ASYMPTOTICALLY MAXIMAL FAMILIES OF HYPERSURFACES IN TORIC VARIETIES

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Abstract. A real algebraic variety is maximal (with respect to the Smith-Thom inequality) if the sum of the Betti numbers (with $\mathbb{Z}_2$ coefficients) of the real part of the variety is equal to the sum of Betti numbers of its complex part. We prove that there exist polytopes that are not Newton polytopes of any maximal hypersurface in the corresponding toric variety. On the other hand we show that for any polytope Δ there are families of hypersurfaces with the Newton polytopes $(\lambda\Delta)_{\lambda\in\mathbb{N}}$ that are asymptotically maximal when λ tends to infinity. We also show that these results generalize to complete intersections.

1. Introduction

In 1876 Harnack showed that the maximal number of connected components of a real algebraic plane projective curve of degree m is $(m-1)(m-2)/2+1$. He also proved that for any positive integer m there exist curves of degree m which are maximal in this sense (i.e. with $(m-1)(m-2)/2+1$ connected components). Harnack's bound is generalized to the case of any real algebraic variety by the Smith-Thom inequality. Let $b_i(V;K)$ be the i^{th} Betti number of a topological space V with coefficients in a field K (i.e. $b_i(V;K) = \dim_K(H_i(V;K))$). Denote by $b_*(V;K)$ the sum of the Betti numbers of V. Let X be a complex algebraic variety equipped with an anti-holomorphic involution c. The real part $\mathbb{R}X$ of X is the fixed point set of c. Then the Smith-Thom inequality states that $b_*(\mathbb{R}X;\mathbb{Z}_2) \leq b_*(X;\mathbb{Z}_2)$. A variety X for which $b_*(\mathbb{R}X;\mathbb{Z}_2) = b_*(X;\mathbb{Z}_2)$ is called a *maximal* variety or *M-variety*. The question "does a given family of real algebraic varieties contain maximal elements?" is one of the problems in topology of real algebraic varieties. For the family of the hypersurfaces of a given degree in $\mathbb{R}P^d$ a positive answer is obtained in [IV04] using the combinatorial Viro method called T-construction (see [Vir84], [Vir04], [Ite97], and Theorem 3.1). This question is, in general, a difficult problem. Indeed we show that Itenberg-Viro's theorem of existence of M-hypersurfaces of any degree in the projective spaces of any dimension cannot be generalized straightforwardly to all projective toric varieties. More precisely, in any dimension greater than or equal to 3 there are polytopes Δ such that no hypersurface in the toric variety X_Δ associated with Δ, with the Newton polytope Δ, is maximal. However, in the 2-dimensional case such a generalization of the Harnack theorem holds (see Section 4).

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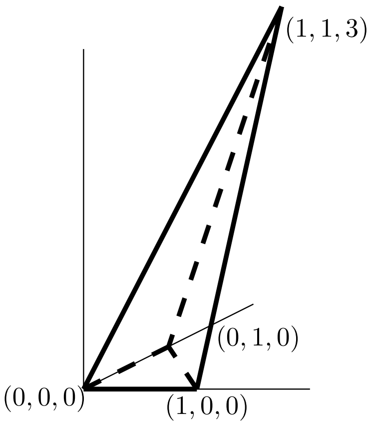

FIGURE 1. Tetrahedron Δ_3.

Let us first consider the 3-dimensional case. Let k be a positive integer number, and Δ_k be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0),(1,0,0),(0,1,0)$, and $(1,1,k)$. Note that the only integer points of Δ_k are its vertices.

Proposition 1.1. *For any odd $k \geq 3$ and any even $k \geq 8$, there is no maximal surface in X_{Δ_k} with the Newton polytope Δ_k.*

It is easy to generalize the above examples in dimension 3 to higher dimensions. From now on by *polytope* we mean a convex polytope with integer vertices in the positive orthant $(\mathbb{R}^+)^d = \{(x_1, \dots x_d) \in \mathbb{R}^d \mid x_1 \geq 0, \dots, x_d \geq 0\}$.

Proposition 1.2. *For any integer $d \geq 3$ there exist d-dimensional polytopes Δ such that no hypersurface in X_Δ with the Newton polytope Δ is maximal.*

It is then natural to tackle the following weaker question. Let Δ be a d-dimensional polytope and $\{\lambda \cdot \Delta\}_{\lambda \in \mathbb{N}}$ the family of the multiples of Δ. Suppose that there exists a collection of polynomials $\{P_\lambda\}_{\lambda \in \mathbb{N}}$ satisfying the following conditions :

(1) the polytope $\lambda \cdot \Delta$ is the Newton polytope of P_λ,
(2) the total Betti numbers $b_*(\mathbb{R}Z_\lambda; \mathbb{Z}_2)$ and $b_*(Z_\lambda; \mathbb{Z}_2)$ are equivalent when λ tends to infinity (here Z_λ denotes the hypersurface in X_Δ defined by P_λ).

In this case we say that the family $\{Z_\lambda\}_{\lambda \in \mathbb{N}}$ is *asymptotically maximal*. Given a d-dimensional polytope Δ in $(\mathbb{R}^+)^d$, does there exist an asymptotically maximal family of hypersurfaces in X_Δ? A positive answer to this question is given here.

Theorem 1.3. *For any polytope Δ there exists an asymptotically maximal family of hypersurfaces $\{Z_\lambda\}_{\lambda \in \mathbb{N}}$ in X_Δ such that for any λ the Newton polytope of Z_λ is $\lambda \cdot \Delta$.*

The above statements have generalizations to complete intersections in projective toric varieties. As a counterpart for Propositions 1.1 and 1.2 we show that, for any integer d greater than 2 there exist polytopes $\Delta_d \subset (\mathbb{R}^+)^d$ of dimension d such that the hypersurfaces defining a maximal complete intersection in X_{Δ_d} cannot all have the Newton polytope Δ_d.

Proposition 1.4. *For any positive integers d and k such that $k \leq d$ there exists a d-dimensional polytope Δ_d such that k hypersurfaces defining a maximal complete intersection in X_{Δ_d} cannot all have the Newton polytope Δ_d.*

On the other hand the following theorem is a counterpart of Theorem 1.3 for complete intersections. Let Δ be a d-dimensional polytope in $\mathbb{R}^d$, and k be an integer such that $1 \leq k \leq d$. Knudsen-Mumford's theorem (see [KKMSD67] p.161 and Theorem 3.2) asserts that there exists a positive integer l such that $l \cdot \Delta$ admits a convex primitive triangulation. Let $\lambda_1, \cdots, \lambda_k$ be k positive integers. Denote by Δ_{λ_i} the polytope $\lambda_i l \cdot \Delta$. Let $\{(\lambda_{1,m}, \cdots, \lambda_{k,m})\}_{m\in\mathbb{N}}$ be a sequence of k-tuples of positive integers such that $\lambda_{i,m}$ tends to infinity for any $i = 1, \dots, k$. Let $\{(Z_{\lambda_{1,m}}, \cdots, Z_{\lambda_{k,m}})\}_m$ be a sequence of k-tuples of algebraic hypersurfaces in X_Δ such that $Z_{\lambda_{i,m}}$ has the Newton polytope $\Delta_{\lambda_{i,m}}$. Assume that for any natural number m the variety $Y_m = Z_{1,m} \cap \cdots \cap Z_{k,m}$ is a complete intersection.

Definition 1.5. Under the above hypotheses, the family $\{Y_m\}_{m\in\mathbb{N}}$ is called *asymptotically maximal* if $b_*(\mathbb{R}Y_m; \mathbb{Z}_2)$ is equivalent to $b_*(Y_m; \mathbb{Z}_2)$ when m tends to infinity.

Theorem 1.6. *Let Δ be a d-dimensional polytope, and k be an integer number satisfying $1 \leq k \leq d$. Let $\{(\lambda_{1,m}, \cdots, \lambda_{k,m})\}_{m\in\mathbb{N}}$ be a sequence of k-tuples of natural numbers such that $\lambda_{i,m}$ tends to infinity for any $i = 1, \dots, k$. Then, there exists a sequence of k-tuples $\{(Z_{\lambda_{1,m}}, \cdots, Z_{\lambda_{k,m}})\}_{m\in\mathbb{N}}$ of algebraic hypersurfaces in X_Δ such that*

1. *$Z_{\lambda_{i,m}}$ has the Newton polytope $\Delta_{\lambda_{i,m}}$*
2. *for any natural number m, the variety $Y_m = Z_{1,m} \cap \cdots \cap Z_{k,m}$ is a complete intersection,*
3. *the family $\{Y_m\}_{m\in\mathbb{N}}$ is asymptotically maximal.*

Organization of the material. We first describe combinatorial patchworking and recall some results we will use. In Section 3 we describe Itenberg and Viro construction of asymptotically maximal hypersurfaces in projective spaces. We then prove the existence of asymptotically maximal families of hypersurfaces for any Newton polytope (Theorem 1.3). Proposition 1.1 and Proposition 1.4 are proved respectively in Section 4 and in Section 5. Finally, Section 6 is devoted to the existence of asymptotically maximal families of complete intersections. We describe there Itenberg and Viro construction of asymptotically maximal complete intersections in projective spaces and we prove Theorem 1.6.

