An introduction to differential topology and surgery theory

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Introduction

These notes are based on a course that was taught at Durham University during the fall of 2018. The initial goal was to provide an introduction to differential topology and, depending on the audience, to learn some surgery theory. As it turned out, the audience was already familiar with basic manifold theory and so surgery became the main focus of the course. Nevertheless, as a remnant of the initial objective, the use of homology was avoided for as long as possible.

The course decomposed into two parts. The first part assumed little background and introduced some basic differential topology (manifolds, tangent spaces, immersions), algebraic topology (higher homotopy groups and vector bundles) and briefly discusses the Smale's sphere eversion. The second part consisted of an introduction to surgery theory and described surgery below the middle dimension, the surgery obstruction in even dimensions and an application to knot theory (Alexander polynomial one knots are topologically slice).

These notes no doubt still contain some inaccuracies. Hopefully, they will be polished in 2019.

Chapter 1

Differential topology

The goal of this chapter is to discuss some classical topics and results in differential topology. As a motivating result, we work towards the proof of Smale's sphere eversion theorem. The chapter is organized as follows. In Section 1.1, we introduce smooth manifolds and their tangent spaces as well as immersions and embeddings. In Section 1.2, we define vector bundles and discuss some classification results. Finally, in Section 1.3, we outline the proof of the sphere eversion theorem.

1.1 Smooth manifolds and their tangent spaces

This first section is introductory: it defines smooth manifolds, tangent spaces and immersions. We will mostly follow Tu's textbook "An introduction to manifolds" [Tu11]; other classical references include [GP74, Hir76, Wal16].

1.1.1 Smooth manifolds

In this subsection, we define smooth manifolds and smooth maps between them. Our main reference is [Tu11, Sections 5 and 6].

A *topological manifold* is a second countable Hausdorff topological space M such that for all $p \in M$, there exists an open set $p \in U$ and a map $\varphi\colon U \to \mathbb{R}^n$ which is a homeomorphism onto a open subset of $\mathbb{R}^n$ (i.e. M is *locally Euclidean)*. The pair (U, φ) is called a *chart*. A topological manifold M is n*-dimensional if it is locally homeomorphic to* $\mathbb{R}^n$*.*

Remark 1.1.1. For the dimension of a manifold to be well defined, we need to know that for $n \neq m$, an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$. This is a non-trivial fact (known as *invariance of domain)*, a proof can be found in Hatcher [Hat02, Theorem 2B.3 and Corollary 2B.4].

Example 1.1.2. We now give some examples of topological manifolds.

1. The euclidean space $\mathbb{R}^n$ is a topological n-manifold: it is covered by a single chart $(\mathbb{R}^n, \mathrm{id})$.
2. The unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is a topological 2-manifold (the same is true for S^n). Endow S^2 with the induced topology from $\mathbb{R}^3$. Since $\mathbb{R}^3$ is Hausdorff

and second countable, so is S^2. The 2-sphere can be covered by the following six charts

$$\begin{aligned}
U_1 &= \{(x,y,z) \in S^2 \mid x > 0\}, \quad \varphi_1(x,y,z) = (y,z).\\
U_2 &= \{(x,y,z) \in S^2 \mid x < 0\}, \quad \varphi_2(x,y,z) = (y,z),\\
U_3 &= \{(x,y,z) \in S^2 \mid y > 0\}, \quad \varphi_3(x,y,z) = (x,z),\\
U_4 &= \{(x,y,z) \in S^2 \mid y < 0\}, \quad \varphi_4(x,y,z) = (x,z),\\
U_5 &= \{(x,y,z) \in S^2 \mid z > 0\}, \quad \varphi_5(x,y,z) = (x,y),\\
U_6 &= \{(x,y,z) \in S^2 \mid z < 0\}, \quad \varphi_6(x,y,z) = (x,y).
\end{aligned}$$

This is clearly an open cover. We check that the φ_i are homeomorphisms onto their image. We do this for $i = 4$ but the other cases are analogous. The image of the continuous map φ_4 consists of pairs $(x,z) \in \mathbb{R}^2$ such that $x^2 + y^2 + z^2 = 1$ for some $y < 0$. Since $x < 0$, a continuous inverse is given by mapping (x,z) to $(x, -\sqrt{1-x^2-z^2}, z)$.

Exercise 1.1.3. Show that the plus sign "+", viewed as a subspace of $\mathbb{R}^2$ with the subspace topology, is not a topological manifold. The solution can be found in [Tu11, Example 5.4].

From now on, we will be interested in topological manifolds with an additional "differentiable structure". We will always use the words "smooth" and "C^∞" interchangeably. Furthermore, diffeomorphisms are assumed to be smooth.

Definition 1. A *smooth n-dimensional manifold* is a n-dimensional topological manifold whose family $\mathscr{A} = (U_i, \varphi_i)_{i\in I}$ of charts satisfies:

1. the family $(U_i)_{i\in I}$ is an open cover of M, meaning that $M = \bigcup_{i\in I} U_i$;
2. the charts are *smoothly compatible*: for every $i, j \in I$, the following map is a diffeomorphism:
$$\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j);$$
3. if a chart (U, φ) is smoothly compatible with all charts of $\mathscr{A}$, then (U, φ) belongs to $\mathscr{A}$.

A familly $\mathscr{A}$ of charts satisyfing the first two conditions of Definition 1 is called an *atlas*. The third condition ensures that the atlas is *maximal*.

Proposition 1.1.4. *Any atlas on a locally Euclidean space is contained in a unique maximal atlas.*

A proof of Proposition 1.1.4 can be found in [Tu11, Proposition 5.19]. In practice, Proposition 1.1.4 means implies that we need not check the maximality assumption.

Example 1.1.5. We argue that the examples of Example 1.1.2 are in fact smooth manifolds.

1. The euclidean space $\mathbb{R}^n$ is a smooth n-manifold. An atlas is given by $(\mathbb{R}^n, \mathrm{id})$.
2. The sphere S^n is a smooth n-manifold. We prove this for $n = 2$. The charts described in Example 1.1.2 clearly cover S^2 and so we need only check that they are smoothly compatible. We only show that the following map is smooth
$$\varphi_1 \circ \varphi_4^{-1} \colon \varphi_4(U_1 \cap U_4) \to \varphi_1(U_1 \cap U_4).$$
Recall that $\varphi_1(x,y,z) = (y,z)$ and $\varphi_4(x,y,z) = (x,z)$. Looking at the definitions of U_1 and U_4, we see that
$$\begin{aligned}
\varphi_4(U_1 \cap U_4) &= \{(x,z) \in \mathbb{R}_+ \times \mathbb{R} \mid x^2 + z^2 = 1 - y^2 \text{ for some } y < 0\},\\
\varphi_1(U_1 \cap U_4) &= \{(y,z) \in \mathbb{R}_- \times \mathbb{R} \mid y^2 + z^2 = 1 - x^2 \text{ for some } x > 0\}.
\end{aligned}$$
Recall that the inverse of $\varphi_4 \colon U_4 \to \varphi(U_4)$ is given by $\varphi_4^{-1}(x,z) = (x, \sqrt{1-x^2-z^2}, z)$. We deduce that for $(x,z) \in \varphi_4(U_1 \cap U_4)$, there is a $y < 0$ such that $y^2 = 1 - x^2 - z^2$. Thus we have $\varphi_1 \circ \varphi_4^{-1}(x,z) = (-\sqrt{1-x^2-z^2}, z)$. This map is smooth since $-\sqrt{1-x^2-z^2} = y < 0$.

Exercise 1.1.6. Prove the following facts about smooth manifolds.

1. Any open subset of a smooth manifold is a smooth manifold. Deduce that $\mathrm{GL}(n,\mathbb{R})$ is a smooth manifold. The solution can be found in [Tu11, Examples 5.12 and 5.15].
2. The product of two smooth manifolds is a smooth manifold. The solution can be found in [Tu11, Example 5.17]. Deduce that the n-torus $\mathbb{T}^n := S^1 \times \ldots \times S^1$ is a smooth manifold.

Remark 1.1.7. There are topological manifolds that do not admit any smooth structure: the first example was produced by Kervaire [Ker60]. A nice introductory account of smoothing theory can be found in [Sco05, pages 207-224].

Next, we define the notion of a smooth map between manifolds.

Definition 2. Let M and N be smooth manifolds. A continuous map $F\colon M \to N$ is *smooth at* $p \in M$ if there are charts (U,φ) and (V,ψ) around p and $F(p)$ such that $\psi \circ F \circ \varphi^{-1}$ is smooth. F is *smooth* if it smooth at every $p \in M$. A *diffeomorphism* is a smooth bijective map whose inverse is also smooth.

A concrete exercise (together with a solution) involving the verification that a given map is smooth can be found in [Tu11, Example 6.19]

1.1.2 Tangent spaces

In the Euclidean space $\mathbb{R}^n$, we think of the "tangent space at p" as the set of all vectors emanating from p. Such a "tangent vector" can be described by a vector in $\mathbb{R}^n$, and so the tangent space of $\mathbb{R}^n$ at p is just a copy of $\mathbb{R}^n$ which we attach to p. In this subsection, we define tangent space to an arbitrary manifold at a point. We give a definition using derivations, but several equivalent (more geometric) formulations also exist. We closely follow [Tu11, Section 8].

Let M be an n-manifold and let U,V be two open neighborhoods of a point $p \in M$. Two functions $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ are *equivalent* if they agree on some subset of U and V that contains p. A *germ at* $p \in M$ is an equivalence class of functions $f\colon U \to \mathbb{R}$, where U is an open set of M. The set of all germs at p is denoted by $C_p^\infty(M)$. Observe that, if U is an open set containing p, then $C_p^\infty(U) = C_p^\infty(M)$: we are dealing with a *local* notion.

Definition 3. Let M be a manifold and let $p \in M$. A *tangent vector at* p (or *derivation*) is a $\mathbb{R}$-linear map $D\colon C_p^\infty(M) \to \mathbb{R}$ that satisfies $D(fg) = (Df)g(p) + f(p)Dg$. The *tangent space of* M *at* p is defined as the set of all tangent vector at p.

The upshot is that Definition 3 associates to each point $p \in M$ a vector space T_pM. As we shall see below, this vector space is in fact n-dimensional. As a consequence, Definition 3 generalizes the concept of the "tangent space to $\mathbb{R}^n$ at p" that we described above. Note also that if U is an open set containing p, then $T_pU = T_pM$.

Notation 1.1.8. We use $r^1,\ldots,r^n$ to denote the *coordinates* of $\mathbb{R}^n$, i.e. each $r^i\colon \mathbb{R}^n \to \mathbb{R}$ is the standard projection. Writing a chart (U,φ) in *local coordinates* consists of writing (U,φ) as $(U,x^1,\ldots,x^n)$, where $x^i := r^i \circ \varphi\colon U \to \mathbb{R}$. For such a local chart, the tangent vector $\frac{\partial}{\partial x^i}|_p\colon C_p^\infty(M) \to \mathbb{R}$ is defined as follows:

$$\begin{aligned}\frac{\partial}{\partial x^i}\Big|_p\colon C_p^\infty(M) &\to \mathbb{R}\\ f &\mapsto \frac{\partial}{\partial r^i}\Big|_{\varphi(p)}(f\varphi^{-1}) =: \frac{\partial f}{\partial x^i}(p).\end{aligned}$$

To check that $\frac{\partial}{\partial x^i}\big|_p$ is indeed a derivation, use the corresponding property for the partial derivatives in $\mathbb{R}^n$. Observe that in Euclidean space (with the chart $(\mathbb{R}^n,\mathrm{id})$), the coordinates $x^i = r^i$ are the "usual" coordinates in $\mathbb{R}^n$. In particular, $\frac{\partial}{\partial x^i}\big|_p$ is the usual partial derivative.

The next proposition describes the tangent space at a point of $\mathbb{R}^n$.

Proposition 1.1.9. *Given $p \in \mathbb{R}^n$, the tangent vectors $\frac{\partial}{\partial r^1}\big|_p, \ldots, \frac{\partial}{\partial r^n}\big|_p$ form a basis of $T_p\mathbb{R}^n$ and, in fact, the following assignement is a linear isomorphism:*

$$\begin{aligned}\Phi \colon \mathbb{R}^n &\to T_p\mathbb{R}^n \\ v &\mapsto \sum_i v^i \frac{\partial}{\partial r^i}\Big|_p\end{aligned}$$

Proof. We saw in Example 1.1.8 that in $(\mathbb{R}^n, \mathrm{id})$, we have $x^i = r^i$, but we keep the former notation. The map Φ is clearly linear and so we first prove injectivity. Suppose that $\Phi_v \equiv 0$. To show that $v = 0$, we show that $v^j = 0$ for each j. Applying Φ_v to the coordinate function x^j, using $\Phi_v \equiv 0$, the definition of Φ_v and basic calculus in $\mathbb{R}^n$, we obtain the desired result:

$$0 = \Phi_v(x^j) = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p(x^j) = \sum_i v^i \delta_i^j = v^j.$$

Next, we prove surjectivity. Given a tangent vector $D \in T_pM$, our goal is to find $v \in \mathbb{R}^n$ such that $D(f) = \Phi_v(f)$ for every germ $f \in C_p^\infty(M)$. We claim that setting $v^i := D(x^i)$ for $i = 1, \ldots, n$ produces a v that satisfies this property. Given a germ $f \in C_p^\infty(M)$, Taylor's theorem with remainder (see e.g. [Tu11, Lemma 1.4]) implies that there are smooth functions $g_i \colon \mathbb{R}^n \to \mathbb{R}$ such that $g_i(p) = \frac{\partial f}{\partial x^i}(p)$ and

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x). \tag{1.1}$$

Before applying D to this equality, we note that derivations vanish on constant functions: if c is such a function, then the $\mathbb{R}$-linearity of D implies that $D(c) = cD(1)$ and the Leibniz rule gives $D(1) = D(1 \cdot 1) = 2D(1)$. Applying D to the Taylor expansion displayed in (1.1), using that derivations are $\mathbb{R}$-linear, vanish on constants (so that $D(f(p)) = 0$ and $D(p^i) = 0$ for $i = 1, \ldots, n$) and satisfy the Leibniz rule, we obtain

$$D(f(x)) = \sum_i (Dx^i) g_i(p) + \sum_i (p^i - p^i) D(g_i(x)) = \sum_i (Dx^i) \frac{\partial f}{\partial x^i}(p) = D_v(f(x)).$$

It follows that $Df = \Phi_v(f)$ for arbitrary $f \in C_p^\infty(M)$ and consequently $D \equiv \Phi_v$, as claimed. This concludes the proof of the proposition. □

We now define smooth maps between manifolds.

Definition 4. The *differential* of a smooth map $F \colon M \to N$ at $p \in M$ is defined as

$$\begin{aligned}T_pF \colon T_pM &\to T_{F(p)}N \\ (X_p) &\mapsto (f \mapsto X_p(f \circ F)).\end{aligned}$$

The following properties of the differential are left as exercises. The solutions are very short and can be found in [Tu11, pages 88-89].

Exercise 1.1.10. Given smooth maps $F \colon M \to N, G \colon N \to X$ and $p \in M$, show that

1. T_pF is linear;
2. $T_p(G \circ F) = T_{F(p)}(G) \circ T_p(F)$ (this is the *chain rule*) and $T_p(\mathrm{id}_M) = \mathrm{id}_{T_pM}$.

Deduce that if F is a diffeomorphism, then T_pF is a linear isomorphism.

Before making this definition more concrete, we show that the dimension of the tangent spaces to an n-manifold are n-dimensional.

