THE SIMPLICIAL VOLUME OF CLOSED MANIFOLDS COVERED BY $\mathbb{H}^2 \times \mathbb{H}^2$

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Abstract. We compute the value of the simplicial volume for closed, oriented Riemannian manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$ explicitly, thus in particular for products of closed hyperbolic surfaces. This gives the first exact value of a nonvanishing simplicial volume for a manifold not admitting a hyperbolic structure.

1. Introduction

Our main result is the computation of the Gromov norm, that is the sup norm, of the Riemannian volume form on the product $\mathbb{H}^2 \times \mathbb{H}^2$, where $\mathbb{H}^2$ denotes the hyperbolic plane of constant curvature -1:

Main Theorem. *Let $\omega_{\mathbb{H}^2\times\mathbb{H}^2} \in H_c^4(\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}, \mathbb{R})$ be the image, under the Van Est isomorphism, of the Riemannian volume form on $\mathbb{H}^2 \times \mathbb{H}^2$. Then*

$$\|\omega_{\mathbb{H}^2\times\mathbb{H}^2}\|_\infty = \frac{2}{3}\pi^2.$$

Recall that the simplicial volume $\|M\|$ of a closed, oriented manifold M is a topological invariant introduced by Gromov in [Gr82] and is defined as

$$\|M\| = \inf\left\{ \Sigma|a_\sigma| \mid \Sigma a_\sigma\sigma \text{ represents the real fundamental class } [M]\right\}.$$

As an immediate consequence of our main theorem, we obtain in Theorem 1 below the explicit proportionality constant relating simplicial volume and volume of closed, oriented, Riemannian manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$. Indeed, we prove in [Bu06, Theorem 2] that the proportionality constant of the proportionality principle for closed, oriented locally symmetric space of noncompact type $M^n = \Gamma\backslash G/K$ is precisely the Gromov norm of the volume form in $H_c^n(G, \mathbb{R})$. Explicit finite values of the proportionality constant were up to now only known for hyperbolic manifolds ([Gr82],[Th78]): It is in this case equal to the maximum volume of ideal geodesic simplices in $\mathbb{H}^n$, a constant which has been computed explicitly up to dimension $n = 6$ only.

Theorem 1. *Let M be a closed, oriented Riemannian manifold whose universal cover $\widetilde{M}$ is isometric to $\mathbb{H}^2 \times \mathbb{H}^2$. Then*

$$\|M\| = \frac{3}{2\pi^2}\mathrm{Vol}(M).$$

In view of Hirzebruch's proportionality principle [Hi56] relating the volume and the Euler characteristic χ of locally symmetric spaces of noncompact type, the conclusion of Theorem 1 can be rewritten as

$$\|M\| = 6\chi(M).$$

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In particular, using the result of Ivanov and Turaev [IvTu82] that the sup norm of the Euler class ϵ of flat $\mathrm{SL}(4,\mathbb{R})$-bundles satisfies the inequality $\|\epsilon\|_\infty \leq 1/2^4$, the following consequence of Theorem 1 is immediate:

Corollary 2. *Let M be a closed, oriented Riemannian manifold whose universal cover $\widetilde{M}$ is isometric to $\mathbb{H}^2\times\mathbb{H}^2$. Let ξ be an $\mathrm{SL}(4,\mathbb{R})$-bundle over M. If ξ admits a flat structure, then*

$$\chi(\xi)=\langle\epsilon(\xi),[M]\rangle\leq\frac{3}{2^3}\chi(M).$$

Note that if M is a product of hyperbolic surfaces and the bundle ξ is a product of flat bundles, then the stronger inequality $\chi(\xi)\leq\frac{1}{2^2}\chi(M)$ follows from Milnor's celebrated inequality [Mi58]. This is probably the correct bound in the general case also. However, Corollary 2 is good enough to conclude that the considered manifolds do not admit an affine structure. Indeed, if this was the case, the $\mathrm{SL}(4,\mathbb{R})$-bundle associated to the tangent bundle TM could be endowed with a flat structure, and we would get the impossible inequality $\chi(M)=\chi(TM)\leq(3/2^3)\chi(M)$.

Another consequence of Theorem 1 is the following product formula:

Corollary 3. *Let M and N be closed, oriented surfaces. Then*

$$\|M\times N\|=\frac{3}{2}\|M\|\cdot\|N\|.$$

Since the simplicial volume of a surface Σ_g of genus $g\geq 1$ is equal to $\|\Sigma_g\|=4(g-1)=2|\chi(\Sigma_g)|$, we obtain, for $g,h\geq 1$,

$$\|\Sigma_g\times\Sigma_h\|=24\cdot(g-1)(h-1),$$

which gives the first exact value of a nonvanishing simplicial volume for a manifold not admitting a constant curvature metric.

Proof of Corollary 3. If either of M or N is the 2-sphere or the 2-torus, then both sides of the equality vanish trivially. If M and N are endowed with a hyperbolic structure, then the proportionality principle for 2-dimensional hyperbolic manifolds gives us $\pi\|M\|=\mathrm{Vol}(M)$ and $\pi\|N\|=\mathrm{Vol}(N)$, so that by Theorem 1,

$$\|M\times N\|=\frac{3}{2\pi^2}\mathrm{Vol}(M\times N)=\frac{3}{2\pi^2}\mathrm{Vol}(M)\mathrm{Vol}(N)=\frac{3}{2}\|M\|\cdot\|N\|,$$

as claimed. □

This is the first instance of an exact product formula for the simplicial volume. Previously known were the rather elementary inequalities

$$\|M\|\cdot\|N\|\leq\|M\times N\|\leq\binom{m+n}{m}\|M\|\cdot\|N\|,$$

where M and N are any closed, oriented manifolds of dimension m and n respectively. Furthermore, when M and N are hyperbolic surfaces, the upper bound of $6\cdot\|M\|\cdot\|N\|$ was improved to $3,25\cdot\|M\|\cdot\|N\|$ by Bowen et al. in [Bo&al04] by exhibiting explicit triangulations of products of polygons. The authors also give lower bounds for the minimal number of simplices in such triangulations, which we improve in [Bu07] using the explicit cocycle Θ representing the volume form $\omega_{\mathbb{H}^2\times\mathbb{H}^2}$. Since those triangulations produce fundamental cycles for the fundamental class $[M\times N]$ which in view of Corollary 3 have strictly greater ℓ^1-norm than $\|M\times N\|$, this indicates the existence of triangulations of products of hyperbolic surfaces not arising from triangulations of fundamental domains of the form of a product of polygons. (Note however that the lower bound for the simplicial volume given in [Bo&al04] is incorrect as it relies on the invalid Lemma 2.7.)

In the same way as a hyperbolic surface M can be covered by precisely $\|M\|$ ideal triangles, one could ask if such a covering can be found for products of hyperbolic surfaces. In fact, the cocycle representing $\omega_{\mathbb{H}^2\times\mathbb{H}^2}$ that we exhibit gives natural candidates for the building blocks of such an ideal tessellation, since it takes extremal values on very specific 5-tuples of points of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$.

The present computations are used in [LoSa07] to give the exact value of the simplicial volume of Hilbert modular surfaces. Those are *open*, finite volume, $\mathbb{Q}$-rank 1 manifolds with universal cover isometric to $\mathbb{H}^2 \times \mathbb{H}^2$.

This paper is structured as follows: In Section 2, we recall the definition of continuous cohomology, and give an explicit cocycle Θ representing the volume form $\omega_{\mathbb{H}^2\times\mathbb{H}^2}$ in Proposition 4. In Section 3, we introduce continuous bounded cohomology, and show how our Main Theorem reduces to computing the norm of our explicit representative Θ in the cohomology group $H^4_{c,b}(H,\widetilde{\mathbb{R}})$, where H is the (full) isometry group of $\mathbb{H}^2 \times \mathbb{H}^2$ and $\widetilde{\mathbb{R}}$ is the real line endowed with the action of H given by orientation. In Section 4, we compute the norm of Θ both as a cocycle in Proposition 7 and as a cohomology class of $H^4_{c,b}(H,\widetilde{\mathbb{R}})$ in Theorem 6. Finally, we prove that the comparison map $H^4_{c,b}(H,\widetilde{\mathbb{R}}) \to H^4_c(H,\widetilde{\mathbb{R}})$ is an isomorphism in Section 5.

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2. The volume form $\omega_{\mathbb{H}^2\times\mathbb{H}^2}$ in $H^4_c(\mathrm{PSL}_2\mathbb{R}\times\mathrm{PSL}_2\mathbb{R},\mathbb{R})$

Let G be a topological group and E a G-module. Recall (for example from [Gui80] or [BoWa00]) that the continuous cohomology $H^*_c(G,E)$ of G with coefficients in E can be computed as the cohomology of the cocomplex $C^*(G,E)^G$ endowed with its natural symmetric coboundary operator, where

$$C^q(G,E) = \left\{ f : G^{q+1} \longrightarrow \mathbb{R} \;\middle|\; f \text{ is alternating, measurable} \right\},$$

and $C^q(G,E)^G$ denotes the subspace of G-invariant cochains, where the action of G on $C^q(G,E)$ is given by

$$(g\cdot f)(g_0,...,g_q) = g \cdot f(g^{-1}g_0,...,g^{-1}g_q),$$

for every $(g_0,...,g_q)$ in G^{q+1}, f in $C^q(G,E)$ and g in G.

Let now G be a Lie group, $K < G$ a maximal compact subgroup and $X = G/K$ the associated symmetric space. The Van Est isomorphism

$$\mathcal{J} : A^*(X,E)^G \xrightarrow{\cong} H^*_c(G,E)$$

between the G-invariant E-valued differential forms on X (where the G -action on $A^*(G,E)$ is defined analogously to that on $C^*_c(G,E)$) and the continuous cohomology of G with coefficients in E is both natural and multiplicative. Note furthermore that Dupont gave it an explicit description at the cochain level in [Du76].