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2. Preliminaries

2.1. **Toric varieties.** We fix here some conventions and notations, the construction of toric varieties we use is based on the one described in [Ful93]. Let Δ be a polytope, p a vertex of Δ, and $\Gamma_1, \cdots, \Gamma_k$ the facets of Δ containing p. To p we associate the cone σ_p generated by the minimal integer inner normal vectors of $\Gamma_1, \cdots, \Gamma_k$. The inner normal fan $\mathfrak{E}_\Delta$ is the fan whose d-dimensional cones are the cones σ_p for all vertices p of Δ. The toric variety X_Δ associated to Δ is the toric variety $X(\mathfrak{E}_\Delta)$ associated to the fan $\mathfrak{E}_\Delta$ (see [Ful93]).

2.2. **Combinatorial patchworking.** By a *subdivision* of a polytope we mean a subdivision in convex polytopes (with integer vertices). A subdivision τ of a polytope Δ of dimension d is called *convex* if there exists a convex piecewise-linear function $\Phi : \Delta \to \mathbb{R}$ whose domains of linearity coincide with the d-dimensional polytopes of τ.

Let us briefly describe the *combinatorial patchworking*, also called *T-construction*, which is a particular case of the Viro method. A more detailed exposition can be found in [IV04] (see also [Vir04] or [GKZ94] p. 385).

Given a triple (Δ, τ, D), where Δ is a polytope, τ a convex triangulation of Δ, and D a distribution of signs at the vertices of τ, the combinatorial patchworking, produces an algebraic hypersurface Z in X_Δ.

Let Δ be a d-dimensional polytope $(\mathbb{R}^+)^d$ and τ be a convex triangulation of Δ. Denote by $s_{(i)}$ the reflection with respect to the coordinate hyperplane $x_i = 0$ in $\mathbb{R}^d$. Consider the union Δ^* of all copies of Δ under the compositions of reflections $s_{(i)}$ and extend τ to a triangulation τ^* of Δ^* by means of these reflections. Let $D(\tau)$ be a sign distribution at the vertices of the triangulation τ (i.e. each vertex is labelled with $+$ or $-$). We extend $D(\tau)$ to a distribution of signs at the vertices of τ^* using the following rule : for a vertex a of τ^*, one has $\mathrm{sign}(s_{(i)}(a)) = \mathrm{sign}(a)$ if the i-th coordinate of a is even, and $\mathrm{sign}(s_{(i)}(a)) = -\mathrm{sign}(a)$, otherwise.

Let σ be a d-dimensional simplex of τ^* with vertices of different signs, and E be the hyperplane piece which is the convex hull of the middle points of the edges of σ with endpoints of opposite signs. We separate vertices of σ labelled with $+$ from vertices labelled with $-$ by E. The union of all these hyperplane pieces forms a piecewise-linear hypersurface H.

For any facet Γ of Δ^*, let N^Γ be a vector normal to Γ. Let F be a face of Δ^* and $\Gamma_1, \ldots, \Gamma_k$ be the facets containing F. Let L be the linear space spanned by $N^{\Gamma_1}, \ldots, N^{\Gamma_k}$. For any $v = (v_1, \ldots, v_d) \in L \cap \mathbb{Z}^d$ identify F with ${s_{(1)}}^{v_1} \circ {s_{(2)}}^{v_2} \circ \cdots \circ {s_{(d)}}^{v_d}(F)$. Denote by $\widetilde{\Delta}$ the result of the identifications. The variety $\widetilde{\Delta}$ is homeomorphic to the real part $\mathbb{R}X_\Delta$ of X_Δ (see, for example, [GKZ94] Theorem 5.4 p. 383 or [Stu94b] Proposition 2).

Denote by $\widetilde{H}$ the image of H in $\widetilde{\Delta}$. Let Q be a polynomial with the Newton polytope Δ. It defines a hypersurface Z_0 in the torus $(\mathbb{C}^*)^d$ contained in X_Δ. The closure Z of Z_0 in X_Δ is the hypersurface defined by Q in X_Δ. We call Δ the *Newton polytope* of Z.

Theorem 2.1 (T-construction, O. Viro)**.** *Under the hypotheses made above, there exists a hypersurface Z in X_Δ with the Newton polytope Δ and a homeomorphism $h : \mathbb{R}X_\Delta \to \widetilde{\Delta}$ such that $h(\mathbb{R}Z) = \widetilde{H}$.*

The hypersurface Z in the above theorem is called a *real algebraic T-hypersurface.* A d-dimensional simplex with integer vertices is called *primitive* if its volume is equal to $\frac{1}{d!}$. A triangulation τ of a d-dimensional polytope is *primitive* if every d-simplex of the triangulation is primitive. Let Δ be a d-dimensional polytope. We call *lattice volume* of Δ and denote by $\mathrm{Vol}(\Delta)$ the volume normalized so that a primitive d-simplex has volume 1. The usual volume is denoted by $\mathrm{vol}(\Delta)$. If Δ is a d-dimensional polytope, then $\mathrm{Vol}(\Delta) = d!\,\mathrm{vol}(\Delta)$.

2.3. **Sturmfels' theorem for complete intersections.** In [Stu94b] B. Sturmfels proposed a combinatorial construction producing complete intersections. In fact, Sturmfels' construction is an extended version of the combinatorial patchworking. We quote here this theorem in the particular case we need. For the general statement and the proof we refer to [Stu94b].

Let Δ_0 be a d-dimensional polytope and $\lambda_1, \cdots, \lambda_k$ positive integers, where $k \leq d$. Denote by Δ_i the polytope $\lambda_i \cdot \Delta_0$ and by Δ the Minkowski sum $\Delta_1 + \cdots + \Delta_k$. Let ν_i be a piecewise-linear convex function on Δ_i defining a triangulation τ_i with integer vertices. For each Δ_i, choose a distribution of signs D_i at the vertices of τ_i.

The initial data of the procedure of construction of a complete intersection using Sturmfels' theorem are the polytopes Δ_i, the functions ν_i and the sign distributions D_i. Apply the T-construction for each triple (Δ_i, τ_i, D_i) to construct the hypersurfaces S_i. Let D_i^* be the sign distribution at the vertices of τ_i^*.

The functions $\nu_1, \cdots, \nu_k$ define a convex decomposition of Δ in the following way (see [Stu94b], [Stu94a] or [Bih02]). Let $\bar{\Delta}_i$ be the convex hull of the set $\{(x, \nu_i(x)), x \in \Delta_i\}$ in $\mathbb{R}^d \times \mathbb{R}$. Let $\bar{\Delta} \subset \mathbb{R}^d \times \mathbb{R}$ be the Minkowski sum $\bar{\Delta}_1 + \cdots + \bar{\Delta}_k$ and denote by G the lower part of the boundary of $\bar{\Delta}$. Let ν be the piecewise-linear convex function of graph G defined on Δ (i.e. G is the union of facets of $\bar{\Delta}$ whose inner normal vectors have positive last coordinate). The function ν defines a convex subdivision δ of Δ whose d-dimensional polytopes are the domains of linearity of ν. Let Γ be a polytope in δ and $\bar{\Gamma}$ its image by ν. Then $\bar{\Gamma}$ can be uniquely written as the Minkowski sum $\bar{\Gamma}_1 + \cdots + \bar{\Gamma}_k$ where $\bar{\Gamma}_i$ is a face of $\bar{\Delta}_i$ for $i = 1, \cdots, k$. This induces a decomposition of Γ as a Minkowski sum $\Gamma = \Gamma_1 + \cdots + \Gamma_k$ such that $\nu_i(\Gamma_i) = \bar{\Gamma}_i$. Sturmfels' theorem requires the following genericity condition on the functions ν_i.

Definition 2.2. The k-tuple $\nu_1, \cdots, \nu_k$ is said *sufficiently generic* if for any polytope Γ of δ, $\dim \bar{\Gamma} = \dim \bar{\Gamma}_1 + \cdots + \dim \bar{\Gamma}_k$, where $\bar{\Gamma} = \bar{\Gamma}_1 + \cdots + \bar{\Gamma}_k$ is the unique way to write $\bar{\Gamma}$ as the Minkowski sum of faces of $\bar{\Delta}_1, \cdots, \bar{\Delta}_n$.

We call *mixed subdivision* a subdivision δ obtained as above from triangulations $\tau_1, \cdots, \tau_k$ and sufficiently generic convex functions $\nu_1, \cdots, \nu_k$. A mixed subdivision δ is equipped with a decomposition of each of its polytopes Γ as a Minkowski sum $\Gamma = \Gamma_1 + \cdots + \Gamma_k$, where Γ_i is a simplex of τ_i. Two mixed subdivisions are considered as equal if and only if they coincide as polyhedral subdivisions, and each polytope of these subdivisions has the same decomposition into a Minkowski sum in both of them.

Extend δ to a subdivision δ^* of Δ^* by means of the reflections with respect to coordinate hyperplanes. The extension of the sign distribution to δ^* is as follows. Let v be a vertex of δ^*, and let $v_1, \cdots, v_k$ be the vertices of $\tau_1^*, \cdots, \tau_k^*$ corresponding to v. Then

$$\epsilon_j(v) = \mathrm{sign}(v_j).$$

For $j \in \{1, \cdots, k\}$ construct the hypersurface $\widetilde{S_j}$ in the following way. For any polytope Γ' in δ^*, consider its symmetric copy Γ in δ. There is a unique way to write $\Gamma = \Gamma_1 + \cdots + \Gamma_k$ with Γ_i in τ_i such that $\nu_1(\Gamma_1) + \cdots + \nu_k(\Gamma_k) = \nu(\Gamma)$. For $i \in \{1, \cdots, k\}$ let Γ'_i be the symmetric copy of Γ_i in τ_i^* such that $\Gamma' = \Gamma'_1 + \cdots + \Gamma'_k$. Define the hypersurface S_j^* in Δ^* by $S_j^* \cap \Gamma' = \Gamma'_1 + \cdots + S_j \cap \Gamma'_j + \cdots + \Gamma'_k$ for all Γ' in δ^*. Let $\widetilde{S_j}$ be the image of S_j^* in $\widetilde{\Delta}$.