Proposition 1.1.11. *Given a chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ containing p, the tangent vectors $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p$ form a basis of T_pM. In particular, the vector space T_pM is n-dimensional.*

Proof. Using Exercise 1.1.10, we know that since φ is diffeomorphism onto its image, its differential induces a linear isomorphism $T_pM \to T_{\varphi(p)}\mathbb{R}^n$. We claim that $T_p\varphi$ maps $\frac{\partial}{\partial x^i}|_p$ to $\frac{\partial}{\partial r^i}|_{\varphi(p)}$; indeed for any germ $f \in C^\infty_{\varphi(p)}(\mathbb{R}^n)$, we use the definition of the differential and the definition of the partial derivative to obtain

$$T_p\varphi(\frac{\partial}{\partial x^i}\big|_p)(f) = \frac{\partial}{\partial x^i}\big|_p(f \circ \varphi) = \frac{\partial}{\partial r^i}\big|_{\varphi(p)}(f \circ \varphi \circ \varphi^{-1}) = \frac{\partial}{\partial r^i}\big|_{\varphi(p)}(f).$$

As $T_p\varphi$ is a linear isomorphism and as Proposition 1.1.9 implies that the $\frac{\partial}{\partial r^1}\big|_{\varphi(p)}, \ldots, \frac{\partial}{\partial r^n}\big|_{\varphi(p)}$ form a basis of $T_{\varphi(p)}\mathbb{R}^n$, we deduce that $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p$ form a basis of T_pM. This concludes the proof of the proposition. □

We conclude by deriving the expression of the differential in local coordinates.

Remark 1.1.12. Let $F\colon M^m \to N^n$ be a smooth map. Pick charts $(U, x^1, \ldots, x^m)$ and $(V, y^1, \ldots, y^n)$ around p and $F(p)$. To understand T_pF in local coordinates, we study its image on the basis vectors $\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p$ of T_pM. Since $\frac{\partial}{\partial y^1}|_{F(p)}, \ldots, \frac{\partial}{\partial y^n}|_{F(p)}$ is a basis for $T_{F(p)}N$, for each j, we can write $T_pF(\frac{\partial}{\partial x^j}|_p) = \sum_k a^k_j \frac{\partial}{\partial y^k}|_{F(p)}$ for some $a^k_j \in \mathbb{R}$. Evaluating both sides on y^i, we get $\frac{\partial(y^i \circ F)}{\partial x^j}|_p = a^i_j$, from which we deduce that

$$T_pF(\frac{\partial}{\partial x^j}\big|_p) = \sum_k \frac{\partial F^k}{\partial x^j}\frac{\partial}{\partial y^k}\big|_{F(p)}.$$

A more geometric approach to the tangent space and differential is presented in [Tu11, Subsection 8.7]. The advantage of derivations is the simplicity of the formulas and the elementary proofs of several properties.

1.1.3 Immersions and embeddings

In this short subsection, we describe immersions and embeddings. We also introduce regular homotopies and define eversions. References include [Tu11, Subsection 8.8] and [Ada93].

Definition 5. A smooth map $f\colon M \to N$ is an *immersion* if $T_xf\colon T_xM \to T_{f(x)}N$ is injective at every point $x \in M$. An immersion is an *embedding* if it is injective and is a homeomorphism onto its image. A *submanifold* is the image of an embedding.

Any map $S^1 \to \mathbb{R}^2$ with image the figure eight "8" is an immersion but not an embedding: the double point leads to a failure of the injectivity condition. The next example shows that an immersion need not be embedding, even if it is injective.

Example 1.1.13. Consider the map $\beta\colon (-\pi, \pi) \to \mathbb{R}^2, t \mapsto (\sin(2t), \sin(t))$ depicted in Figure 1.1 below. β is injective and, calculating derivatives, one sees that it is an immersion. Observe that $\beta((-\pi, \pi))$ is compact (e.g. because it is bounded and contains all its limit points). Since $(-\pi, \pi)$ is not compact, we deduce that β can not be an embedding.

Embeddings, by definition, are injective immersions, but thanks to Example 1.1.13, we know that the converse need not be true. The next proposition shows that the situation simplifies considerably if the domain is compact.

Proposition 1.1.14. *If M is compact, then any injective immersion $f\colon M \to N$ is an embedding.*

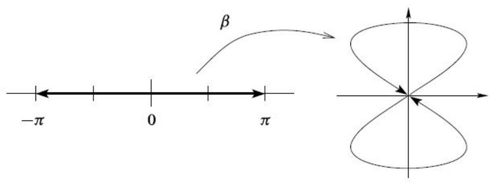


Figure 1.1: The image of the map β is an injective immersion but not an embedding.

Proof. Since f is an injective immersion, we need only show that f is a homeomorphism on its image. A standard result in general topology shows that an injective continuous map from a compact space to a Hausdorff space is a homeomorphism onto its image. Since M is compact (by assumption) and N is Hausdorff (it is a manifold), we can apply this result, concluding the proof of the proposition. $\square$

Very informally, Proposition 1.1.14 shows that embeddings defined on a compact manifold should be thought of as "immersions with no multiple points". Next, one might wonder whether an immersion can be deformed into an embedding. To make this question more precise, we introduce some additional terminology.

Definition 6. Two immersions $f, g\colon M \to N$ are *regular homotopic* if there is a homotopy $H\colon M \times [0,1] \to N$ such that each $h_t := H(-,t)\colon M \to N$ is an immersion.

Immersions will often be denoted as $M \looparrowright N$. The next example describes regular homotopy classes of immersions $S^1 \looparrowright \mathbb{R}^2$.

Example 1.1.15. Intuitively, any immersion $S^1 \looparrowright \mathbb{R}^2$ with image the figure eight "8" should not be regular homotopic to the the standardly embedded circle: if we try to remove the kink by making it smaller, then we run into a cusp. Therefore, while both immersions are homotopic they should not be regular homotopic.

To make this intuition more precise, recall that the *winding number* of an oriented closed immersed curve $f\colon S^1 \looparrowright \mathbb{R}^2$ is equal to the total number of counterclockwise turns that the unit normal vector to the curve makes as we travel along $f(S^1)$. The winding number has several more formal definitions: for instance, it is the equal to the degree of the Gauss map of $f(S^1)$. In particular, the winding number is a regular homotopy invariant. Since the immersed curve "8" has vanishing winding number (regardless of the orientation) but the standard embedding of S^1 has winding number ± 1, these immersions are not regular homotopic.

The *Whitney-Graustein* theorem states that two immersions $f, g\colon S^1 \looparrowright \mathbb{R}^2$ are regular homotopic if and only if they have the same winding number [Whi37]. While Whitney's original proof can also be found in [Ada93, Chapter 0], we refer to [Gei09] for a particular short (*and* enlightening) proof using contact geometry. Summarizing, the space of regular homotopy classes of immersions $S^1 \looparrowright \mathbb{R}^2$ is in bijective correspondence with $\mathbb{Z}$.

In Section 1.3, we shall be concerned with immersions of $S^{n-1} \looparrowright \mathbb{R}^n$. Namely, Smale's result is concerned with the space of regular homotopy classes of immersions $S^{n-1} \to \mathbb{R}^n$.

Definition 7. An *eversion* of the 2-sphere is a regular homotopy from the standard embedding $\iota\colon S^2 \to \mathbb{R}^3$ to the antipodal embedding $-\iota\colon S^2 \to \mathbb{R}^3$.

The antipodal map on S^n is a composition of $(-1)^{n+1}$ reflections, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$. In particular, the antipodal map in $S^2 \subset \mathbb{R}^3$ is an orientation *reversing* homeomorphism, while the antipodal map on $S^1 \subset \mathbb{R}^2$ is orientation *preserving.*

Remark 1.1.16. Suppose we imagine taking the standard embedding ι of the 2-sphere in $\mathbb{R}^3$ and coloring the outside surface red and the inside blue. Since the antipodal map is orientation reversing, $-\iota(S^2)$ represents the sphere with the outside blue and the inside red. Therefore Definition 7 truthfully corresponds to our intuitive notion of "turning the sphere inside out".

As we shall see in Section 1.3 below, Smale showed that eversions of the 2-sphere do exist.

Remark 1.1.17. Moving down one dimension, the Whitney-Graustein theorem ensures that there is no regular homotopy between the standard embedding $\iota\colon S^1 \to \mathbb{R}^2$ and $\iota(S^1)$ with the reverse orientation. This makes Smale's result all the more surprising.

To show that an eversion does exist, Smale proved that the space of regular homotopy classes of immersions $S^2 \looparrowright \mathbb{R}^3$ is trivial. In order to understand this result, we now discuss vector bundles.

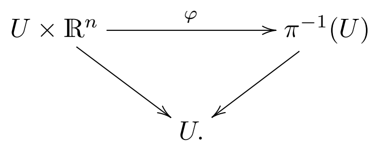
1.2 Vector bundles and homotopy groups

In Subsection 1.2.1, we introduce vector bundles and provide several examples. In Subsection 1.2.2, we define homotopy groups and prove several basic results on the subject. Finally in Subsection 1.2.3, we discuss the homotopy classification of vector bundles.

1.2.1 Vector bundles: definitions and examples

We introduce vector bundles. In our examples, we emphasize the tangent and normal bundles over a manifold. References include [Tu11, Section 12.3], as well as [MS74, Hus94].

Definition 8. A *rank n vector bundle* consists of spaces E (the *total space*) and B (the *base space*), a continuous surjection $\pi\colon E \to B$ (the *projection*) such that for every $x \in B$ each *fiber* $E_x := \pi^{-1}(x)$ is an n-dimensional real vector space and the following *local triviality condition* holds: for every $x \in B$, there is an open neighborhood $U \subset X$ containing x and homeomorphism $\varphi\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that for all $y \in U$, the map $\varphi(y, -)\colon \mathbb{R}^n \to E_y$ is a linear isomorphism that makes the following diagram commute:



Next, we provide some basic examples of vector bundles.

Example 1.2.1.
1. The *trivial rank n vector bundle* over a space B is the product $B \times \mathbb{R}^n$. When the base space is understand, we shall sometime denote this vector bundle by ϵ^n or $\underline{\mathbb{R}}^n$.
2. The *infinite Mobius band* is a rank 1 vector bundle over the circle $S^1 = [0,1]/0 \sim 1$: the total space is $\mathscr{M} := [0,1] \times \mathbb{R}/\sim$, with $(0,x) \sim (1,-x)$ for all $x \in \mathbb{R}$, and the projection map $\mathscr{M} \to S^1$ is defined as $\pi(x,v) = x$.

Note that vector bundles are a particular case of "fiber bundles".

Remark 1.2.2. Let F be a topological space. In Definition 8, if we replace all occurrences of $\mathbb{R}^n$ by F and remove all conditions involving linearity, then we obtain the notion of a "fiber bundle with fiber F". For instance, the (finite) Mobius band is an example of a $[0,1]$-bundle, and vector bundles are "linear" $\mathbb{R}^n$-bundles. In the following sections, we will frequently encounter disc bundles.

Next, we continue with further examples of vector bundles.

Example 1.2.3.
1. We describe the tangent bundle TM of a smooth manifold M. As a set, the total space TM is the disjoint union $TM = \bigsqcup_{p \in M} T_pM$ and the projection is the surjective map $\pi\colon TM \to M$ mapping $v \in T_xM$ to x. Next, we endow TM with the the structure of a $2n$-manifold and $\pi\colon TM \to M$ with the structure of a rank n vector bundle. Given an

open set $U \subset M$, note that $TU = \pi^{-1}(U) = \bigsqcup_{p \in U} T_pM$ and, for every chart (U, φ) of M, consider the map

$$\begin{aligned}\psi\colon \pi^{-1}(U) &\to \varphi(U) \times \mathbb{R}^n \\ v &\mapsto (\varphi(\pi(v)), T_{\pi(v)}\varphi(v)).\end{aligned}$$

The atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ of the manifold M therefore gives rise to a collection of maps $\{\psi_\alpha \colon \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n\}_\alpha$. We declare a set $O \subset TM$ to be open if $\psi_\alpha(O \cap \pi^{-1}(U_\alpha))$ is open in $\varphi_\alpha(U_\alpha) \cap \mathbb{R}^n$ for all α. It can then be checked that this defines a topology on TM, that each ψ_α is a homeomorphism (with inverse $(\varphi(p), v) \mapsto \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}|_p$) and that the collection $\{(\pi^{-1}(U_\alpha), \psi_\alpha)\}_\alpha$ provides a smooth atlas for TM. This collection of maps also endows $\pi\colon TM \to M$ with the structure of a rank n vector bundle. We refer to [Tu11, Section 12] for further details.

2. We describe the normal bundle $\nu(f)$ of an embedding $f\colon M^m \to \mathbb{R}^n$. Use $\langle -, - \rangle$ to denote the standard inner product on $\mathbb{R}^n$. Since f is an embedding, $T_xf\colon T_xM \to T_{f(x)}\mathbb{R}^n$ is injective. In order to simplify notations, we think of T_xM as a subspace of $\mathbb{R}^n \cong T_{f(x)}\mathbb{R}^n$ (i.e. we write T_xM instead of $T_xf(T_xM)$). As a set, the total space of $\nu(f)$ is defined as

$$\nu(f) = \{(x, v) \in M \times \mathbb{R}^n \mid v \in (T_xM)^\perp\},$$

while the projection map $\pi\colon \nu(f) \to M$ is given by projection on the first component. Next, we endow $\nu(f)$ with the structure of an n-manifold and explain why $\pi\colon \nu(f) \to M$ is a rank $n - m$ vector bundle. Fix a chart (U, φ) containing a point $p \in M$. Next, choose m maps $v_i\colon U \to TU$ so that for each $x \in U$, the $v_i(x)$ are a basis for T_xU; such v_i exist and can for instance be obtained by setting $v_i(x) = (T_x\varphi)^{-1}(b_i)$, where $b_1, \ldots, b_m$ denote the canonical basis of $\mathbb{R}^m$. Taking $x = p$, we complete our basis of T_pM to a basis $v_1(p), \ldots, v_m(p), e_1, \ldots, e_{n-m}$ of $\mathbb{R}^n$. These vectors still form a basis for $\mathbb{R}^n \cong T_{f(x)}\mathbb{R}^n$ for all $x \in U'$ in a small enough neighborhood $U' \subset U$ of p. Set $e_i(x) = e_i$ for $x \in U'$. Using the Gram-Schmidt process, we can assume that $v_1(p), \ldots, v_m(p), e_1(p), \ldots, e_{n-m}(p)$ is an orthonormal basis. Consider the following map:

$$\begin{aligned}\psi\colon \pi^{-1}(U') &\to \varphi(U) \times \mathbb{R}^{n-m} \\ (x, v) &\mapsto (\varphi(x), v \cdot e_1(x), v \cdot e_{n-m}(x)).\end{aligned}$$

Observe that ψ has a smooth inverse given by $(y, \lambda) \mapsto (\varphi^{-1}(y), \sum_{i=1}^{n-m} \lambda^i e_i(x))$. This construction can be shown to endow $\nu(f)$ with the structure of an n-manifold and π with structure of a rank $n - m$ vector bundle.

For simplicity, we chose to work with manifolds embedded in $N = \mathbb{R}^n$. However, the definition of the normal bundle $\nu(f)$ can be carried out for embeddings $\varphi\colon M \to N$, with N a Riemannian manifold. As we shall see in Exercise 1.2.8 below, $\nu(f)$ can also be described without making use of a metric on N.

Normal bundles will play a major role in Chapter 2, so we record the following deep fact for later use. A proof can be found in [Bre93, Theorem II.11.14] as well as in [MS74, Theorem 11.1].

Theorem 1.2.4 (Tububular neighbourhood theorem)**.** *An embedding $f\colon N^n \to M^m$ extends to a codimension 0 embedding $\nu(f) \to M$ of the total space of the normal bundle.*

Theorem 1.2.4 establishes the existence of tubular neighborhoods around submanifolds.

