1, \ldots, n$, then we obtain the relation (3.3) among $e_n, w_1, \ldots, w_n$. It is then easy to see that the near circuit $\{0, \ell e_n, \ldots, k\ell e_n, w_1, \ldots, w_n\}$ is primitive (for any integer $k$) if we assume that $N$ and $\ell$ are coprime, or $\ell = 1$ if $N = 0$.

## 4. Upper bounds for near circuits

We first give a version of Viro's construction for univariate polynomials that takes multiplicities into account. We then use this to establish upper bounds for the number of real roots of polynomials of the form (3.5) by studying the total variation in the number of real roots of a pertubation of the eliminant.

4.1. **Viro univariate polynomials.** Consider a univariate Viro polynomial

$$f_t(y) \;=\; \sum_{p=p_0}^{d} \phi_p(t)\, y^p \,,$$

where $t$ is a positive real number, and each coefficient $\phi_p(t)$ is a finite sum $\sum_{q \in I_p} c_{p,q} t^q$ with $c_{p,q} \in \mathbb{R}$ and $q$ a rational number. Write $f$ for the function of $y$ and $t$ defined by $f_t$.

Let $P$ be the convex hull of the points $(p, q)$ for $p_0 \leq p \leq d$ and $q \in I_p$. Assume that $P$ has dimension 2. Its lower hull $L$ is the union of the edges $L_1, \ldots, L_l$ of $P$ whose inner normals have positive second coordinate. Let $I_i$ be the image of $L_i$ under the projection to the first axis. Then the intervals $I_1, \ldots, I_l$ subdivide the Newton segment $[p_0, d]$ of $f_t$.

Let $f^{(i)}$ be the facial subpolynomial of $f$ for the face $L_i$. That is, $f^{(i)}$ is the sum of terms $c_{p,q} y^p$ such that $(p, q) \in L_i$. Suppose that $L_i$ is the graph of $y \mapsto a_i y + b_i$ over $I_i$, Expanding $f_t(yt^{-a_i})/t^{b_i}$ in powers of $t$ gives

$$\frac{f_t(yt^{-a_i})}{t^{b_i}} \;=\; f^{(i)}(y) + t^{A_i} d^{(i)}(y) + h^{(i)}(y, t) \,, \quad i = 1, \ldots, l, \tag{4.1}$$

where $t^{A_i}d^{(i)}(y)$ collects the terms with smallest positive power of $t$ and $h^{(i)}(y,t)$ collects the remaining terms (whose powers of $t$ exceed $A_i$). Then $f^{(i)}(y)$ has Newton segment $I_i$ and its number of non-zero roots counted with multiplicities is $|I_i|$, the length of the interval $I_i$.

**Proposition 4.1.** *Assume that for any non-zero root $\rho$ of $f^{(i)}$, $i=1,\dots,l$, either $\rho$ is a simple root of $f^{(i)}$, or else $d^{(i)}(\rho)\neq 0$. Then there exists $t_0>0$ such for $0<t<t_0$, the univariate polynomial $f_t(y)$ has only simple non-zero roots, with*

$$r \;=\; \sum c(\rho)$$

*non-zero real roots, where the sum is over $i=1,\dots,l$ and then all non-zero real roots $\rho$ of $f^{(i)}$ where*

$$c(\rho) \;=\; \begin{cases} 1 & \text{if the multiplicity } m \text{ of } \rho \text{ is odd,}\\ 0 & \text{if } m \text{ is even and } f^{(i)}(y)/d^{(i)}(\rho)>0, \text{ for } y \text{ near } \rho,\\ 2 & \text{if } m \text{ is even and } f^{(i)}(y)/d^{(i)}(\rho)<0, \text{ for } y \text{ near } \rho.\end{cases}$$

In particular, if the non-zero roots of $f^{(1)},\dots,f^{(l)}$ are simple, then the number of non-zero real roots of $f_t$ for $t>0$ small enough equals the total number of non-zero real roots of $f^{(1)},\dots,f^{(l)}$. This is the usual version of Viro's theorem for univariate polynomials.

*Proof.* For each root $\rho\neq 0$ of $f^{(i)}(y)$ of multiplicity $m$, there will be $m$ roots near $\rho$ to

$$f^{(i)}(y)+t^{A_i}d^{(i)}(y)+h^{(i)}(y,t)\,,$$

for $t>0$ sufficiently small. This gives $|I_i|$ roots to $f_t(yt^{-a_i})/t^{b_i}$, and thus all solutions to $f_t$ in $\mathbb{C}^*$, at least when $t>0$ is sufficiently small. Indeed, let $K\subset\mathbb{C}^*$ be a compact set containing the non-zero roots of the facial polynomials $f^{(1)}(y),\dots,f^{(l)}(y)$. Then, for $t>0$ sufficiently small, $K$ contains the $|I_i|$ roots to $f_t(yt^{-a_i})/t^{b_i}$ that we just constructed. The compact sets $t^{-a_1}K,\dots,t^{-a_l}K$ are pairwise disjoint for $t>0$ sufficiently small, and this gives $|I_1|+\cdots+|I_l|=d-p_0$ non-zero simple roots of $f_t$ for $t>0$ small enough. But this accounts for all the non-zero simple roots of $f_t$.