Theorem 2.3 (B. Sturmfels)**.** *With the above notation, there exist hypersurfaces Z_i with the Newton polytopes Δ_i, respectively, and a homeomorphism $f : \mathbb{R}X_\Delta \to \widetilde{\Delta}$ such that the hypersurfaces Z_i define a complete intersection Y in X_Δ, and f sends $\mathbb{R}Z_i$ (resp., $\mathbb{R}Y$) onto $\widetilde{S}_i$. (resp., $\cap_{j=1\cdots k}\widetilde{S_j}$).*

2.3.1. *Cayley trick.* Instead of constructing the complete intersection in the Minkowski sum of Newton polytopes, it is convenient to use so-called Cayley trick (see, for example, [Stu94a]).

Let $\Delta_1, \ldots, \Delta_k$ be convex polytopes with integer vertices in $\mathbb{R}^d$ ($k \le d$). For any $i = 1, \ldots, k$ put

$$\hat{\Delta}_i = \{(x_1, \ldots, x_{k+d}) \in \mathbb{R}^{k+d} \mid x_i = 1; x_j = 0 \text{ if } j \le k \text{ and } j \ne i; \\ (x_{k+1}, \ldots, x_{k+d}) \in \Delta_i\}.$$

The convex hull of $\hat{\Delta}_1, \ldots, \hat{\Delta}_k$ in $\mathbb{R}^{k+d}$ is called *Cayley polytope* and is denoted by $C(\Delta_1, \ldots, \Delta_k)$. The intersection of $C(\Delta_1, \ldots, \Delta_k)$ with the subspace $B \subset \mathbb{R}^{k+d}$ defined by $x_1 = \ldots = x_k = 1/k$ is naturally identified with the Minkowski sum Δ of $\Delta_1, \ldots, \Delta_k$ multiplied by $1/k$. Thus, any triangulation of the Cayley polytope $C(\Delta_1, \ldots, \Delta_k)$ induces a subdivision of the Minkowski sum of $\Delta_1, \ldots, \Delta_k$.

The following lemma can be found, for example, in [Stu94a].

Lemma 2.4. *The correspondence described above establishes a bijection between the set of convex triangulations with integer vertices of $C(\Delta_1, \ldots, \Delta_k)$ and the set of mixed subdivisions of the Minkowski sum of $\Delta_1, \ldots, \Delta_k$.* $\square$

Denote by C^* the union of the symmetric copies of $C(\Delta_1, \ldots, \Delta_k)$ under the reflections $s_{(i)}$, $i = k+1, \ldots, k+n$, where $s_{(i)}$ is the reflection of $\mathbb{R}^{k+d}$ with respect to the hyperplane $\{x_i = 0\}$, and compositions of these reflections.

Choose a convex triangulation τ of $C(\Delta_1, \ldots, \Delta_k)$ having integer vertices and a distribution of signs at the vertices of τ. Extend the triangulation τ to a symmetric triangulation τ^* of C^* and the distribution of signs at the vertices of τ to a distribution at the vertices of the extended triangulation by the same rule as in Subsection 2.2: passing from a vertex to its mirror image with respect to a coordinate hyperplane we preserve its sign if the distance from the vertex to the hyperplane is even, and change the sign if the distance is odd.

For any $(k+d-1)$-dimensional simplex γ of τ^* and any $j = 1, \ldots, k$ denote by γ_j the maximal face of γ which belongs to a symmetric copy of $\hat{\Delta}_j$. Let $K_j(\gamma)$ be the convex hull of the middle points of the edges of γ_j having endpoints of opposite signs, and let $H(\gamma)$ be the intersection of the join $K_1(\gamma) * \ldots * K_k(\gamma)$ with B. Denote by H the union of the intersections $H(\gamma)$, where γ runs over all the $(k+d-1)$-dimensional simplices of τ^*, and denote by $\widetilde{H}$ the image of H in $\widetilde{(\frac{1}{k}\Delta)}$.

The following statement is an immediate corollary of Theorem 2.3.

Proposition 2.5. *Assume that all the polytopes $\Delta_1, \ldots, \Delta_k$ are multiples of the same polytope Π with integer vertices. Then, there exist nonsingular real hypersurfaces $Z_1, \ldots, Z_k$ in X_Π with the Newton polytopes $\Delta_1, \ldots, \Delta_k$, respectively, and a homeomorphism $f : \mathbb{R}X_\Pi \to \widetilde{(\frac{1}{k}\Delta)}$ such that the hypersurfaces $Z_1, \ldots, Z_k$ define a complete intersection Y in X_Π and f maps the set of real points $\mathbb{R}Y$ of Y onto $\widetilde{H}$.* $\square$

2.4. **Formulae for the Betti numbers.** V. Danilov and A. Khovanskii [DK87] computed the Hodge numbers of a smooth hypersurface in a toric variety X_Δ in terms of the polytope Δ involving in particular the coefficients of the Ehrhart polynomial of Δ (see [Ehr94] or [Ehr67]). Our aim being to investigate asymptotical behaviors of certain families of hypersurfaces or complete intersections, we need only the simpler results that are quoted below.

Definition 2.6. A d-dimensional polytope Δ is simple if for each vertex a of Δ, the number of edges of Δ containing a is d.

Let $l^*(\Delta)$ be the number of integer points in the interior of Δ (i.e. $l^*(\Delta) = \#(\mathbb{Z}^d \cap (\Delta \setminus \partial\Delta))$). The following statement can be found in [DK87] Section 5.11.

Lemma 2.7. *Let Δ be a 3-dimensional simple polytope, and Z be an algebraic hypersurface of X_Δ with the Newton polytope Δ. Then $b_*(Z;\mathbb{C}) = l^*(2\Delta) - 2l^*(\Delta) - \sum_{\Gamma\in\mathcal{F}_2(\Delta)}(l^*(\Gamma)-1) - 1$.*

The following two propositions can be derived from Khovanskii's results (see [Hov77] and [Hov78]) or can be found in [Mik].

Proposition 2.8. *Let Δ be a polytope, and $\{Z_\lambda\}_{\lambda\in\mathbb{N}}$ be a family of algebraic hypersurfaces in X_Δ with the Newton polytopes $\lambda\cdot\Delta$. Then $b_*(Z_\lambda;\mathbb{Z}_2)$ is equivalent to $\mathrm{Vol}(\lambda\cdot\Delta)$ when λ tends to infinity.*

Denote by $\mathrm{Vol}(\Delta_1,\cdots,\Delta_k)$ the mixed volume of the polytopes $\Delta_1,\cdots,\Delta_k$. We choose a normalization of the mixed volume in such a way that for a primitive simplex σ we have $\mathrm{Vol}(\sigma,\cdots,\sigma) = 1$.

Proposition 2.9. *Let Δ be a d-dimensional polytope, and k be a positive integer satisfying $k \le d$. Assume that for any collection $\lambda_1,\cdots,\lambda_k$ of positive integers we have a collection of k hypersurfaces $Z_{\lambda_1},\cdots,Z_{\lambda_k}$ in X_Δ with the Newton polytopes $\lambda_1\cdot\Delta,\dots,\lambda_k\cdot\Delta$, respectively, such that $Z_{\lambda_1},\cdots,Z_{\lambda_k}$ define a complete intersection $Y_{\lambda_1,\dots,\lambda_k}$ in X_Δ. Then $b_*(Y_{\lambda_1,\dots,\lambda_k};\mathbb{Z}_2)$ is equivalent to $\mathrm{Vol}(\lambda_1\cdot\Delta,\cdots,\lambda_k\cdot\Delta)$ when λ_i tends to infinity for all i.*

We also use the following result of Khovanskii on the Euler characteristic of a complete intersection in the torus $(\mathbb{C}^*)^d$ (see [Hov78]).

Theorem 2.10 (A. Khovanskii)**.** *Let Y be a complete intersection in $(\mathbb{C}^*)^d$ defined by polynomials $P_1,\cdots,P_k$ with the Newton polytopes $\Delta_1,\cdots,\Delta_k$, respectively. Then, the Euler characteristic of Y is the homogeneous term of degree d of*

$$\Delta_1(1+\Delta_1)^{-1}\cdot\ \cdots\ \cdot\Delta_k(1+\Delta_k)^{-1},$$

where the product of d polytopes stands for their mixed volume and $(1+\Delta_i)^{-1}$ stands for the series $\sum_{j=0}^{\infty}(-1)^j(\Delta_i)^j$.

In the case of two 3-dimensional polytopes we use the following direct consequence of Theorem 2.10.