Remark 1.2.5. For an embedding $f\colon M^m \to N^n$, a *tubular neighborhood* of N in M consists of a D^{n-m}-bundle $B \to V$ (recall Remark 1.2.2) together with an embedding $\psi\colon B \to N$ extending the map taking the center of each disc to the corresponding point of M. Theorem 1.2.4 establishes the existence of a tubular neighborhood: endow N with a Riemannian metric and take B to be the disc bundle obtained from $\nu(f)$ consisting of vectors of at most unit length.

Next, we describe operations on vector bundles. In brief, for every operation involving vector spaces, such as the direct sum, the dual vector space, the tensor product, or exterior powers, there is a corresponding operation for vector bundles over fixed base space.

Example 1.2.6. Given vector bundles $\pi_E \colon E \to B$ and $\pi_F \colon F \to B$, we describe the vector bundle $\pi \colon E \oplus F \to B$ whose fiber over a point $x \in B$ is $E_x \oplus F_x$. As a set, the total space of the direct sum $E \oplus F$ (sometimes called *Whitney sum*) is

$$E \oplus F = \bigsqcup_{x \in B} E_x \oplus F_x,$$

and the projection is $\pi(v,w) = x$ for $(v,w) \in E_x \oplus F_x$. Alternatively, the projection can be described as $\pi(v,w) = \pi_E(v) = \pi_F(w)$. We show that $E \oplus F$ has the structure of a vector bundle. Assume that E (resp. F) has rank n (resp. m). Given $x \in B$, since E and B are vector bundles, there is an open neighborhood U of x as well as homeomorphisms

$$\varphi_E \colon \pi_E^{-1}(U) \to U \times \mathbb{R}^n,$$
$$\varphi_F \colon \pi_F^{-1}(U) \to U \times \mathbb{R}^m.$$

Let $\mathrm{proj}_{2,n}$ (resp. $\mathrm{proj}_{2,m}$) be the projection of $U \times \mathbb{R}^n$ (resp. $U \times \mathbb{R}^m$) on the second coordinate. Define a local trivialisation for $E \oplus F$ by considering

$$\begin{aligned} \varphi \colon \pi^{-1}(U) &\to U \times (\mathbb{R}^n \oplus \mathbb{R}^m) \\ (v,w) &\mapsto (\pi_E(v), (\mathrm{proj}_{2,n} \circ \varphi_E(v), \mathrm{proj}_{2,m} \circ \varphi_F(w))). \end{aligned}$$

This shows that $E \oplus F$ is a rank $n+m$ vector bundle.

Similarly to Example 1.2.6, given vector bundles $E \to B$ and $F \to B$, there are vector bundles $E \otimes F \to B$ and $\mathrm{Hom}(E,F)$ with respective fiber over a point $x \in B$ given by $E_x \otimes_{\mathbb{R}} F_x$ and $\mathrm{Hom}_{\mathbb{R}}(E_x, F_x)$. Similarly, if F is a *subbundle* of E (i.e. if F_x is a subspace of E_x for each $x \in B$), then one can form the *quotient bundle* E/F with fiber E_x/F_x.

Given vector bundles $\pi_E \colon E \to B$ and $\pi_F \colon F \to B$, a *homomorphism* of vector bundles is a continuous map $f \colon E \to F$ such that $\pi_F f = \pi_E$ (i.e. f restricts to a map on the fibers) and such that $f \colon E_x \to F_x$ is linear for each $x \in B$. A homomorphism $f \colon E \to F$ is an *isomorphism* if there exists a homomorphism $g \colon F \to E$ such that $g \circ f = \mathrm{id}_E$ and $f \circ g = \mathrm{id}_F$.

Exercise 1.2.7. Let $f \colon E \to B$ be a homomorphism of vector bundles. Show that f is an isomorphism if and only if f restricts to a linear isomorphism on each fiber.

The next exercise shows that the normal bundle of an embedding $M \to N$ can be defined without endowing N with a Riemannian metric.

Exercise 1.2.8. Let $f \colon M \to N$ be an embedding. Use $TN|_{f(M)} \to M$ to denote the vector bundle with underlying set $\bigsqcup_{x \in M} T_x f(T_x M)$ (more formally, this is the pullback bundle $f^*(TN)$, see Subsection 1.2.3 below). Show that if N is endowed with a Riemanian metric (or just take $N = \mathbb{R}^n$ with the standard inner product), then the quotient bundle $TN|_{f(M)}/TM$ is isomorphic to the vector bundle $\nu(f)$ described in Example 1.2.3.

A rank n vector bundle over B is *trivial* if it is isomorphic to the trivial rank n bundle $B \times \mathbb{R}^n$. A *section* of a vector bundle $\pi \colon E \to B$ is a map $s \colon B \to E$ such that $\pi \circ s = \mathrm{id}_B$. Reformulating, for each $x \in B$, the definition of a section ensures that $s(x)$ lies in E_x. Sections of the tangent bundle are called *vector fields*.

The next remark presents a useful criterion to prove that a vector bundle is trivial.

Lemma 1.2.9. *A rank n vector bundle $\pi \colon E \to B$ is trivial if and only if there sections $s_1, \ldots, s_n$ such that for each $x \in B$, the vectors $s_1(x), \ldots, s_n(x)$ are linearly independent.*

Proof. Assume that the bundle $E \to B$ is trivial and choose an isomorphism $\Phi\colon B \times \mathbb{R}^n \to E$. Define $r_i\colon B \to B \times \mathbb{R}^n$ as $r_i(x) = (x, e_i)$, where e_i is the i-th canonical basis vector of $\mathbb{R}^n$. For $i = 1, \ldots, n$ the maps $s_i\colon B \xrightarrow{r_i} B \times \mathbb{R}^n \xrightarrow{\Phi} E$ provide the desired sections. To prove the converse, assume that we have n pointwise linearly independent sections $s_i\colon B \to \mathbb{R}^n$; our goal is to define a bundle isomorphism $\Phi\colon E \to B \times \mathbb{R}^n$. Given $v \in E$, set $x = \pi(v)$ and note that $s_1(x), \ldots, s_n(x)$ form a basis of E_x. As a consequence, we can write $v = \sum_{i=1}^n v^i s_i(x)$ for some $v^i \in \mathbb{R}$ and define $\Phi(v) := (x, v^1, \ldots, v^n)$. It is immediate that Φ is a bundle homomorphism that restricts to a linear isomorphism on the fibers. Exercise 1.2.7 now implies that Φ is a bundle isomorphism (alternatively, it is easy to show that the inverse of Φ is given by $(x, v^1, \ldots, v^n) \mapsto \sum_{i=1}^n v^i s_i(x)$). This concludes the proof of the lemma. □

We conclude this subsection with a concrete example of these notions.

Example 1.2.10. Show that the normal bundle of the standard embedding $\iota\colon S^n \to \mathbb{R}^{n+1}$ is trivial. This can be done either by constructing an explicit bundle isomorphism $\nu(\iota) \to S^n \times \mathbb{R}$ or, thanks to Lemma 1.2.9, by constructing a nowhere vanishing section $s\colon S^n \to \nu(\iota)$.

1.2.2 Some homotopy theory

This subsection provides a very brief introduction to higher homotopy groups of a space. We assume familiarity with CW-complexes and emphasis is put on the results and methods that will be used in surgery theory. We prove the long exact sequence of the pair and sate (without proof) Whitehead's theorem as well as cellular approximation. These results provide means and motivation for the idea of "killing a homotopy class". References include [Hat02, Section 4].

Given spaces X and Y endowed with fixed basepoints $x_0 \in X$ and $y_0 \in Y$, a map $f\colon X \to Y$ is *based* (or *pointed)* if $f(x_0) = y_0$. A homotopy $f_t\colon X \to Y$ is *based* if $f_t(x_0) = y_0$ for each $t \in [0, 1]$. We use $[(X, x_0), (Y, y_0)]$ to denote the set of based homotopy classes of based maps from (X, x_0) to (Y, y_0). Fix a basepoint s_0 of S^n.

Definition 9. The *n-th homotopy group* of a based space (X, x_0) consists of based homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, in other words,

$$\pi_n(X, x_0) := [(S^n, s_0), (X, x_0)].$$

The group law is defined as follows: for $f, g\colon S^n \to X$, the sum $f + g$ is the composition $S^n \xrightarrow{c} S^n \vee S^n \to X$, where c collapses the equatorial $S^{n-1} \subset S^n$ to a point, and we choose the basepoint of S^n to lie in this S^{n-1}. It can be shown that $(\pi_n(X, x_0), +)$ is a group [Hat02, Section 4.1], where a map represents the zero element if and only if it is homotopic to the constant map at x_0. For $n \geq 2$, the *higher homotopy groups* $\pi_n(X, x_0)$ are abelian [Hat02, Section 4.1], while for $n = 1$ one recovers the fundamental group.

A based map $f\colon (X, x_0) \to (Y, y_0)$ gives rise to an *induced map* $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ via the well defined assignment $[\varphi] \mapsto [f \circ \varphi]$.

Exercise 1.2.11. Show that the induced maps satisfy $(g \circ f)_* = g_* \circ f_*$. Deduce that if (X, y_0) and (Y, y_0) are homotopy equivalent (in the based sense), then $\pi_n(X, x_0)$ and $\pi_n(Y, y_0)$ are isomorphic. This implies that the homotopy groups of a contractible space are trivial.

Contrarily to the homology groups, the homotopy groups of a space are very difficult to compute: for instance the homotopy groups of spheres are not known in general.

The next result assumes some familiarity with covering space theory.

Proposition 1.2.12. *A covering space projection $p\colon (\widehat{X}, \widehat{x}_0) \to (X, x_0)$ induces isomorphisms $p_*\colon \pi_n(\widehat{X}, \widehat{x}_0) \xrightarrow{\cong} \pi_n(X, x_0)$ for $n \geq 2$.*

Proof. We start with surjectivity. A based map $f\colon (Y, y_0) \to (X, x_0)$ lifts to $(\widehat{X}, \widehat{x}_0)$ if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widehat{X}, \widehat{x}_0))$ [Hat02, Proposition 1.33]. Since S^n is simply connected for $n \geq 2$, any based map $S^n \to X$ lifts to $\widehat{X}$, proving the surjectivity of p_*. Next, we prove injectivity. If $h_t\colon Y \to X$ is a homotopy and $\widetilde{h}_0$ is a lift of h_0, then there exists a unique homotopy $\widetilde{h}_t$ of h_0 that lifts h_t [Hat02, Proposition 1.30]. Now suppose that $f, g\colon (S^n, s_0) \to (\widehat{X}, \widehat{x}_0)$ are two based maps such that $p \circ f, p \circ g\colon S^n \to X$ are based homotopic. Let h_t be the homotopy between $h_0 = p \circ f$ and $h_1 = p \circ g$. We now apply the aforementioned homotopy lifting result with $\widetilde{h}_0 = f$: the result is a homotopy $\widetilde{h}_t$ from f to a lift $\widetilde{f}_1$ of $p \circ g$. Since all maps are based, the uniqueness of lifts [Hat02, Proposition 1.34], implies that $\widetilde{f}_1 = g$. This concludes the proof of the injectivity of p_* and thus the proof of the proposition. □

We deduce some computations of higher homotopy groups (all maps are based, although we sometimes omit the base points from the notation).

Example 1.2.13. Here are some applications of Proposition 1.2.12:

1. The higher homotopy groups of S^1 are trivial since the universal cover of S^1 is $\mathbb{R}$. The higher homotopy groups of the n-torus $\mathbb{T}^n$ are trivial since the universal cover of $\mathbb{T}^n$ is $\mathbb{R}^n$.
2. For $g \geq 2$, the higher homotopy groups of the closed genus g surface Σ_g all vanish: the universal cover of Σ_g is the hyperbolic plane.
3. The universal cover of $SO(3)$ is $SU(2)$ so $\pi_k(SO(3)) \cong \pi_k(SU(2))$ for $k \geq 2$. Since $SU(2)$ is homeomorphic to S^3 and since we will see in Corollary 1.2.20 below that $\pi_2(S^3) = 0$, we deduce that $\pi_2(SO(3)) = 0$. This also follows from the fact that $\pi_2(G) = 0$ for any compact connected Lie group.

A powerful computational tool in homology theory is the long exact sequence of a pair. We now describe the corresponding exact sequence for homotopy groups. Recall that a *pair of spaces* is a pair (X, A), where X is a space and $A \subset X$. If X is based, then we assume that A contains the basepoint. A (based) *map of pairs* $f\colon (X, A) \to (Y, B)$ is a (based) map $f\colon X \to Y$ such that $f(A) \subset B$. Recall that two maps $f, g\colon (X, A) \to (Y, B)$ are (based) *homotopic rel* A if they are homotopic by a (based) homotopy $f_t\colon X \to Y$ such that $f_t|_A$ is independent of t. We use $[(X, A, x_0), (Y, B, y_0)]$ to denote the corresponding set of based rel A homotopy classes of based maps. Fix a basepoint $s_0 \in D^n$ that lies in ∂D^n.

Definition 10. The *relative homotopy group* of a based pair (X, A, x_0) consists of the rel ∂D^n homotopy classes of maps $(D^n, \partial D^n, s_0) \to (X, A, x_0)$:

$$\pi_n(X, A, x_0) := [(D^n, \partial D^n, s_0), (X, A, x_0)].$$

The sum is once again obtained via the map $c\colon D^n \to D^n \vee D^n$ that collapses the equatorial D^{n-1} to a point. Next, we describe the maps that appear in the exact sequence of the pair.

Given a based pair (X, A, x_0), the inclusion $A \to X$ induces a map $i\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ for each n. Next, thinking of $\pi_n(X, x_0)$ as $[(D^n, \partial D^n, x_0), (X, x_0, x_0)]$, the inclusion $\{x_0\} \to A$ induces a canonical map $j\colon \pi_n(X, x_0) \to \pi_n(X, A, x_0)$. Finally, the connecting homomorphism $\partial\colon \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is defined by sending a map $f\colon (D^n, \partial D^n, x_0) \to (X, A, x_0)$ to the restriction $f|_{\partial D^n}\colon S^{n-1} \to A$. It can be checked that ∂ is well defined.

The next theorem describes the long exact sequence of a pair (X, A). [1]

Theorem 1.2.14. *Given a based pair (X, A, x_0), there is a long exact sequence*

$$\ldots \to \pi_n(A, x_0) \xrightarrow{i} \pi_n(X, x_0) \xrightarrow{j} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \ldots$$

[1] Recall that given a collection of groups G_n and group homomorphisms $f_n\colon G_n \to G_{n-1}$, the sequence

$$\ldots G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} G_{n-2} \to \ldots$$

is *exact* if $\ker(f_{n-1}) = \operatorname{im}(f_n)$ for each n.

Proof. To prove exactness, we must establish the equalities $\mathrm{im}(\partial) = \ker(i)$ and $\mathrm{im}(i_*) = \ker(j)$ and $\mathrm{im}(j) = \ker(\partial)$. In what follows, we will show the first two equalities, but refer to [Hat02, proof of Theorem 4.3] for the last equality. To abbreviate notations, we will often omit basepoints.

We first show that $\mathrm{im}(\partial) = \ker(i)$. To show the inclusion $\mathrm{im}(\partial) \subset \ker(i)$, we start with a relative map $f\colon (D^n, \partial D^n) \to (X, A)$ representing an element $[f]$ of $\pi_n(X, A)$, we must show that $(i \circ \partial)[f]$ is zero, i.e. that $i \circ \partial(f)$ is nullhomotopic. By definition, $\partial([f])$ is represented by the map $f| := f|_{\partial D^n}\colon S^{n-1} \to A$ and therefore $i \circ \partial$ is represented by the the composition $i \circ f|\colon S^n \to X$. By definition $i \circ f|$ extends over D^n and is therefore nullhomotopic, as desired.