The volume form in $\mathbb{H}^2$. Let $\omega_{\mathbb{H}^2}$ denote the volume form in $A^2(\mathbb{H}^2,\mathbb{R})^{\mathrm{PSL}_2\mathbb{R}}$. While $\mathcal{J}(\omega_{\mathbb{H}^2}) \in H^2_c(\mathrm{PSL}_2\mathbb{R},\mathbb{R})$ can, by Dupont's description of the Van Est isomorphism be represented by the cocycle sending a triple of points (g_0,g_1,g_2) in $(\mathrm{PSL}_2\mathbb{R})^3$ to the signed volume of the geodesic triangle with vertices (g_0x,g_1x,g_2x), for some fixed point x in $\mathbb{H}^2$, let us now describe another cocycle representing

$\mathcal{J}(\omega_{\mathbb{H}^2})$. Define

$$\begin{array}{rccl} \mathrm{Or}: & \left(S^1\right)^3 & \longrightarrow & \mathbb{R} \\ & (\xi_0,\xi_1,\xi_2) & \longmapsto & \begin{cases} +1 & \text{if } \xi_0,\xi_1,\xi_2 \text{ are positively oriented,} \\ -1 & \text{if } \xi_0,\xi_1,\xi_2 \text{ are negatively oriented,} \\ 0 & \text{if } \xi_i=\xi_j \text{ for } i\neq j. \end{cases} \end{array}$$

Fix a point ξ in S^1 and let $\mathrm{Or}_\xi : (\mathrm{PSL}_2\mathbb{R})^3 \to \mathbb{R}$ be the cocycle defined by

$$\mathrm{Or}_\xi(g_0,g_1,g_2) = \mathrm{Or}(g_0\xi,g_1\xi,g_2\xi).$$

It is well known and easy to check that

$$\mathcal{J}(\omega_{\mathbb{H}^2}) = \pi[\mathrm{Or}_\xi].$$

Thus, the cocycle $\pi\mathrm{Or}_\xi$ representing $\mathcal{J}(\omega_{\mathbb{H}^2})$ can be thought of as sending a triple of points (g_0,g_1,g_2) to the signed volume of the *ideal* geodesic triangle with vertices $(g_0\xi,g_1\xi,g_2\xi)$. It is the limit of the above described cocycle when $x\in\mathbb{H}^2$ tends to $\xi\in\partial\mathbb{H}^2=S^1$.

The volume form in $\mathbb{H}^2\times\mathbb{H}^2$. Let $\omega_{\mathbb{H}^2\times\mathbb{H}^2}$ denote the volume form in $A^4(\mathbb{H}^2\times\mathbb{H}^2,\mathbb{R})^{\mathrm{PSL}_2\mathbb{R}\times\mathrm{PSL}_2\mathbb{R}}$. We will abuse notation and write $p_i : Y\times Y\to Y$, for $i=1,2$, for the projections on the first and second factors for $Y=\mathbb{H}^2$, $Y=\partial\mathbb{H}^2$ or $Y=\mathrm{PSL}_2\mathbb{R}$. Which of those spaces is meant should be clear from the context. For $i=1,2$, set

$$\begin{aligned} \omega_i = p_i^*(\omega_{\mathbb{H}^2}) &\in A^2(\mathbb{H}^2\times\mathbb{H}^2,\mathbb{R})^{\mathrm{PSL}_2\mathbb{R}\times\mathrm{PSL}_2\mathbb{R}} \quad \text{and} \\ \mathrm{Or}_i = p_i^*(\mathrm{Or}) &: (S^1\times S^1)^3 \longrightarrow \mathbb{R}. \end{aligned}$$

Recall that the standard cup product $f_1\cup f_2$ of a p-cochain $f_1 : Y^{p+1}\to\mathbb{R}$ and a q-cochain $f_2 : Y^{q+1}\to\mathbb{R}$ is the *nonalternating* $(p+q)$-cochain sending the $(p+q+1)$-tuple $(y_0,...,y_{p+q})$ to the product $f_1(y_0,...,y_p)\cdot f_2(y_p,...,y_{p+q})$. Moreover, given a non necessarily alternating p-cochain $f : Y^{p+1}\to\mathbb{R}$, its alternation is the alternating cochain $\mathrm{Alt}(f) : Y^{p+1}\to\mathbb{R}$ defined by

$$\mathrm{Alt}(f)(y_0,...,y_p) = \frac{1}{(p+1)!}\sum_{\sigma\in\mathrm{Sym}(p+1)} \mathrm{sign}(\sigma) f(y_{\sigma(0)},...,y_{\sigma(p)}),$$

for every $(y_0,...,y_p)$ in Y^{p+1}. Set

$$\Theta = \mathrm{Alt}(\mathrm{Or}_1\cup\mathrm{Or}_2) : (S^1\times S^1)^5 \longrightarrow \mathbb{R}.$$

Fix a point ξ in S^1 and let $\Theta_\xi : (\mathrm{PSL}_2\mathbb{R}\times\mathrm{PSL}_2\mathbb{R})^5\to\mathbb{R}$ be the cocycle defined by

$$\Theta_\xi((g_0,h_0),...,(g_4,h_4)) = \Theta((g_0\xi,h_0\xi),...,(g_4\xi,h_4\xi)),$$

for every $((g_0,h_0),...,(g_4,h_4))$ in $(\mathrm{PSL}_2\mathbb{R}\times\mathrm{PSL}_2\mathbb{R})^5$. Note that by construction,

$$\Theta_\xi = \mathrm{Alt}(p_1^*(\mathrm{Or}_\xi)\cup p_2^*(\mathrm{Or}_\xi)).$$

Proposition 4. $\mathcal{J}(\omega_{\mathbb{H}^2\times\mathbb{H}^2}) = \pi^2\left[\Theta_\xi\right]$.

Proof. By definition of the Riemannian product metric, the volume form on the product, is the wedge product of the volume forms on the factors, so that $\omega_{\mathbb{H}^2\times\mathbb{H}^2} = \omega_1\wedge\omega_2$. Since the van Est isomorphism $\mathcal{J}$ is multiplicative, we thus have

$$\mathcal{J}(\omega_{\mathbb{H}^2\times\mathbb{H}^2}) = \mathcal{J}(\omega_1\wedge\omega_2) = \mathcal{J}(\omega_1)\cup\mathcal{J}(\omega_2).$$

But by naturality of $\mathcal{J}$, we now obtain, for $i=1,2$,

$$\mathcal{J}(\omega_i) = \mathcal{J}(p_i^*(\omega_{\mathbb{H}^2})) = p_i^*(\mathcal{J}(\omega_{\mathbb{H}^2})) = p_i^*(\pi[\mathrm{Or}_\xi]).$$

In particular, we get

$$\mathcal{J}(\omega_{\mathbb{H}^2\times\mathbb{H}^2}) = \pi^2[p_1^*(\mathrm{Or}_\xi)]\cup[p_2^*(\mathrm{Or}_\xi)] = \pi^2\left[\Theta_\xi\right],$$

since the cup product is given, at the cochain level, by alternating the standard cup product. □

3. The continuous (bounded) cohomology of $H = \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$

For more details on continuous, bounded cohomology, we invite the reader to consult [Mo01]. Let G be a topological group and E a Banach G-module. Recall that the continuous cohomology of G with coefficients in E was defined in the previous section as the cohomology of the cocomplex $C_c^*(G,E)^G$. Now that E is moreover assumed to be a Banach space, with norm $\|-\|_E$, say, we can consider the sup norm

$$\|f\|_\infty = \sup\left\{ \|f(g_0,...,g_q)\|_E \;\middle|\; (g_0,...,g_q) \in G^{q+1}\right\}$$

of any cochain f in $C^q(G,E)^G$. Clearly, the coboundary operator restricts to the cocomplex $C_b^*(G,E)^G$ of bounded G-invariant cochains, where

$$C_b^q(G,E) = \{f \in C_c^q(G) \mid \|f\|_\infty < +\infty\},$$

and the *continuous bounded cohomology* $H_{c,b}^*(G,E)$ *of* G *with coefficients in* E is defined as the cohomology of this cocomplex. The inclusion of cocomplexes $C_{c,b}^*(G,E)^G \subset C_c^*(G,E)^G$ induces a *comparison map* $c : H_{c,b}^*(G,E) \to H_c^*(G,E)$. The sup norm defines both a seminorm on $H_c^*(G,E)$ and $H_{c,b}^*(G,E)$ and we continue to denote those by $\|-\|_\infty$. (Note that on $H_c^*(G,E)$ we allow the value $+\infty$.) We will abuse terminology and refer to those seminorms as (sup) norms. By definition, we have for any α in $H_c^q(G,E)$:

$$\|\alpha\|_\infty = \inf\left\{ \|\alpha_b\|_\infty \;\middle|\; \alpha_b \in H_{c,b}^q(G,E),\ c(\alpha_b) = \alpha\right\},$$

where the right hand side of the above equation is understood to be equal to infinity when the infimum is taken over the empty set.

Let now G be a Lie group, $K < G$ a maximal compact subgroup and $X = G/K$ the associated symmetric space. We have already come across, in the previous section, a very convenient cocomplex for the computation of $H_c^*(G,E)$, namely the degenerate cocomplex $A^*(X,E)^G$ (its differential is zero since G-invariant forms are always closed). Let us now describe another useful cocomplex for both continuous and continuous, bounded cohomology: Define

$$C_c^q(X,E) = \left\{ f : X^{q+1} \to E \;\middle|\; f \text{ continuous, alternating}\right\}$$

and as above, let $C_{c,b}^q(X,E)$ denote its subspace of bounded cochains, that is the subspace of $C_c^q(X,E)$ consisting of elements with finite sup norm. The coboundary operator on $C_c^*(X,E)$, which is the canonical symmetric operator, clearly restricts to $C_{c,b}^*(X,E)$. The action of G on $C_c^*(X,E)$ is defined analogously to the one on $C^*(G,E)$. It is a standard fact, that the continuous cohomology $H_c^*(G,E)$ of G with coefficients in E is isomorphic to the cohomology of the cocomplex $\left(C_c^*(X,E)^G,\delta\right)$ (see for example [Gui80, Chapitre III, Proposition 2.3] for a proof). As for the bounded case, it is proven in [Mo01, Corollary 7.4.10] that the continuous bounded cohomology $H_{c,b}^*(G,E)$ of G with coefficients in E is isomorphic to the cohomology of the cocomplex $\left(C_{c,b}^*(X,E)^G,\delta\right)$. Furthermore, the comparison map $c : H_{c,b}^*(G,E) \to H_c^*(G,E)$ is induced by the natural inclusion $C_{c,b}^*(X,E) \subset C_c^*(X,E)$ and the sup norm of $C_c^*(X,E)$ induces the same seminorms on the cohomology groups $H_c^*(G,E)$ and $H_{c,b}^*(G,E)$.

Set $G = \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$ and $H = \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$ and note that G is a subgroup of H of index 8. Indeed, $\mathrm{PSL}_2\mathbb{R}$ has index 2 in $\mathrm{Isom}(\mathbb{H}^2)$, so that G has index 4 in the product $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$ and the latter group together with the isometry τ of $\mathbb{H}^2 \times \mathbb{H}^2$ permuting the factors (i.e. $\tau(x,y) = (y,x)$ for (x,y) in $\mathbb{H}^2 \times \mathbb{H}^2$) generate

H. Consider the absolute value norm on $\mathbb{R}$ and denote by $\widetilde{\mathbb{R}}$ the Banach space $\mathbb{R}$ endowed with the following action of H: an element h of H acts by multiplication by $+1$, respectively -1, if h preserves, resp. reverses, the orientation in $\mathbb{H}^2 \times \mathbb{H}^2$. Observe that restricted to G, this action is trivial, and we denote by $\mathbb{R}$, the Banach space $\mathbb{R}$ endowed with the trivial action of G.