We now determine how many roots of $f_t(yt^{-a_i})/t^{b_i}$ are real. Roots close to $\rho$ are real only if $\rho$ is real, and then the number of such real roots is determined by the first two terms $f^{(i)}(y)+t^{A_i}d^{(i)}(y)$ in $t$, as $d^{(i)}(\rho)\neq 0$. But this polynomial has $c(\rho)$ real roots near $\rho$.

4.2. **Upper bounds.** We give upper bounds on the number of real roots of a generic polynomial system with support a primitive near circuit

$$\mathcal{C} \;=\; \{0,\ell e_n,2\ell e_n,\dots,k\ell e_n,\ w_1,\dots,w_n\}\,.$$

As explained in Section 3, it suffices to bound the roots of a polynomial $f$ of the form (3.5). Consider the polynomial $f_t(y)$ depending on a real parameter $t\neq 0$ defined by

$$f_t(y) \;:=\; t\cdot y^N\prod_{i=1}^{p}(g_i(y^\ell))^{\lambda_i} \;-\; \prod_{i=p+1}^{\nu}(g_i(y^\ell))^{\lambda_i}\,,$$

where $g_1,\dots,g_\nu$ are generic polynomials of degree $k$. We will study how the number of real roots of $f_t$ can vary as $t$ runs from $\infty$ to 1. Note that $f_1$ is our original eliminant $f$.

**Remark 4.2.** If $p \neq 0$ then $f_t(x_n)$ is the eliminant of the system

$$\begin{cases} x^{w_1} &=& t^{1/\lambda_1} \cdot g_1(x_n^\ell)\,, \\ x^{w_i} &=& g_i(x_n^\ell) \quad i = 2, \ldots, n\,. \end{cases}$$

If $p = 0$, then $f_t$ is $t^2$ times the eliminant of the system

$$\begin{cases} x^{w_1} &=& t^{-1/\lambda_1} \cdot g_1(x_n^\ell)\,, \\ x^{w_i} &=& g_i(x_n^\ell) \quad i = 2, \ldots, \ldots, n\,. \end{cases}$$

For any integer $a$, define $\overline{a} \in \{0, 1, 2\}$ by

$$\overline{a} \;:=\; \begin{cases} 2 & \text{if } a \text{ is positive and even} \\ 1 & \text{if } a \text{ is positive and odd} \\ 0 & \text{otherwise} \end{cases}$$

A root $\rho$ of a univariate polynomial is *singular* if it has multiplicity greater than 1.

Let $\chi(Y)$ be the boolean truth value of $Y$, so that $\chi(0 > 1) = 0$, but $\chi(0 < 1) = 1$.

**Proposition 4.3.** *The total sum of the multiplicities of the non-zero singular real roots of $f_t$ for $t \in \mathbb{R}^*$ is no more than $2k\overline{\ell}\nu - 2\overline{\ell}\,(\chi(\delta = 0) + \chi(N = 0))$.*

*Moreover, if $\ell$ is even, so that $N$ is odd due to the primitivity of $\mathcal{C}$, then the total sum of the multiplicities of the non-zero singular real roots of $f_t$ for $t > 0$ is equal to the corresponding sum for $t < 0$. Hence, both numbers are no more $2k\nu$.*

*Proof.* Write $f_t = tF - G$, where $F$ and $G$ are the two terms of $f$. Let $\rho$ be a non-zero root of $f_t$ for some $t \neq 0$. Then $F(\rho)G(\rho) \neq 0$, as the roots of $g_1, \ldots, g_\nu$ are distinct. Note that $t = G(\rho)/F(\rho)$. Then $\rho$ is a singular root of $f_t$ if and only if

$$(F'G - FG')(\rho) \;=\; 0\,.$$

If $N \neq 0$, then the polynomial $F'G - FG'$ factors as

$$(F'G - FG')(y) \;=\; \left( y^{N-1} \prod_{i=1}^{\nu} (g_i(y^\ell))^{\lambda_i - 1} \right) \cdot H(y) \tag{4.2}$$

where $H$ is the polynomial defined by

$$H(y) \;=\; \prod_{i=1}^{\nu} g_i(y^\ell) \cdot \left( N + \ell y^\ell \cdot D(y^\ell) \right)\,,$$

with

$$D(z) \;=\; \sum_{i=1}^{p} \lambda_i \cdot \frac{g_i'(z)}{g_i(z)} \;-\; \sum_{i=p+1}^{\nu} \lambda_i \cdot \frac{g_i'(z)}{g_i(z)}\,.$$

(If $N = 0$, then $y^{N-1}$ is replaced by $y^{\ell-1}$, and the last factor in $H$ is simply $\ell D(y^\ell)$.)

Thus $H(y) = h(y^\ell)$, where $h$ is a polynomial of degree $k\nu - (\chi(\delta = 0) + \chi(N = 0))$ with a non-zero constant term, as the $g_i$ are generic. If $\rho$ is a non-zero singular real root of $f_t$ for $t \neq 0$ then $h(\rho^\ell) = 0$. Thus the total number of non-zero singular real roots of $f_t$ for $t \in \mathbb{R}^*$ is at most $k\overline{\ell}\nu - \overline{\ell}\,(\chi(\delta = 0) + \chi(N = 0))$.