Next, we prove the reverse inclusion, namely $\ker(i) \subset \mathrm{im}(\partial)$. Let $f\colon (D^{n-1}, S^{n-2}) \to (A, x_0)$ be a representative of $\pi_{n-1}(A)$ such that $i([f])$ is zero in $\pi_{n-1}(X)$. In other words, we know that the composition $i \circ f\colon (D^{n-1}, S^{n-2}) \to (A, x_0) \to (X, x_0)$ is nullhomotopic. Thus, we obtain a rel ∂D^{n-1} homotopy $H\colon D^{n-1} \times [0, 1] \to X$ between $i \circ f$ and the constant map c_{x_0}. Since this map is rel S^{n-1}, each $H(-, t)$ must map S^{n-1} into $\{x_0\}$. Viewing D^n as $D^{n-1} \times [0, 1]$, it only remains to show H induces a map of pairs $H\colon (D^n, \partial D^n) \to (X, A)$ such that $\partial([H]) = [f]$. Note that $\partial D^n = \partial(D^{n-1} \times [0, 1])$ can be viewed as $(S^{n-2} \times [0, 1]) \cup (D^{n-1} \times \{0, 1\})$. Since we have a homotopy rel S^{n-2} between the composition $i \circ f$ and the constant map c_{x_0}, we see that H maps $(S^{n-2} \times I) \cup (D^{n-1} \times \{1\})$ to $\{x_0\}$, while the restriction of H to $D^{n-1} \times \{0\}$ is precisely f. We have therefore proved that $\partial([H]) = [f]$, concluding the proof of the equality $\mathrm{im}(\partial) = \ker(i)$.

The equality $\mathrm{im}(i) = \ker(j)$, will follow from the following "compression criterion".

Claim. *A based map $f\colon (D^n, S^{n-1}) \to (X, A)$ represents zero in $\pi_n(X, A)$ if and only if it is homotopic rel S^{n-1} to a map $g\colon (D^n, S^{n-1}) \to (X, A)$ with image contained in A.*

Proof. We start with the "only if" direction, namely given a map $f\colon (D^n, \partial D^n) \to (X, A)$ representing zero in $\pi_n(X, A)$, and construct a map g with the desired properties. We know that there is a rel ∂D^n homotopy $F\colon D^n \times [0, 1] \to X$ between f and the constant map c_{x_0}. In other words, F is such that $F|_{D^n \times \{0\}} = f$ and $F|_{D^n \times \{1\}} = c_{x_0}$ and $F|_{S^{n-1} \times [0,1]} \subset A$. We set $f_1 := F|\colon D^n \simeq D^n \times \{1\} \cup S^{n-1} \times I \to A$. Since the image of f_1 lies in A, it is enough to show that f is homotopic to g. The homotopy f_t is given by the restriction of F to $D^n \simeq D^n \times \{t\} \cup S^{n-1} \times [0, t] \to X$: by construction, we have $f_1 = g$ and f_0 (which is defined on $D^n \times \{0\}$) is equal to f.

Next, we prove the converse. Assume that f is homotopic rel S^{n-1} to a map g with image in A. We show that g represents zero in $\pi_n(X, A)$, i.e. that it is nullhomotopic rel ∂D^{n-1}. As $g(D^n) \subset A$, we have a map $g\colon (D^n, S^{n-1}) \to (A, A)$. This map is nullhomotopic since D^n is contractible. This concludes the proof of the claim. □

We now prove the equality $\mathrm{im}(i) = \ker(j)$. We start by showing $\mathrm{im}(i) \subset \ker(j)$. Let $f\colon (D^n, \partial D^n) \to (A, x_0) \to (X, x_0)$ be a representative of an element in $\mathrm{im}(i) \subset \pi_n(X)$. By definition, $(j \circ i)(f)\colon D^n \to X$ has image in A. The compression lemma therefore ensures that this element is zero in $\pi_n(X, A)$. We conclude by proving the reverse inclusion, namely $\ker(j) \subset \mathrm{im}(i)$. Assume that $f\colon (D^n, \partial D^n) \to (X, x_0)$ maps to zero in the relative group $\pi_n(X, A)$. By the compression lemma, this means that f is homotopic rel ∂D^n to a map g with image in A. But as such a g lies in $\mathrm{im}(i)$, this establishes the equality $\mathrm{im}(i) = \ker(j)$ and concludes the proof of the theorem. □

The *mapping cylinder* of a a map $f\colon X \to Y$ is defined as $M_f = X \times [0, 1] \cup_f Y$. More explicitly, this space is $(X \times [0, 1]) \sqcup Y/\sim$, where the equivalence relation is $(x, 1) \sim f(x)$ for every $x \in X$. It is known that M_f deformation retracts onto Y. The map $f\colon X \to Y$ factors as $f = r \circ i$, where $r\colon M_f \to Y$ is the aforementioned homotopy equivalence and $i\colon X \to M_f$ is the canonical inclusion $x \mapsto (x, 0)$. Since X is a subspace of M_f, we may consider the pair (M_f, X).

Definition 11. The *homotopy group of a map* $f\colon X \to Y$ is defined as

$$\pi_n(f) := \pi_n(M_f, X).$$

Since these homotopy groups will be frequently used in Chapter 2 below, we record some facts for later reference.

Remark 1.2.15. Let $f\colon X \to Y$ be a map.

1. Elements of $\pi_n(f)$ can be represented by maps $(D^n, S^{n-1}) \to (M_f, X)$. Since M_f is homotopy equivalent to Y, it follows that elements of $\pi_n(f)$ can be represented by a pair of maps $S^{n-1} \to X$ and $D^n \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X \\ \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle f} \\ D^n & \longrightarrow & Y. \end{array}$$

2. Given a map $X \to Y$, the mapping cylinder construction produces a "long exact sequence of the pair (Y, X) " even when X is not a subspace of Y. Indeed, replacing Y by the mapping cylinder M_f and applying Theorem 1.2.14 leads to the following exact sequence:

$$\ldots \to \pi_n(X) \to \pi_n(Y) \to \pi_n(f) \to \ldots.$$

3. A map f is *n-connected* if it induces an isomorphism on the first $n-1$ homotopy groups and a surjection on π_n. An application of the long exact sequence (recall Theorem 1.2.14) shows that f is n-connected if and only if $\pi_k(f) = 0$ for all $k \leq n$.

Recall that a map $f\colon X \to Y$ between CW-complexes is *cellular* if $f(X^n) \subset Y^n$. We now state two important theorems in homotopy theory. We refer to [Hat02, Theorem 4.5 and 4.8] for proofs, but present some applications instead.

Theorem 1.2.16. *Let X and Y be connected CW complexes.*

1. Whitehead's theorem*: If $f\colon X \to Y$ induces an isomorphism $\pi_n(X) \to \pi_n(Y)$ for each n, then f is a homotopy equivalence.*

2. Cellular approximation*: every map $f\colon X \to Y$ is homotopic to a cellular map.*

We deduce that attaching cells of dimension $n+1$ or greater does not affect the first $n-1$ homotopy groups.

Corollary 1.2.17. *If X is obtained from A by attaching cells of dimension $n+1$ or greater, then the inclusion $A \to X$ is n-connected. In particular, (X, X^n) is n-connected.*

Proof. Using the long exact sequence, we must show that (X, A) is n-connected i.e. that $\pi_i(X, A) = 0$ for $i < n$. Let $f\colon (D^i, \partial D^i) \to (X, A)$ be a representative of $\pi_i(X, A)$. Using the (relative version of) cellular approximation, we can assume that $f(D^i) \subset X^i$. Since $i \leq n$, the assumption on the pair (X, A) implies that $X^i \subset A$. Applying the compression criterion, we deduce that $[f] = 0$ in $\pi_n(X, A)$. This concludes the proof of the corollary. □

Corollary 1.2.17 is useful to "kill" a given homotopy classes.

Corollary 1.2.18. *Let X be a CW complex and let $[f] \in \pi_n(X)$. There exists a space X' containing X such that $\pi_i(X') \cong \pi_i(X)$ for $i \leq n-1$ and $\pi_n(X') \cong \pi_n(X)/\langle [f] \rangle$.*

Proof. Set $X' = X \cup_f e^{n+1}$ and use the Corollary 1.2.17 to deduce that $\pi_i(X') \cong \pi_i(X)$ for $i \leq n-1$ and that $i\colon \pi_n(X) \to \pi_n(X')$ is surjective. To see that $[f]$ belongs to $\ker(i)$, note that that in X', the map $i \circ f$ extends over the $n+1$-cell. Some additional work shows that $\ker(i) = \langle \omega \rangle$. This concludes the proof of the corollary. □

Next, show how to kill relative classes.

Corollary 1.2.19. *Let $f\colon X \to Y$ be a map of CW complexes and let $\omega \in \pi_{n+1}(f)$. There exists a CW complex X' containing X such that $\pi_i(X') \cong \pi_i(X)$ for $i \leq n$ and $\pi_{n+1}(f') \cong \pi_{n+1}(f)/\langle\omega\rangle$.*

Proof. Represent ω by a pair (q, Q) with $q\colon S^n \to X$ and $Q\colon D^{n+1} \to Y$. Glue a $k+1$-cell to X along q yielding X' and set $f' = f \cup Q$. Since we attached a cell of dimension $n+1$, we see that $\pi_i(X', X) = 0$ for $i \leq n$. To show that $\pi_i(f) = \pi_i(f')$ for $i \leq n$, consider the following commutative diagram in which the vertical maps are inclusion induced:

$$\begin{array}{ccccccccccc}
\longrightarrow & \pi_i(X) & \xrightarrow{f_*} & \pi_i(Y) & \longrightarrow & \pi_i(f) & \longrightarrow & \pi_{i-1}(X) & \xrightarrow{f_*} & \pi_i(Y) & \longrightarrow \\
 & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle=} & & \downarrow & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle=} & \\
\longrightarrow & \pi_i(X') & \xrightarrow{f'_*} & \pi_i(Y) & \longrightarrow & \pi_i(f') & \longrightarrow & \pi_{i-1}(X') & \xrightarrow{f'_*} & \pi_i(Y) & \longrightarrow .
\end{array}$$

For $i < n-1$, we know that the ι maps are isomorphisms and so we can apply the five lemma. For $i = n$, the right hand side ι map is an isomorphism while the left hand side one is only a surjection. This is nevertheless enough to once again apply the five lemma, concluding the proof that $\pi_i(X') \cong \pi_i(X)$ for $i \leq n$.

Using cellular approximation for pairs, we see that $\pi_{n+1}(f) \to \pi_{n+1}(f')$ is surjective. The fact that ω maps to zero in $\pi_{k+1}(f')$ follows from the compression lemma and because the map $Q\colon D^{n+1} \to Y$ factors through $X' = X \cup_q D^{n+1}$. Some additional work shows that $\ker(i) = \langle\omega\rangle$. This concludes the proof of the corollary. □

We conclude this subsection by proving that $\pi_k(S^n) = 0$ for $k < n$.

Corollary 1.2.20. *For $k < n$, we have $\pi_k(S^n) = 0$.*

Proof. Give S^n and S^k are given their usual CW structures with 0-cells as basepoints. Using cellular approximation, every basepoint-preserving map $S^n \to S^k$ can be homotoped fixing the basepoint to be cellular and hence constant if $k < n$. This concludes the proof of the corollary. □

1.2.3 Vector bundles: the homotopy classification

The goal in this subsection is state the classification of vector bundles over a fixed base space. We choose to only sketch the proof of the main results: the goal of this course is after all to provide an introduction to surgery theory; proving these results in details would be an ideal topic for a course on vector bundles and characteristic classes. Additionally, we also use the classification of vector bundles to improve our understanding of the normal and tangent bundles.

We start by discussing the pullback construction. The *pullback* of a vector bundle $\pi\colon E \to B$ along a map $f\colon X \to B$ is the vector bundle $f^*(E) \to X$ with total space

$$f^*(E) = \{(x, e) \in X \times E \mid f(x) = \pi(e)\}$$

and bundle projection $(x, e) \mapsto x$. We will not check that $f^*(E)$ is a vector bundle but instead observe that the fiber above $x \in X$ is $\{x\} \times E_{f(x)}$. The vector space structure on this fiber is induced by the one on $E_{f(x)}$. Note furthermore that $f\colon X \to B$ lifts to a map $\widetilde{f}\colon f^*(E) \to E$ by setting $\widetilde{f}(x, e) = e$ and one obtains the following commutative diagram:

$$\begin{array}{ccc}
f^*(E) & \xrightarrow{\widetilde{f}} & E \\
\downarrow & & \downarrow{\scriptstyle\pi} \\
X & \xrightarrow{f} & B.
\end{array}$$

It follows that $\widetilde{f}$ is linear on the fibers and, conversely, it is not difficult to show that $f^*(E) \to X$ satisfies the corresponding universal property: if $p\colon F \to X$ is a vector bundle and if $\varphi\colon F \to E$ is fiberwise linear and satisfies $f \circ p = \pi \circ \varphi$, then $p\colon F \to X$ is isomorphic to $f^*(E) \to X$.

The next exercise shows that we have already encountered a pullback bundle.

Exercise 1.2.21. Given an embedding $f\colon M \to N$, show that the restricted bundle $TN|_{f(M)} \to M$ described in Exercise 1.2.8 is isomorphic to the pullback bundle $f^*(TN) \to M$. Use Exercise 1.2.8 to deduce that $TM \oplus \nu(f) \cong f^*(TN)$.

The next exercise establishes some basic properties of the pullback construction.

Exercise 1.2.22. Prove the vector bundle isomorphisms $\mathrm{id}^*(E) \cong E$ and $(g \circ f)^*(E) \cong f^*(g^*(E))$. Show that the pullback of a trivial bundle is trivial.

We now move towards the classification of vector bundles.

Definition 12. A *universal vector bundle* is a rank n vector bundle $\gamma_n \to BO(n)$ that satisfies the following property: for every rank n vector bundle $E \to X$ over a CW complex X, there is a unique (up to homotopy) map $f\colon X \to BO(n)$ such that $E \cong f^*(BO(n))$.

Temporarily assuming the existence of classifying spaces, we use Definition 12 to provide a homotopy theoretic interpretation of the set $\mathrm{Vect}_n(X)$ of isomorphism classes of rank n vector bundles over a CW complex X.

Theorem 1.2.23. *For any CW complex X, the pullback construction determines a bijection*

$$[X, BO(n)] \xrightarrow{\cong} \mathrm{Vect}_n(X).$$

Proof. We need only check that the assignment is well defined: the bijective correspondence immediately follows from the definition of a universal bundle, recall Definition 12. To check that that the assignment is well defined, we need the following fact: if $E \to B$ is a vector bundle and if $f, g\colon X \to B$ are homotopic, then $f^*(E)$ and $g^*(E)$ are isomorphic. A proof can be found in [Hat, Therorem 1.6]; we only mention the key idea. Given a homotopy $F\colon X \times I \to B$, the restrictions of $F^*(E)$ over $X \times \{0\}$ and $X \times \{1\}$ are $f_0^*(E)$ and $f_1^*(E)$. The result then follows from the following statement whose proof can be found in [Hat, Proposition 1.7]: the restriction of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is a manifold. This concludes the proof sketch. □

It is worth mentioning that both Definition 12 and Theorem 1.2.23 hold whenever X is a paracompact space. We refer to [Hat, Appendix of Section 1.2] for a discussion of paracompactness.

Next, we sketch the proof of the existence of universal vector bundles.