The subgroup inclusion $G < H$ induces maps $i : H_c^*(H, \widetilde{\mathbb{R}}) \to H_c^*(G, \mathbb{R})$ and $i_b : H_{c,b}^*(H, \widetilde{\mathbb{R}}) \to H_{c,b}^*(G, \mathbb{R})$ which are realized, at the cochain level, by the canonical inclusions of cocomplexes $C_c^*(\mathbb{H}^2 \times \mathbb{H}^2, \widetilde{\mathbb{R}})^H \subset C_c^*(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R})^G$ and $C_{c,b}^*(\mathbb{H}^2 \times \mathbb{H}^2, \widetilde{\mathbb{R}})^H \subset C_{c,b}^*(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R})^G$ respectively. In particular, both i and i_b can not increase norms.

Averaging the value of a G-invariant cocycle on a fundamental domain for H/G, it is readily seen that both i and i_b admit left inverses. In fact, those transfer maps can be described explicitly as follows: Fix an orientation reversing isometry σ of $\mathbb{H}^2$ and let σ_1 (respectively σ_2) be the orientation reversing isometry of $\mathbb{H}^2 \times \mathbb{H}^2$ acting as σ (resp. the identity) on the first factor and the identity (resp. σ) on the second factor. As above, let $\tau \in H$ be the orientation preserving isometry permuting the two factors in $\mathbb{H}^2 \times \mathbb{H}^2$. Define

$$m : C_c^q(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R})^G \longrightarrow C_c^*(\mathbb{H}^2 \times \mathbb{H}^2, \widetilde{\mathbb{R}})^H$$

as

$$\begin{aligned} m(f)(z_0, ..., z_q) = \frac{1}{8}\, [& f(z_0, ..., z_q) - f(\sigma_1 z_0, ..., \sigma_1 z_q) \\ & - f(\sigma_2 z_0, ..., \sigma_2 z_q) + f(\sigma_1\sigma_2 z_0, ..., \sigma_1\sigma_2 z_q) \\ & + f(\tau z_0, ..., \tau z_q) - f(\sigma_1\tau z_0, ..., \sigma_1\tau z_q) \\ & - f(\sigma_2\tau z_0, ..., \sigma_2\tau z_q) + f(\sigma_1\sigma_2\tau z_0, ..., \sigma_1\sigma_2\tau z_q)]\,, \end{aligned} \tag{1}$$

for every f in $C^q(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R})^G$ and $(z_0, ..., z_q)$ in $\left(\mathbb{H}^2 \times \mathbb{H}^2\right)^{q+1}$. To check that m is well defined, we need to verify that $m(f)$ is H-invariant whenever f is G-invariant: First, note that since τ has order 2, $m(f)$ is invariant with respect to τ. Second, we compute

$$\begin{aligned} m(f)(\sigma_1 z_0, ..., \sigma_1 z_q) = \frac{1}{8}\, [& f(\sigma_1 z_0, ..., \sigma_1 z_q) - f(\sigma_1^2 z_0, ..., \sigma_1^2 z_q) \\ & - f(\sigma_2\sigma_1 z_0, ..., \sigma_2\sigma_1 z_q) + f(\sigma_1\sigma_2\sigma_1 z_0, ..., \sigma_1\sigma_2\sigma_1 z_q) \\ & + f(\tau\sigma_1 z_0, ..., \tau\sigma_1 z_q) - f(\sigma_1\tau\sigma_1 z_0, ..., \sigma_1\tau\sigma_1 z_q) \\ & - f(\sigma_2\tau\sigma_1 z_0, ..., \sigma_2\tau\sigma_1 z_q) + f(\sigma_1\sigma_2\tau\sigma_1 z_0, ..., \sigma_1\sigma_2\tau\sigma_1 z_q)]\,. \end{aligned}$$

Using the facts that σ_1 commutes with σ_2, that $\sigma_1\tau = \tau\sigma_2$, that both σ_1^2 and σ_2^2 belong to G and that f is G-invariant, we have

$$\begin{aligned} f(\sigma_1^2 z_0, ..., \sigma_1^2 z_q) &= f(z_0, ..., z_q), \\ f(\sigma_2\sigma_1 z_0, ..., \sigma_2\sigma_1 z_q) &= f(\sigma_1\sigma_2 z_0, ..., \sigma_1\sigma_2 z_q), \\ f(\sigma_1\sigma_2\sigma_1 z_0, ..., \sigma_1\sigma_2\sigma_1 z_q) &= f(\sigma_2 z_0, ..., \sigma_2 z_q), \\ f(\tau\sigma_1 z_0, ..., \tau\sigma_1 z_q) &= f(\sigma_2\tau z_0, ..., \sigma_2\tau z_q), \\ f(\sigma_1\tau\sigma_1 z_0, ..., \sigma_1\tau\sigma_1 z_q) &= f(\sigma_1\sigma_2\tau z_0, ..., \sigma_1\sigma_2\tau z_q), \\ f(\sigma_2\tau\sigma_1 z_0, ..., \sigma_2\tau\sigma_1 z_q) &= f(\tau z_0, ..., \tau z_q), \\ f(\sigma_1\sigma_2\tau\sigma_1 z_0, ..., \sigma_1\sigma_2\tau\sigma_1 z_q) &= f(\sigma_1\tau z_0, ..., \sigma_1\tau z_q). \end{aligned}$$

Hence, the above expression for $m(f)(\sigma_1 z_0, ..., \sigma_1 z_q)$ is equal to $-m(f)(z_0, ..., z_q)$, which proves the invariance of $m(f)$ with respect to σ_1. The invariance with respect to σ_2 is proven symmetrically. Third, let g be an isometry in G and observe that since G is normal in H, there exists g_1, g_2 and g_3 in G such that $\sigma_1 g = g_1\sigma_1$,

$\sigma_2 g = g_2\sigma_2$ and $\sigma_1\sigma_2 g = g_3\sigma_1\sigma_2$. For k in G, define $\overline{k}$ in G as $\overline{k} = \tau k \tau$. Note that if $k = (k_1, k_2) \in \mathrm{PSL}_2\mathbb{R}\times\mathrm{PSL}_2\mathbb{R}$, then $\overline{k} = (k_2, k_1)$. We now have

$$\begin{aligned}
m(f)(gz_0, ..., gz_q) &= \frac{1}{4}\,[f(gz_0, ..., gz_q) - f(\sigma_1 gz_0, ..., \sigma_1 gz_q)\\
&\qquad - f(\sigma_2 gz_0, ..., \sigma_2 gz_q) + f(\sigma_1\sigma_2 gz_0, ..., \sigma_1\sigma_2 gz_q)\\
&\qquad + f(\tau gz_0, ..., \tau gz_q) - f(\sigma_1\tau gz_0, ..., \sigma_1\tau gz_q)\\
&\qquad - f(\sigma_2\tau gz_0, ..., \sigma_2\tau gz_q) + f(\sigma_1\sigma_2\tau gz_0, ..., \sigma_1\sigma_2\tau gz_q)]\\
&= \frac{1}{4}\,[f(gz_0, ..., gz_q) - f(g_1\sigma_1 z_0, ..., g_1\sigma_1 z_q)\\
&\qquad - f(g_2\sigma_2 z_0, ..., g_2\sigma_2 z_q) + f(g_3\sigma_1\sigma_2 z_0, ..., g_3\sigma_1\sigma_2 z_q)\\
&\qquad + f(\overline{g}\tau z_0, ..., \overline{g}\tau z_q) - f(\overline{g}_2\sigma_1\tau z_0, ..., \overline{g}_2\sigma_1\tau z_q)\\
&\qquad - f(\overline{g}_1\sigma_2\tau z_0, ..., \overline{g}_1\sigma_2\tau z_q) + f(\overline{g}_3\sigma_1\sigma_2\tau z_0, ..., \overline{g}_3\sigma_1\sigma_2\tau z_q)]\\
&= m(f)(z_0, ..., z_q),
\end{aligned}$$

where for the last equality we have used eight times the G-invariance of f. Finally, the H-invariance of $m(f)$ follows from that H is generated by σ_1, σ_2, τ and G. Observe also that, by the G-invariance of f, the definition of $m(f)$ is independent of the choice of σ.

It is readily seen that m is a cochain map which moreover restricts to a map

$$m_b : C^q_{c,b}(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R})^G \longrightarrow C^q_{c,b}(\mathbb{H}^2 \times \mathbb{H}^2, \widetilde{\mathbb{R}})^H$$

between the respective bounded cocomplexes. In particular, m and m_b induce maps, which we still denote by m and m_b between the corresponding cohomology groups. It is clear that neither m nor m_b can increase norms. Furthermore, since both the inclusion of cocomplexes $C^*_c(\mathbb{H}^2 \times \mathbb{H}^2, \widetilde{\mathbb{R}})^H \subset C^*_c(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R})^G$ composed with m and $C^*_{c,b}(\mathbb{H}^2 \times \mathbb{H}^2, \widetilde{\mathbb{R}})^H \subset C^*_{c,b}(\mathbb{H}^2 \times \mathbb{H}^2, \mathbb{R})^G$ composed with m_b are the identity maps, we have obtained a commutative diagram

$$\begin{tikzcd}
H^*_c(H, \widetilde{\mathbb{R}}) \arrow[r, hook, "i"] \arrow[rr, bend left, "\mathrm{Id}"] & H^*_c(G, \mathbb{R}) \arrow[r, two heads, "m"] & H^*_c(H, \widetilde{\mathbb{R}}) \\
H^*_{c,b}(H, \widetilde{\mathbb{R}}) \arrow[u, "c"] \arrow[r, hook, "i_b"] \arrow[rr, bend right, "\mathrm{Id}"'] & H^*_{c,b}(G, \mathbb{R}) \arrow[u, "c"] \arrow[r, two heads, "m_b"] & H^*_{c,b}(H, \widetilde{\mathbb{R}}). \arrow[u, "c"]
\end{tikzcd}$$

Going back to the definition of the above cohomology groups in terms of the cocomplexes $C^*(G, \mathbb{R})^G$ and $C^*(H, \widetilde{\mathbb{R}})^H$ and their bounded subcocomplexes, we see that, for any ξ in S^1, the cochain $\Theta_\xi \in C^4(G, \mathbb{R})^G$ encountered in the previous section has sup norm $\|\Theta_\xi\|_\infty \le 1$ and hence belongs to $C^4_b(G, \mathbb{R})^G$. (In fact, we will show in Proposition 7 that $\|\Theta_\xi\|_\infty = \|\Theta\|_\infty = 2/3$.) Denote by $[\Theta] \in H^4_c(G, \mathbb{R})$ and $[\Theta]_b \in H^4_{c,b}(G, \mathbb{R})$ the corresponding cohomology classes.