A root $\rho$ of $f_t$ has multiplicity $m \geq 2$ for some $t \neq 0$ if and only if $tF^{(i)}(\rho) - G^{(i)}(\rho) = 0$ for $i = 0, \ldots, m-1$ and $tF^{(m)}(\rho) - G^{(m)}(\rho) \neq 0$. This is equivalent to the system

$$\begin{array}{rcll} F^{(i)}(\rho) \cdot G^{(j)}(\rho) - F^{(j)}(\rho) \cdot G^{(i)}(\rho) & = & 0 & \text{if } \ 0 \leq i, j \leq m-1, \\ F^{(i)}(\rho) \cdot G^{(j)}(\rho) - F^{(j)}(\rho) \cdot G^{(i)}(\rho) & \neq & 0 & \text{if } i = m \text{ and } 0 \leq j \leq m-1. \end{array} \tag{4.3}$$

Solving (4.2) for $H$ and using the expression of the $k$th derivative of $F'G - FG'$ as a sum of polynomials of the form $F^{(i)}G^{(j)} - F^{(j)}G^{(i)}$, we can use (4.3) to deduce that $\rho$ is a root of multiplicity $m \geq 2$ of $f_t$ for some $t \neq 0$ if and only if $\rho$ is a root of multiplicity $m-1$ of $H$. Thus the total sum of the multiplicities of the non-zero singular real roots of $f_t$ for $t \in \mathbb{R}^*$ is

$$\sum_{\rho \text{ a real root of } H} (m_\rho(H) + 1)\,, \tag{4.4}$$

where $m_\rho(H)$ is the multiplicity of the root $\rho$ of $H$. We see that (4.4) is bounded by $2k\overline{\ell}\nu - 2\overline{\ell}\,(\chi(\delta = 0) + \chi(N = 0))$ with equality when all roots of $h$ are real and simple (and positive if $\ell$ is even) so that the singular real roots of $f_t$ for $t \in \mathbb{R}^*$ are real double roots.

Finally, the statement concerning the case $\ell$ even and $N$ odd is obvious after noting that in this case the (non-zero) real roots of $h(y^\ell)$ come in pairs $(\rho, -\rho)$, the function $G/F$ is an odd function, and $N$ and $\delta$ are both odd integer numbers.

**Remark 4.4.** If $\{k\ell e_n, w_1, \ldots, w_n\}$ are affinely independent and $N \neq 0$, then the difference $\delta$ (3.6) of the degrees of the terms of $f$ is non-zero, and the polynomials $f_t$ have the same Newton segment for $t \neq 0$. Thus the number of real roots of $f_t$ can change only if $t$ passes through 0, or through a value $c \neq 0$ such that $f_c$ has a singular root.

If the difference $\delta = 0$, then there is one number $t_\infty$ for which the degree of $f_{t_\infty}$ drops. If necessary, we may perturb coefficients of one $g_i$ so that the number of real roots of $f$ does not change, and the degree of $f_{t_\infty}$ drops by one. This will result in no net change in the number of real roots of $f_t$ as $t$ passes through $t_\infty$, for the root which 'disappears' in $f_{t_\infty}$ is a real root at infinity. Similarly, if $N = 0$, then we may assume that there is one number $t_0$ for which the constant term of $f_{t_0}$ vanishes. Perturbing again if necessary results in no net change in the number of non-zero real roots of $f_t$ as $t$ passes through $t_0$.

Thus, the number of values $c$ where the number of real roots of $f_t$ changes is finite by Proposition 4.3, and hence it makes sense to define the numbers

$$r_{-\infty}\,, \quad r_{0-}\,, \quad r_{0+}\,, \quad r_{+\infty}$$

as the numbers of real roots of $f_t$ as $t$ tends to $-\infty$, 0 by negative values, 0 by positive values and $+\infty$, respectively.

Recall that $\chi(Y)$ denotes the boolean truth value of $Y$.

**Proposition 4.5.** *We have*

$$\begin{aligned}
\frac{r_{0+}+r_{0-}}{2} &\leq k\overline{\ell}(\nu-p)+\chi(\delta>0)\\
\frac{r_{+\infty}+r_{-\infty}}{2} &\leq k\overline{\ell}p+\chi(N>0)+\chi(\delta<0)\\
\frac{|r_{0+}-r_{0-}|}{2} &\leq k\overline{\ell}\left(\sum_{i=p+1}^{\nu}\overline{\lambda_i}\right)-k\overline{\ell}(\nu-p)+\chi(\delta>0 \textit{ is even})\\
\frac{|r_{+\infty}-r_{-\infty}|}{2} &\leq k\overline{\ell}\left(\sum_{i=1}^{p}\overline{\lambda_i}\right)-k\overline{\ell}p+\chi(N>0 \textit{ is even})+\chi(\delta<0 \textit{ is even})
\end{aligned}$$

*Furthermore, if $l$ is even and $N$ is odd, we have*

$$\frac{r_{0+}+r_{+\infty}}{2} \leq k\nu+1$$

*Proof.* As in the proof of Proposition 4.3, write $f_t = tF - G$. We apply Proposition 4.1 (and its proof) to $f_t$ and $f_{-t} = -tF - G$ to estimate $r_{0+}$ and $r_{0-}$, respectively.

Let $P$ be the common Newton polygon of $f_t(y)$ and $f_{-t}(y)$, as polynomials in $y$ and $t$. Projecting the lower faces of $P$ onto the first ($y$) coordinate axis gives a single interval $I_1$ if $\delta \leq 0$, or the union of two intervals $I_1$ and $I_2$ if $\delta > 0$. Here, $I_1 = [0, \deg(G)]$, the Newton segment of $G$ and $I_2 = [\deg(G), \deg(F)]$, which has length $\delta$. For both $f_t(y)$ and $f_{-t}(y)$, the polynomial $f^{(1)}$ corresponding to $I_1$ is just $G$. If $\delta > 0$, then the polynomial $f^{(2)}$ corresponding to $I_2$ is the binomial $\pm M_F - M_G$, which is the difference of the highest degree terms of $\pm F$ and $G$.