Theorem 1.2.24. *A universal rank n vector bundle exists.*

Proof. We construct the bundle $\gamma_n \to BO(n)$ but do not show that it is universal; we refer the interested reader to [Hat, Theorem 1.16] for a proof. Let $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ denote the space of n-dimensional vector subspaces of $\mathbb{R}^{n+k}$. Fix an integer n and observe that there is a canonical inclusion $\mathrm{Gr}_n(\mathbb{R}^{n+k}) \to \mathrm{Gr}_n(\mathbb{R}^{n+k+1})$: an n-dimensional vector subspace of $\mathbb{R}^{n+k}$ can also be viewed as a vector subspace of $\mathbb{R}^{n+k+1}$. The infinite Grassmannian $BO(n) := \mathrm{Gr}_n(\mathbb{R}^\infty)$ is the direct limit (or union) over k. Next, the *tautological bundle* over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ has total space

$$\gamma_{n,k} := \{(V, x) \in \mathrm{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \mid x \in V\}.$$

and projection map $\pi(V, x) = V$. Some work shows that this a rank n-vector bundle [Hat, Lemma 1.15]. Setting $\gamma_n := \lim_k \gamma_{n,k}$ provides the desired rank n bundle over $BO(n)$. It can once again be shown that $\gamma_n \to \mathrm{Gr}_n$ is a vector bundle [Hat, Lemma 1.15], and the proof of universality can be found in [Hat, Theorem 1.16]. This concludes our proof sketch. □

As a corollary of Theorem 1.2.23, we can classify vector bundles over contractible spaces.

Corollary 1.2.25. *Any vector bundle over a contractible space is trivial.*

Proof. This is immediate: if $X \simeq \{x_0\}$ is contractible, then $\mathrm{Vect}_n(X)$ corresponds bijectively to the set $[X, BO(n)] \simeq [x_0, BO(n)] \simeq \{*\}$. □

Corollary 1.2.25 provides some further insight into the normal bundle of submanifolds of $\mathbb{R}^n$.

Example 1.2.26. We investigate the relation between the tangent bundle and the normal bundle.

1. For any embedding $f\colon M \to \mathbb{R}^n$, the direct sum $TM \oplus \nu(f)$ is trivial: this follows from the isomorphism $f^*(T\mathbb{R}^n) \cong TM \oplus \nu(f)$ (recall Exercise 1.2.21), from the fact that any bundle over $\mathbb{R}^n$ is trivial (by Corollary 1.2.25) and since the pullback of a trivial bundle is trivial (recall Exercise 1.2.22).

2. The tangent bundle of the sphere S^r satisfies $TS^r \oplus \epsilon \cong \epsilon^{r+1}$, where ϵ^k denotes the rank k trivial bundle over S^r. To see this, embed S^r in $\mathbb{R}^{r+1}$ via the standard embedding f. Since we now know that $TS^r \oplus \nu(f)$ is trivial, the conclusion follows from Example 1.2.10:
$$\epsilon^{r+1} = TS^r \oplus \nu(f) = TS^r \oplus \epsilon^1.$$

3. Two vector bundles E_1 and E_2 over B are *stably isomorphic* if $E_1 \oplus \epsilon^m$ is isomorphic to $E_2 \oplus \ \epsilon^n$; this time ϵ^k denotes the trivial rank n vector bundle over B. For instance, we saw above that TS^r is *stably trivial.* Use $\mathrm{Vect}_{\mathrm{stab}}(B)$ to denote the set of stable isomorphism classes of vector bundles over B and set $BO := \lim_n BO(n)$, just as in Theorem 1.2.23, it can be shown that there is an isomorphism
$$\mathrm{Vect}_{\mathrm{stab}}(B) \cong [B, BO].$$
This transition to homotopy allows for convenient reformulations of naturally occurring questions: for instance, a stably trivial rank n vector bundle over S^r defines an element in $\ker(\pi_r(BO(n)) \to \pi_r(BO))$. If this kernel happens to be trivial, then every stably trivial rank n vector bundle over S^r is trivial.

We conclude this subection by using Theorem 1.2.23 to study vector bundles over spheres.

Example 1.2.27. Using Theorem 1.2.23, we know that rank n vector bundles over the k-sphere. are classified by $\pi_k(BO(n))$, and this latter group is known to be $\pi_{k-1}(O(n))$. [2] For instance, we deduce that rank 1 vector bundles over S^1 are classified by $\pi_0(O(1)) \cong \mathbb{Z}_2$: the trivial bundle and the infinite Mobius band of Example 1.2.1 are the only examples.

We conclude by sketching a more geometric proof of the isomorphism $\mathrm{Vect}_k(S^n) \cong \pi_{k-1}(O(n))$. Any bundle E over S^n restricts to two bundles E_+ and E_- over the upper and lower hemispheres D^n_+ and D^n_-. Using Corollary 1.2.25, these bundles are trivial. The vector bundle $E \to S^n$ is recovered by gluing $D^n_+ \times \mathbb{R}^k$ and $D^n_- \times \mathbb{R}^k$ along the equatorial S^{n-1} via a *clutching map* $f\colon S^{k-1} \to GL_n(\mathbb{R})$. Conversely, a clutching map f can be used to obtain a bundle $S^n \to E$ by gluing two bundles $D^n \times \mathbb{R}^k$ along the equatorial S^{n-1}. It turns out that this construction only depends on the homotopy class of f. The isomorphism $\mathrm{Vect}_k(S^n) \cong \pi_{k-1}(O(n))$ follows since $GL_n(\mathbb{R})$ deformation retracts onto $O(n)$.

1.3 Turning the sphere inside out

In this section, we return to eversions. Namely relying on the deep Smale-Hirsch theorem, we show that regular homotopy classes of immersions $S^n \looparrowright \mathbb{R}^{n+1}$ correspond bijectively to elements of $\pi_n(SO(n+1))$. Setting $n = 2$, we deduce that all immersions $S^2 \looparrowright \mathbb{R}^3$ are regular homotopic and, in particular, eversions exist.

[2] This follows from the long exact sequence of the fibration applied to the universal $O(n)$-bundle over $BO(n)$.

Recall that two immersions $f_0, f_1 \colon M \looparrowright N$ are regularly homotopic if there is a homotopy $f_t \colon M \to N$ between f_0 and f_1 such that each f_t is an immersion. We use $\pi_0(\mathrm{Imm}(M,N))$ to denote the set of regular homotopy classes of immersions $M \to N$. We also write $\pi_0(\mathrm{Mono}(M,N))$ for the set of homotopy classes of bundle monomorphisms $TM \to TN$.

The following theorem will be referred to as the *Smale-Hirsch theorem*.

Theorem 1.3.1. *Let M be an m-manifold and let N be an n-manifold. If $m < n$, then the differential induces a bijection between regular homotopy classes of immersions $M \to N$ and homotopy classes of bundle monomorphisms $TM \to TN$:*

$$T \colon \pi_0(\mathrm{Imm}(M,N)) \xrightarrow{\cong} \pi_0(\mathrm{Mono}(TM,TN)).$$

Note that Smale originally classified regular homotopy classes of immersions $S^k \looparrowright \mathbb{R}^n$ [Sma59] (establishing the existence of an eversion as a corollary), while Hirsch generalized this result to arbitrary manifolds [Hir59]. We do not present Hirsch's proof but only show how it gives rise to the classification of regular homotopy classes of immersions $S^n \looparrowright \mathbb{R}^{n+1}$. The exposition is inspired by [Now].

Theorem 1.3.2. *Regular homotopy classes of immersions $S^n \looparrowright \mathbb{R}^{n+1}$ correspond bijectivetly to elements of $\pi_n(SO(n+1))$.*

Proof. Using the Smale-Hirsch theorem (Theorem 1.3.1), we need only establish a bijective correspondence between $\pi_0(\mathrm{Mono}(TS^n, T\mathbb{R}^{n+1}))$ and $\pi_n(SO(n+1)) = [S^n, SO(n+1)]$. First, note that the space $\mathrm{Mono}(TS^n, T\mathbb{R}^{n+1})$ is homotopy equivalent to

$$X := \{f \colon TS^n \to \mathbb{R}^{n+1} \mid f \text{ is a linear monomorphism on each fiber of } TS^n\}.$$

To see this, contract the first $\mathbb{R}^{n+1}$ factor of $T\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ to a point. Note that the space X can also be understood as $\mathrm{Mono}(TS^n, \{p\} \times \mathbb{R}^{n+1})$ where $\{p\} \times \mathbb{R}^{n+1}$ is the trivial rank $n+1$ vector bundle over $\{p\}$. Next, we argue that X is homotopy equivalent to the space

$$Y := \mathrm{Map}(S^n, GL^+_{n+1}(\mathbb{R})).$$

Fix a basepoint $b_0 \colon TS^n \to \mathbb{R}^{n+1}$ of X. In order to define a map $\Phi \colon X \to Y$, we must send an element $b \colon TS^n \to \mathbb{R}^{n+1}$ of X to a map $\Phi(b) \colon S^n \to GL^+_{n+1}(\mathbb{R})$. By definition of X, for any $x \in S^n$, we have linear injections $b_0(x), b(x) \colon T_xS^n \hookrightarrow \mathbb{R}^{n+1}$. Define $\Phi(b)(x)$ as the unique matrix $A \in GL^+_{n+1}(\mathbb{R})$ such that $Ab_0(x) = b(x)$ and A maps a unit vector perpendicular to $b_0(x)(T_xS^n)$ into a unit vector perpendicular to $b(x)(T_xS^n)$. In order to define the homotopy inverse $\Psi \colon Y \to X$ of Φ, given an element $f \colon S^n \to GL^+_{n+1}(\mathbb{R})$ of Y, we define a map $\Psi(f) \colon TS^n \to \mathbb{R}^{n+1}$ by mapping $v \in T_xS^n$ to the element $\Psi(f)(v) = f(x) \circ b_0(x)(v)$ of $\mathbb{R}^{n+1}$. It can be checked that Φ and Ψ are homotopy inverses.

Additional work shows that these bijective correspondences descend to (regular) homotopy classes, establishing a bijection between $\mathrm{Mono}(TS^n, T\mathbb{R}^{n+1})$ and $\pi_n(GL^+_{n+1}(\mathbb{R})) = [S^n, GL^+_{n+1}(\mathbb{R})]$. The Gram-Schmidt process implies that $GL^+_{n+1}(\mathbb{R})$ deformation retracts onto $SO(n+1)$ and therefore $\pi_n(GL^+_{n+1}(\mathbb{R})) \cong \pi_n(SO(n+1))$. This concludes the proof of the theorem. □

Setting $n = 1$, we obtain the Whitney-Graustein theorem (recall Example 1.1.15) while setting $n = 2$, we obtain existence of a sphere eversion.

Corollary 1.3.3. *The set of regular homotopy classes of immersions $S^2 \looparrowright \mathbb{R}^3$ is trivial. In particular the standardly embedded sphere $S^2 \subset \mathbb{R}^3$ can be turned inside out.*

Proof. The first assertion follows from Theorem 1.3.2 and Example 1.2.13 which showed that $\pi_2(SO(3))$ is trivial. Since an eversion is a regular homotopy from the standard embedding $\iota \colon S^2 \to \mathbb{R}^3$ to the antipodal embedding $-\iota \colon S^2 \to \mathbb{R}^3$, the second assertion follows immediately: we know that all immersions $S^2 \looparrowright \mathbb{R}^3$ are regularly homotopic. This concludes the proof of the corollary. □

Chapter 2

Surgery theory

One of the motivations of surgery theory is to decide whether two manifolds are diffeomorphic. Before describing this classification program, we introduce some terminology. If M and N are two n-manifolds, then a *cobordism* is an $n+1$-manifold whose boundary is $\partial W = M \sqcup N$. A cobordism (W, M, N) is an *h-cobordism* if the inclusions $M \to W$ and $N \to W$ are homotopy equivalences. Here are the main steps in the surgery program to decide whether two closed manifolds M and N are diffeomorphic:

1. construct a homotopy equivalence $f \colon M \to N$;
2. construct a cobordism $(W; M, N)$ and a map $F \colon W \colon N \to N \times [0,1]$ with $F|_M = f, F|_N = \mathrm{id}_N$;
3. modify W and F relative boundary so that F becomes a homotopy equivalence.

If M^n and N^n are simply connected and $n \geq 5$, then this program ensures that M and N are diffeomorphic: Smale's h-cobordism theorem states that every h-cobordism W is diffeomorphic to $M \times [0,1]$ [Sma62]. In general, the s-cobordism theorem states that W is diffeomorphic to $M \times\ [0,1]$ if and only if W is an s-cobordism (i.e. the inclusions $M \to W$ and $N \to W$ are *simple* homotopy equivalences). An account of such results (and how they imply the generalized Poincaré conjecture) can be found in [LÖ2].

The aim of this chapter is to focus on the third step of the surgery program and study the question whether a map $f \colon M \to X$ "be modified into" a homotopy equivalence. The "modification" is called "surgery" and is discussed in Section 2.1; we also define normal maps and show that surgery below the middle dimension is always possible. In Section 2.2, we restrict to even-dimensional manifolds and discuss the obstruction to carrying out the third step of the surgery program.

2.1 Surgery below the middle dimension

For simplicity, we will assume that M and X are manifolds, with M closed. [1] We wish to know whether a map $f \colon M \to X$ can be modified into a homotopy equivalence. By Whitehead's theorem (Theorem 1.2.16), it is sufficient to kill all homotopy groups $\pi_n(f)$. While Corollary 1.2.18 described a procedure to kill these homotopy groups by attaching cells, the outcome need not be a manifold. In a nutshell, surgery is the analogue of "attaching cells", but within the category of manifolds.

Definition 13. Let M^n be a closed manifold and let $\phi \colon S^r \times D^{n-r} \to M$ be an embedding. The operation of removing the interior of the image of ϕ and attaching $D^{r+1} \times \partial D^{n-r}$ is called *surgery*.

[1] In general, M need not be closed and X is assumed to be a *Poincaré complex.*

The resulting manifold M' is called the *effect of the surgery*:

$$M' = M \setminus \phi(S^r \times D^{n-r}) \cup_\phi D^{r+1} \times \partial D^{n-r}.$$

The *trace of the surgery* is the following cobordism between M and M':

$$W := M \times [0,1] \cup_\varphi D^{r+1} \times D^{n-r}.$$

We say that W is obtained from $M \times [0,1]$ by *adding an n-dimensional $(r+1)$-handle* via the *attaching map ϕ*.

Let us illustrate Definition 13 with a low dimensional example.

Example 2.1.1. We describe 0-surgery and 1-surgery on a surface M^2. The effect of 0-surgery is "to add a tube to M": namely, we remove two disks from M (i.e. $S^0 \times D^2$) and add the tube $D^1 \times S^1$. The effect of 1-surgery is to remove a cylinder $S^1 \times D^1$ and to add two disks $S^0 \times D^2$. Observe that if we do 1-surgery on a neighborhood of the meridian of the tube added by a 0-surgery, then we recover the initial surface.

The next sections will show that surgery is really the "manifold analogue" of attaching cells to a CW complex. In Subsection 2.1.1, we further discuss surgery and introduce normal maps, while in Subsection 2.1.3 in discuss surgery below the middle dimension.

2.1.1 Surgery and its effect on homotopy groups

The aim of this subsection is to improve our understanding of surgery and its effect on (relative) homotopy groups. We also motivate and introduce the notion of a normal map. References include [LÖ2, Ran02, Wal70, GS99, Wal16, CLMa].

An embedding $\varphi\colon S^r \times D^{n-r} \to M^n$ determines an embedding $\varphi_0\colon S^r \times \{0\} \to M$. In order to discuss the degree to which φ_0 determines φ, we introduce some terminology: a *framing* of an embedding $\varphi_0\colon S^r \to M$ with trivial normal bundle $\nu(\varphi_0)$ consists of an identification of $\nu(\varphi_0)$ with $S^r \times \mathbb{R}^{n-r}$.

We establish that a framed embedding $S^r \to M^n$ give rise to embeddings $S^r \times D^{n-r} \to M$.