Since H acts on $S^1 \times S^1 = \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$, we can as well extend our definition of Θ_ξ to a cocycle on H which maps a 5-tuple $(h_0, ..., h_4)$ in H^5 to $\Theta(h_0(\xi, \xi), ..., h_4(\xi, \xi))$. The extended Θ_ξ is clearly H-invariant and hence belongs to $C^4(H, \widetilde{\mathbb{R}})^H$ and also to $C^4_b(H, \widetilde{\mathbb{R}})^H$. Denote by $[\Theta]^H \in H^4_c(H, \widetilde{\mathbb{R}})$ and $[\Theta]^H_b \in H^4_{c,b}(H, \widetilde{\mathbb{R}})$ the corresponding cohomology classes.

Because the four cohomology classes $[\Theta]$, $[\Theta]^H$, $[\Theta]_b$ and $[\Theta]^H_b$ are the equivalence classes of the same cocycle Θ_ξ, it is obvious that

$$\begin{array}{lll}
c([\Theta]_b) = [\Theta]\,, & c([\Theta]^H_b) = [\Theta]^H & \text{and}\\
i([\Theta]^H) = [\Theta]\,, & i([\Theta]^H_b) = [\Theta]_b\,. &
\end{array}$$

Proposition 5. $\|[\Theta]\|_\infty = \left\|[\Theta]^H\right\|_\infty$.

Proof. We use the facts that both i and m can not increase norms and that $m \circ i$ is the identity on $H^4_c(H, \widetilde{\mathbb{R}})$ to obtain

$$\left\|[\Theta]^H\right\|_\infty = \left\|(m \circ i)\left([\Theta]^H\right)\right\|_\infty \leq \left\|i\left([\Theta]^H\right)\right\|_\infty \leq \left\|[\Theta]^H\right\|_\infty.$$

In particular, all the above inequalities are equalities. The proposition is now immediate from that $i\left([\Theta]^H\right) = [\Theta]$. $\square$

Theorem 6. *The comparison map $c : H^*_{cb}(H, \widetilde{\mathbb{R}}) \to H^*_c(H, \widetilde{\mathbb{R}})$ is an isomorphism in degree 4 sending $[\Theta]^H_b$ to $[\Theta]^H$.*

Note that if the comparison map is an isomorphism, it is automatically isometric. Theorem 6 is proven in Section 5 in a constructive way: given a cocycle representing a cohomology class of degree 4 in the kernel of the comparison map, we can explicitly express it as a coboundary. This isomorphism allows us to reduce the proof of our Main Theorem to the computation of the norm of Θ in $H^*_{cb}(H, \widetilde{\mathbb{R}})$, as stated in the next theorem, which we will prove in Section 4.

Theorem 6. *The norm of Θ in $H^*_{cb}(H, \widetilde{\mathbb{R}})$ is equal to $\left\|[\Theta]^H_b\right\|_\infty = 2/3$.*

Proof of Main Theorem. Recall that we showed in Proposition 4 that $\omega_{\mathbb{H}^2 \times \mathbb{H}^2} \in H^4_c(G, \mathbb{R})$ (which was there denoted by $\mathcal{J}(\omega_{\mathbb{H}^2 \times \mathbb{H}^2})$) is equal to $\pi^2[\Theta_\xi] = \pi^2[\Theta]$. Applying successively Proposition 5, Theorem 6 and Theorem 6, we obtain

$$\frac{\|\omega_{\mathbb{H}^2 \times \mathbb{H}^2}\|}{\pi^2} = \|[\Theta]\|_\infty = \left\|[\Theta]^H\right\|_\infty = \left\|[\Theta]^H_b\right\|_\infty = 2/3.$$

$\square$

We now introduce yet another cocomplex for the computation of the continuous, bounded cohomology groups: Let $C^q_b(S^1 \times S^1, \widetilde{\mathbb{R}})$ denote the space of alternating, measurable, bounded, real-valued functions on $(S^1 \times S^1)^{q+1}$ endowed with its natural symmetric coboundary operator δ. The action of H on $C^q_b(S^1 \times S^1, \widetilde{\mathbb{R}})$ is defined analogously to the one of H on $C^q(H, \widetilde{\mathbb{R}})$. Its subspaces of H-invariant and G-invariant functions are denoted by $C^q_b(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ and $C^q_b(S^1 \times S^1, \mathbb{R})^G$ respectively. It is proven in [Mo01, Corollary 7.5.9] that the cohomology of the latter cocomplexes are isomorphic to $H^*_{cb}(H, \widetilde{\mathbb{R}})$ and $H^*_{cb}(G, \mathbb{R})$ respectively, that the map $i_b : H^*_{cb}(H, \widetilde{\mathbb{R}}) \to H^*_{cb}(G, \mathbb{R})$ is realized at the cochain level by the canonical inclusion $C^q_b(S^1 \times S^1, \widetilde{\mathbb{R}})^H \subset C^q_b(S^1 \times S^1, \mathbb{R})^G$, and that the sup norm on $C^q_b(S^1 \times S^1, \widetilde{\mathbb{R}})$ gives rise to the desired seminorms on the continuous, bounded cohomology groups. Furthermore, it is easy to verify, that the map $m_b : H^*_{cb}(G, \mathbb{R}) \to H^*_{cb}(H, \widetilde{\mathbb{R}})$ is realized at the cochain level by the map

$$m_b : C^q_b(S^1 \times S^1, \mathbb{R})^G \longrightarrow C^q_b(S^1 \times S^1, \widetilde{\mathbb{R}})^H$$

defined exactly as in (5) except that f is now taken in $C^q_b(S^1 \times S^1, \mathbb{R})^G$ and $(z_0, ..., z_q)$ in $\left(S^1 \times S^1\right)^{q+1}$. Finally, note that the cohomology classes $[\Theta]^H_b$ and $[\Theta]_b$ are represented in the cocomplexes $C^q_b(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ and $C^q_b(S^1 \times S^1, \mathbb{R})^G$ by the cocycle $\Theta : (S^1 \times S^1)^5 \to \mathbb{R}$.

4. The norm of Θ

Recall that $\Theta \in C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ is defined as the cocycle

$$\Theta = \mathrm{Alt}(\mathrm{Or}_1 \cup \mathrm{Or}_2),$$

where Or_1 and Or_2 are the pullbacks under the first and second projection respectively of the orientation cocycle on S^1.

Proposition 7. $\|\Theta\|_\infty = 2/3$.

Proof. By definition, we have, for any 5-tuple $((x_0, y_0), ..., (x_4, y_4))$ in $(S^1 \times S^1)^5$, that $\Theta((x_0, y_0), ..., (x_4, y_4))$ is equal to

$$\frac{1}{120} \sum_{\sigma \in \mathrm{Sym}(5)} \mathrm{sign}(\sigma) \mathrm{Or}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) \cdot \mathrm{Or}(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}).$$

Set $\tau = (0\ 1\ ...\ 4)$ and observe that every permutation $\sigma \in \mathrm{Sym}(5)$ can be written uniquely as $\sigma = \tau^k \circ \alpha$, where $\alpha \in \mathrm{Sym}(5)$ maps 2 to 0, and k is an integer between 0 and 4. Now, exploiting the fact that Or is alternating, we can rewrite the above expression for $\Theta((x_0, y_0), ..., (x_4, y_4))$ as

$$\begin{aligned} \frac{1}{30} \sum_{\substack{\tau=(0\ 1...4)^k \\ k\in\{0,1,...,4\}}} \Big[&\mathrm{Or}\left(x_{\tau(0)}, x_{\tau(1)}, x_{\tau(2)}\right) \cdot \mathrm{Or}(y_{\tau(0)}, y_{\tau(3)}, y_{\tau(4)}) \\ &+ \mathrm{Or}\left(x_{\tau(0)}, x_{\tau(3)}, x_{\tau(4)}\right) \cdot \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(2)}) \\ &- \mathrm{Or}\left(x_{\tau(0)}, x_{\tau(1)}, x_{\tau(3)}\right) \cdot \mathrm{Or}(y_{\tau(0)}, y_{\tau(2)}, y_{\tau(4)}) \\ &- \mathrm{Or}\left(x_{\tau(0)}, x_{\tau(2)}, x_{\tau(4)}\right) \cdot \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(3)}) \\ &+ \mathrm{Or}\left(x_{\tau(0)}, x_{\tau(1)}, x_{\tau(4)}\right) \cdot \mathrm{Or}(y_{\tau(0)}, y_{\tau(2)}, y_{\tau(3)}) \\ &+ \mathrm{Or}\left(x_{\tau(0)}, x_{\tau(2)}, x_{\tau(3)}\right) \cdot \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(4)}) \Big]. \end{aligned} \tag{2}$$

Let us now compute the absolute value of the evaluation of Θ on an arbitrary 5-tuple $((x_0, y_0), ..., (x_4, y_4))$ in $(S^1 \times S^1)^5$. If the x_i's are all distinct, we can, since Θ is alternating, up to permuting the x_i's assume that they are cyclically ordered according to their numbering. Thus, $\mathrm{Or}\,(x_i, x_j, x_k) = +1$ whenever $0 \le i < j < k \le 4$, and all the orientation cocycles involving the x_i's in the expression (2) are equal to $+1$, so that $\Theta((x_0, y_0), ..., (x_4, y_4))$ is equal to

$$\begin{aligned} &\frac{1}{30} \sum_{\substack{\tau=(0\ 1...4)^k \\ k\in\{0,1,...,4\}}} \Big[\mathrm{Or}(y_{\tau(0)}, y_{\tau(3)}, y_{\tau(4)}) + \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(2)}) \\ &\qquad - \mathrm{Or}(y_{\tau(0)}, y_{\tau(2)}, y_{\tau(4)}) - \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(3)}) \\ &\qquad + \mathrm{Or}(y_{\tau(0)}, y_{\tau(2)}, y_{\tau(3)}) + \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(4)}) \Big] \\ = &\frac{1}{30} \sum_{\substack{\tau=(0\ 1...4)^k \\ k\in\{0,1,...,4\}}} \Big[\mathrm{Or}(y_{\tau(2)}, y_{\tau(3)}, y_{\tau(4)}) + \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(2)}) \\ &\qquad - \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(3)}) + \mathrm{Or}(y_{\tau(0)}, y_{\tau(1)}, y_{\tau(4)}) \Big], \end{aligned}$$

where we have used the cocycle relation

$$\begin{aligned} 0 &= \delta \mathrm{Or}(y_{\tau(0)}, y_{\tau(2)}, y_{\tau(3)}, y_{\tau(4)}) \\ &= \mathrm{Or}(y_{\tau(2)}, y_{\tau(3)}, y_{\tau(4)}) - \mathrm{Or}(y_{\tau(0)}, y_{\tau(3)}, y_{\tau(4)}) \\ &\quad + \mathrm{Or}(y_{\tau(0)}, y_{\tau(2)}, y_{\tau(4)}) - \mathrm{Or}(y_{\tau(0)}, y_{\tau(2)}, y_{\tau(3)}). \end{aligned}$$

It is now immediate that $|\Theta((x_0,y_0),...,(x_4,y_4))| \leq 2/3$, since the last expression for $\Theta((x_0,y_0),...,(x_4,y_4))$ is a sum of $5\cdot 4 = 20$ elements admitting the values $\pm 1/30$ and 0.

