Simplicial volume of products and fiber bundles

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Abstract. We give new lower bounds for the simplicial volume of fiber bundles, when the fiber is a surface, improving the lower bounds of Hoster and Kotschick [Proc. of the AMS Vol 129 Nr 4 (2001)]. Our bounds are new also in the product case. Furthermore, for fiber bundles E with fiber F over a base space B we show that the simplicial volume of E is greater or equal to the simplicial volume of the product $F \times B$ when E has dimension smaller or equal to 4.

1. Introduction

Let M be a closed oriented m-dimensional manifold. In his seminal paper [**3**], Gromov introduced the homotopy invariant *simplicial volume* of M for its many connections to Riemannian geometry. Recall that it is defined as the infimum of the L^1-seminorm of the real valued fundamental class $[M] \in H_m(M, \mathbb{R})$ of M, that is

$$\|M\| = \inf \left\{ \Sigma \left| a_\sigma \right| : [M] \text{ is represented by } \Sigma a_\sigma \sigma \right\}.$$

It is easy to see that manifolds admitting self maps of degree greater or equal to 2 have to have vanishing simplicial volume. Thus the simplicial volume of spheres and tori is zero. More generally, manifolds having amenable fundamental group have vanishing simplicial volume. The first nontrivial examples are given by surfaces Σ_g of genus $g \geq 2$ which have $\|\Sigma_g\| = 2\left|\chi(\Sigma_g)\right| = 4(g-1)$. If Σ_g is endowed with a hyperbolic structure, then it follows that the simplicial volume is proportional to the volume of Σ_g. This phenomenon generalizes to all Riemannian manifolds, and is known as Gromov-Thurston's Proportionality Principle [**3, 7**]. For hyperbolic manifolds, the proportionality constant is equal to the supremum of the volumes of geodesic simplices in the hyperbolic space. In particular, since this constant is finite, this shows that the volume of hyperbolic manifolds is a homotopy invariant, generalizing also to odd dimensions this fundamental consequence of Gauss-Bonnet's Theorem.

In this note, we will investigate the behavior of this classical invariant under natural operations such as products and fiber bundles. In particular, we will give

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new lower bounds for the value of the simplicial volume of surface bundles, improving the known bounds of Hoster and Kotschick [**4**]. The present bounds are new also in the product case.

1.1. Products. Let M, N be closed oriented manifolds of respective dimensions m, n. The simplicial volume of the product $M \times N$ has lower and upper bounds as multiples of the product of the simplicial volumes:

$$\|M\| \, \|N\| \leq \|M \times N\| \leq \begin{pmatrix} m+n \\ m \end{pmatrix} \|M\| \, \|N\| \,. \tag{1.1}$$

This was first observed by Gromov [**3**]. We will recall below how to use bounded cohomology to prove the lower inequality. The upper inequality relies on the fact that the product $\Delta^m \times \Delta^n$ of the standard m-simplex Δ^m and the standard n-simplex Δ^n can be canonically triangulated in $(m+n)!/(m!n!)$ simplices.

The inequalities in (1.1) are interesting since they show that the simplicial volume of a product is zero if and only if the simplicial volume of one of the factor is zero, and it is often important to understand if the simplicial volume vanishes or not. However, the given bounds are most probably never sharp when they are not zero. For example, it is shown in [**2**] that if M and N are surfaces, then

$$\|M \times N\| = \frac{3}{2} \|M\| \, \|N\| \,. \tag{1.2}$$

When one of the factor is a surface, we improve the lower inequality in (1.1) by an asymptotic factor of 2:

Theorem 1.1. *Let F be an oriented closed surface, and B an oriented closed manifold of dimension $p-2$. Then*

$$\|F \times N\| \geq \begin{cases} 2\frac{(p-1)}{p} \|F\| \, \|N\| & \text{if } p \text{ is even,} \\ \\ 2\frac{p}{p+1} \|F\| \, \|N\| & \text{if } p \text{ is odd.} \end{cases}$$

Those lower bounds are sharp for $p = 4$, but again, it is very likely that they are not sharp in all other nontrivial cases. For example, we could show that the simplicial volume of the product of three surfaces is greater or equal to $45/11$ times the product of the simplicial volume of the factors. For the product of four and five surfaces, Laurent Bartholdi found the amusing factors $105/4$ and $14175/227$ respectively for the lower bounds (computed by computer).

Theorem 1.1 will follow from Theorem 1.2 below.