Proposition 2.1.2. *Let M be an n-manifold. A framed embedding $\varphi_0\colon S^r \hookrightarrow M^n$ determines an embedding $\varphi\colon S^r \times D^{n-r} \hookrightarrow M$ such that $\varphi \times 0 = \varphi_0$. In particular, in order to perform a surgery on M, we need only specify a framed embedding $\varphi_0\colon S^r \hookrightarrow M$.*

Proof. This is essentially Remark 1.2.5, whose content we recall. The tubular neighborhood theorem (Theorem 1.2.4) ensures the existence of an embedding $\nu(\varphi_0) \hookrightarrow X$. Combining this with the choice of a framing $\nu(\varphi_0) \cong S^r \times \mathbb{R}^{n-r}$ therefore produces an embedding $S^r \times \mathbb{R}^{n-r} \hookrightarrow X$. The proposition now follows by taking the unit disc bundle. □

In fact, the embedding $\varphi\colon S^r \times D^{n-r} \hookrightarrow M$ described in Proposition 1.2.11 is determined up to isotopy by the data of φ and the framing f: if $\varphi'\colon S^r \times D^{n-r} \to M^n$ is another embedding with $\varphi'|_{\partial D^r \times \{0\}} = \varphi_0$ and $T\varphi'|_{0 \times T_0 D^{n-k}} = f^{-1}$, then φ' is isotopic to φ. Since the trace of the surgery $(M \times [0,1]) \cup_\varphi h$ only depends on the isotopy class of φ (e.g [GS99, Chapter 4.1]), we deduce that the effect of surgery is determined by the framed embedding (φ_0, f).

The previous discussion implies that if we do surgery along φ, then we obtain a homotopy class $[\varphi_0] \in \pi_r(M)$. We use this observation to investigate the effect of surgery on the homotopy groups of M.

Proposition 2.1.3. *Let W^{n+1} be the trace of an r-surgery obtained on M^n along φ_0 and let M' be the effect of this surgery. The following statements hold:*

1. *The inclusion $M \to W$ is r-connected and $\pi_r(W) = \pi_r(M)/\langle[\varphi_0]\rangle$.*

2. *The inclusion $M' \to W$ is $n-r-1$-connected.*

In particular, if $k < r$ and $k < n-r-1$, then $\pi_k(M) \cong \pi_k(M')$.

Proof. We first observe that W has the homotopy of $M \cup e^{r+1}$: there is a deformation retraction of $W = M \times [0,1] \cup_\varphi D^{r+1} \times D^{n-r}$ onto $M \times [0,1] \cup_{\varphi|_{S^r \times \{0\}}} D^{r+1} \times \{0\}$. Therefore, up to homotopy, attaching an $(r+1)$-handle is the same as attaching an $(r+1)$-cell. The first assertion follows from Corollary 1.2.17. To prove the second assertion, observe that the manifold M is obtained from $M' = M \setminus (\partial D^{r+1} \times D^{n-r}) \cup D^{r+1} \times \partial D^{n-r}$ by performing an $(n-r-1)$-surgery along $D^{r+1} \times \partial D^{n-r}$. The second assertion is now a consequence of the first, and the last is a consequence of the two first. This concludes the proof of the proposition. □

Proposition 2.1.3 shows that surgery on an embedding $\varphi_0 \colon S^r \to M$ kills $[\varphi_0] \in \pi_r(M)$. Consequently, we can kill represent $x \in \pi_n(M)$ provided x is represented by a framed embedding. However, what we really want is to kill homotopy groups of a map. Consequently, we fix a smooth map $f \colon M \to X$ and suppose we do surgery on an embedding $\varphi \colon S^r \times D^{n-r} \to M$. Observe that f induces a map $f' \colon M' \to X$ if φ extends to an embedding $\Phi \colon D^{r+1} \times D^{n-r} \to X$ that makes the following diagram commutes:

$$\begin{array}{ccc} S^r \times D^{n-r} & \xrightarrow{\varphi} & M \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ D^{r+1} \times D^{n-r} & \xrightarrow{\Phi} & X. \end{array} \tag{2.1}$$

Indeed, given such an extension Φ of φ, we can define $f' \colon M' \to X$ as $f \cup \Phi_|$, and in fact f extends to a map $F := f \cup \Phi \colon W \to X$ on the trace of the surgery along φ. Set $\Phi_0 := \Phi|_{D^{r+1} \times \{0\}}$ and note that the pair (φ_0, Φ_0) defines a homotopy class in $\pi_r(f)$ which we call the *homotopy class defined by the pair* (φ, Φ).

The next proposition describes the effect of surgery on the relative homotopy groups of f.

Proposition 2.1.4. *Let $f \colon M^n \to X$ be a map and fix an integer $r \geq 0$ so that $2(r+1) \leq n$. Let $\varphi \colon S^r \times D^{n-r} \to M$ be an embedding with an extension to a map $\Phi \colon D^{r+1} \times D^{n-r} \to X$ making (2.1) commute. Using $x \in \pi_{r+1}(f)$ to denote the homotopy class defined by (φ, Φ) and f' the result of surgery on φ, then*

$$\pi_{r+1}(f') = \pi_{r+1}(f)/\langle x \rangle \text{ and } \pi_j(f') = \pi_j(f) \text{ for } j \leq r.$$

Proof. We saw in the proof of Proposition 2.1.3 that up to homotopy, the trace W of the surgery along φ is obtained both by adding an $(r+1)$-cell to $M \times [0,1]$ and by adding an $(n-r)$-cell to $M' \times [0,1]$. As we explained above, the assumptions of the lemma guarantee that f extends to a map $F := f \cup \Phi \colon W \to X$. Combining these two observations, Corollary 1.2.19 implies that $\pi_{r+1}(F) = \pi_{r+1}(f)/\langle x \rangle$, as well as $\pi_j(f) \cong \pi_j(F)$ for $j \leq r$ and $\pi_j(f') \cong \pi_j(F)$ for $j \leq n-r-1$. It follows that $\pi_j(f) \cong \pi_j(f')$ for $2j \leq n-1$.

We can now conclude. Recall that we assume that $2(r+1) \leq n$ and that the statements of the proposition involve π_j with $j \leq r+1$. For $j \leq r$, we have $2j \leq 2r \leq n-2 \leq n-1$ and so in this case, we have $\pi_j(f) \cong \pi(f')$. For $j = r+1$, we have $\pi_{r+1}(F) = \pi_{r+1}(f)/\langle x \rangle$ and $\pi_{r+1}(f') \cong \pi_{r+1}(F)$. It follows that $\pi_{r+1}(f') = \pi_{r+1}(f)/\langle x \rangle$ and this concludes the proof of the proposition. □

2.1.2 Motivating the definition of a normal map

Next, we wish to use inductively Proposition 2.1.4 in order to kill elements of $\pi_r(f)$ (with the hope of converting f into a homotopy equivalence). To carry this out, we must be able to represent $x \in \pi_r(f)$ by a pair (φ_0, Φ_0) with $\varphi_0 \colon S^r \to M$ a framed embedding. Arranging that φ_0 is an embedding follows from classical results of Whitney (that we will state in Subsection 2.1.3 below) and we therefore focus the framing issue. References include [LÖ2, Ran02, Wal70, Wal16, CLMa].

We first forget about Φ_0 and establish some necessary and sufficient conditions for an embedding $S^r \to M^n$ to have (stably) trivial normal bundle.

Lemma 2.1.5. *Assume $\varphi_0 \colon S^r \to M^n$ is an embedding.*

1. *If $\nu(\varphi_0)$ is trivial, then $\varphi_0^*(TM)$ is trivial.*
2. *If $\varphi_0^*(TM)$ is stably trivial, then $\nu(\varphi_0)$ is stably trivial.*
3. *If $2r < n$ and $\nu(\varphi_0)$ is stably trivial, then $\nu(\varphi_0)$ is trivial.*

Proof. The first two assertions follow from the fact that $\varphi_0^*(TM) = \nu(\varphi_0) \oplus TS^r$ (Example 1.2.26) together with the fact that TS^r is stably trivial (Example 1.2.26)

We now prove the third assertion. Recall from the third item of Example 1.2.10 that stable isomorphism classes of bundles over S^r are classified by $[S^r, BO] = \pi_r(BO)$. Since we assumed $\nu(\varphi_0)$ to be stably trivial, $\nu(\varphi_0)$ is trivial in $\pi_r(BO)$. On the other hand, $\nu(\varphi_0)$ is a rank $n-r$ vector bundle and therefore defines an element in $\pi_r(BO(n-r))$ (recall Theorem 1.2.23). It is therefore enough to show that for $2r < n$, the kernel $\ker(\pi_r(BO(n-r)) \to \pi_r(BO))$ is trivial. This further reduces to showing that for $2r < n$, we have $\pi_r(BO(n-r)) \cong \pi_r(BO(n-r+k))$ for $k > 0$. Corollary 1.2.20 implies that $\pi_j(S^{n-r}) = 0$ for $j < n-r$. Using the assumption $2r < n$, the long exact sequence of the fibration $S^{n-r} \to BO(n-r+1) \to BO(n-r)$ gives an isomorphism $\pi_r(BO(n-r)) \cong \pi_r(BO(n-r+1))$. The procedure can then be iterated. This concludes the proof of the third assertion [2] and therefore the proof of the proposition. □

Lemma 2.1.5 shows that the task of framing an embedding φ_0 is therefore intimately tied with the tangent bundle TM. For $f \colon M \to X$ and $[(\varphi_0, \Phi_0)] \in \pi_r(f)$, the next proposition provides a sufficient condition for the triviality of $\varphi_0^*(TM)$.

Lemma 2.1.6. *Assume that $TM \cong f^*(\xi)$ for some vector bundle over ξ over X. If $x \in \pi_{r+1}(f)$ is represented by a pair (φ, Φ) with $\varphi_0 \colon S^r \to M$ an embedding, then $\varphi_0^*(TM)$ is trivial, and therefore $\nu(\varphi_0)$ is stably trivial.*

Proof. First, note that our assumption implies that $\varphi_0^*(TM) \cong \varphi_0^* f^*(\xi) = (f \circ \varphi_0)^*(\xi)$. Next, observe that $f \circ \varphi_0 \colon S^r \to X$ is nullhomotopic since Φ_0 provides an extension to D^{r+1} (recall (2.1)). Since $f \circ \varphi_0$ is nullhomotopic, we deduce that $\varphi_0^*(TM) = (f \circ \varphi_0)^*(\xi)$ is trivial. The last assertion follows from Lemma 2.1.5. This concludes the proof of the lemma. □

Given $x \in \pi_r(f)$, Lemma 2.1.6 establishes that if $TM = f^*(\xi)$, then $\nu(\varphi_0)$ is stably trivial (and even trivial if $2r < n$ by the last item of Lemma 2.1.6). Note however that the condition $TM = f^*(\xi)$ is overkill: it implies that $\varphi_0^*(TM)$ is trivial, while only stable triviality is needed (recall the second item of Lemma 2.1.5).

The next definition provides a sharper condition on f that nevertheless ensures that embeddings of spheres can be stably framed.

Definition 14. A *normal map* (f, b) consists of a map $f \colon M \to X$ together with a stable isomorphism $f^*(\xi) \cong TM$ i.e. a bundle map $b \colon TM \oplus \epsilon^u \to \xi \oplus \epsilon^v$ covering f:

$$\begin{array}{ccc} TM \oplus \epsilon^u & \xrightarrow{b} & \xi \oplus \epsilon^v \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X. \end{array}$$

Before returning to framed embeddings, we provide a common reformulation of Definition 14.

[2] Observe that this statement was about vector bundles over spheres; we only stated it for the normal bundle of an embedding for the sake of concreteness.

Exercise 2.1.7. A normal map can also be thought of as a pair (f, b), where b is a stable trivialisation of $TM \oplus f^*(\eta)$ for some bundle η over X. To see this, use the fact that for every vector bundle η over a compact space X, there is a bundle ξ over X so that $\xi \oplus \eta$ is trivial.

The next result shows that a normal map f control the normal bundle of elements in $\pi_r(f)$.

Proposition 2.1.8. *Let $f\colon M^n \to X$ be a normal map. If $x \in \pi_r(f)$ can be represented by a pair (φ_0, Φ_0) with φ_0 an embedding, then*

1. *$\nu(\varphi_0)$ is stably framed.*
2. *$\nu(\varphi_0)$ is framed, provided we additionally assume that $2r < n$.*

Proof. The second assertion follows from the first thanks to the third item of Lemma 2.1.5. We now prove the first assertion. As in the proof of Lemma 2.1.5, we have $\varphi_0^*(TM) \cong \nu(\varphi_0) \oplus \epsilon$, and it is therefore enough to show that $\varphi_0^*(TM)$ is stably framed. Since f is a normal map (and using Exercise 2.1.7), we have a stable framing of $TM \oplus f^*(\xi)$. Pulling back by φ_0, we obtain a stable framing of $\varphi_0^*(TM) \oplus \varphi_0^*(f^*(\xi))$. It therefore only remains to show that $\varphi_0^*(f^*(\xi))$ is trivial. Since $[(\varphi_0, \Phi_0)]$ belongs to $\pi_{r+1}(f)$, we deduce that $f \circ \varphi_0$ factors through D^{r+1} and is therefore nullhomotopic. This implies that $\varphi_0^*(f^*(\xi))$ is trivial, concluding the proof of the proposition. □

At this point, we know that below the middle dimension, we can use the normal data of f to represent classes of $\pi_r(f)$ by framed embedded spheres. As a consequence, we can perform surgery on one element. To repeat the process, we need to ensure that the result $f'\colon M' \to X$ of surgery on a map $f\colon M \to X$ is still a normal map.

Since this verification will occupy most of next section, we start with a reality check: we show that homotopy equivalences are normal maps, (it is therefore conceivable that successive surgeries on a normal map will lead to a homotopy equivalence).

Lemma 2.1.9. *If $f\colon M \to X$ is a homotopy equivalence, then it is a normal map .*

Proof. Since the manifold M is a compact, a theorem of Whitney ensures the existence of an embedding $\iota\colon M \to \mathbb{R}^n$ for some large n. Let $g\colon X \to M$ be the homotopy inverse of f and consider the vector bundle $\xi := g^*(\nu(i_n))$. Using the definition of ξ, the fact that $g \circ f \sim \mathrm{id}$, the fact that homotopic maps induce isomorphic pullback bundles, as well as Example, we obtain

$$TM \oplus f^*(\xi) = TM \oplus (g \circ f)^*(\nu(i_n)) = TM \oplus \nu(i_n) = \epsilon^n,$$

This shows that f is a normal map and therefore concludes the proof of the lemma. □

Observe that in the proof of Lemma 2.1.9, the dimension of the euclidean space $\mathbb{R}^n$ in which we embedded M was immaterial. This suggests another equivalent definition of a normal map which makes use of the so-called "stable normal bundle". To define this concept, note that given any two embeddings $\iota_1\colon M \hookrightarrow \mathbb{R}^{n_1}$ and $\iota_1\colon M \hookrightarrow \mathbb{R}^{n_w}$, there exists $N \geq n_1, n_2$ such that the compositions $\iota_1\colon M \hookrightarrow \mathbb{R}^{n_1} \subset \mathbb{R}^N$ and $\iota_2\colon M \hookrightarrow \mathbb{R}^{n_2} \subset \mathbb{R}^N$ are regular homotopic. This implies that the classifying maps $\nu(i_1)\colon M \to BO(N)$ and $\nu(i_2)\colon M \to BO(N)$ are homotopic. As a consequence, we obtain a well defined map $\nu\colon M \to BO$, and the corresponding "stable bundle" νM is called the *stable normal bundle* of M.

The next exercise provides a definition of normal maps which relies on the stable normal bundle instead of tangent bundles.

Exercise 2.1.10. A normal map can be thought of as a pair (f, b), where $b\colon \nu M \to \xi$ is a bundle map covering $f\colon M \to X$ for some stable bundle ξ. To show this, use the fact that TM and νM are stable inverses.