If the x_i's are not all distinct, then we can without loss of generality assume that $x_0 = x_1$. If the y_i's were all distinct, then by symmetry we could apply the above argument to show that $|\Theta((x_0,y_0),...,(x_4,y_4))| \leq 2/3$. Let us thus assume that the y_i's are not all distinct. If $y_0 = y_1$ then $(x_0,y_0) = (x_1,y_1)$ and hence $\Theta((x_0,y_0),...,(x_4,y_4)) = 0$ since Θ is alternating. We can now, again without loss of generality assume that $y_2 = y_k$, for k in $\{0,3,4\}$. In the expression (2) there are exactly 9 summands which have as a factor $\mathrm{Or}(x_0,x_1,x_j)$, up to permutation of the entries, for $j \geq 2$, and hence vanish. Furthermore, the summand $\mathrm{Or}(x_0,x_3,x_4)\cdot \mathrm{Or}(y_1,y_2,y_k)$ (which exists in (2) again up to permutation of the entries) also vanishes and clearly has not yet been counted among the summands having a factor of the form $\mathrm{Or}(x_0,x_1,x_j)$, so at least 10 of the 30 summands in (2) vanish and

$$|\Theta((x_0,y_0),...,(x_4,y_4))| \leq 2/3.$$

We have thus proven $\|\Theta\|_\infty \leq \frac{2}{3}$. To prove equality, observe that if $x_0,...,x_4$ in S^1 are positively cyclically ordered according to their numbering and $y_0,...,y_4$ in S^1 are ε-cyclically ordered, for ε in $\{-1,+1\}$, according to their numbering, then

$$\Theta((x_0,y_0),(x_1,y_2),(x_2,y_4),(x_3,y_1),(x_4,y_3)) = \varepsilon\cdot\frac{2}{3},$$

which finishes the proof of the proposition. $\square$

Let us now state and prove three easy lemmas which will furthermore be useful again in the next section. For the moment, they will allow us a better understanding of the spaces $C_b^*(S^1\times S^1,\widetilde{\mathbb{R}})^H$ in low degree.

Lemma 8. *Let f be a cochain in $C_b^q(S^1\times S^1,\widetilde{\mathbb{R}})^H$ and let $\underline{z} = ((x_0,y_0),...,(x_q,y_q))$ be a $(q+1)$-tuple in $(S^1\times S^1)^{q+1}$. If there exists σ in $\mathrm{Sym}(q+1)$ such that the permutations $x_i\mapsto x_{\sigma(i)}$ and $y_i\mapsto y_{\sigma(i)}$, for $0\leq i\leq q$, can be realized by isometries g and h of $\mathbb{H}^2$ respectively, then*

$$f(\underline{z}) = \mathrm{sign}(\sigma)\mathrm{sign}(g)\mathrm{sign}(h)f(\underline{z}),$$

where $\mathrm{sign}(k) = +1$, respectively -1, if k is an orientation preserving, resp. reversing, isometry of $\mathbb{H}^2$.

Proof. On the one hand, we have, since f is alternating,

$$f(\underline{z}) = \mathrm{sign}(\sigma)f((x_{\sigma(0)},y_{\sigma(0)}),...,(x_{\sigma(q)},y_{\sigma(q)})).$$

On the other hand, using the H-invariance of f, we get

$$f(\underline{z}) = \mathrm{sign}(g)\mathrm{sign}(h)f((gx_0,hy_0),...,(gx_q,hy_q)).$$

But by assumption, $(x_{\sigma(i)},y_{\sigma(i)}) = (gx_i,hy_i)$, for every $0\leq i\leq q$, and the lemma follows. $\square$

Lemma 9. *Let x_0,x_1,x_2,x_3 be distinct points on S^1. Denote by $\langle x_i,x_j\rangle$, for $i\neq j$, the geodesic in $\mathbb{H}^2$ between x_i and x_j in $\partial\mathbb{H}^2 = S^1$.*

(1) *If $\langle x_0,x_1\rangle\cap\langle x_2,x_3\rangle\neq\varnothing$, then there exists an orientation preserving isometry of $\mathbb{H}^2$ realizing the permutation $(0\ 1)(2\ 3)$.*
(2) *If $\langle x_0,x_1\rangle\cap\langle x_2,x_3\rangle=\varnothing$, then there exists an orientation reversing isometry of $\mathbb{H}^2$ realizing the permutation $(0\ 1)(2\ 3)$.*

Proof. This is elementary from hyperbolic geometry:

(1) Since the points are all distinct, if the geodesics intersect, they intersect in precisely one point. Then the rotation by π centered at the intersection realizes the permutation $x_0 \leftrightarrow x_1, x_2 \leftrightarrow x_3$ and clearly preserves orientation.
(2) There exists a unique geodesic γ perpendicular to both $\langle x_0, x_1\rangle$ and $\langle x_2, x_3\rangle$. The reflection along γ is a reversing orientation isometry of $\mathbb{H}^2$ realizing the permutations $x_0 \leftrightarrow x_1, x_2 \leftrightarrow x_3$.

□

It is easy to conclude, from Lemma 8, that $C_b^q(S^1 \times S^1, \widetilde{\mathbb{R}})^H = 0$ for $0 \leq q \leq 2$, and consequently also $H_{c,b}^q(H, \widetilde{\mathbb{R}}) = 0$ for $0 \leq q \leq 2$. In degree 3, the space of cochain $C_b^3(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ is not zero, but we have the following useful vanishing criterion:

Lemma 10. *Let f be a cochain in $C_b^3(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ and let $\underline{z} = ((x_0, y_0), ..., (x_3, y_3))$ be a 4-tuple in $(S^1 \times S^1)^4$ such that $\langle x_0, x_2\rangle \cap \langle x_1, x_3\rangle \neq \varnothing$ and $\langle y_0, y_2\rangle \cap \langle y_1, y_3\rangle = \varnothing$. Then*

$$f(\underline{z}) = 0.$$

Proof. By Lemma 9 there exists an orientation preserving isometry g of $\mathbb{H}^2$ realizing the permutation $(0\ 2)(1\ 3)$ of the points x_0, x_1, x_2, x_3 and an orientation reversing isometry h of $\mathbb{H}^2$ realizing the permutation $(0\ 2)(1\ 3)$ of the points y_0, y_1, y_2, y_3. Thus, by Lemma 8 we obtain $f(\underline{z}) = -f(\underline{z})$, and the lemma is proven. □

Theorem 6. $\left\|[\Theta]_b^H\right\|_\infty = 2/3$.

Proof. From Proposition 7, one equality is already immediate, namely $\left\|[\Theta]_b^H\right\|_\infty \leq \|\Theta\|_\infty = 2/3$. For the other inequality, let $b \in C_b^3(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ be an arbitrary cochain. As in the end of the proof of Proposition 7, let $x_0, ..., x_4$, and respectively $y_0, ..., y_4$, be positively cyclically ordered points in S^1 and consider the 5-tuple

$$((x_0, y_0), (x_1, y_2), (x_2, y_4), (x_3, y_1), (x_4, y_3))$$

(which we already know has value 2/3 on Θ). Whatever coordinate one removes from this given 5-tuple, the remaining 4-tuple satisfies the conditions of Lemma 10, so that

$$\delta b((x_0, y_0), (x_1, y_2), (x_2, y_4), (x_3, y_1), (x_4, y_3)) = 0.$$

In particular, we obtain

$$\begin{aligned}\|\Theta + \delta b\|_\infty &\geq |(\Theta + \delta b)\left((x_0, y_0), (x_1, y_2), (x_2, y_4), (x_3, y_1), (x_4, y_3)\right)| \\ &= |\Theta((x_0, y_0), (x_1, y_2), (x_2, y_4), (x_3, y_1), (x_4, y_3))| = 2/3,\end{aligned}$$

and hence

$$\left\|[\Theta]_b^H\right\|_\infty = \inf\left\{ \|\Theta + \delta b\|_\infty \middle|\ b \in C_b^3(S^1 \times S^1, \widetilde{\mathbb{R}})^H \right\} \geq 2/3,$$

which finishes the proof of the theorem. □

5. The comparison map $c : H_{cb}^*(H, \widetilde{\mathbb{R}}) \to H_c^*(H, \widetilde{\mathbb{R}})$

In this last section, we prove Theorem 6, that is, we prove that the comparison map $c : H_{cb}^*(H, \widetilde{\mathbb{R}}) \to H_c^*(H, \widetilde{\mathbb{R}})$ is an isomorphism in degree 4 sending $[\Theta]_b^H$ to $[\Theta]^H$. Since $H_c^*(H, \widetilde{\mathbb{R}})$ injects in the 1-dimensional cohomology group $H_c^4(G, \mathbb{R})$ and contains the nonzero class $[\Theta]^H$, it is clearly also 1-dimensional, generated by $[\Theta]^H$. Moreover, we have already seen that $c([\Theta]_b^H) = [\Theta]^H$. Thus, it only remains to prove that the comparison map is injective in degree 4.

Let $p_i : S^1 \times S^1 \to S^1$, for $i = 1, 2$, denote the projection on the first and second factor respectively. For any $(q+1)$-tuple $(z_0, ..., z_q)$ in $(S^1 \times S^1)^{q+1}$, define

$$n_i(z_0, ..., z_q) = \sharp\{p_i(z_0), ..., p_i(z_q)\},$$

for $i = 1, 2$. We will now prove inductively on (n_1, n_2) that if a cocycle $f \in C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ represents a cohomology class which is mapped to zero by the comparison map, then $f = \delta h$ on 5-tuples $\underline{z}$ satisfying $n_1(\underline{z}) \leq n_1$, $n_2(\underline{z}) \leq n_2$. Observe that the fact that $c([f]) = 0$ will only be used in Step 1, where we show that $f = \delta h$ on 5-tuples $\underline{z}$ verifying $n_1(\underline{z}) = n_2(\underline{z}) = 3$. Thus, Step 2 and Step 3 amount to proving that a cocycle vanishing on 5-tuples $\underline{z}$ satisfying $n_1(\underline{z}) = n_2(\underline{z}) = 3$ is a coboundary.