Both binomials $\pm M_F - M_G$ have only simple non-zero roots. The polynomial $f^{(1)} = G$ has singular roots if any of $\lambda_{p+1}, \ldots, \lambda_\nu$ are not equal to 1. Since $F$ and $G$ have no common root, the assumptions of Proposition 4.1 are fulfilled for both $f_t$ and $f_{-t}$. The numbers $r_{0+}$ and $r_{0-}$, hence $r_{0+} + r_{0-}$ and $|r_{0+} - r_{0-}|$, are sums of contributions of the non-zero real roots of $G$ and of the non-zero real roots of $\pm M_F - M_G$.

Consider contributions from roots $\rho$ of $G$, which satisfy $\rho^\ell = \zeta$, where $\zeta$ is a root of some $g_i$ for $p < i \leq \nu$. This has multiplicity $\lambda_i$. If $\lambda_i$ is odd, then $\rho$ contributes 1 to both $r_{0\pm}$ and hence 2 to $r_{0+} + r_{0-}$ and 0 to $|r_{0+} - r_{0-}|$. If $\lambda_i$ is even, then $\rho$ contributes 2 or 0 to $r_{0+}$, depending upon the sign of $G/F$ near $\rho$. Replacing $t$ by $-t$, shows that it contributes 2 or 0 to $r_{0-}$, depending upon the sign of $-G/F$ near $\rho$. Thus $\rho$ contributes 2 both to $r_{0+} + r_{0-}$ and to $|r_{0+} - r_{0-}|$.

Suppose now that $\delta > 0$. Then each non-zero real root of the binomial $\pm M_F - M_G$ is simple and thus contributes 1 to $r_{0\pm}$. Both binomials have only one non-zero real root if $\delta$ is odd. If $\delta$ is even, then $M_F - M_G$ (resp. $-M_F - M_G$) has 0 or 2 (resp. 2 or 0) real roots according as the product of the coefficients of $M_F$ and $M_G$ is positive or negative. It follows that the roots of these binomials contribute 2 to $r_{0+} + r_{0-}$, and contribute 0 or 2 to $|r_{0+} - r_{0-}|$ according as $\delta$ is odd or even, respectively. Summing up all contributions gives the desired upper bounds for $r_{0+} + r_{0-}$ and $|r_{0+} - r_{0-}|$.

The upper bounds for $r_{+\infty} - r_{-\infty}$ and $|r_{+\infty} - r_{-\infty}|$ are obtained in exactly the same way if we use the polynomial $g_t(y) = F(y) - tG(y)$ instead of $f_t(y)$. The Newton polygon

$Q$ of $g_t$ is the reflection of $P$ in the horizontal line of height $1/2$, so that the lower faces of $Q$ are the upper faces of $P$. They project to 1, 2, or 3 intervals on the first coordinate axis, $I_1 = [0, N]$ (if $N \neq 0$), $I_2 = [N, \deg(F)]$, and $I_3 = [\deg(F), \deg(G)]$, if $\delta < 0$. The polynomial $f^{(2)}$ corresponding to $I_2$ is just $F$, and the other polynomials are binomials.

Finally, assume that $\ell$ is even, $N$ is odd, and let us prove the last inequality. Using the facts that the non-zero real roots of $G$ come in pairs $(\rho, -\rho)$, the function $G/F$ is an odd function, and that $N$ and $\delta$ are both odd integers, we obtain $r_{0+} \leq 2k(\nu - p) + \chi(\delta > 0)$. Similarly, using the polynomial $g_t(y) = F(y) - tG(y)$ instead of $f_t(y)$, we obtain that $r_{+\infty} \leq 2kp + 1 + \chi(\delta < 0)$. Suming up these two inequalities gives the result.

**Theorem 4.6.** *The number $r$ of real solutions to a generic system with support the near circuit $\mathcal{C} = \{0, \ell e_n, 2\ell e_n, \ldots, k\ell e_n,\ w_1, \ldots, w_n\}$ satisfies the following inequalities*

$$\begin{aligned} r \ &\leq\ 2k\overline{\ell}p + k\overline{\ell}\left(\textstyle\sum_{i=p+1}^{\nu} \overline{\lambda_i}\right) + \chi(N > 0) + 1 - \chi(\delta > 0 \text{ is odd}) \\ &\qquad -\chi(\delta = 0) - \overline{\ell}\,(\chi(\delta = 0) + \chi(N = 0))\ , \end{aligned} \tag{4.5}$$

*and*

$$\begin{aligned} r \ &\leq\ 2k\overline{\ell}(\nu - p) + k\overline{\ell}\left(\textstyle\sum_{i=1}^{p} \overline{\lambda_i}\right) + \chi(N > 0 \text{ is even}) + 1 - \chi(\delta < 0 \text{ is odd}) \\ &\qquad -\chi(\delta = 0) - \overline{\ell}\,(\chi(\delta = 0) + \chi(N = 0))\ , \end{aligned} \tag{4.6}$$

*where $k, \ell, N, \nu$, and $\lambda_i$ are defined in Section* 3.2.