Corollary 2.11. *Let Δ be a simple 3-dimensional polytope and λ_1 and λ_2 be positive integers. For $i = 1,2$ put $\Delta_i = \lambda_i\cdot\Delta$. Let Y be a complete intersection in X_Δ defined by polynomials P_1 and P_2 with the Newton polytopes Δ_1 and Δ_2, respectively. Then, $b_*(Y;\mathbb{C}) = (\lambda_1^2\lambda_2 + \lambda_2^2\lambda_1)\,\mathrm{Vol}(\Delta) - \sum_{\Gamma\in\mathcal{F}_2(\Delta)}\lambda_1\lambda_2\,\mathrm{Vol}(\Gamma) + 4$.*

Proof. By Theorem 2.10, the Euler characteristic $\chi(Y)$ of Y is given by $\chi(Y) = -(\lambda_1^2\lambda_2+\lambda_2^2\lambda_1)\,\mathrm{Vol}(\Delta) + \sum_{\Gamma\in\mathcal{F}_2(\Delta)}\lambda_1\lambda_2\,\mathrm{Vol}(\Gamma)$. Since $b_*(Y;\mathbb{C}) = -\chi(Y)+4$, we have the desired result. □

3. Asymptotically maximal families of hypersurfaces

3.1. **Auxiliary statements.** This section is devoted to the proof of Theorem 1.3 on existence of asymptotically maximal families of hypersurfaces. The proof is based on two important results.

In [IV04] I. Itenberg and O. Viro, using the T-construction, proved that there exist M-hypersurfaces of any degree in the projective space of any dimension.

Theorem 3.1 (I. Itenberg and O. Viro). *Let d and m be natural numbers, and T_1^d be a primitive d-dimensional simplex. Put $T_m^d = m \cdot T_1^d$. Then, there exists a primitive convex triangulation $\tau_{T_m^d}$ of T_m^d and a sign distribution $D(\tau_{T_m^d})$ at the vertices of $\tau_{T_m^d}$ such that the T-hypersurface Z_d^m obtained via the combinatorial patchworking from $\tau_{T_m^d}$ and $D(\tau_{T_m^d})$ is maximal.*

The second important result we use is due to F. Knudsen and D. Mumford [KKMSD67].

Theorem 3.2 (F. Knudsen and D. Mumford). *Let Δ be a polytope. There exists a positive integer l such that $l \cdot \Delta$ admits a convex primitive triangulation.*

In the sequel, when there is no ambiguity on the triangulation of a polytope Δ and the sign distribution chosen, we denote by H_Δ the piecewise-linear hypersurface in Δ_* obtained by T-construction, $\widetilde{H_\Delta}$ its image in $\widetilde{\Delta}$, and Z_Δ the corresponding hypersurface in X_Δ.

3.2. **Itenberg-Viro asymptotical construction.** In fact, we use only the following asymptotical version of Theorem 3.1.

Theorem 3.3 (I. Itenberg and O. Viro). *For any positive integers m and d such that $m \geq d+1$, there exists a hypersurface X of degree m in $\mathbb{R}P^d$ such that*

$$b_*(\mathbb{R}X; \mathbb{Z}_2) \geq (m-2)(m-3)\dots(m-d-1).$$

The proof of this asymptotical version is much simpler than the proof of Theorem 3.1. It can be extracted from [IV04] and was communicated to us by the authors of [IV04]. We reproduce their proof here for the completeness.

3.2.1. *Proof of Theorem 3.3.* We describe a triangulation τ of the standard simplex $T = T_m^d$ and a distribution of signs at the integer points of T which provide via the combinatorial patchworking theorem a hypersurface with the properties formulated in Theorem 3.3.

To construct the triangulation τ, we use induction on d. If $d = 1$, the triangulation of $[0, m]$ is formed by m intervals $[0, 1]$, ..., $[m-1, m]$ for any m. Assume that for all natural $k < d$ the triangulations of the standard k-dimensional simplices of all sizes are constructed and consider the d-dimensional one of size m.

Denote by $x_1, \dots, x_d$ the coordinates in $\mathbb{R}^d$. Let $T_j^{d-1} = T \cap \{x_d = m-j\}$ and T_j be the image of T_j^{d-1} under the orthogonal projection to the coordinate hyperplane $\{x_d = 0\}$. Numerate the vertices of each simplex $T_1, \dots, T_{m-1}, T_m = T_m^{d-1}$ as follows: assign 1 to the vertex at the origin and $i+1$ to the vertex with nonzero coordinate at the i-th place. Assign to the vertices of $T_1^{d-1}, \dots, T_{m-1}^{d-1}$ the numbers of their projections. A triangulation of each simplex $T_0, \dots, T_{m-1}$ is constructed. Take the corresponding triangulations in the simplices T_j^{d-1}.

Let l be a nonnegative integer not greater than $d-1$. If $m-j$ is even, denote by $T_j^{(l)}$ the l-face of T_j^{d-1} which is the convex hull of the vertices with numbers $1, \dots, l+1$. If $m-j$ is odd denote by $T_j^{(l)}$ the l-face of T_j^{d-1} which is the convex hull of the vertices with numbers $d-l, \dots, d$.

Now for any integer $0 \le j \le m-1$ and any integer $0 \le l \le d-1$, take the join $T_{j+1}^{(l)} * T_j^{(d-1-l)}$. The triangulations of $T_{j+1}^{(l)}$ and $T_j^{(d-1-l)}$ define a triangulation of $T_{j+1}^{(l)} * T_j^{(d-1-l)}$. This gives rise to the desired triangulation τ of T. One can see that τ is convex.

The distribution of signs at the vertices of τ is given by the following rule. The vertex gets the sign "+" if the sum of its coordinates is even, and it gets the sign "−" otherwise.

Lemma 3.4. *For the hypersurface X of degree m in $\mathbb{R}P^d$ provided according to the combinatorial patchworking theorem by the triangulation τ and the distribution of signs defined above, one has*

$$b_*(\mathbb{R}X; \mathbb{Z}_2) \ge \begin{cases} (m-2)(m-3)\dots(m-d-1), & \text{if } m \ge d+1, \\ 0, & \text{otherwise.} \end{cases}$$

To prove Lemma 3.4 we define a collection of cycles c_i, $i \in I$ of $\widetilde{H}$ (in fact, any c_i is also a cycle of the hypersurface $H \subset T_*$, and moreover, of the hypersurface $H \cap (\mathbb{R}^*)^d$). The cycles c_i are called *narrow*.

The collection of narrow cycles c_i is constructed together with a collection of *dual cycles* b_i. Any dual cycle b_i is a $(d-1-p)$-cycle in $\widetilde{T} \setminus \widetilde{H}$ (where p is the dimension of c_i) composed by simplices of τ_* and representing a homological class such that its linking number with any p-dimensional narrow cycle c_k is δ_{ik}.

Let us fix some notations. For any simplex $T_j^{(l)}$ (where $1 \le j \le m$ and $0 \le l \le d-1$), denote by $(T_j^{(l)})_*$ the union of the symmetric copies of $T_j^{(l)}$ under the reflections with respect to coordinate hyperplanes $\{x_i = 0\}$, where $i = 1, \dots, l$, if $m-j$ is even, and $i = d-l, \dots, d-1$, if $m-j$ is odd, and compositions of these reflections.

Any simplex $T_j^{(l)}$ is naturally identified with the standard simplex T_j^l in $\mathbb{R}^l$ with vertices $(0, \dots, 0)$, $(j, 0, \dots, 0)$, $\dots, (0, \dots, 0, j)$ via the linear map $\mathcal{L}_j^l : T_j^{(l)} \to T_j^l$ sending

1. the vertex with number i of $T_j^{(l)}$ to the vertex of T_j^l with the same number, if $m-j$ is even,
2. the vertex with number i of $T_j^{(l)}$ to the vertex of T_j^l with the number $i-d+l+1$, if $m-j$ is odd.

It is easy to see that $\mathcal{L}_j^l$ is simplicial with respect to the chosen triangulations of $T_j^{(l)}$ and T_j^l. The natural extension of $\mathcal{L}_j^l$ to $(T_j^{(l)})_*$ identifies $(T_j^{(l)})_*$ with $(T_j^l)_*$ and respects the chosen triangulations.

By *a symmetry* we mean a composition of reflections with respect to coordinate hyperplanes. Let $s_{(i)}$ be the reflection of $\mathbb{R}^d$ with respect to the hyperplane $\{x_i = 0\}$, $i = 1, \dots, d$. Denote by s_j^l the symmetry of $(T_j^{l+1})_*$ which is identical if $m-j$ is even, and coincides with the restriction of $s_{(d-l-1)} \circ \dots \circ s_{(d-1)}$ on $(T_j^{l+1})_*$ if $m-j$ is odd.

The narrow cycles and their dual cycles are defined below using induction on d. For $d = 1$ the narrow cycles are the pairs of points

$$(1/2, 3/2), \dots, ((2m-5)/2, (2m-3)/2).$$

The dual cycles are pairs of vertices

$$(1, m-1), (2, m), (3, m+1), \dots, (m-2, m),$$

if m is even, and pairs of vertices

$$(1, m), (2, m-1), (3, m), \dots, (m-2, m),$$

if m is odd.

Assume that for all natural m and all natural $k < d$ the narrow cycles c_i in the hypersurface $\widetilde{H} \subset \widetilde{T}_m^k$ and the dual cycles b_i in $\widetilde{T}_m^k \setminus \widetilde{H}$ are constructed. The narrow cycles of the hypersurface in $\widetilde{T}_m^d$ are divided into 3 families.

Horizontal Cycles. The initial data for constructing a cycle of the first family consist of an integer j satisfying inequality $1 \leq j \leq m-1$ and a narrow cycle of the hypersurface in T_*^{d-1} constructed at the previous step. In the copy $(T_j^{d-1})_*$ of T_*^{d-1}, take the copy c of this cycle and b of its dual cycle.