1.2. Fiber bundles. For fiber bundles, one cannot expect upper bounds as in (1.1) in general, since there exists 3-dimensional manifolds M admitting a hyperbolic structure (hence with $\|M\| \neq 0$) which fiber over the circle (and $\left\|S^1\right\| = 0$). For the lower bound, Hoster and Kotschick showed in [**4**] that if E is an oriented surface bundle with fiber an oriented surface F and base space B, then

$$\|E\| \geq \|F\| \, \|B\| \,.$$

For other fiber spaces, no lower bound seems to be known. We improve here the lower bounds of Hoster and Kotschick by an asymptotic factor of 2:

THEOREM 1.2. *Let E be an oriented surface bundle with fiber an oriented surface F over an oriented closed manifold B of dimension $p-2$. Then*

$$\|E\| \geq \begin{cases} 2\frac{(p-1)}{p}\,\|F\|\,\|N\| & \textit{if } p \textit{ is even,} \\ \\ 2\frac{p}{p+1}\,\|F\|\,\|N\| & \textit{if } p \textit{ is odd.} \end{cases}$$

COROLLARY 1.3. Let E be a fiber bundle with fiber F over a closed oriented manifold B. If $\dim(E) \leq 4$, then

$$\|E\| \geq \|F \times B\|\,.$$

The only nontrivial case of the corollary is when $\dim(F) = \dim(B) = 2$, which immediately follows from Theorem 1.2 and Formula (1.2).

Since the simplicial volume tends to be bigger for more complicated manifolds, it seems reasonable to expect that Corollary 1.3 further holds without any restriction on the dimension of E.

We already saw an example of a fiber bundle for which $\|B\| = 0$ but $\|E\| \neq 0$, namely a 3-dimensional manifold E fibering over the circle. To the question whether the same can happen for the fiber, that is, if there can exist bundles E with $\|E\| \neq 0$, for which the fiber satisfies $\|F\| = 0$, we have only the partial negative answer:

LEMMA 1.4. *Suppose that E is a fiber bundle for which the fundamental group of the fiber is amenable. Then $\|E\| = 0$.*

This is a straightforward consequence of a corollary of Gromov [**3**, Section 3.1] of the difficult Vanishing Theorem. Indeed, it immediately follows from the fact that if a closed manifold X can be mapped into a manifold Y with $\dim(Y) < \dim(X)$ such that the preimage of every point of Y has an "amenable" (see [**3**] for a definition) neighborhood in X, then $\|X\| = 0$.

The proof of Theorem 1.2 is given in Section 3.

2. Bounded cohomology and a cocycle norm inequality

Simplicial volumes are in practice mostly computed through the dual L^∞-seminorm on real valued singular (or group) cohomology. The dual L^∞-norm of a singular cochain c is defined as

$$\|c\|_\infty = \sup\left\{|c(\sigma)| : \ \sigma : \Delta^q \to M \text{ is a singular simplex}\right\}.$$

The cohomology of the subcocomplex of bounded singular cochains is by definition the bounded cohomology $H_b^*(M)$ of M. The inclusion of cocomplexes induces a comparison map $c : H_b^*(M) \to H^*(M)$ on the cohomology groups. The L^∞-norm on the space of cochains induces seminorms on $H_b^*(M)$ and on $H^*(M)$ (where we allow the value $+\infty$ on the latter cohomology group): The seminorm of a cohomology class β is defined as the infimum of the L^∞-norm of the singular cocycles representing β. It is a straightforward consequence of Hahn-Banach Theorem (see [**3**] or [**1**]) that if $\beta_M \in H^m(M)$ is dual to the (real valued) fundamental cycle $[M] \in H_m(M)$, then

$$\|M\| = \frac{1}{\|\beta_M\|_\infty}. \tag{2.1}$$

Similarly, the sup norm can be considered on the space of $\pi_1(M)$-invariant cochains $c : \pi_1(M)^{q+1} \to \mathbb{R}$, and the cohomology of the subcocomplex of bounded cochains gives the bounded group cohomology $H_b^*(\pi_1(M))$. Again, the inclusion of cocomplexes induces a comparison map $c : H_b^*(\pi_1(M)) \to H^*(\pi_1(M))$, and seminorms are defined as above on the group cohomology.

There are natural maps $H^*(\pi_1(M)) \to H^*(M)$ and $H_b^*(\pi_1(M)) \to H_b^*(M)$. While the former map is surely not an isomorphism in general, it is a remarkable theorem of Gromov [**3**], that the latter map is in fact an isometric isomorphism. Thus, if $\|M\| > 0$, then $\beta_M \in H^m(M)$ is in the image of $H_b^m(\pi_1(M)) \cong H_b^m(M) \to H^m(M)$. In particular the seminorm $\|\beta_M\|_\infty$ is equal to the infimum of the sup norm of all bounded $\pi_1(M)$-invariant group cocycles $b : \pi_1(M)^{m+1} \to \mathbb{R}$ representing a cohomology class being sent to β_M. We refer the reader to [**3**] for more details.