*Moreover, if $\ell$ is even and $N$ is odd, then we have*

$$r \leq 2k\nu + 1 \tag{4.7}$$

**Remark 4.7.** Since $\lambda_1, \ldots, \lambda_\nu$ are relatively prime, we see that the absolute upper bound for such a near circuit with $\ell$ odd is

$$k(2\nu - 1) + 2\,,$$

and this can be obtained if exactly one $\lambda_i$ is odd, $N > 0$ and even, and $\delta$ is even and non-zero. This upper bound is maximized when $\{0, e_n, w_1, \ldots, w_n\}$ forms a non degenerate circuit, that is, if no proper subset is affinely dependent. If $\ell$ is even, the corresponding absolute upper bound is $2k\nu + 1$, and is also maximized when $\{0, e_n, w_1, \ldots, w_n\}$ forms a non degenerate circuit.

*Proof of Theorem* 4.6. Let $f$ be the univariate eliminant of a generic polynomial system with support $\mathcal{C}$, which has the form (3.5). For an interval $I \subset \mathbb{R}$, let $\Delta_I$ be the (positive) variation in the number of non-zero real roots of $f_t$ for $t \in I$. As noted in Remark 4.4, the number of non-zero real roots of $f_t$ can change only if $t$ passes through 0, or if $t$ passes through a value $c \neq 0$ such that $f_c$ has a real singular root. Passing through the value $t = 0$, the variation of the number of real roots of $f_t$ is at most $|r_{0+} - r_{0-}|$.

Recall that $f = f_t$ for $t = 1$ and that $f$ has no singular roots. Considering the path from $t$ close to $-\infty$ to $t = 1$, we obtain

$$r \ \leq\ r_{-\infty} + \Delta_{(-\infty,0)} + |r_{0+} - r_{0-}| + \Delta_{(0,1)}\,.$$

Considering the path from $t$ close to $+\infty$ to $t=1$, we obtain

$$r \;\leq\; r_{+\infty} + \Delta_{(1,+\infty)}\,.$$

Combining these two inequalities yields

$$r \;\leq\; \frac{r_{+\infty}+r_{-\infty}+|r_{0+}-r_{0-}|}{2} + \frac{\Delta_{\mathbb{R}^*}}{2}\,. \tag{4.8}$$

The number $\Delta_{\mathbb{R}^*}$ is at most the total multiplicity of the singular real roots of $f_t$ for $t\neq 0$. By Proposition 4.3, this is at most $2k\overline{\ell}\nu - 2\overline{\ell}\,(\chi(\delta=0)+\chi(N=0))$. The inequality (4.5) follows then from (4.8) and Proposition 4.5.

Using the polynomial $g_t = F - tG$ in place of $f_t = tF - G$, leads to

$$r \;\leq\; \frac{r_{0+}+r_{0-}+|r_{+\infty}-r_{-\infty}|}{2} + \frac{\Delta_{\mathbb{R}^*}}{2}. \tag{4.9}$$

The inequality (4.6) is then obtained using Proposition 4.5.

Finally, considering the paths from $t=0$ to $t=1$, and from $t$ close to $+\infty$ to $t=1$ gives

$$r \;\leq\; \frac{r_{0+}+r_{+\infty}}{2} + \frac{\Delta_{(0,+\infty)}}{2}. \tag{4.10}$$

For $\ell$ even and $N$ odd, the inequality (4.7) comes then from the corresponding statements in Proposition 4.3 and Proposition 4.5.

## 5. Constructions and sharp upper bounds

We now construct polynomials $f$ having the form (3.5) with many real roots. In some cases, this achieves the upper bound of Theorem 4.6 for the maximal number of real solutions to generic systems with support a given near circuit.

**Theorem 5.1.** *Let $\mathcal{C}$ be a primitive near circuit with $k$, $\ell$, $N$, and $\lambda_i$ as in Section 3.2. Suppose that $d_1,\ldots,d_\nu$ are nonnegative integers with $d_i\leq k$ such that*

$$\ell\sum_{i=1}^{\nu} d_i\lambda_i \;<\; N + k\ell\sum_{i=1}^{p}\lambda_i\,. \tag{5.1}$$

*If $\ell$ is odd, then there is a generic polynomial system with support $\mathcal{C}$ having*

$$\sum_{i=1}^{\nu} d_i\,\overline{\lambda_i} \;+\; \overline{d}$$

*real solutions, where $d$ is the (positive) difference of the two sides of (5.1). If $\ell$ is even, hence $N$ is odd, then there is a generic polynomial system with support $\mathcal{C}$ having*

$$2\sum_{i=1}^{\nu} d_i + 1$$

*real solutions.*

We use Proposition 4.1 to determine the number of real roots for small $t > 0$ of a polynomial of the form

$$f_t(x) \;=\; t^a \prod_{\zeta\in R_1} (t^{-b}x^\ell - \zeta)^{m(\zeta)} \;-\; x^\mu \prod_{\zeta\in R_2} (\zeta - x^\ell)^{m(\zeta)}\,,$$

where $\mu, a, b$ are positive integers, $R_1$ and $R_2$ are disjoint sets of positive real numbers, and $m(\zeta)$ is a positive integer for $\zeta \in R_1 \cup R_2$. Set $\mu_i := \ell \cdot \sum_{\zeta\in R_i} m(\zeta)$.