There exists exactly one symmetric copy of T_{j+1}^0 incident to b. It is T_{j+1}^0 itself, if $m-j$ is odd, and either T_{j+1}^0, or $s_{(d-1)}(T_{j+1}^0)$, if $m-j$ is even. If the sign of the symmetric copy $s(T_{j+1}^0)$ of T_{j+1}^0 incident to b is opposite to the sign of c, we include c in the collection of narrow cycles of $\widetilde{H}$. Otherwise take $s_{(d)}(c)$ as a narrow cycle of $\widetilde{H}$. The dual cycle of c (resp., $s_{(d)}(c)$) is the suspension of b (resp., $s_{(d)}(b)$) with the vertex $s(T_{j+1}^0)$ (resp., $s_{(d)}(s(T_{j+1}^0))$) and with the vertex $s(T_{j-1}^0)$ (resp., $s_{(d)}(s(T_{j-1}^0))$).

Co-Horizontal Cycles. The initial data for constructing a cycle of the second family are the same as in the case of the horizontal cycles: the data consist of an integer j satisfying inequality $1 \leq j \leq m-1$ and a narrow cycle of the hypersurface in T_*^{d-1}.

In the copy $(T_j^{d-1})_*$ of T_*^{d-1}, take the copy c of this cycle and b of its dual cycle. If the sign of the symmetric copy $s(T_{j+1}^0)$ of T_{j+1}^0 incident to b coincides with the sign of c, take b as dual cycle of a narrow cycle of $\widetilde{H}$. Otherwise take $s_{(d)}(b)$. The corresponding narrow cycle is a suspension of c (resp., $s_{(d)}(c)$).

Join Cycles. The initial data consist of integers j and l satisfying inequalities $1 \leq j \leq m-1$, $1 \leq l \leq d-2$, the copy $c_1 \subset (T_{j+1}^l)_*$ of a narrow cycle of the hypersurface in $(T_{j+1}^l)_*$, the copy $c_2 \subset (T_j^{d-1-l})_*$ of a narrow cycle of the hypersurface in $(T_j^{d-1-l})_*$ and the copies $b_1 \subset (T_{j+1}^l)_*$ and $b_2 \subset (T_j^{d-1-l})_*$ of the dual cycles of these narrow cycles.

One of the joins $b_1 * b_2$ and $s_{j+1}^l(b_1) * s_j^{d-1-l}(b_2)$, belongs to τ_*; denote it by J. If the signs of c_1 and c_2 coincide, take J as the dual cycle of a cycle of $\widetilde{H}$. Otherwise take $s_{(d)}(J)$. The corresponding narrow cycle is either $c_1 * c_2$, or $s_{j+1}^l(c_1) * s_j^{d-1-l}(c_2)$, or $s_{(d)}(c_1 * c_2)$, or $s_{(d)}(s_{j+1}^l(c_1) * s_j^{d-1-l}(c_2))$.

Proof of Lemma 3.4. Both c_i and b_i with $i \in I$ are $\mathbb{Z}_2$-cycles homologous to zero in $\widetilde{T}$, which is homeomorphic to the projective space of dimension d. The sum of dimensions of c_i and b_i is $d-1$. Thus we can consider the linking number of c_i with $i \in I$ and b_k, $k \in I$ taking values in $\mathbb{Z}_2$. Each c_i bounds an obvious ball in $\widetilde{T}$. This ball meets b_i in a single point transversally and is disjoint with b_k for $k \neq i$ and $i, k \in I$. Hence the linking number of c_i and b_k is δ_{ik}.

Therefore the collections of homology classes realized in $\widetilde{T} \setminus \widetilde{H}$ and $\widetilde{H}$ by $b_i, i \in I$ and $c_i, i \in I$, respectively, generate subspaces of $H_*(\widetilde{T} \setminus \widetilde{H}; \mathbb{Z}_2)$ and $H_*(\widetilde{H}; \mathbb{Z}_2)$ and are dual bases of the subspaces with respect to the restriction of the Alexander duality. Hence c_i with $i \in I$ realize linearly independent Z_2-homology classes of $\widetilde{H}$.

It remains to show that the number of narrow cycles is at least

$$(m-2)(m-3)\dots(m-d-1),$$

if $m \geq d+1$. The statement can be proved by induction on d. The base $d = 1$ is evident. To prove the induction step notice, first, that the statement is evidently true for $m = d+1$. Now, we use the induction on m and obtain the required statement from the inequality

$$\begin{aligned} &(m-3)(m-4)\dots(m-d-2) + 2(m-3)(m-4)\dots(m-d-1) \\ &+ \sum_{k=1}^{d-2} [(m-2)(m-3)\dots(m-k-1)]\,[(m-3)(m-4)\dots(m-d+k-1)] \\ &\geq (m-2)(m-3)\dots(m-d-1). \end{aligned}$$

This finishes the proofs of Lemma 3.4 and Theorem 3.3. □

Remark 3.5. The family of hypersurfaces in $\mathbb{R}P^d$ constructed in Theorem 3.3 is asymptotically maximal.

Proof. Indeed, the total Betti number of a nonsingular hypersurface of degree m in $\mathbb{C}P^d$ is equal to $\frac{(m-1)^{d+1}-(-1)^{d+1}}{m} + d + (-1)^{d+1}$. This number is equivalent to $(m-2)(m-3)\dots(m-d-1)$ when m tends to infinity. □

3.3. **Proof of Theorem 1.3.** For a positive integer λ put $\Delta_\lambda = \lambda \cdot \Delta$. Let l be a positive integer such that Δ_l admits a primitive convex triangulation τ (see Theorem 3.2). Denote by ν a function certifying the convexity of τ. Let τ_λ be the triangulation of $\Delta_{\lambda l}$ obtained from τ by multiplication of its simplices by λ.

We can assume that $\lambda > d+1$. Let δ be a d-dimensional simplex of τ. The convex hull of the interior integer points of $\lambda \cdot \delta$ is a d-dimensional simplex $(\lambda - (d+1)) \cdot \delta$. Put $\delta_\lambda = \lambda \cdot \delta$ and $\delta'_\lambda = (\lambda - (d+1)) \cdot \delta$. For any d-dimensional simplex δ_λ of τ_λ, apply the construction of Lemma 3.4 to the convex hull δ'_λ of the interior integer points of δ_λ. Complete the triangulation of δ'_λ to a convex triangulation of δ_λ whose only extra vertices are the vertices of δ_λ in the following way. Let $\nu_{\lambda-(d+1)}$ be a convex piecewise-linear function certifying the convexity of the triangulation of δ'_λ. Define a convex function ν^δ_λ on δ_λ choosing the values of $\nu_{\lambda-(d+1)}$ at the integer points of δ'_λ and the value v at the vertices of δ_λ, where v is large enough (the graph of ν^δ_λ is the lower part of the convex hull of the defined points in $\delta_\lambda \times \mathbb{R}$). Note that ν^δ_λ restricted to δ'_λ coincides with $\nu_{\lambda-(d+1)}$. If the decomposition defined by ν^δ_λ is not a triangulation, we slightly perturb $\nu_{\lambda-(d+1)}$ (without changing the triangulation of δ'_λ) to break the polytopes of the subdivision which are not simplices. Denote by τ^δ_λ the obtained triangulation of δ_λ.

The only vertices of τ_λ^δ in $\delta_\lambda \setminus \delta'_\lambda$ are the vertices of δ_λ. One can choose the same value v of the functions ν_λ^δ at the vertices of all the d-dimensional simplices δ of τ_λ. Hence, the functions ν_λ^δ can be glued together to form a piecewise-linear function ν_λ on $\Delta_{\lambda l}$ which is, by construction, convex on each d-dimensional simplex of τ_λ. Let ν' be a function certifying the convexity of τ_λ. Then, for sufficiently small $\epsilon > 0$ the function $\nu = \nu' + \epsilon\nu_\lambda$ certifies the convexity of the triangulation obtained by gluing the triangulations of the d-dimensional simplices of τ_λ. Thus, one gets a convex triangulation τ_λ^l of $\Delta_{\lambda l}$. Choose a sign distribution $D(\tau_\lambda^l)$ at the vertices of τ_λ^l in such a way that on each simplex δ'_λ the distribution coincides with the one Lemma 3.4. Let $Z_{\Delta_{\lambda l}}$ be the hypersurface obtained via the combinatorial patchworking from τ_λ^l and $D(\tau_\lambda^l)$.

Proposition 3.6. *The family of hypersurfaces $Z_{\Delta_{\lambda l}}$ of X_Δ constructed above is asymptotically maximal.*

Proof. The total Betti number of $Z_{\Delta_{\lambda l}}$ is equivalent to $\mathrm{Vol}(\Delta_{\lambda l})$ when λ tends to infinity (see Proposition 2.8). For each d-dimensional simplex δ of τ_λ consider the narrow cycles of $H_{\Delta_{\lambda l}} \cap (\delta'_\lambda)_*$ which are constructed in the proof of Lemma 3.4. Since the narrow cycles are constructed with the dual cycles, the union of the obtained collections of narrow cycles consists of linearly independent cycles. Thus, $b_*(\mathbb{R}Z_{\Delta_{\lambda l}}; \mathbb{Z}_2) \geq \mathrm{Vol}(\Delta_l)n_\lambda$, where n_λ is the number of narrow cycles in each δ'_λ. Since $n_\lambda \sim \mathrm{Vol}(\delta'_\lambda)$, we have $n_\lambda \sim \mathrm{Vol}(\delta_\lambda)$. So, $b_*(\mathbb{R}Z_{\Delta_{\lambda l}}; \mathbb{Z}_2)$ is equivalent to $\mathrm{Vol}(\Delta_l)\,\mathrm{Vol}(\delta_\lambda)$. The latter number is equal to $\mathrm{Vol}(\Delta_{\lambda l})$. □

4. Newton polytopes without maximal hypersurfaces

Before giving the proof Proposition 1.1 let us consider the lower dimensional cases. Clearly, if Δ is an interval $[a, b]$ in $\mathbb{R}$, where a and b are nonnegative integers, then there exists a maximal 0-dimensional subvariety in $\mathbb{C}P^1 = X_\Delta$ with the Newton polygon Δ.