More generally, the L^∞-norm is also defined for arbitrary q-cochains $c : Y^{q+1} \to \mathbb{R}$ as the supremum of the absolute value of the evaluation of c on $(q+1)$-tuples of points in Y^{q+1}. A q-cochain is said to be a q-cocycle if the $(q+2)$-cochain δc vanishes, where δ denotes the homogeneous coboundary operator defined as

$$\delta c(y_0, ..., y_{q+1}) = \sum_{i=0}^{q+1} (-1)^i c(y_0, ..., \widehat{y_i}, ..., y_{q+1}),$$

for $(q+2)$-tuples $(y_0, ..., y_{q+1}) \in Y^{q+2}$. The cup product of a p-cochain $b : Y^{p+1} \to \mathbb{R}$ and a q-cochain $c : Y^{q+1} \to \mathbb{R}$ is defined to be the $(p+q)$-cochian given as

$$b \cup c(y_0, ..., y_{p+q}) = b(y_0, ..., y_p) c(y_p, ..., y_{p+q}),$$

for $(q+p+1)$-tuples $(y_0, ..., y_{p+q}) \in Y^{p+q+1}$. Note the obvious upper bound

$$\|b \cup c\|_\infty \leq \|b\|_\infty \|c\|_\infty$$

inducing the lower bound in (1.1) for the simplicial volume of products. Stronger bounds can be obtained by alternating the cup product. If $c : Y^{q+1} \to \mathbb{R}$ is a q-cochain, define a cochain $\mathrm{Alt}(c) : Y^{q+1} \to \mathbb{R}$ by alternating c, that is,

$$\mathrm{Alt}(c)(y_0, ..., y_q) = \frac{1}{(q+1)!} \sum_{\sigma \in S_{q+1}} \mathrm{sign}(\sigma) c(y_{\sigma(0)}, ..., y_{\sigma(q)}).$$

The orientation cocycle on the circle S^1 is defined as follows. Choose an orientation on S^1 and define $\mathrm{Or} : (S^1)^3 \to \mathbb{R}$ as

$$\mathrm{Or}(x_0, x_1, x_2) = \begin{cases} +1 & \text{if } x_0, x_1, x_2 \text{ are cyclically positively oriented,} \\ -1 & \text{if } x_0, x_1, x_2 \text{ are cyclically negatively oriented,} \\ 0 & \text{if the points } x_0, x_1, x_2 \text{ are not all distinct.} \end{cases}$$

It is straightforward to check that Or is an alternating cocycle.

Proposition 2.1. *Let $c : Y^{p-1} \to \mathbb{R}$ be a $(p-2)$-cocycle on Y. Then*

$$\|\mathrm{Alt}(\mathrm{Or} \cup c)\|_\infty \leq \begin{cases} \frac{k}{2(k-1)} \|c\|_\infty & \text{if } k \text{ is even,} \\ \\ \frac{k+1}{2k} \|c\|_\infty & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Since $\mathrm{Alt}(\mathrm{Or} \cup c) = \mathrm{Alt}(\mathrm{Or} \cup \mathrm{Alt}(c))$ and $\|\mathrm{Alt}(c)\|_\infty \leq \|c\|_\infty$, we can without loss of generality assume that c is alternating. Let $(z_0, ..., z_p)$ be a $(k+1)$-tuple of points $z_i = (x_i, y_i) \in S^1 \times Y$. We show that the evaluation of $\mathrm{Alt}(\mathrm{Or} \cup c)$ on $(z_0, ..., z_p)$ is bounded as in the statement of the proposition.

If there exists $x_i = x_j$ with $i \neq j$, define x_i^-, respectively x_i^+, to be points on S^1, obtained from x_i by moving x_i in the negative, respectively positive direction, and close enough to x_i so that no other point x_k, for $0 \leq k \leq p$, $k \neq i$, lies between x_i and x_i^-, or x_i and x_i^+. (Although it could be that $x_k = x_i$ for some $k \neq i$.) Note that for all $0 \leq k, \ell \leq p$, we have

$$\mathrm{Or}(x_i, x_k, x_\ell) = \frac{1}{2}\left[\mathrm{Or}(x_i^-, x_k, x_\ell) + \mathrm{Or}(x_i^+, x_k, x_\ell)\right].$$