2.1.3 The surgery step and surgery below the middle dimension

At this point, we know that if $f\colon M^n \to X$ is a normal map and $x \in \pi_{r+1}(f)$ is represented by a pair (φ, Φ) with $\varphi\colon S^r \times D^{n-r} \to M$ an embedding, then f extends to a map $F\colon W \to X$ on the trace of the surgery and $\pi_{r+1}(F) = \pi_{r+1}(f)/\langle x\rangle$. In order to repeat the process, we must ensure that on the effect M' of the surgery, the resulting map $f'\colon M' \to X$ is still normal. References include [LÖ2, Ran02, Wal70, Wal16, CLMa].

The first step is to show that for $r \leq n-2$, a class $x \in \pi_{r+1}(f)$ canonically determines a regular homotopy class of immersions $S^r \times D^{n-r} \to M$. This dispenses with the need of systematically assuming that "$x \in \pi_r(f)$ is represented by (φ, Φ) with φ_0 an immersion".

Proposition 2.1.11. *Let $f\colon M^n \to X$ be a normal map and let x be an element of $\pi_{r+1}(f)$.*

1. *The relative homotopy class x determines a regular homotopy class $[\varphi_x\colon S^r \times D^{n-r} \looparrowright M]$ of immersions provided $r \leq n-2$.*
2. *If $2r < n$, then the regular homotopy class $[\varphi_x]$ contains an embedding.*

Furthermore, an immersion $g\colon S^r \times D^{n-r} \to M$ belongs to $[\varphi_x]$ if and only if g satisfies the following three conditions:

- *There is a map $G\colon D^{r+1} \times D^{n-r}$ such that g can be fitted into the diagram*

$$\begin{array}{ccc} S^r \times D^{n-r} & \xrightarrow{g} & M \\ \big\downarrow & & \big\downarrow f \\ D^{r+1} \times D^{n-r} & \xrightarrow{G} & X. \end{array} \tag{2.2}$$

- $[(g|_{S^r\times 0}, G_{D^{r+1}\times 0})] = x$ *in* $\pi_{r+1}(f)$
- *The diagram (2.2) is covered by diagram of vector bundle maps*

$$\begin{array}{ccc} T(S^r \times D^{n-r}) \oplus \epsilon^a & \xrightarrow{\overline{g}} & TM \oplus \epsilon^a \\ \big\downarrow & & \big\downarrow \\ T(D^{r+1} \times D^{n-r}) \oplus \epsilon^{a-1} & \xrightarrow{\overline{G}} & \xi \oplus \epsilon^a, \end{array} \tag{2.3}$$

where the rightmost map is induced by the bundle data of the normal map f and the leftmost map arises from the outer normal vector field of $S^r = \partial D^{r+1}$.

Proof. Choose a pair (φ_0, Φ_0) representing the element $x \in \pi_{k+1}(f)$. Assume that φ_0 and Φ_0 are smooth (they need not be immersions) and extend these maps to obtain a commutative diagram

$$\begin{array}{ccc} S^r \times D^{n-r} & \xrightarrow{\varphi} & M \\ \big\downarrow i & & \big\downarrow f \\ D^{r+1} \times D^{n-r} & \xrightarrow{\Phi} & X. \end{array}$$

Since f is a normal map, we have stable trivialisation of $TM \oplus f^*(\xi)$. Since $D^{r+1} \times D^{n-r}$ is contractible, we have a trivialisation of $T(D^{r+1} \times D^{n-r}) \oplus \Phi^*(\xi)$. These stable trivilisations respectively induce bundle maps $TM \oplus \epsilon^a \to \xi \oplus \epsilon^a$ and $T(D^{r+1} \times D^{n-r}) \oplus \epsilon^a \to \xi \oplus \epsilon^a$ (for large enough a) that respectively cover f and Φ. To get a map $T(S^r \times D^{n-r}) \oplus \epsilon^a \to TM \oplus \epsilon^a$ covering φ, we must stably trivialize $T(S^r \times D^{n-r}) \oplus \varphi^*(TM)$. We stably trivialize the first summand by adding ϵ in order to obtain the trivial bundle $T(D^{r+1} \times D^{n-r})$. To stably trivialize the second summand, first observe that

$$\varphi^*(TM \oplus f^*(\xi)) = \varphi^*(TM) \oplus \varphi^*(f^*(\xi)) = \varphi^*(TM) \oplus i(\Phi^*(\xi)).$$

We stably trivialize $\varphi^*(TM \oplus f^*(\xi))$ using the normal data of f and, for $i(\Phi^*(\xi))$ we once again use that $D^{r+1} \times D^{n-r}$ is itrivial. We have therefore obtained our stable trivialisation of $T(S^r \times D^{n-r}) \oplus \varphi^*(TM)$. By construction, all the resulting bundle maps then fit into the desired commutative diagram:

$$\begin{array}{ccc} T(S^r \times D^{n-r}) \oplus \epsilon^a & \longrightarrow & TM \oplus \epsilon^a \\ \downarrow & & \downarrow \\ T(D^{r+1} \times D^{n-r}) \oplus \epsilon^{a-1} & \longrightarrow & \xi \oplus \epsilon^a. \end{array}$$

One can argue that the top map is a bundle monomorphism and therefore $x \in \pi_{r+1}(f)$ determines (a homotopy class of) a stable bundle monomorphism $T(S^r \times D^{n-r}) \oplus \epsilon^a \to TM \oplus \epsilon^a$. Since we assumed that $r \leq n-2$, a stable version of the Smale-Hirsch theorem (recall Theorem 1.3.1) states that there is an isomorphism

$$\pi_0(\mathrm{Imm}(T(S^r \times D^{n-r}), TM) \cong \mathrm{colim}_{a \to \infty} \pi_0(\mathrm{Mono}(T(S^r \times D^{n-r}) \oplus \epsilon^a, TM \oplus \epsilon^a)).$$

As a consequence, the relative homotopy class $x \in \pi_{r+1}(f)$ determines a regular homotopy class of immersions $\varphi_x \colon S^r \times D^{n-r} \to M$, concluding the proof of the first assertion.

We sketch the proof of the last assertions. If $2r < n$, then the *strong Whitney embedding theorem* states that any smooth map $N^r \to M^n$ is homotopic to an embedding. The second assertion follows by applying this theorem to extract an embedding from the regular homotopy class $[\varphi_x]$. We move on to the last assertion. The construction of φ_x shows that if g is regular homotopic to φ_x, then it must satisfy the three listed requirements. To prove the converse, the idea is that the commutativity of the diagram in (2.3) determines the stable homotopy class of the bundle monomorphism $\overline{g}$ and therefore the regular homotopy class of g, details can be found in [CLMa, Theorem 3.93]. This concludes the proof of the proposition. □

The next proposition shows that the normal data extends across the trace of a surgery.

Proposition 2.1.12. *Let $(f, b) \colon M \to X$ be a normal map and let $x \in \pi_{r+1}(f)$ be a relative homotopy class. If $\varphi \colon S^r \times D^{n-r} \to M$ is an embedding that lies in the regular homotopy class determined by x, then (f, b) induces a normal map on the trace W of the surgery along φ.*

Proof. Since φ belongs to the regular homotopy class determined by x, Proposition 2.1.11 ensures the existence of a map $\Phi \colon D^{r+1} \times D^{n-r} \to X$ that fits into the following diagram:

$$\begin{array}{ccc} S^r \times D^{n-r} & \xrightarrow{\varphi} & M \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f} \\ D^{r+1} \times D^{n-r} & \xrightarrow{\Phi} & X. \end{array}$$

This diagram implies that f extends to a map $F := f \cup \Phi \colon W \to X$ on the trace of the surgery. It remains to shows that F is normal i.e. that the bundle map $b \colon TM \oplus \epsilon^a \to \xi \oplus \epsilon^a$ extends to a bundle map $B \colon TW \oplus \epsilon^b \to \xi \oplus \epsilon^b$ covering F. The bundle map b certainly extends to a bundle map $T(M \times [0,1]) \oplus \epsilon^a \to X \oplus \epsilon^a$. Since $D^{r+1} \times D^{n-r}$ is contractible, we also obtain a bundle map $T(D^{r+1} \times D^{n-r}) \oplus \epsilon^b \to \xi \oplus \epsilon^b$, covering Φ (just as in the proof of Proposition 2.1.11). In order to obtain a well defined map on TW, it is necessary and sufficient that these bundle maps agree over $S^r \times D^{n-r}$ i.e. that the following diagram commutes:

$$\begin{array}{ccc} T(S^r \times D^{n-r}) \oplus \epsilon^a & \longrightarrow & TM \oplus \epsilon^a \\ \downarrow & & \downarrow \\ T(D^{r+1} \times D^{n-r}) \oplus \epsilon^{a-1} & \longrightarrow & \xi \oplus \epsilon^a. \end{array}$$

Proposition 2.1.11 ensures that this commutativity follows from the fact that φ belongs to the regular homotopy class determined by x. This concludes the proof of the proposition. □

Motivated by Proposition 2.1.12, we make the following definition.

Definition 15. For $i = 1, 2$, two normal maps $(f_i, b_i)\colon M_i^n \to X$ are *normal bordant* if there exists a cobordism W between M_1 and M_2, and a normal map $(F, B)\colon W \to X$ such that F restricts to f_i on M_i, and such that the stable trivialisation B of $TW \oplus F^*(\xi)$ restricts the stable trivialisation b_i of $TM_i \oplus f_i^*(\xi)$ over M_i.

Using this terminology, if we perform surgery along a normal map $f\colon M \to X$ along an embedding satisfying the condition of Proposition 2.1.12, then the effect $f'\colon M' \to X$ of the surgery is normal bordant to f.

We can now prove the main result of this section, namely *surgery below the middle dimension.*

Theorem 2.1.13. *If $2r < n$, then every normal map $(f, b)\colon M^n \to X$ is normal bordant to an r-connected normal map.*

Proof. Let $x \in \pi_{r+1}(f)$ be a relative homotopy class. Using Proposition 2.1.11, x determines a regular homotopy class $[\varphi_x]$ of immersions. Since $2r < n$, this regular homotopy class contains an embedding $\varphi\colon S^r \times D^{n-r} \to M$. We can therefore perform surgery on φ, and Proposition 2.1.12 shows that we obtain a normal map $f'\colon M' \to X$ on the effect of the surgery. Proposition 2.1.4 implies that this process does not affect $\pi_k(f)$ for $k \leq r$ but replaces $\pi_{r+1}(f)$ by $\pi_{r+1}(f)/\langle x\rangle$. Using these facts, we proceed inductively on r. Since $\pi_{r+1}(f)$ is a finitely generated $\mathbb{Z}[\pi_1(X)]$-module [CLMa, Lemma 3.81], we can kill the $\mathbb{Z}[\pi_1(X)]$-generators of $\pi_{r+1}(f)$ one by one. Since $\pi_k(f) = \pi_k(f')$ for $k \leq r$, once we have killed a certain relative homotopy class, it remains dead. This concludes the proof of the theorem. □

2.2 The even-dimensional surgery obstruction

Let us recall the current situation: our goal is to determine whether a fixed normal map $f\colon M \to X$ is normal bordant to a map $f'\colon M' \to X$ such that $\pi_j(f) = 0$ for each j: Whitehead's theorem (Theorem 1.2.16) would then ensure that f' is a homotopy equivalence. Theorem 2.1.13 shows that we can assume that f is r-connected with $2r < n$. As we shall in see in Proposition 2.2.2 below, for $2r$-dimensional manifolds, we need only focus on killing $\pi_{r+1}(f)$, and this will be the goal of the remainder of the section.

In order to embark on this venture, we need to collect a few facts from homology theory. If the reader is not familiar with the subject, he need not worry: the "degree one" condition will not play a major role in the proofs below and homology will only used in these introductory paragraphs.

Remark 2.2.1. Given closed n-manifolds manifolds M, X, a map $f\colon M \to X$ is of *degree* k if $f_*[M] = k[X]$, where $[M] \in H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ and $[X] \in H_n(X; \mathbb{Z}) \cong \mathbb{Z}$ denote the respective fundamental classes. Because of the two following properties, we shall from now on assume that all our normal maps are of degree 1:

1. if f is a homotopy equivalence, then it has degree ± 1;
2. if f and f' are bordant, then they have the same degree.

The first assertion follows from the multiplicativity of degrees and its homotopy invariance: if $g\colon X \to M$ is the homotopy inverse of f, then $1 = \deg(\mathrm{id}_X) = \deg(g \circ f) = \deg(g)\deg(f)$. To prove the second property, note that if (W^{n+1}, M, M') is a cobordism and $F\colon W \to X$ extends f and f', then $0 = F([W]) = f_*([M]) - f'_*([M'])$; here the first equality follows from the fact that X is n-dimensional.

The next proposition shows that for an even dimensional manifold M^{2k}, our only remaining task is to kill $\pi_{k+1}(f)$. The proof requires some familiarity with homology theory and covering spaces; if the reader is not familiar with these topics, he may read the statement of the proposition and skip its proof.

Proposition 2.2.2. *Let $f\colon M^{2k} \to X$ be a degree one map. If f is $(k+1)$-connected, then f is a homotopy equivalence.*

Proof. Let $\widetilde{M}$ and $\widetilde{X}$ be the universal covers of M and X. Since f induces an isomorphism on fundamental groups, it lifts to a map $\widetilde{f}\colon \widetilde{M} \to \widetilde{X}$. Recall from Proposition 1.2.12 that we have $\pi_j(\widetilde{M}) = \pi_j(M)$ and $\pi_j(\widetilde{X}) = \pi_j(X)$ for $j \geq 2$. Using the long exact sequences of (X, M) and $(\widetilde{X}, \widetilde{M})$ (recall the second item of Remark 1.2.15) and the five lemma we deduce that for all j we have the isomorphisms

$$\pi_j(\widetilde{f}) = \pi_j(f).$$

Since $\widetilde{M}$ is simply connected, the relative Hurewicz theorem implies that the first non-zero $\pi_n(\widetilde{f})$ is equal to the first non-zero $H_n(\widetilde{f})$. It is therefore enough to show that the latter all vanish: this will imply that $\pi_j(f) = \pi_n(\widetilde{f}) = 0$ for all n, and Whitehead's theorem (Theorem 1.2.16) will allow us to conclude that f is a homotopy equivalence.

We now show that all $H_j(\widetilde{f})$ vanish. Consider the *surgery kernels* $K_j(f) := \ker(\widetilde{f}_*\colon H_j(\widetilde{M}) \to H_j(\widetilde{X}))$ and $K^j(f) := \operatorname{coker}(\widetilde{f}^*\colon H_j(\widetilde{X}) \to H_j(\widetilde{M}))$. Using the long exact sequence of the pair (M, X) and Poincaré duality for M and X, we deduce that $\widetilde{f}$ is a surjection and therefore $K_j(f) \cong H_{j+1}(\widetilde{f})$. Combining this fact with Hurewicz's theorem, we deduce that for $j \leq k$, we have

$$0 = \pi_{j+1}(\widetilde{f}) \cong H_{j+1}(\widetilde{f}) = K_j(f). \tag{2.4}$$

Since $K_j(f) = 0$ for $j \leq k$, some homological algebra shows that $K^j(f) = 0$ for $j \leq k$. Since f is of degree 1, we can show that $\widetilde{f}_*$ is in fact a split surjection and a quick diagram chase proves that Poincaré duality restricts to the surgery kernels. Combining these observations, it follows that $H_{j+1}(\widetilde{f}) = K_j(f) = 0$ for all j, as desired. This concludes the proof of the proposition. □

Let $f\colon M^{2k} \to X$ be a degree one normal map. To show that f is normal bordant to a homotopy equivalence, Theorem 2.1.13 and Proposition 2.2.2 show that we can assume f to be k-connected and study $\pi_{k+1}(f)$. Thanks to Proposition 2.1.11, elements of $\pi_{k+1}(f)$ determine regular homotopy classes of framed immersions $S^k \to M$. If we could regular homotope such framed immersions to framed embeddings, then we would perform surgery, and f would be normal bordant to a homotopy equivalence.