If Δ is a polygon in the first quadrant of $\mathbb{R}^2$, then again there exists a maximal curve in X_Δ with the Newton polygon Δ. Such a curve can be constructed by the combinatorial patchworking: it suffices to take as initial data a primitive convex triangulation of Δ equipped with the following distribution of signs: an integer point (i, j) of Δ gets the sign "-" if i and j are both even, and gets the sign "+", otherwise (see for example [Ite95], [IV96], [Haa98]).

Proof of Proposition 1.1. The proof of Proposition 1.1 relies on the estimation of the Betti numbers of the complex and real parts of a real algebraic surface Z_k in X_{Δ_k} with the Newton polytope Δ_k. The Betti numbers $b_*(Z_k; \mathbb{C})$ are given by Lemma 2.7. We have $b_*(Z_k; \mathbb{C}) = l^*(2\Delta_k) - 2l^*(\Delta_k) - \sum_{\Gamma \in \mathcal{F}_2(\Delta_k)}(l^*(\Gamma) - 1) - 1$. Since $l^*(2\Delta_k) = k-1$ and $l^*(\Delta_k) = 0$, we get $b_*(Z_k; \mathbb{C}) = k+2$. Thus, $b_*(Z_k; \mathbb{Z}_2) \geq k + 2$.

To estimate $b_*(\mathbb{R}Z_k; \mathbb{Z}_2)$ we consider two cases. If k is odd, Δ_k is an elementary tetrahedron, and $\mathbb{R}Z_k$ is homeomorphic to the projective plane. Thus, in this case, $b_*(\mathbb{R}Z_k; \mathbb{Z}_2) = 3$.

If k is even, Δ_k has either 6 or 8 nonempty symmetric copies. In the first case $\mathbb{R}Z_k$ is homeomorphic to three spheres with some points identified. Each of the spheres has 4 marked points. Pairs of marked points are identified in the following way.

Two marked points of each sphere are identified with two marked points of another sphere, and the two other marked points are identified with the marked points of the remaining sphere. Then the Euler characteristic is zero and $b_*(\mathbb{R}Z_k;\mathbb{Z}_2) = 8$. In the case of 8 nonempty symmetric copies, $\mathbb{R}Z_k$ is homeomorphic to four spheres with some points identified. Each sphere has three marked points. Pairs of marked points are identified in the following way: on each sphere the three marked points are identified with marked points of three different spheres. Thus the Euler characteristic is 2 and we also have $b_*(\mathbb{R}Z_k;\mathbb{Z}_2) = 8$.

Thus, for k even greater than or equal to 8 and for k odd greater than or equal to 3, there is no maximal surface in X_{Δ_k} with the Newton polytope Δ_k.

Proof of Proposition 1.2. Fix an integer $d \geq 3$ and consider a family $\{\sigma_k\}_{k\in\mathbb{N}}$ of d-dimensional simplices in $\mathbb{R}^d$ such that their vertices are their only integer points and $\mathrm{Vol}(\sigma_k) = k$. For example, one can take for σ_k the simplex in $\mathbb{R}^d$ with vertices

$$(0,0,\ldots,0,0),\ (1,0,\ldots,0,0),\ (0,1,\ldots,0,0),\ \ldots,(0,0,\ldots,1,0),$$
$$\text{and } (1,1,\ldots,1,k).$$

Let Z_k be any hypersurface in X_{σ_k}. By Proposition 2.8 $b_*(Z_k;\mathbb{C})$ tends to infinity when k does, and so does $b_*(Z_k;\mathbb{Z}_2)$. Meanwhile, $b_*(\mathbb{R}Z_k;\mathbb{Z}_2)$ is bounded (for example, by the number of simplices in σ_k^*). So there exists a number k_0 such that for any integer $k > k_0$ and any hypersurface Z_k in X_{σ_k} one has $b_*(\mathbb{R}Z_k;\mathbb{Z}_2) < b_*(Z_k;\mathbb{Z}_2)$. □

5. Newton polytopes without maximal complete intersection

Let us first consider the case of complete intersections of two surfaces. Let Δ_k be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0),(1,0,0),(0,1,0)$ and $(1,1,k)$.

Proposition 5.1. *Let $k \geq 5$ be an integer, and Z_1 and Z_2 be real algebraic surfaces in X_{Δ_k} with the Newton polytope Δ_k. Assume that Z_1 and Z_2 define a complete intersection Y_k in X_{Δ_k}. Then Y_k is not maximal.*

The proof of Proposition 5.1 relies on the estimation of the Betti numbers of the complex and real parts of the complete intersection Y_k of two surfaces whose Newton polytopes coincide with Δ_k.

Lemma 5.2. *Let Y_k be the complete intersection of two surfaces in X_{Δ_k} whose Newton polytopes coincide with Δ_k. Then $b_*(Y_k;\mathbb{C}) = 2k$.*

Proof. By Corollary 2.11, we have

$$b_*(Y_k;\mathbb{C}) = 2\,\mathrm{Vol}(\Delta_k) - \sum_{\Gamma\in\mathcal{F}_2(\Delta_k)} \mathrm{Vol}(\Gamma) + 4.$$

So, we get $b_*(Y_k;\mathbb{C}) = 2k$. □

Proof of Proposition 5.1. According to Lemma 5.2, we have $b_*(Y_k;\mathbb{C}) = 2k$. Thus, $b_*(Y_k;\mathbb{Z}_2) \geq 2k$.

Let f_1 and f_2 be the polynomials defining the two surfaces. Then,

$$f_l(x,y,z) = a_l x + b_l y + c_l z^k + d_l \ (l = 1,2)$$

for some (a_l, b_l, c_l, d_l) in $\mathbb{R}^4$. The change of variables $\Lambda_k : x \mapsto x$, $\Lambda_k : y \mapsto y$, $\Lambda_k : z \mapsto z^{\frac{1}{k}}$ is a diffeomorphism of the first octant $(\mathbb{R}_+^*)^3$, where $\mathbb{R}_+^* = \{x \in \mathbb{R} : x > 0\}$. Let Q_i be another octant, and ϕ_i be the diffeomorphism from Q_i to $(\mathbb{R}_+^*)^3$ defined by $\phi_i(x, y, z) = (|x|, |y|, |z|)$. Then $\psi_i = \phi_i^{-1} \circ \Lambda_k \circ \phi_i$ is a diffeomorphism from Q_i to itself. The diffeomorphism ψ_i maps the zeros of f_l to the zeroes of $\psi_{i*}(f_l)$ and $\psi_{i*}(f_l)(x, y, z) = a_l x + b_l y + c_l z + d_l$. Thus, in each octant, $\mathbb{R}Y_k$ is diffeomorphic to the intersection of two plans. Hence, the number of connected components of Y_k is at most 4. So, $\mathbb{R}Y_k$ is not maximal for $k \geq 5$. □

The example above should be compared with the following result in dimension 2 which is probably well known but that I couldn't find in the literature.

Proposition 5.3. *Let Δ be a two-dimensional polygon. For any positive integers λ_1 and λ_2 there exist algebraic curves C_1 et C_2 in X_Δ such that*

- *the Newton polygons of C_1 et C_2 are $\lambda_1 \cdot \Delta$ and $\lambda_2 \cdot \Delta$, respectively,*
- *the curves C_1 et C_2 define a 0-dimensional maximal complete intersection in X_Δ.*

Proof. We use here the Cayley trick. Take any primitive convex triangulation τ of Δ. By homothety, τ induces a triangulation τ_i on $\lambda_i \cdot \Delta$. Put $\Delta_i = \lambda_i \cdot \Delta$. Consider the following subdivision δ_0 of the Cayley polytope $C(\Delta_1, \Delta_2)$. In the faces $\hat{\Delta}_1$ and $\hat{\Delta}_2$ of $C(\Delta_1, \Delta_2)$ corresponding to Δ_1 and Δ_2 take the triangulations τ_1 and τ_2, respectively. Each 3-dimensional polytope of the subdivision δ_0 is the convex hull of a triangle of τ_1 and a triangle of τ_2 which are the multiples of the same triangle of τ. Since τ is convex, δ_0 is also convex. Let ν_0 be a convex function certifying the convexity of δ_0, and let ν_1 be the convex function defined by $\nu_1(0, 1, x, y) = C_1 y + C_2 x$ with $C_1 > C_2 > 0$ and $\nu_1(1, 0, x, y) = 0$. Put $\nu_3 = \nu_1 + \nu_2$. If C_1 is sufficiently small, the function ν_3 induces the following refinement δ_1 of δ_0. Each 3-dimensional polytope of δ_0 is subdivided into two cones whose bases are triangles in $\hat{\Delta}_1$ and $\hat{\Delta}_2$, respectively, and a join J of two edges: one in $\hat{\Delta}_1$ and the other one in $\hat{\Delta}_2$. Take any convex primitive triangulations τ_1' and τ_2' refining τ_1 and τ_2, respectively. They define a convex primitive refinement δ_2 of δ_1. Choose a sign distribution at the vertices of δ_2 and apply the procedure of the combinatorial patchworking. Let J be a join of the decomposition δ_1 described above. It is triangulated into primitive tetrahedra t_i and has lattice volume $\lambda_1\lambda_2$. Each t_i has a symmetric copy containing a point of the T-complete intersection constructed. Thus, the number of intersection points obtained is $\lambda_1\lambda_2 \operatorname{Vol}(\Delta)$ and the complete intersection constructed is maximal. □