Step 0. *Let f be a cochain in $C_b^q(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ and $\underline{z}$ be a $(q+1)$-tuple in $(S^1 \times S^1)^{q+1}$. If $n_1(\underline{z}) \leq 2$ or $n_2(\underline{z}) \leq 2$, then $f(\underline{z}) = 0$.*

Proof. By symmetry, it is enough to treat the case $n_1(\underline{z}) \leq 2$. If $n_1(\underline{z}) \leq 2$, for $\underline{z} = ((x_0, y_0), ..., (x_q, y_q))$ in $(S^1 \times S^1)^{q+1}$, then there exists an orientation reversing isometry of $\mathbb{H}^2$ fixing $x_0, ..., x_q$, while the identity fixes $y_0, ..., y_q$. In particular, by Lemma 8, $f(\underline{z}) = 0$. □

Let now $f \in C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ be a cocycle satisfying $c([f]) = 0$. Define $h_1 : (S^1 \times S^1)^4 \to \mathbb{R}$ as

$$h_1((x_1, y_1), ..., (x_4, y_4)) = \begin{cases} f((x_i, y_j), (x_1, y_1), ..., (x_4, y_4)), \\ \qquad \text{if } \exists\ i \neq i',\ j \neq j' \text{ with } x_i = x_{i'},\ y_j = y_{j'}, \\ 0, \qquad \text{otherwise.} \end{cases}$$

Let us check that h_1 is well defined: If the condition $x_i = x_{i'}$ is satisfied for different pairs $i_1 \neq i_1'$ and $i_2 \neq i_2'$ (thus $\{i_1, i_1'\} \neq \{i_2, i_2'\}$), then $\sharp\{x_1, ..., x_4\} \leq 2$ and

$$n_1((x_{i_\ell}, y_j), (x_1, y_1), ..., (x_4, y_4)) \leq 2, \quad \text{for } \ell = 1, 2.$$

By Step 0, this now implies that f vanishes on both of those 5-tuples and hence $h_1((x_1, y_1), ..., (x_4, y_4)) = 0$ is well defined. The case when the condition $y_j = y_{j'}$ is satisfied for different pairs of indices is treated symmetrically. Observe furthermore that h_1 belongs to $C_b^3(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ because f belongs to $C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$.

Step 1. *Set $f_1 = f - \delta h_1 \in C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$. If $\underline{z} \in (S^1 \times S^1)^5$ satisfies $n_1(\underline{z}) = n_2(\underline{z}) = 3$, then $f_1(\underline{z}) = 0$.*

Proof. Because f is alternating and H-invariant, and since $\mathrm{Isom}(\mathbb{H}^2)$ acts transitively on oriented triples of distinct points of S^1, the value of f on 5-tuples $\underline{z}$ with $n_1(\underline{z}) = n_2(\underline{z}) = 3$ only depends on the configuration of the coordinates of $\underline{z}$. There are, up to permutation, five such configurations. Thus, there exists $\lambda_0, ..., \lambda_4$ in $\mathbb{R}$ such that for every triple (x_0, x_1, x_2) and (y_0, y_1, y_2) of distinct points of S^1, the following equalities hold:

$$\begin{aligned} f((x_0, y_0), (x_0, y_1), (x_1, y_0), (x_1, y_1), (x_2, y_2)) &= \lambda_0 \cdot \mathrm{Or}(x_0, x_1, x_2) \cdot \mathrm{Or}(y_0, y_1, y_2), \\ f((x_0, y_0), (x_1, y_0), (x_2, y_0), (x_2, y_1), (x_2, y_2)) &= \lambda_1 \cdot \mathrm{Or}(x_0, x_1, x_2) \cdot \mathrm{Or}(y_0, y_1, y_2), \\ -f((x_0, y_0), (x_1, y_0), (x_1, y_1), (x_2, y_1), (x_2, y_2)) &= \lambda_2 \cdot \mathrm{Or}(x_0, x_1, x_2) \cdot \mathrm{Or}(y_0, y_1, y_2), \\ f((x_0, y_0), (x_1, y_0), (x_1, y_1), (x_1, y_2), (x_2, y_2)) &= \lambda_3 \cdot \mathrm{Or}(x_0, x_1, x_2) \cdot \mathrm{Or}(y_0, y_1, y_2), \\ f((x_0, y_0), (x_0, y_1), (x_1, y_1), (x_2, y_1), (x_2, y_2)) &= \lambda_4 \cdot \mathrm{Or}(x_0, x_1, x_2) \cdot \mathrm{Or}(y_0, y_1, y_2). \end{aligned}$$

Note that by Step 0, the above relations also hold when $n_1(\underline{z}) \leq 2$ or $n_2(\underline{z}) \leq 2$ since both sides of the equations are then equal to 0.

We start by invoking Lemma 8 to show that $\lambda_0 = 0$: The even permutation exchanging the first with the second and the third with the fourth coordinate of

$$((x_0, y_0), (x_0, y_1), (x_1, y_0), (x_1, y_1), (x_2, y_2))$$

is realized on the first factor by the identity and on the second by the reversing orientation isometry permuting y_0 with y_1 and fixing y_2. In particular, f has to vanish on this 5-tuple.

Furthermore, note that $\lambda_3 = \lambda_4$ since f is invariant under the orientation preserving isometry τ of $\mathbb{H}^2 \times \mathbb{H}^2$ permuting the two factors.

From the cocycle relation $\delta f = 0$, we compute

$$\begin{aligned} 0 &= \delta f((x_0, y_0), (x_1, y_0), (x_1, y_1), (x_2, y_1), (x_2, y_2), (x_2, y_0)) \\ &= (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\mathrm{Or}(x_0, x_1, x_2) \cdot \mathrm{Or}(y_0, y_1, y_2), \end{aligned}$$

and we see that

$$\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = 2\lambda_3. \tag{3}$$

Claim 11. *If $f \in C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ is such that $c([f]) = 0$, then*

$$2\,(\lambda_1 + \lambda_2) + \lambda_3 + \lambda_4 = 2\,(\lambda_1 + \lambda_2 + \lambda_3) = 0.$$

Proof of Claim. Let

$$\Gamma_2 = \langle a_1, b_1, a_2, b_2 |\ [a_1, b_1][a_2, b_2] = 1\rangle < \mathrm{PSL}_2\mathbb{R}$$

be a representation of the fundamental group Γ_2 of the genus 2 surface Σ_2 in $\mathrm{PSL}_2\mathbb{R}$. Note that

$$\begin{aligned} z = {} & (1, a_1, b_1) + (1, a_1 b_1, a_2) + (1, a_1 b_1 a_2, b_2) \\ & - (1, b_2, a_2) - (1, b_2 a_2, b_1) - (1, b_2 a_2 b_1, a_1) \end{aligned}$$

is a cycle in $C_2(\Gamma_2) \hookrightarrow C_2(\mathrm{PSL}_2\mathbb{R})$ representing the fundamental class $[\Sigma_2] \in H_2(\Sigma_2) \cong H_2(\Gamma_2)$. Recall that given two 2-chains (g_0, g_1, g_2) and (k_0, k_1, k_2) in $C_2(\mathrm{PSL}_2\mathbb{R})$, their product $(g_0, g_1, g_2) \times (k_0, k_1, k_2)$ in $C_4(\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R})$ is defined as the 4-chain

$$\begin{aligned} & ((g_0, k_0), (g_0, k_1), (g_0, k_2), (g_1, k_2), (g_2, k_2)) \\ - & ((g_0, k_0), (g_0, k_1), (g_1, k_1), (g_1, k_2), (g_2, k_2)) \\ + & ((g_0, k_0), (g_0, k_1), (g_1, k_1), (g_2, k_1), (g_2, k_2)) \\ + & ((g_0, k_0), (g_1, k_0), (g_1, k_1), (g_1, k_2), (g_2, k_2)) \\ - & ((g_0, k_0), (g_1, k_0), (g_1, k_1), (g_2, k_1), (g_2, k_2)) \\ + & ((g_0, k_0), (g_1, k_0), (g_2, k_0), (g_2, k_1), (g_2, k_2)). \end{aligned}$$

Thus, $z \times z$ is a 4-cycle in $C_4(G) \hookrightarrow C_4(H)$.

For any cocycle f in $C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$, the cohomology class $c([f]) \in H_{cb}^4(H, \widetilde{\mathbb{R}})$ is represented in $C^4(H, \widetilde{\mathbb{R}})$ by the cocycle

$$\begin{array}{cccc} f_\xi : & H^5 & \longrightarrow & \mathbb{R} \\ & (h_0, ..., h_4) & \longmapsto & f(h_0(\xi, \xi), ..., h_4(\xi, \xi)), \end{array}$$

where ξ is a fixed base point in S^1. For any (g_0, g_1, g_2) and (k_0, k_1, k_2) in $C_2(\mathrm{PSL}_2\mathbb{R})$, we have

$$f_\xi((g_0, g_1, g_2) \times (k_0, k_1, k_2)) = (2(\lambda_1 + \lambda_2) + \lambda_3 + \lambda_4)\mathrm{Or}(g_0\xi, g_1\xi, g_2\xi) \cdot \mathrm{Or}(k_0\xi, k_1\xi, k_2\xi). \tag{4}$$

Upon conjugating Γ_2, we can without loss of generality assume that $a_1\xi = \xi$. Now, remember that, as seen in Section 2, Or_ξ is a cocycle in $C^2(\mathrm{PSL}_2\mathbb{R}, \mathbb{R})$ representing $(1/\pi)\omega_{\mathbb{H}^2}$. In particular, its evaluation on the fundamental class $[\Sigma_2]$ is equal to $(1/\pi) \cdot \mathrm{Vol}(\Sigma_2) = 4$, so that

$$\begin{aligned} 4 = \langle \mathrm{Or}_\xi, z \rangle &= \mathrm{Or}(\xi, a_1\xi, b_1\xi) + \mathrm{Or}(\xi, a_1 b_1\xi, a_2\xi) + \mathrm{Or}(\xi, a_1 b_1 a_2\xi, b_2\xi) \\ &\quad - \mathrm{Or}(\xi, b_2\xi, a_2\xi) - \mathrm{Or}(\xi, b_2 a_2\xi, b_1\xi) - \mathrm{Or}(\xi, b_2 a_2 b_1\xi, a_1\xi) \\ &= \mathrm{Or}(\xi, a_1 b_1\xi, a_2\xi) + \mathrm{Or}(\xi, a_1 b_1 a_2\xi, b_2\xi) - \mathrm{Or}(\xi, b_2\xi, a_2\xi) - \mathrm{Or}(\xi, b_2 a_2\xi, b_1\xi), \end{aligned} \tag{5}$$

since $a_1\xi = \xi$. Because the cocycle Or takes its values in $\{-1, 0, +1\}$, it is now immediate that

$$\mathrm{Or}(\xi, a_1b_1\xi, a_2\xi) = \mathrm{Or}(\xi, a_1b_1a_2\xi, b_2\xi) = -\mathrm{Or}(\xi, b_2\xi, a_2\xi) = -\mathrm{Or}(\xi, b_2a_2\xi, b_1\xi) = 1.$$

Note that alternatively, the above equalities can be checked directly by studying the action of Γ_2 on $\partial\mathbb{H}^2$.