Thus, setting $z_i^- = (x_i^-, y_i)$ and $z_i^+ = (x_i^+, y_i)$, it follows that the evaluation of $\mathrm{Alt}(\mathrm{Or} \cup c)$ on $(z_0, ..., z_i, ..., z_p)$ is equal to

$$\frac{1}{2}\left[\mathrm{Alt}(\mathrm{Or} \cup c)(z_0, ..., z_i^-, ..., z_p) + \mathrm{Alt}(\mathrm{Or} \cup c)(z_0, ..., , z_i^+, ..., z_p)\right].$$

In particular, the evaluation of $\mathrm{Alt}(\mathrm{Or}\cup c)$ on $(z_0, ..., z_p)$ is bounded by the maximum between $\mathrm{Alt}(\mathrm{Or}\cup c)(z_0, ..., z_i^-, ..., z_p)$ and $\mathrm{Alt}(\mathrm{Or}\cup c)(z_0, ..., z_i^+, ..., z_p)$. By induction, we can thus without loss of generality assume that the x_i's are all distincts.

Since $\mathrm{Alt}(\mathrm{Or} \cup c)$ is alternating, up to permuting the points z_i, we can suppose that the x_i are positively cyclically ordered on S^1. In other words, $\mathrm{Or}(x_i, x_j, x_k) = +1$ whenever $0 \leq i < j < k \leq p$. By definition, we have that the evaluation of $\mathrm{Alt}(\mathrm{Or} \cup c)$ on $(z_0, ..., z_p)$ is equal to

$$\frac{1}{(p+1)!}\sum_{\sigma \in S_{p+1}} \mathrm{sign}(\sigma)\mathrm{Or}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})c(y_{\sigma(2)}, ..., y_{\sigma(p)}).$$

Translating the summation by the permutation $(0\ 2)$, which is of odd order but will, after permuting $x_{\sigma(0)}$ and $x_{\sigma(2)}$, change the sign in $\mathrm{Or}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})$, we see that we can rewrite the sum as

$$\frac{1}{(p+1)!}\sum_{\sigma \in S_{p+1}} \mathrm{sign}(\sigma)\mathrm{Or}(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})c(y_{\sigma(0)}, y_{\sigma(3),} ..., y_{\sigma(p)}).$$

Since every permutation in $\mathrm{Sym}(p+1)$ can be written as the composition of a permutation of $\{1, ..., p\}$ and a power of the cycle $\tau = (0\ 1\ ...\ p)$, we decompose the sum according to its value on $\sigma(0)$ as

$$\frac{1}{p+1}\sum_{k=0}^{p}\frac{(-1)^k}{p!}\sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\mathrm{Or}(x_{\tau^k(0)}, x_{\tau^k\sigma(1)}, x_{\tau^k\sigma(2)})c(y_{\tau^k(0)}, y_{\tau^k\sigma(3)} ..., y_{\tau^k\sigma(p)}).$$

We will now show that the summand corresponding to $\tau = \mathrm{id}$ satisfies the inequality

$$\frac{1}{p!}\sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\mathrm{Or}(x_0, x_{\sigma(1)}, x_{\sigma(2)})c(y_0, y_{\sigma(3)} ..., y_{\sigma(p)}) \leq \begin{cases} \frac{p}{2(p-1)}\left\|c\right\|_\infty & \text{if } p \text{ is even,} \\ \\ \frac{p+1}{2p}\left\|c\right\|_\infty & \text{if } p \text{ is odd.}\end{cases}$$

and by symmetry, the proposition will follow. As both Or and c are alternating, we can average only over those permutations for which $\sigma(1) < \sigma(2)$ and $\sigma(3) < ... < \sigma(p)$ (the so called $(2, p-2)$-shuffles). Writing $\sigma(1) = i$ and $\sigma(2) = j$, we see that since $\mathrm{Or}(x_0, x_i, x_j) = +1$, the left hand side of the previous inequality becomes

$$\frac{2}{p(p-1)}\sum_{i=1}^{p-1}(-1)^{i+1}\sum_{j=i+1}^{p}(-1)^j c(y_0, y_1, ..., \widehat{y_i}, ..., \widehat{y_j}, ..., y_p).$$

In view of the cocycle relation for c, we can rewrite the sum over j as

$$c(y_3, ..., \widehat{y_i}, ..., , y_p) + \sum_{j=1}^{i-1} (-1)^j c(y_0, y_1, ..., \widehat{y_j}, ..., \widehat{y_i}, ..., , y_p).$$