**Lemma 5.2.** *Suppose that $\mu_1 < a\ell/b < \mu$, and we have further that*

(1) *if $\zeta, \zeta' \in R_1$ with $m(\zeta)$ even and $m(\zeta')$ odd, then $\zeta' < \zeta$, and*
(2) *if $\zeta, \zeta' \in R_2$ with $m(\zeta)$ even and $m(\zeta')$ odd, then $\zeta' > \zeta$.*

*Let $e$ (respectively $o$) be the number of $\zeta \in R_1 \cup R_2$ such that $m(\zeta)$ is even (respectively odd). If $\ell$ is odd, then, for $t > 0$ sufficiently small, $f_t$ has exactly $2e + o + \overline{\mu - \mu_1}$ simple non-zero real roots. If $\ell$ is even and $\mu$ is odd, then, for $t > 0$ sufficiently small, $f_t$ has exactly $2e + 2o + 1$ simple non-zero real roots.*

*Proof.* We use the notation of Proposition 4.1. The inequalities $\mu_1 < a\ell/b < \mu$ imply that the lower hull of $P$ consists of three segments whose projection onto the first coordinate axis are the intervals $I_1 = [0, \mu_1]$, $I_2 = [\mu_1, \mu]$, and $I_3 = [\mu, \mu + \mu_2]$. The corresponding facial subpolynomials are

$$\begin{aligned}
f^{(1)}(x) &= \prod_{\zeta\in R_1} (x^\ell - \zeta)^{m(\zeta)}\\
f^{(2)}(x) &= x^{\mu_1} \;-\; \Big(\prod_{\zeta\in R_2} \zeta^{m(\zeta)}\Big)\cdot x^\mu\\
f^{(3)}(x) &= -x^\mu \prod_{\zeta\in R_2} (\zeta - x^\ell)^{m(\zeta)}\,.
\end{aligned}$$

Note that $\prod_{\zeta\in R_2} \zeta^{m(\zeta)} > 0$ since $R_2$ consists of positive real numbers. Thus the non-zero roots of the binomial $f^{(2)}$ are simple, with $\overline{\mu - \mu_1}$ of them real. Proposition 4.1 applies as we can see from the expansions (4.1) for $f^{(1)}$ and $f^{(3)}$.

$$\begin{aligned}
f_t(xt^{\frac{b}{\ell}})/t^a &= f^{(1)}(x) \;-\; t^{\frac{b\mu}{\ell}-a} x^\mu \prod_{\zeta\in R_2} (\zeta - x^\ell t^b)^{m(\zeta)}\\
&= f^{(1)}(x) \;-\; t^{\frac{b\mu}{\ell}-a} x^\mu \Big(\prod_{\zeta\in R_2} \zeta^{m(\zeta)}\Big) \;+\; h^{(1)}(x,t)\,.\\
f_t(x) &= f^{(3)}(x) \;+\; t^a \prod_{\zeta\in R_1} (t^{-b}x^\ell - \zeta)^{m(\zeta)}\\
&= f^{(3)}(x) \;+\; t^{a-\frac{b\mu_1}{\ell}} x^{\mu_1} \;+\; h^{(3)}(x,t)\,.
\end{aligned}$$

Assume that $\ell$ is odd. Then the map $x \mapsto x^\ell$ is a bijection from the real roots of $f^{(1)}$ (resp., $f^{(3)}$) to $R_1$ (resp., $R_2$). Conditions (1) and (2) imply that the contribution of any real root $\rho$ such that $\rho^\ell = \zeta \in R_i$ and $m(\zeta)$ is even is equal to 2.

Assume that $\ell$ is even and $\mu$ is odd. Then the real roots of $f^{(1)}$ (resp., $f^{(3)}$) come in pairs $(\rho, -\rho)$ with $\rho^\ell = \zeta \in R_1$ (resp., $R_2$). If $m(\zeta)$ is odd, then the contributions of $\rho$ and $-\rho$ are both equal to 1. If $m(\zeta)$ is even, then one contribution is 2, while the other is 0, as $\mu$ is odd. Finally, note that $\overline{\mu - \mu_1} = 1$ if $\ell$ is even and $\mu$ is odd.

*Proof of Theorem* 5.1. Set

$$\mu \;:=\; N + \ell \sum_{i=1}^{p} (k - d_i)\lambda_i \,.$$

The inequality (5.1) can be rewritten as

$$\ell \sum_{i=p+1}^{\nu} d_i \lambda_i \;<\; \mu \,.$$

Then, by Lemma 5.2 there exist polynomials $h_1, \ldots, h_n$ with distinct roots such that $h_i$ has degree $d_i$ with $d_i$ real roots and the polynomial

$$g(x) \;:=\; x^\mu \prod_{i=1}^{p} (h_i(x^\ell))^{\lambda_i} \;-\; \prod_{i=p+1}^{\nu} (h_i(x^\ell))^{\lambda_i}$$

has either

$$\sum_{i=1}^{\nu} d_i \,\overline{\lambda_i} \;+\; \overline{d}$$

or else

$$2 \sum_{i=1}^{\nu} d_i + 1$$

simple real roots according as $\ell$ is odd, or $\ell$ is even and $N$ is odd, respectively. The polynomial $g(x)$ can be rewritten as

$$g(x) \;=\; x^N \prod_{i=1}^{p} \big(x^{\ell(k-d_i)} h_i(x^\ell)\big)^{\lambda_i} \;-\; \prod_{i=p+1}^{\nu} \big(h_i(x)^\ell\big)^{\lambda_i}$$

If $d_i = k$, set $g_i(x) := h_i(x)$. Otherwise, set

$$\begin{aligned} g_i(x) \;&:=\; \epsilon(1 + x + \cdots + x^{k-d_i-1}) + x^{k-d_i} h_i(x) \qquad 1 \le i \le p \\ g_i(x) \;&:=\; h_i(x) + \epsilon(x^{d_i+1} + \cdots + x^k) \qquad p+1 \le i \le \nu \,. \end{aligned}$$

For sufficiently small $\epsilon > 0$, the polynomial

$$f(x) \;=\; x^N \prod_{i=1}^{p} \big(g_i(x^\ell)\big)^{\lambda_i} - \prod_{i=p+1}^{\nu} \big(g_i(x^\ell)\big)^{\lambda_i}$$

has simple roots and at least the same number of real roots as $g$.