5.1. **Proof of Proposition 1.4.** Consider the simplex σ_k in $\mathbb{R}^d$ with the vertices

$$(0, 0, \ldots, 0, 0),\ (1, 0, \ldots, 0, 0),\ (0, 1, \ldots, 0, 0),\ \ldots, (0, 0, \ldots, 1, 0),$$
$$\text{and } (1, 1, \ldots, 1, k).$$

Let Y_k be a complete intersection of hypersurfaces in X_{σ_k} such that all these hypersurfaces have the Newton polytope σ_k. Proposition 2.9 implies that $b_*(Y_k; \mathbb{Z}_2)$ tends to infinity when k tends to infinity.

Let $f_1, \dots, f_n$ be the polynomials defining the hypersurfaces. Then,

$$f_l(x, y, z) = a_{l,0} + \sum_{i=1}^{d-1} a_{l,i} x_i + a_{l,d} {x_d}^k \ (l = 1, \cdots, n)$$

for some $(a_{l,0}, \cdots, a_{l,d})$ in $\mathbb{R}^{d+1}$. The change of variables $\Lambda_k : x_i \mapsto x_i$ for $i \neq d$, $\Lambda_k : x_d \mapsto {x_d}^{\frac{1}{k}}$ is a diffeomorphism of the first orthant $(\mathbb{R}_+^*)^d$. Let Q_j be another orthant, and ϕ_j be the diffeomorphism from Q_j to $(\mathbb{R}_+^*)^d$ defined by $\phi_j(x_1, \cdots, x_d) = (|x_1|, \cdots, |x_d|)$. Then $\psi_j = \phi_j^{-1} \circ \Lambda_k \circ \phi_j$ is a diffeomorphism from Q_j to itself. The diffeomorphism ψ_j maps the zeros of f_l to the zeroes of ${\psi_j}_*(f_l)$ and ${\psi_j}_*(f_l)(x_1, \dots, x_d) = a_{l,0} + \sum_{i=1}^{d} a_{l,i} x_i$. Thus, in each orthant, Y_k is diffeomorphic to the intersection of n hyperplanes. Hence, $b_*(\mathbb{R}Y_i; \mathbb{Z}_2)$ is bounded.

So, there exists a number k_0 such that for any $k \geq k_0$ and any complete intersection Y_k in X_{σ_k} one has $b_*(\mathbb{R}Y_k; \mathbb{Z}_2) < b_*(Y_k; \mathbb{Z}_2)$. □

6. Asymptotically maximal families of complete intersections

6.1. **Itenberg-Viro asymptotical statement.** The proof of Theorem 1.6 is based on the following result of Itenberg and Viro.

Theorem 6.1 (I. Itenberg and O. Viro). *Let Δ be a primitive d-dimensional simplex. For any k-tuple $\lambda_1, \cdots, \lambda_k$ of natural numbers, there exist piecewise-linear convex functions $\mu_1, \dots, \mu_k$ on $\lambda_1 \cdot \Delta, \dots, \lambda_k \cdot \Delta$, respectively, and sign distributions at the vertices of the corresponding triangulations of $\lambda_1 \cdot \Delta, \dots, \lambda_k \cdot \Delta$ such that the real complete intersection in $X_\Delta = \mathbb{C}P^d$ obtained via Sturmfels' Theorem 2.3 from these data is maximal.*

In fact, as in Section 3, we use only the following asymptotical version of Theorem 6.1.

Theorem 6.2 (I. Itenberg and O. Viro). *For any positive integers k, $m_1, \dots, m_k$ and d such that $k \leq d$ and $m_j \geq d+1$ ($j = 1, \dots, k$), there exists a complete intersection X of multi-degree $(m_1, \dots, m_k)$ in $\mathbb{R}P^d$ such that*

$$b_*(\mathbb{R}X; \mathbb{Z}_2) \geq \sum_{i_1 + \ldots + i_k = d} \left(\prod_{j=1}^{k} (m_j - 2)(m_j - 3) \ldots (m_j - i_j - 1) \right)$$

(the summation is over all possible decompositions $i_1 + \ldots + i_k = d$ of d in a sum of k positive integer numbers).

The proof of this asymptotical version is much simpler than the proof of Theorem 6.1. It can be extracted from [IV04] and was communicated to us by the authors of [IV04]. We reproduce their proof here for the completeness.

Proof of Theorem 6.2.

The notations used here are those of Subsection 3.2.1. Take the standard simplices $T_{m_1}^d, \dots, T_{m_k}^d$ and triangulate the Cayley polytope $C(T_{m_1}^d, \dots, T_{m_k}^d)$ (see Subsection 2.3.1) in the following way. Let $i_1, \dots, i_k$ be nonnegative integers such that $i_1 + \ldots + i_k = d$, and put $i_0 = 0$. For any $j = 1, \dots, k$ consider the face of $T_{m_j}^d$ with the vertices having the numbers

$$i_1 + \ldots + i_{j-1} + 1, \dots, i_1 + \ldots + i_j + 1.$$

Denote by $J_{i_1,\dots,i_k}$ the join of the corresponding faces of $C(T^d_{m_1},\dots,T^d_{m_k})$. The simplices $J_{i_1,\dots,i_k}$ (for all the possible choices of nonnegative integers such that $i_1+\dots+i_k=d$) form a triangulation τ' of $C(T^d_{m_1},\dots,T^d_{m_k})$.

Take for each simplex $T^d_{m_j}$ the triangulation and the distribution of signs described in Subsection 3.2.1. For the simplices $\hat{T}^d_{m_1},\dots,\hat{T}^d_{m_k}$ take the corresponding triangulations and distributions of signs. The triangulations of $\hat{T}^d_{m_1},\dots,\hat{T}^d_{m_k}$ induce a refinement τ of τ'. Notice that τ is a primitive triangulation of $C(T^d_{m_1},\dots,T^d_{m_k})$.

Lemma 6.3. *For the complete intersection X of multi-degree $m_1,\dots,m_k$ in $\mathbb{R}P^d$ provided according to Proposition 2.5 by the triangulation τ and the distribution of signs defined above, one has*

$$b_*(\mathbb{R}X;\mathbb{Z}_2)\geq \sum_{i_1+\dots+i_k=d}\left(\prod_{j=1}^{k}(m_j-2)(m_j-3)\dots(m_j-i_j-1)\right)$$

(the summation is over all the possible decompositions $i_1+\dots+i_k=d$ of d in a sum of k positive integer numbers).

Proof. We define a collection of narrow cycles c_i, $i\in I$ of $\widetilde{H}$. The families of narrow cycles of $\widetilde{H}$ are indexed by the decompositions $i_1+\dots+i_k=d$ of d in a sum of k positive integer numbers.

Fix a decomposition $\mathcal{I}: i_1+\dots+i_k=d$ of d, where $i_1,\ \dots,\ i_k$ are positive integers. The initial data for constructing a narrow cycle of the corresponding family consist of narrow cycles $c_{(j)}\subset \widetilde{H}^{\mathcal{I}}_j$, $j=1,\dots,k$, constructed in Subsection 3.2.1 for the hypersurface $\widetilde{H}^{\mathcal{I}}_j$ in $\widetilde{T}^{i_j}_{m_j}$ produced via the combinatorial patchworking by the triangulation and distribution of signs described in Subsection 3.2.1.

The i_j-dimensional face Δ^{i_j} of $T^d_{m_j}$ with the vertices having the numbers

$$i_1+\dots+i_{j-1}+1,\dots,i_1+\dots+i_j+1$$

are naturally identified with $T^{i_j}_{m_j}$ via the linear map $\mathcal{L}^{i_j}:\Delta^{i_j}\to T^{i_j}_{m_j}$ sending the vertex with number $i_1+\dots+i_{j-1}+r$ of Δ^{i_j} to the vertex with number r of $T^{i_j}_{m_j}$. The map $\mathcal{L}^{i_j}$ is simplicial with respect to the chosen triangulations of Δ^{i_j} and $T^{i_j}_{m_j}$. Denote by $\Delta^{i_j}_*$ the union of the symmetric copies of Δ^{i_j} under the reflections with respect to coordinate hyperplanes $\{x_i=0\}$ in $\mathbb{R}^d$, where $i=i_1+\dots+i_{j-1}+2,\dots,i_1+\dots+i_j+1$, and compositions of these reflections. The natural extension of $\mathcal{L}^{i_j}$ to $\Delta^{i_j}_*$ identifies $\Delta^{i_j}_*$ with $(T^{i_j}_{m_j})_*$ and respects the chosen triangulations. We also denote this extension by $\mathcal{L}^{i_j}$. Denote by $\hat{\Delta}^{i_j}_*$ the union of faces of $\hat{T}^d_{m_j}$ corresponding to $\Delta^{i_j}_*$, and by $\hat{\mathcal{L}}^{i_j}$ the corresponding map from $\hat{\Delta}^{i_j}_*$ to $(T^{i_j}_{m_j})_*$. Put $\hat{c}_{(j)}=(\hat{\mathcal{L}}^{i_j})^{-1}(c_{(j)})$.