The original sum contains $p-i$ summands, and the latter sum contains i summands, we thus get less summands if we replace the original sum by the latter one whenever $2i \leq p$. When $p = 2q$ is even, the whole expression will, up to the factor $2/(p(p-1)$ hence be a sum of

$$1 + 2 + ... + (q-1) + q + (q-1) + ... + 2 + 1 = 2\frac{q(q-1)}{2} + q = q^2 = \frac{p^2}{4}$$

evaluations of c, giving the claimed upper bound

$$2\frac{p^2/4}{p(p-1)} \|c\|_\infty = \frac{p}{2(p-1)} \|c\|_\infty .$$

When $p = 2q+1$ is odd, the whole expression will, up to the factor $2/(p(p-1)$ be a sum of

$$1 + 2 + ... + (q-1) + q + q + (q-1) + ... + 2 + 1 = 2\frac{(q+1)q}{2} = \frac{(p-1)(p+1)}{4}$$

evaluations of c, giving the claimed upper bound

$$2\frac{(p-1)(p+1)/4}{p(p-1)} \|c\|_\infty = \frac{p+1}{2p} \|c\|_\infty .$$

□

3. Surface bundles: proof of Theorem 1.2

The inequality being trivial if the fiber F is the sphere or the torus, as in that case $\|F\| = 0$ (and also $\|E\| = 0$ by Lemma 1.4), we can assume that the genus of F is greater or equal to 2. Also, we suppose that $\|B\| > 0$.

Let $e(T\pi) \in H^2(E)$ be the Euler class of the vertical bundle

$$T\pi = \{ X \in TE | \, \pi_*(X) = 0 \}$$

of the bundle $\pi : E \to B$. Let $\beta_B \in H^{p-2}(B)$ and $\beta_E \in H^p(E)$ denote the duals of the respective fundamental classes $[B] \in H_{p-2}(B)$ and $[E] \in H_p(E)$. as observed in [**4**], we have

$$\beta_E = \frac{1}{\chi(F)} e(T\pi) \cup \pi^*(\beta_B).$$

It is shown in [**6**, Proposition 4.1], that $e(T\pi)$ is the image via $H^{p-2}(\pi_1 E) \to H^{p-2}(E)$ of the pullback by a homomorphism $\rho : \pi_1(E) \to \mathrm{Homeo}_+(S^1)$ of the Euler class in $H^2(\mathrm{Homeo}_+(S^1))$ which can be represented by $\frac{1}{2}\mathrm{Or}$. The homomorphism ρ is obtained by composing the lift of the holonomy $\pi_1 B \to \mathcal{M}_g$ to $\pi_1 E \to \mathcal{M}_{g,*}$, where $\mathcal{M}_g$ and $\mathcal{M}_{g,*}$ denote the mapping class groups $\pi_0(\mathrm{Diff}_+(\Sigma_g))$ and $\pi_0(\mathrm{Diff}_+(\Sigma_g, b_0))$ respectively, with the natural homomorphism

$$\mathcal{M}_{g,*} \longrightarrow \mathrm{Homeo}_+(S^1).$$

See [**6**] for more details.

Since $\|B\| > 0$ it follows that β_B is in the image of $H_b^{p-2}(\pi_1(B)) \cong H_b^{p-2}(B) \to H^{p-2}(B)$. Let $b : (\pi_1(B))^{p-1} \to \mathbb{R}$ be an arbitrary bounded cocycle representing a cohomology class being mapped to β_B. Then the cup product $1/(2\chi(F))\rho^*(\mathrm{Or}) \cup \pi^*(b)$ and hence also its alternation

$$\mathrm{Alt}\left(\frac{1}{2\chi(F)}\rho^*(\mathrm{Or}) \cup \pi^*(b)\right) : \pi_1(E)^{p+1} \longrightarrow \mathbb{R}$$

represent a cohomology class in $H^p(\pi_1 E)$ which is mapped to β_E. We thus get from Proposition 2.1 that

$$\|\beta_E\|_\infty \leq \frac{1}{2\,|\chi(F)|}\,\|\rho^*(\mathrm{Or}) \cup \pi^*(b)\|_\infty \leq \begin{cases} \frac{p}{2(p-1)}\frac{\|b\|_\infty}{\|F\|} & \text{if } p \text{ is even,} \\ \\ \frac{p+1}{2p}\frac{\|b\|_\infty}{\|F\|} & \text{if } p \text{ is odd.} \end{cases}$$

Taking the infimum over all such b's, we get the same inequality with $\|\beta_B\|_\infty$ instead of $\|b\|_\infty$, and the theorem now follows from that $\|E\| = 1/\|\beta_E\|_\infty$ and $\|B\| = 1/\|\beta_B\|_\infty$.

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