Finally, the assumption that $c([f]) = 0$ tells us that f_ξ is a coboundary and hence vanishes on cycles. In particular, we get $f_\xi(z \times z) = 0$. But from (4) and (5), we straightforwardly compute

$$f_\xi(z \times z) = 16 \cdot (2(\lambda_1 + \lambda_2) + \lambda_3 + \lambda_4)\,,$$

which proves the claim. □

Denote by $\lambda'_0, ..., \lambda'_4$ the real numbers in the defining equations for f on the 5-tuples $\underline{z}$ with $n_1(\underline{z}) = n_2(\underline{z}) = 3$ we would obtain by replacing f by f_1. Note that, as for f, we have $\lambda'_0 = 0$. From the definition of f_1 as $f - \delta h_1$, we furthermore obtain

$$\begin{array}{ll} \lambda'_1 = \lambda_1 - \lambda_1 = 0, & \lambda'_2 = \lambda_1 + \lambda_2, \\ \lambda'_3 = \lambda_3, & \lambda'_4 = \lambda_4. \end{array}$$

But from (3) and the claim, it now follows that $\lambda'_j = 0$, for every $0 \leq j \leq 4$, which proves that f_1 vanishes on all 5-tuples $\underline{z}$ with $n_1(\underline{z}) = n_2(\underline{z}) = 3$. □

Define $h_2 : (S^1 \times S^1)^4 \to \mathbb{R}$ as

$$\begin{aligned} &h_2((x_1, y_1), ..., (x_4, y_4)) = \\ &= \begin{cases} \frac{1}{2}\left[f_1((x_i, y_k), (x_1, y_1), ..., (x_4, y_4)) + f_1((x_i, y_\ell), (x_1, y_1), ..., (x_4, y_4))\right], \\ \qquad \text{if } \{i, j, k, \ell\} = \{1, 2, 3, 4\} \text{ and } x_i = x_j, \\ \frac{1}{2}\left[f_1((x_k, y_i), (x_1, y_1), ..., (x_4, y_4)) + f_1((x_\ell, y_i), (x_1, y_1), ..., (x_4, y_4))\right], \\ \qquad \text{if } \{i, j, k, \ell\} = \{1, 2, 3, 4\} \text{ and } y_i = y_j, \\ 0, \quad \text{otherwise.} \end{cases} \end{aligned}$$

To check that h_2 is well defined, we verify that if the first or the second condition are verified by different sets of indices, then h_2 is in both cases defined as 0: If the condition $x_i = x_j$ is satisfied for different pairs $i_1 \neq j_1$ and $i_2 \neq j_2$ (thus $\{i_1, j_1\} \neq \{i_2, j_2\}$), then as in the proof that h_1 is well defined, we get

$$n_1((x_{i_\ell}, y_*), (x_1, y_1), ..., (x_4, y_4)) \leq 2,$$

for $\ell = 1, 2$ and $y_* \in \{y_1, ..., y_4\}$. By Step 0, this implies that f_1 evaluated on those 5-tuples vanishes, and $h_2((x_1, y_1), ..., (x_4, y_4)) = 0$ is well defined. The case when the condition $y_i = y_j$ is satisfied for different pairs of indices is treated symmetrically. Finally, suppose that $\{i, j, k, \ell\} = \{i', j', k', \ell'\} = \{1, 2, 3, 4\}$ with $x_i = x_j$ and $y_{i'} = y_{j'}$. Then we have both

$$\sharp\{x_1, ..., x_4\} \leq 3 \text{ and } \sharp\{y_1, ..., y_4\} \leq 3.$$

In particular, both n_1 and n_2 are at most equal to 3 when evaluated on the 5-tuples appearing in the definition of h_2. Since by Step 1, f_1 vanishes on those 5-tuples, we obtain, in this case also, that $h_2((x_1, y_1), ..., (x_4, y_4)) = 0$ is well defined. Observe furthermore that h_2 belongs to $C^3_b(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ because f_1 belongs to $C^4_b(S^1 \times S^1, \widetilde{\mathbb{R}})^H$.

Step 2. *Set $f_2 = f_1 - \delta h_2 \in C^4_b(S^1 \times S^1, \widetilde{\mathbb{R}})^H$. If $\underline{z} \in (S^1 \times S^1)^5$ satisfies $n_1(\underline{z}) + n_2(\underline{z}) \leq 7$, then $f_2(\underline{z}) = 0$.*

Proof. By Step 0, $f_2(\underline{z}) = 0$ whenever $n_1(\underline{z}) \leq 2$ or $n_2(\underline{z}) \leq 2$. If $n_1(\underline{z}) = n_2(\underline{z}) = 3$, then $f_1(\underline{z}) = 0$ by Step 1. Furthermore, in this case $\delta h_2(\underline{z})$ is also equal to 0 since all the 5-tuples $\underline{z}'$ evaluated on by f_1 in the definition of $h_2(\widehat{\underline{z}^i})$, where $\widehat{\underline{z}^i}$ denotes the 4-tuple obtained from $\underline{z}$ by removing its i-th coordinate, for $0 \leq i \leq 4$, satisfy $n_1(\underline{z}') = n_2(\underline{z}') = 3$. Thus, f_2 vanishes in this case also. By symmetry, it now remains to treat the case $n_1(\underline{z}) = 3$, $n_2(\underline{z}) = 4$.

Up to permutation, we have two possibilities for the first factor:

(1) $x_0 = x_1 = x_2 \neq x_3 \neq x_4$,
(2) $x_0 = x_1 \neq x_2 = x_3 \neq x_4$.

(1) In the first case, we then have, again up to permutation, three options for the second factor:
   (a) $y_0 = y_1$: Trivially, $f_2(\underline{z}) = 0$ since the two first coordinates of $\underline{z}$ are equal.
   (b) $y_0 = y_4$: We consider two subcases:
   - $\langle y_0, y_3 \rangle \cap \langle y_1, y_2 \rangle \neq \varnothing$: By Lemma 9, there exists an orientation preserving isometry h of $\mathbb{H}^2$ exchanging y_0 with y_3 and y_1 with y_2. Furthermore, there exists an orientation reversing isometry g of $\mathbb{H}^2$ with $gx_0 = x_0$, $gx_3 = x_4$ and $gx_4 = x_3$. Since f_2 is alternating and H-invariant, we get on the one hand, applying the even permutation (1 2)(3 4) and the action by (g, h),

$$\begin{aligned} & f_2((x_0, y_0), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0)) \\ &= f_2((x_0, y_0), (x_0, y_2), (x_0, y_1), (x_4, y_0), (x_3, y_3)) \\ &= -f_2((gx_0, hy_0), (gx_0, hy_2), (gx_0, hy_1), (gx_4, hy_0), (gx_3, hy_3)) \\ &= -f_2((x_0, y_3), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0)). \end{aligned} \tag{6}$$

     On the other hand, apply the cocycle relation of f_2 to the 6-tuple

$$\underline{w} = ((x_0, y_3), (x_0, y_0), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0)).$$

     If one removes the 3-rd or the 4-th variable of $\underline{w}$, then the remaining 5-tuple has $n_1 = n_2 = 3$ and thus f_2 vanishes on it. If one removes the 5-th or the 6-th variable of $\underline{w}$, then the remaining 5-tuple has $n_1 = 2$ and here also f_2 vanishes on it. The cocycle relation $\delta f_2(\underline{w})$ hence simplifies to

$$\begin{aligned} & f_2((x_0, y_0), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0)) \\ &= f_2((x_0, y_3), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0)). \end{aligned}$$

     Together with (6), this shows that f_2 vanishes on 5-tuples of the form $((x_0, y_0), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0))$.
   - $\langle y_0, y_3 \rangle \cap \langle y_1, y_2 \rangle = \varnothing$: By Lemma 9, there exists an orientation reversing isometry h of $\mathbb{H}^2$ exchanging y_0 with y_3 and y_1 with y_2. As above, there exists an orientation reversing isometry g of $\mathbb{H}^2$ with $gx_0 = x_0$, $gx_3 = x_4$ and $gx_4 = x_3$. Since f_1 is alternating and H-invariant, we get, applying the even permutation (1 2)(3 4) and the action by (g, h),

$$\begin{aligned} & f_1((x_0, y_0), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0)) \\ &= f_1((x_0, y_0), (x_0, y_2), (x_0, y_1), (x_4, y_0), (x_3, y_3)) \\ &= f_1((gx_0, hy_0), (gx_0, hy_2), (gx_0, hy_1), (gx_4, hy_0), (gx_3, hy_3)) \\ &= f_1((x_0, y_3), (x_0, y_1), (x_0, y_2), (x_3, y_3), (x_4, y_0)). \end{aligned}$$

In particular,

$$\begin{aligned}
&\delta h_2((x_0,y_0),(x_0,y_1),(x_0,y_2),(x_3,y_3),(x_4,y_0))\\
&= h_2((x_0,y_1),(x_0,y_2),(x_3,y_3),(x_4,y_0))\\
&= \frac{1}{2}\left[f_1((x_0,y_3),(x_0,y_1),(x_0,y_2),(x_3,y_3),(x_4,y_0))\right.\\
&\quad \left.+f_1((x_0,y_0),(x_0,y_1),(x_0,y_2),(x_3,y_3),(x_4,y_0))\right]\\
&= f_1((x_0,y_0),(x_0,y_1),(x_0,y_2),(x_3,y_3),(x_4,y_0)),
\end{aligned}$$

and hence f_2 vanishes on this 5-tuple.

(c) $y_3 = y_4$: Our 5-tuple $\underline{z}$ has the form

$$\underline{z} = ((x_0,y_0),(x_0,y_1),(x_0,y_2),(x_3,y_3),(x_4,y_3)).$$

Set

$$\underline{w} = (\underline{z},(x_0,y_3)).$$

The cocycle relation $\delta f_2(\underline{w}) = 0$ gives

$$f_2(\underline{z}) = \sum_{i=0}^{4}(-1)^i f_2(\widehat{\underline{w}}^i),$$

where $\widehat{\underline{w}}^i$ denotes the 5-tuple obtained from $\underline{w}$ by removing its i-th coordinate, for $0 \leq i \leq 4$. But $f_2(\widehat{\underline{w}}^i) = 0$ for $i = 0,1,2$, since in this case $n_1(\widehat{\underline{w}}^i) = n_2(\widehat{\underline{w}}^i) = 3$, and for $i = 3,4$, since then $n_1(\widehat{\underline{w}}^i) = 2$. In particular, $f_2(\underline{z}) = 0$.

(2) In the second case, we have up to permutation, three options for the second factor:

(a) $y_0 = y_1$: Again, trivially, $f_2(\underline{z}) = 0$.