Let $b_{(j)}\subset \widetilde{T}^{i_j}_{m_j}\setminus \widetilde{H}^{\mathcal{I}}_j$ be the dual cycle of $c_{(j)}$. Put $\hat{b}_{(j)}=(\hat{\mathcal{L}}^{i_j})^{-1}(b_{(j)})$. Consider the symmetric copies of $\hat{b}_{(1)},\dots,\hat{b}_{(k)}$ under the reflections with respect to coordinate hyperplanes $\{x_i=0\}$ in $\mathbb{R}^{k+d}$ where $i=k+1,\dots,k+d$, and compositions of these reflections. Among these symmetric copies there exist copies $\hat{b}'_{(1)},\dots,\hat{b}'_{(k)}$ of $\hat{b}_{(1)},\dots,\hat{b}_{(k)}$, respectively, such that

- the join $\hat{b}'_{(1)}*\dots*\hat{b}'_{(k)}$ is the union of simplices of τ_*,
- all the vertices of $\hat{b}'_{(1)}*\dots*\hat{b}'_{(k)}$ have the same sign.

Let $\hat{c}'_{(1)}, \ldots, \hat{c}'_{(k)}$ be the corresponding symmetric copies of $\hat{c}_{(1)}, \ldots, \hat{c}_{(k)}$, respectively. Then, take the intersection $B \cap (\hat{c}'_{(1)} * \ldots * \hat{c}'_{(k)})$ as a narrow cycle of $\widetilde{H}$.

The number of narrow cycles in the family indexed by $\mathcal{I}$ is at least

$$\prod_{j=1}^{k} (m_j - 2)(m_j - 3) \ldots (m_j - i_j - 1).$$

Thus, the total number of constructed narrow cycles in $\widetilde{H}$ is at least

$$\sum_{i_1 + \ldots + i_k = n} \left(\prod_{j=1}^{k} (m_j - 2)(m_j - 3) \ldots (m_j - i_j - 1) \right)$$

(the summation is over all the possible decompositions $i_1 + \ldots + i_k = d$ of d in a sum of k positive integer numbers). The linear independence of the narrow cycles of a hypersurface $H^l_{m_j} \subset T^l_{m_j}$ for any $1 \leq l \leq d$ and any $1 \leq j \leq k$ implies the linear independence of the narrow cycles constructed in $\widetilde{H}$. □

Remark 6.4. Denote by $Y^{\sigma}_{m_1, \ldots, m_k}$ the complete intersection constructed in Lemma 6.3. Then, the family $\{Y^{\sigma}_{m_1, \ldots, m_k}\}_{m_1, \ldots, m_k}$ is asymptotically maximal.

Proof. Note that

$$\sum_{i_1 + \ldots + i_k = d} \left(\prod_{j=1}^{k} (m_j - 2)(m_j - 3) \ldots (m_j - i_j - 1) \right)$$

is equivalent to the mixed volume of $T^d_{m_1}, \ldots, T^d_{m_k}$. Thus, by Proposition 2.9, $b_*(\mathbb{R}Y_{m_1, \ldots, m_k}; \mathbb{Z}_2)$ is equivalent to $b_*(Y_{m_1, \ldots, m_k}; \mathbb{Z}_2)$, when all m_i's tend to infinity.

6.2. **Proof of Theorem 1.6.** Let τ be a primitive convex triangulation of $l \cdot \Delta$, and $(\lambda_1, \cdots, \lambda_k)$ be a k-tuple of positive integers. Denote by Δ_{λ_i} the polytopes $\lambda_i l \cdot \Delta$. We can assume that λ_i is greater than $d + 1$ for any i.

Let δ be a d-dimensional simplex of the triangulation τ. Denote by $\hat{\delta}_1, \ldots, \hat{\delta}_k$ the corresponding simplices in $\hat{\Delta}_{\lambda_1}, \ldots, \hat{\Delta}_{\lambda_k}$, respectively. Subdivide the Cayley polytope $C(\Delta_{\lambda_1}, \ldots, \Delta_{\lambda_k})$ into convex hulls of $\hat{\delta}_1, \ldots, \hat{\delta}_k$, where δ runs over all d-dimensional simplices of τ. For a d-dimensional simplex δ of τ, put $\delta_i = \lambda_i \cdot \delta$ and $\delta'_i = (\lambda_i - (d+1)) \cdot \delta$, where $i = 1, \ldots, k$.

For any d-dimensional simplex δ of τ, take the triangulation of $C(\delta'_1, \ldots, \delta'_k)$ and the distribution of signs at the vertices of this triangulation described in the proof of Theorem 6.1. Extend the triangulations of the Cayley polytopes $C(\delta'_1, \ldots, \delta'_k)$ to a primitive convex triangulation $\hat{\tau}$ of $C(\Delta_{\lambda_1}, \ldots, \Delta_{\lambda_k})$ in the same way as it was done in Subsection 3.3. Extend also the distributions of signs at the integer points of polytopes $C(\delta'_1, \ldots, \delta'_k)$ to some distribution of signs $\hat{D}$ at the vertices of $\hat{\tau}$.

Let $Y_{\lambda_1, \ldots, \lambda_k}$ be the complete intersection in X_Δ obtained via Theorem 2.5 from $\hat{\tau}$ and $\hat{D}$.

Proposition 6.5. *The family of complete intersections $Y_{\lambda_1, \ldots, \lambda_k}$ constructed above is asymptotically maximal.*

Proof. By the construction, we have $b_*(\mathbb{R}Y_{\lambda_1, \ldots, \lambda_k}; \mathbb{Z}_2) \geq \mathrm{Vol}(l \cdot \Delta) \cdot n_{\lambda_1, \ldots, \lambda_k}$, where $n_{\lambda_1, \ldots, \lambda_k}$ is the number of narrow cycles in each $C(\delta'_1, \ldots, \delta'_k)$. Note that $n_{\lambda_1, \ldots, \lambda_k}$

is equivalent to $b_*(\mathbb{R}Y^{\sigma}_{\lambda_1,\dots,\lambda_k};\mathbb{Z}_2)$ when all numbers $\lambda_1,\dots,\lambda_k$ tend to infinity. So, by Proposition 2.9 and Remark 6.4, we obtain that $b_*(\mathbb{R}Y_{\lambda_1,\dots,\lambda_k};\mathbb{Z}_2)$ is equivalent to $b_*(Y_{\lambda_1,\dots,\lambda_k};\mathbb{Z}_2)$ when the numbers $\lambda_1,\dots,\lambda_k$ tend to infinity. □

References

[Bih02] F. Bihan, *Viro method for the construction of real complete intersections*, Adv. Math. **169** (2002), no. 2, 177–186. MR **2003g:**14080

[DK87] V. Danilov and A. Khovanskii, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Math. USSR Izvestiya vol 29 No 2 (1987), 279–298.

[Ehr67] E. Ehrhart, *Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux*, J. Reine Angew. Math. **226** (1967), 1–29. MR 35 #4184

[Ehr94] ———, *Un théorème arithmo-géométrique et ses généralisations*, L'ouvert n. 77 (1994), 33–34.

[Ful93] W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.

[GKZ94] I. Gelfand, M. Kapranov, and A. Zelevinsky, *Discriminent, resultants and multidimensional determinants*, Springer-Verlag, 1994.

[Haa98] B. Haas, *Ph. d. thesis*, 1998.

[Hov77] A. G. Hovanskiĭ, *Newton polyhedra, and toroidal varieties*, Funkcional. Anal. i Priložen. **11** (1977), no. 4, 56–64, 96.

[Hov78] ———, *Newton polyhedra, and the genus of complete intersections*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 1, 51–61.

[Ite95] I. Itenberg, *Counter-examples to Ragsdale conjecture and T-curves*, Real algebraic geometry and topology (East Lansing, MI, 1993), Contemp. Math., vol. 182, Amer. Math. Soc., Providence, RI, 1995, pp. 55–72.

[Ite97] ———, *Topology of real algebraic T-surfaces*, Revista mathematica univ. complutense de Madrid **10, no. supp.** (1997), 131–152.

[IV96] I. Itenberg and O. Viro, *Patchworking algebraic curves disproves the Ragsdale conjecture*, Math. Intelligencer **18** (1996), no. 4, 19–28.

[IV04] ———, *Maximal real algebraic hypersurfaces of projective space*, in preparation (2004).

[KKMSD67] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Springer-Verlag, 1967.

[Mik] G. Mikhalkin, *Maximal real algebraic hypersurfaces (in preparation)*.

[Stu94a] B. Sturmfels, *On the Newton polytope of the resultant*, Journal of Algebraic Combinatorics (1994), no. 3, 207 – 236.

[Stu94b] ———, *Viro's theorem for complete intersections*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), no. 3, 377–386.

[Vir84] O. Viro, *Gluing of plane algebraic curves and construction of curves of degree 6 and 7 (lnm 1060)*, Lecture Notes in Mathematics (1984), 187–200.

[Vir04] ———, *Patchworking real algebraic varieties*, preprint Uppsala University, Available at `http:www.math.uu.se/~oleg/pw.ps`, (2004).

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