**Theorem 5.3.** *Assume that $N > k\ell\sum_{i=p+1}^{\nu}\lambda_i$ and let $m$ be the maximal number of real solutions to a generic system with support the near circuit $\mathcal{C}$.*

*If $\ell$ is even, then $m = 2k\nu + 1$.*

*Suppose now that $\ell$ is odd.*

(1) *If $\lambda_1, \ldots, \lambda_p$ are even, then*

$$m \;=\; 2kp + k\sum_{i=p+1}^{\nu}\overline{\lambda_i} + \overline{\delta}\,.$$

(2) *If exactly one number among $\lambda_1, \ldots, \lambda_p$ is odd, $k = \ell = 1$ and $\delta$ is odd, then*

$$m \;=\; 2p + 1 + \sum_{i=p+1}^{\nu}\overline{\lambda_i}\,.$$

(3) *If $\lambda_{p+1}, \ldots, \lambda_n$ are even, then*

$$m \;=\; 2k(n-p) + k\sum_{i=1}^{p}\overline{\lambda_i} + \overline{N}\,.$$

(4) *If exactly one number among $\lambda_{p+1}, \ldots, \lambda_n$ is odd, $k = \ell = 1$ and $N$ is odd, then*

$$m \;=\; 2(n-p) + 1 + \sum_{i=1}^{p}\overline{\lambda_i}\,.$$

*Proof.* We apply Theorem 5.1 with each $d_i = k$. The case of $\ell$ even is a direct consequence of Theorem 4.6 and Theorem 5.1. Suppose now that $\ell$ is odd and let $B_1$ and $B_2$ be the upper bounds for the number of real solutions to a generic system with support $\mathcal{C}$ which are given in Theorem 4.6 by formulas (4.5) and (4.6), respectively. Set

$$B_- \;:=\; k\sum_{i=1}^{\nu}\overline{\lambda_i} + \overline{d}\,,$$

where $d := N - k\ell\sum_{i=p+1}^{\nu}\lambda_i > 0$. Note that $\delta, N \geq d$, so $\delta, N > 0$. By Theorem 5.1, the number $B_-$ is a lower bound on the maximal number of real solutions of a generic system with support $\mathcal{C}$.

We check that $B_1 \geq B_-$ and analyze the conditions under which $B_1 = B_-$. As $\delta > 0$ and $N > 0$, we have

$$B_1 - B_- \;=\; k\sum_{i=1}^{p}(2 - \overline{\lambda_i}) \;+\; \overline{\delta} - \overline{d}\,.$$

We have

$$\overline{\delta} - \overline{d} \;=\; \begin{cases} \;\;0 & \text{if } k\ell\sum_{i=1}^{p}\lambda_i \text{ is even} \\ \;\;1 & \text{if } k\ell\sum_{i=1}^{p}\lambda_i \text{ is odd and } d \text{ is odd} \\ -1 & \text{if } k\ell\sum_{i=1}^{p}\lambda_i \text{ is odd and } d \text{ is even} \end{cases}$$

If $\overline{\delta} - \overline{d} = 0$, then $B_1 \geq B_-$ with equality only if $\lambda_1, \ldots, \lambda_p$ are even. This proves Part (1).

If $\overline{\delta} - \overline{d} = 1$, then $B_1 > B_-$. Assume now that $\overline{\delta} - \overline{d} = -1$. Then $k\ell\sum_{i=1}^{p}\lambda_i$ is odd, $d$ is even and $B_1 - B_- = k\ell\sum_{i=1}^{p}(2-\overline{\lambda_i}) - 1$. Since $k\ell\sum_{i=1}^{p}\lambda_i$ is odd, $k\ell$ is odd and at least one number among $\lambda_1,\ldots,\lambda_p$ is odd. Thus $B_1 - B_- \geq k\ell - 1$ with equality only if exactly one number among $\lambda_1,\ldots,\lambda_p$ is odd. Part (2) now follows.

Parts (3) and (4) are similar. ∎

**Theorem 5.4.** *Assume that $\lambda_i \in \{1,2\}$ for $i = 1,\ldots,\nu$, $\ell$ is odd, and let $m$ be the maximal number of real solutions to a generic system with support the near circuit $\mathcal{C}$.*

(1) *If $N > k\ell\sum_{i=p+1}^{\nu}\lambda_i$, then*

$$m \;=\; k\sum_{i=1}^{\nu}\lambda_i + \overline{N - k\ell\sum_{i=p+1}^{\nu}\lambda_i}\,.$$

(2) *Suppose that $\ell = 1$. If $N < k\sum_{i=p+1}^{\nu}\lambda_i$, then*

$$m \;=\; v(\mathcal{C}) \;=\; \max\left\{k\sum_{i=p+1}^{\nu}\lambda_i\ , N + k\sum_{i=1}^{p}\lambda_i\right\}\ .$$

*Proof.* For Part (1), the number $m$ equals the upper bound given by Descartes's rule of signs when applied to a polynomial of the form (3.5). Theorem 5.1 with each $d_i = k$ implies the existence of a polynomial with this form with $m$ real roots.