(b) $y_0 = y_4$: Our 5-tuple $\underline{z}$ has the form

$$\underline{z} = ((x_0,y_0),(x_0,y_1),(x_2,y_2),(x_2,y_3),(x_4,y_0)).$$

Set

$$\underline{w} = (\underline{z},(x_2,y_0)).$$

The cocycle relation $\delta f_2(\underline{w}) = 0$ gives

$$f_2(\underline{z}) = \sum_{i=0}^{4}(-1)^i f_2(\widehat{\underline{w}}^i).$$

But $f_2(\widehat{\underline{w}}^i) = 0$ for $i = 1,2,3$, since in this case $n_1(\widehat{\underline{w}}^i) = n_2(\widehat{\underline{w}}^i) = 3$, and for $i = 4$, since then $n_1(\widehat{\underline{w}}^i) = 2$. Finally, for $i = 0$, we have $(n_1(\widehat{\underline{w}}^i), n_2(\widehat{\underline{w}}^i)) = (3,4)$, but $\widehat{\underline{w}}^i$ is of the form treated in (1) since its first coordinates consists of the 5-tuple (x_0,x_2,x_2,x_4,x_2) and hence vanishes when evaluated on f_2. It follows that $f_2(\underline{z}) = 0$.

(c) $y_0 = y_2$: Our 5-tuple $\underline{z}$ has the form

$$\underline{z} = ((x_0,y_0),(x_0,y_1),(x_2,y_0),(x_2,y_3),(x_4,y_4)).$$

Set

$$\underline{w} = (\underline{z},(x_2,y_4)).$$

The cocycle relation $\delta f_2(\underline{w}) = 0$ gives

$$f_2(\underline{z}) = \sum_{i=0}^{4}(-1)^i f_2(\widehat{\underline{w}}^i).$$

We see that: for $i = 1,3$, $n_1(\widehat{\underline{w}}^i) = n_2(\widehat{\underline{w}}^i) = 3$; for $i = 4$, $n_1(\widehat{\underline{w}}^i) = 2$; for $i = 0$, $(n_1(\widehat{\underline{w}}^i), n_2(\widehat{\underline{w}}^i)) = (3,4)$, but $\widehat{\underline{w}}^i$ is of the form treated in (1) since its first coordinates consists of the 5-tuple $(x_0, x_2, x_2, x_4, x_2)$; for $i = 2$, again $(n_1(\widehat{\underline{w}}^i), n_2(\widehat{\underline{w}}^i)) = (3, 4)$, but

$$\widehat{\underline{w}}^0 = ((x_0, y_0), (x_0, y_1), (x_2, y_3), (x_4, y_4), (x_2, y_4))$$

is of the form treated in (2*b*). In all those cases, we thus obtain $f_2(\widehat{\underline{w}}^i) = 0$ and hence $f_2(\underline{z}) = 0$.

□

Define $h_3 : (S^1 \times S^1)^4 \to \mathbb{R}$ as

$$h_3((x_1, y_1), ..., (x_4, y_4)) = \frac{1}{12} \sum_{i,j=1}^{4} f_2((x_i, y_j), (x_1, y_1), ..., (x_4, y_4)).$$

Observe furthermore that h_3 belongs to $C_b^3(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ because f_2 belongs to $C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$.

Step 3. *Set $f_3 = f_2 - \delta h_3 \in C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$. For any $\underline{z} \in (S^1 \times S^1)^5$, we have $f_3(\underline{z}) = 0$.*

Proof. We start with a preliminary computation.

Claim 12. *For any $((x_1, y_1), ..., (x_4, y_4))$ in $\left(S^1 \times S^1\right)^4$ and any $1 \leq i \neq j \leq 4$ and $1 \leq i' \neq j' \leq 4$, we have*

$$f_2((x_i, y_j), (x_1, y_1), ..., (x_4, y_4)) = f_2((x_{i'}, y_{j'}), (x_1, y_1), ..., (x_4, y_4)).$$

Proof of Claim. By symmetry, if we prove the claim for $i = i'$, then it is also proven for $j = j'$. Furthermore, the general case then follows since if $i \neq j'$, we can go from (i, j) to (i, j') and then to (i', j'), and similarly if $i' \neq j$. Finally, if $i = j'$ and $i' = j$, there exists $k \neq i, j$ so that we can go from (i, j) to (i, k) to (i', k) and to (i', j'). Thus, it is now enough to prove the claim for $i = i'$.

Consider the 6-tuple

$$\underline{z} = ((x_i, y_j), (x_{i'}, y_{j'}), (x_1, y_1), ..., (x_4, y_4)).$$

From the cocycle relation $\delta f_2(\underline{z})$ we see that the claim would follow from the equality

$$0 = \sum_{k=1}^{4} (-1)^k f_2((x_i, y_j), (x_{i'}, y_{j'}), (x_1, y_1), ..., \widehat{(x_k, y_k)}, ..., (x_4, y_4)). \tag{7}$$

But n_1 evaluated on those 5-tuple is smaller or equal to 4 when $k = i$ and to 3 otherwise, while n_2 is smaller or equal to 4 when $k = j$ or j' and to 3 otherwise. Because k can not simultaneously be equal to i and j or j' it follows that $n_1 + n_2 \leq 7$ on all of the 5-tuples appearing in (7), so that, by Step 2, each of the summand in (7) is equal to 0. □

Note that it follows that

$$h_3((x_1, y_1), ..., (x_4, y_4)) = f_2((x_i, y_j), (x_1, y_1), ..., (x_4, y_4)), \tag{8}$$

for any $1 \leq i \neq j \leq 4$.

Observe that by Step 2, both f_2 and δh_3 vanish on 5-tuples $\underline{z}$ satisfying $n_1(\underline{z}) + n_2(\underline{z}) \leq 7$, so that the same holds for f_3. We now will prove step by step, that f_3 also vanishes on 5-tuples $\underline{z}$ with $(n_1(\underline{z}), n_2(\underline{z})) = (3, 5), (4, 4), (4, 5)$ and $(5, 5)$. In all but one subcase, the strategy is the same as in most of the proof of Step 2: 1) Start with an arbitrary 5-tuple with given $(n_1(\underline{z}), n_2(\underline{z}))$. 2) Apply the cocycle relation

$\delta f_3 = 0$ to an appropriately chosen 6-tuple $\underline{w} = (\underline{z}, (x, y))$, for (x, y) in $S^1 \times S^1$, so that

$$f_3(\underline{z}) = \sum_{i=0}^{4} (-1)^i f_3(\widehat{\underline{w}}^i),$$

where $\widehat{\underline{w}}^i$ denotes, for $0 \leq i \leq 5$, the 5-tuple obtained from $\underline{w}$ by removing its i-th coordinate. (In particular, $\widehat{\underline{w}}^5 = \underline{z}$.) 3) Show that $f_3(\widehat{\underline{w}}^i) = 0$ for $0 \leq i \leq 4$.

(3,5): Let $\underline{z}$ be a 5-tuple with $n_1(\underline{z}) = 3$ and $n_2(\underline{z}) = 5$. We distinguish two subcases:

- The 5-tuple $\underline{z}$ has, up to permutation, the form
$$\underline{z} = ((x_0, y_0), (x_1, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4)).$$
Set
$$\underline{w} = (\underline{z}, (x_2, y_0)).$$
We have $f_3(\widehat{\underline{w}}^i) = 0$ for $i = 0, 1$ because $n_1(\widehat{\underline{w}}^i) = 2$, and for $i = 2, 3, 4$ because $n_1(\widehat{\underline{w}}^i) + n_2(\widehat{\underline{w}}^i) = 3 + 4$.
- The 5-tuple $\underline{z}$ has, up to permutation, the form
$$\underline{z} = ((x_0, y_0), (x_0, y_1), (x_2, y_2), (x_2, y_3), (x_4, y_4)).$$
Set
$$\underline{w} = (\underline{z}, (x_0, y_4)).$$
We have $n_1(\widehat{\underline{w}}^i) = 3$ for $i = 0, 1, 2, 3$ and $n_1(\widehat{\underline{w}}^i) = 2$ for $i = 4$. Also, $n_2(\widehat{\underline{w}}^i) = 4$ for $i = 0, 1, 2, 3$ and $n_2(\widehat{\underline{w}}^i) = 5$ for $i = 4$. In any case, $n_1(\widehat{\underline{w}}^i) + n_2(\widehat{\underline{w}}^i) = 7$ so that $f_3(\widehat{\underline{w}}^i) = 0$ for every i.

Note that it follows that $f_3(\underline{z}) = 0$ whenever $n_1(\underline{z}) = 3$ or $n_2(\underline{z}) = 3$.

(4,4): We distinguish two subcases:

- The 5-tuple $\underline{z}$ has, up to permutation, the form
$$\underline{z} = ((x_1, y_2), (x_1, y_1), ..., (x_4, y_4)).$$
In this case we have
$$\delta h_3(\underline{z}) = h_3((x_1, y_1), ..., (x_4, y_4)) = f_2(\underline{z}),$$
where the last equality follows from (8). In particular, $f_3(\underline{z}) = f_2(\underline{z}) - \delta h_3(\underline{z}) = 0$, as desired.
- The 5-tuple $\underline{z}$ has, up to permutation, the form
$$\underline{z} = ((x_0, y_0), (x_0, y_1), (x_2, y_2), (x_3, y_2), (x_4, y_4)).$$
Set
$$\underline{w} = (\underline{z}, (x_0, y_2)).$$
We have $f_3(\widehat{\underline{w}}^i) = 0$ for $i = 0, 1$ because $n_2(\widehat{\underline{w}}^i) = 3$, and for $i = 2, 3, 4$ because $n_1(\widehat{\underline{w}}^i) = 3$.

(4,5): We can assume that $\underline{z}$ has the form
$$\underline{z} = ((x_0, y_0), (x_0, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)).$$
Set
$$\underline{w} = (\underline{z}, (x_3, y_4)).$$
We have $f_3(\widehat{\underline{w}}^i) = 0$ for $i = 0, 1, 3$ because $n_1(\widehat{\underline{w}}^i) = n_2(\widehat{\underline{w}}^i) = 4$, and for $i = 2, 4$ because $n_1(\widehat{\underline{w}}^i) = 3$.

(5,5): Let finally $\underline{z}$ be a generic 5-tuple. Set
$$\underline{w} = (\underline{z}, (x_3, y_4)).$$
We have $f_3(\widehat{\underline{w}}^i) = 0$ for $i = 0, 1, 2, 3$ because $n_2(\widehat{\underline{w}}^i) = 4$, and for $i = 4$ because $n_1(\widehat{\underline{w}}^i) = 4$.

$\square$

In conclusion, the arbitrary cocycle $f \in C_b^4(S^1 \times S^1, \widetilde{\mathbb{R}})^H$ satisfying $c([f]) = 0$ we started with is a coboundary since

$$f = f_1 + \delta h_1 = f_2 + \delta h_2 + \delta h_1 = f_3 + \delta h_3 + \delta h_2 + \delta h_1 = \delta(h_1 + h_2 + h_3),$$

and Theorem 6 is hence proven.

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