For Part (2), we have that $N < k\sum_{i=p+1}^{\nu}\lambda_i$. If we also have $N + k\sum_{i=1}^{p}\lambda_i > k\sum_{i=p+1}^{\nu}\lambda_i$, then there exist nonnegative integers $d_1,\ldots,d_\nu \leq k$ such that

$$d \;=\; N + k\sum_{i=1}^{p}\lambda_i - \left(\sum_{i=1}^{\nu}\lambda_i d_i\right) \;\in\; \{1,2\}\,,$$

as $\lambda_i \in \{1,2\}$. By Theorem 5.1, there exists a polynomial of the form (3.5) having $\sum_{i=1}^{\nu}\lambda_i d_i + d = N + k\sum_{i=1}^{p}\lambda_i$ non-zero real roots.

Finally, suppose that $N + k\sum_{i=1}^{p}\lambda_i \leq k\sum_{i=p+1}^{\nu}\lambda_i$. Consider a polynomial

$$f_t(x) \;=\; t\cdot x^N\prod_{i=1}^{p}(g_i(x))^{\lambda_i} - \prod_{i=p+1}^{\nu}(g_i(x))^{\lambda_i}\,,$$

where $g_1,\ldots,g_\nu$ are polynomials of degree $k$ with non-zero constant terms. The lower part of the Newton polygon of $f_t(x)$ consists of a single segment projecting onto $[0, k\sum_{i=p+1}^{\nu}\lambda_i]$. Hence, if Proposition 4.1 applies, the number of real roots of $f_t$ for $t > 0$ small enough is the sum of contributions of the non-zero real roots of $g_{p+1},\ldots,g_\nu$. Choosing polynomials $g_1,\ldots,g_\nu$ with distinct roots satisfying conditions

(1) If $i,j > p$, then $g_i$ has $k$ positive roots, and if $\lambda_i$ is odd and $\lambda_j$ is even, then the leading coefficient of $g_i$ is positive and every root of $g_i$ is less than every root of $g_j$.
(2) The polynomials $g_1,\ldots,g_p$ are positive at each root of of $g_{p+1},\ldots,g_\nu$.

By Proposition 4.1, $f_t$ has $k\sum_{i=p+1}^{\nu}\lambda_i$ non-zero real roots for $t > 0$ small enough. ∎

**Remark 5.5.** The example of Section 2 is a special case of Part(1) of Theorem 5.4. Indeed, in Section 2, we have

$$w_0 = e_n,\ w_i = e_i\ i = 1, \ldots, n{-}1, \text{ and } w_n = \sum \epsilon_i e_i \ + \ le_n\,,$$

so that

$$N \ = \ l,\ \nu \ = \ |\epsilon| + 1 \ = \ \sum_i \lambda_i\ , \text{ and } p \ = \ |\epsilon|\,.$$

Then the maximum number of Part(1) of Theorem 5.4 is

$$m \ = \ k(|\epsilon| + 1) + \overline{l - k}\,,$$

which is what we found in Section 1.

**Theorem 5.6.** *The number of real roots of a generic system with support a primitive near circuit $\mathcal{C} = \{0, \ell e_n, \ldots, k\ell e_n, w_1, \ldots, w_n\}$ in $\mathbb{R}^n$ is at most $(2\nu - 1)k + 2$ if $\ell$ is odd, or $2k\nu + 1$ if $\ell$ is even. Moreover, these bounds are sharp.*

*Proof.* As the numbers $\lambda_1, \ldots, \lambda_\nu$ are coprime, at least one is odd. Since $\overline{\lambda_j} \leq 2$, the upper bound for odd $\ell$ follows from Theorem 4.6 (see Remark 4.7). The sharpness of this bound follows from Theorem 5.4 for a primitive near circuit with all $\lambda_1, \ldots, \lambda_\nu$ but one equal to 2 and one which is equal to 1, and where and $N - k\ell\sum_{i=p+1}^{\nu} \lambda_i$ is positive and even (Example 3.6 shows that such a near circuit exists).

The bound $2k\nu + 1$ for $\ell$ even comes from Theorem 4.6, its sharpness follows from Theorem 5.3.

A near circuit with $k = 1$ is just a circuit. The following result is a particular case of the previous one.

**Theorem 5.7.** *The number of real roots of a generic system with support a primitive circuit $\mathcal{C} = \{0, \ell e_n, , w_1, \ldots, w_n\}$ in $\mathbb{R}^n$ is at most $2\nu + 1$, and this bound this sharp.*

*The absolute upper bound for the number of real roots of a generic system with support a primitive circuit in $\mathbb{R}^n$ is $2n + 1$, and this bound is sharp. Moreover, this bound can be attained only for non-degenerate circuits.*

*Proof.* We only need to prove the last sentence as the others are corollaries of Theorem 5.6. For this, we note that the bound $2\nu + 1$ is obtained when $\delta$ and $N$ are non-zero. Hence the absolute bound $2n + 1$ is obtained when $\nu = n$, $\delta$ and $N$ are non-zero, which is exactly the case of a non-degenerate circuit.

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