2.2.1 The intersection and self intersection numbers of immersed spheres

Given a k-connected degree one normal map $f\colon M^{2k} \to X$, our goal is to decide whether an element $x \in \pi_{k+1}(f)$ can be represented by a pair (φ, Φ) with $\varphi\colon S^k \to M$ a framed embedding. Since we saw in Proposition 2.1.11 that x determines the regular homotopy class of a framed immersion, the goal of this subsection is to decide whether an immersion is regular homotopic to an embedding. The main tools in this process are the intersections and self-intersections numbers. References include [Ran02, Section 11], [Wal70, page 57] and [CLMa].

Fix a manifold M^{2k}. A *double point* between two maps $\varphi_1\colon S^k \to M$ and $\varphi_2\colon S^k \to M$ consists of a pair $(x_1, x_2) \in S^k \times S^k$ such that $\varphi_1(x_1) = \varphi_2(x_2)$. We often refer to $p := \varphi_1(x_1) = \varphi_2(x_2)$ as the double point. A double point (x_1, x_2) is *transverse* if the following linear map is an isomorphism

$$(T_{x_1}\varphi_1, T_{x_2}\varphi_2)\colon T_{x_1}N \oplus T_{x_2}N \to T_pM.$$

Given two immersions $\varphi_i\colon S^k \to M$ that intersect transversely in a finite set of double points (as depicted schematically in Figure 2.1 below), our goal is to associate to each intersection point p a sign ϵ_p and an element $g_p \in \pi_1(M)$. The sign ϵ_p is the sign of the intersection of $\varphi_1(S^k)$ and $\varphi_2(S^k)$ at p; we briefly recall this concept.

Remark 2.2.3. Assume that $\varphi_1, \varphi_2\colon S^k \to M^{2k}$ are two immersions that intersect transversely at $p = \varphi_1(x_1) = \varphi_2(x_2)$. Pick bases $v_1, \ldots, v_k$ of $T_{x_1}S^k$ and $w_1, \ldots, w_k$ for $T_{x_2}S^k$ so that

$\mathscr{B}_p := (T_{x_1}\varphi_1(v_1), \dots, T_{x_1}\varphi_1(v_k), T_{x_2}\varphi_2(w_1), \dots, T_{x_2}\varphi_2(w_k))$ is a basis of T_pM. The *algebraic intersection* of φ_1 and φ_2 at p is defined as the sign of the ordered basis $\mathscr{B}_p$.

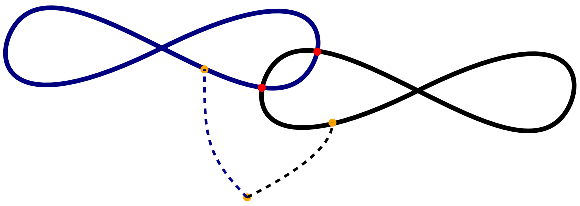


Figure 2.1: A sketch of two based immersions $\varphi_1, \varphi_2 \colon S^k \to M^{2k}$ that intersect transversely in two double points.

In order to describe the group element $g_p \in \pi_1(M)$, we need to fix basepoints m in M^{2k} and s in S^k and introduce some terminology.

Definition 16. A *based immersion* is a pair (φ, γ) with $\varphi \colon S^k \to M$ an immersion, and γ a path from m to $\varphi(s)$.

From now on, all immersions are assumed to be based. To each double point between φ_1 and φ_2, the element $g_p \in \pi_1(M)$ is defined as follows: for $i = 1, 2$ choose paths $\eta_i \subset \varphi_i(S^k)$ from $\varphi_i(s)$ to p and set

$$g_p := \gamma_2 \cdot \eta_2 \cdot \overline{\eta}_1 \cdot \overline{\gamma}_1.$$

In other words, g_p is loop at m that starts along the path γ_2 to the basepoint $\varphi_2(s)$ of $\varphi_2(S_2)$, round $\varphi_2(S_2)$ to p (using η_2), round S_1 to $\varphi_1(s)$ (using the reverse path $\overline{\eta}_1$) and back along the path γ_1 to m. This is illustrated in Figure 2.2 below. Given a double point p, we shall sometimes refer to $\varepsilon_p g_p$ as the *index* of p.

The first main definition of this subsection is the following.

Definition 17. The *intersection number* of two based immersions $\varphi_1 \colon S^k \to M^{2k}$ and $\varphi_2 \colon S^k \to M^{2k}$ is obtained by summing the indices of all double points of φ_1 with φ_2:

$$\lambda(\varphi_1, \varphi_2) = \sum_p \epsilon_p g_p \in \mathbb{Z}[\pi_1(M)].$$

The fact that λ is independent of the choice of η_1, η_2 follows because S^k is simply connected for $k \geq 2$. In fact, the crucial point is that for $i = 1, 2$ the subgroup $\varphi_{i*}(\pi_1(S^k)) \subset \pi_1(M)$ is trivial (leading to a definition for the $k = 1$ case). In turn, this is equivalent to asking that the immersions φ_i lifts to the universal cover of M.

Remark 2.2.4. More generally, one can define an intersection number for immersions $\varphi_i \colon N_i \to M$ that lift to the universal cover (resp. G-covers) of M provided $\varphi_{i*}(\pi_1(N_i))$ is trivial in $\pi_1(M)$ (resp G). We choose not to pursue these generalizations further and work only with immersed spheres $S^k \to M^{2k}$ with $k \geq 2$. Curious readers are referred to [Ran02, Chapter 11].

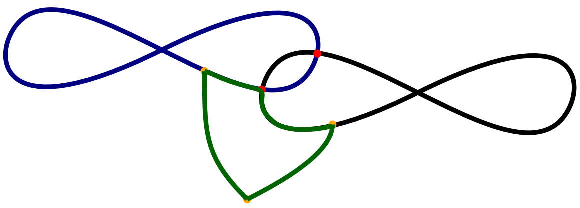


Figure 2.2: A sketch of the group element $g_p \in \pi_1(M)$ associated to a transverse double point p.

It can be shown that the *intersection form* λ is well defined on the set $I_k(M)$ of based regular homotopy classes of based immersed spheres. This leads to the following observation.

Remark 2.2.5. If a framed based immersion φ is regular homotopic to a framed embedding g, then $\lambda(\varphi, \varphi) = 0$. To see this, let g' be the embedded sphere obtained from g by pushing along a nowhere-zero section of $\nu(g)$. Since we just saw that λ is invariant under regular homotopy, we have $\lambda(\varphi, \varphi) = \lambda(g, g') = 0$.

Remark 2.2.5 shows that the vanishing of the intersection numbers provides a necessary condition for an immersion to be regular homotopic to an embedding. Unfortunately, the next example shows that this condition is not sufficient.

Example 2.2.6. The immersion f of S^1 as a figure eight "8" in $M = S^2 = \mathbb{R}^2 \cup \{\infty\}$ has intersection number $\lambda(f, f) = 0$ but it is not regular homotopic to an embedding (e.g. using the Whitney-Graustein theorem, recall Remark 1.1.17). This example can be generalized to arbitrary dimensions as explained in [Ran02, Proposition 7.12].

A sufficient condition to obtain an embedding is obtained by studying self intersections. The self intersection number of a single based immersed sphere $f\colon S^k \to M$ is defined similarly to λ. For each transverse double point p in S^k, two branches of S^k cross at p and, after *choosing* an order of these branches, we can define the index $\epsilon_p g_p$ exactly as in the definition of λ, and consider

$$f \mapsto \sum_p \epsilon_p g_p.$$

Unfortunately, since there is no canonical choice of ordering for the aforementioned branches, this quantity is not well defined in $\mathbb{Z}[\pi_1(M)]$ (switching the order of branches that meet at p changes the sign and group element by $\epsilon_p \leftrightarrow (-1)^k \epsilon_p$ and $g_p \leftrightarrow g_p^{-1}$). On the other hand, it will be well defined in the following quotient which cancels out the effect of swapping the order of the branches:

$$Q_{(-1)^k}(\mathbb{Z}[\pi_1(M)]) := \mathbb{Z}[\pi_1(M)]/\{x \sim (-1)^k \overline{x} \mid x \in \mathbb{Z}[\pi_1(M)]\}.$$

Here we used the following notation: we write $^-\colon \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\pi_1(M)]$ for the involution obtained by extending $\mathbb{Z}$-linearly the involution $g \mapsto g^{-1}$ of $\pi_1(M)$ to $\mathbb{Z}[\pi_1(M)]$.

We can now define the self intersection number of a based immersed sphere.

Definition 18. The *self intersection number* of a based immersion $f\colon S^k \to M^{2k}$ is obtained by summing the indices of all double points:

$$\mu(f) := \sum_p \epsilon_p g_p \in Q_{(-1)^k}(\mathbb{Z}[\pi_1(M)]).$$

By construction, the choice of $Q_{(-1)^k}(\mathbb{Z}[\pi_1(M)])$ as a target ensures that μ is well defined. Some additional work shows that μ is unchanged under regular homotopy.

Remark 2.2.7. We have seen that both λ and μ are well defined on the set $I_k(M)$ of based regular homotopy classes of based immersed spheres. In fact, $I_k(M)$ is a group and even a $\mathbb{Z}[\pi_1(M)]$-module; we deliberately avoid giving further details and instead refer to [CLMa].

The next result (often referred to as the *Wall embedding theorem*) shows that in high dimensions, self intersections are the only obstruction for a regular homotopy class of an immersion to contain an embedding.

Theorem 2.2.8. *Let $f\colon S^k \to M^{2k}$ be a based immersion. If $k \geq 3$, then the following statements are equivalent:*

1. *f is based regular homotopic to a based embedding.*
2. *$\mu(f) = 0$.*

Proof. If f is based regular homotopic to a based embedding g, then (since embeddings are injective immersions), we deduce that $\mu(f) = \mu(g) = 0$. The proof of the converse relies on the Whitney trick. Since $\mu(f) = 0$, we can find a pair of double points p, p' with opposite signs but same group elements: $\epsilon_p = -\epsilon_{p'}$ and $g_p = g_{p'}$. Use D_f to denote the double point sets of f. Since $k \geq 3$, the Whitney trick makes it possible to change f within its based regular homotopy class to a based immersion g with $D_g = D_f \setminus \{p, p'\}$ and $\mu(g) = \mu(f)$. The proof is concluded by induction (and because injective immersions of compact manifolds are embeddings by Proposition 1.1.14). □

We now return to our initial question. Is a given degree one normal map $f\colon M \to X$ normal bordant to a degree one normal map $f'\colon M' \to X$ with $\pi_i(f') = 0$ for $i \leq k+1$?

The next result recalls the relation between the relative homotopy group $\pi_{k+1}(f)$ and the set $I_k(M)$ of based regular homotopy classes of based immersions $S^k \to M$.

Lemma 2.2.9. *Let $f\colon M^{2k} \to X$ be a normal map. There is a map $\alpha\colon \pi_{k+1}(f) \to I_k(M)$ that fits into the following commutative diagram:*

$$\begin{array}{ccc} \pi_{k+1}(f) & \xrightarrow{\alpha} & I_k(M) \\ & {\scriptstyle\partial}\searrow \quad \swarrow{\scriptstyle\iota} & \\ & \pi_k(M). & \end{array} \tag{2.5}$$

Here, ∂ is the connecting homomorphism in the long exact sequence of homotopy groups (recall Theorem 1.2.14) and ι is the map that sends a regular homotopy class of an immersion to its underlying homotopy class.

Proof. Since f is a normal map, Proposition 2.1.11 implies that $x \in \pi_{k+1}(f)$ determines a regular homotopy class $[\varphi_x\colon S^k \times D^{n-k} \to M]$ of immersions. Proposition 2.1.11 also implies that if φ belongs to $[\varphi_x]$ then it fits into a diagram of the form

$$\begin{array}{ccc} S^k \times D^{n-k} & \xrightarrow{\varphi} & M \\ \downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} \times D^{n-k} & \xrightarrow{\Phi} & X, \end{array}$$

and $[(\varphi|_{S^k \times 0}, \Phi|_{D^{k+1} \times 0})] = x$. We set $\alpha(x) := [\varphi|_{S^k \times 0}]$. The commutativity of the diagram in (2.5) follows readily. This concludes the proof of the lemma. □

Slightly abusing notations, we also use λ and μ to denote the maps on $\pi_{k+1}(f)$ obtained by respectively precomposing λ and μ with $\alpha \times \alpha$ and α. If we assume that f is a k-connected degree one normal map, then Proposition 2.2.2 and Lemma 2.2.9 produce a map

$$K_k(M) \cong \pi_{k+1}(f) \xrightarrow{\alpha} I_k(M).$$

Here recall that $K_k(M) := \ker(H_k(\widetilde{M}; \mathbb{Z}) \to H_k(\widetilde{X}; \mathbb{Z}))$. Note that since $\pi_1(M)$ acts on the universal cover $\widetilde{M}$, this *surgery kernel* is a left $\mathbb{Z}[\pi_1(M)]$-module. Summarizing, given a k-connected degree one normal map $f\colon M \to X$, we get a triple $(K_k(M), \lambda, \mu)$.

The next proposition collects some properties of this *kernel form.*

Proposition 2.2.10. *The kernel form $(K_k(M), \lambda, \mu)$ of a k-connected degree one normal map $f\colon M^{2k} \to X$ satisfies the following properties for all $x, y, x_1, x_2, y_1, y_2 \in K_k(M)$, all $r, s \in \mathbb{Z}[\pi_1(M]$ and all $g \in \pi_1(M)$:*

1. *$K_k(M)$ is a finitely generated stably free $\mathbb{Z}[\pi_1(M)]$-module: there exist free finitely generated $\mathbb{Z}[\pi_1(M)]$-modules F_1, F_2 such that $K_k(M) \oplus F_1 = F_2$.*
2. *λ is sesquilinear: $\lambda(x_1, rx_2 + sy_2) = \overline{r}\lambda(x_1, x_2) + \overline{s}\lambda(x_1, y_2)$ and $\lambda(rx_1 + sy_1, x_2) = \lambda(x_1, x_2)r + \lambda(y_1, x_2)s$.*

3. λ *is* $(-1)^k$*-Hermitian:* $\lambda(y,x) = (-1)^k\overline{\lambda(x,y)}$.

4. λ *is nonsingular: the map* $x \mapsto \lambda(x,-)$ *is a* $\mathbb{Z}[\pi_1(M)]$*-linear isomorphism.*

5. $\mu(gx) = g\mu(x)\overline{g}$

6. *Using* $\mathrm{pr}\colon \mathbb{Z}[\pi_1(M)] \to Q_{(-1)^k}(\mathbb{Z}[\pi_1(M)])$ *to denote the quotient map, one has*

$$\mu(x+y) - \mu(x) - \mu(y) = \mathrm{pr}(\lambda(x,y)).$$

7. *Let* $\alpha\colon K_k(M) \to I_k(M)$ *be the map described in Lemma 2.2.9. If the normal bundle of* $\alpha(x)$ *is trivial, then*

$$\lambda(x,x) = \mu(x) + (-1)^k\overline{\mu(x)}.$$

Proof. We only provide a proof sketch and some references. The proof of the first assertion can be found in [CLMa, Lemma 4.56]. To show that that the *adjoint* $x \mapsto \lambda(x,-)$ of λ is an isomorphism, the idea is to show that it coincides with the composition of Poincaré dualiy and an evaluation map (both of which are isomorphisms): [3]

$$K_k(M) \to K^k(M) \to \overline{\mathrm{Hom}_{\mathbb{Z}[\pi_1(M)]}(K_k(M), \mathbb{Z}[\pi_1(M)])}.$$

The remaining statements can be shown direcly on the level of the set $I_k(M)$ of based immersions. The point that we have glossed over is that $I_k(M)$ is a $\mathbb{Z}[\pi_1(M)]$-module (recall Remark 2.2.7) and α is a $\mathbb{Z}[\pi_1(M)]$-linear map. The $(-1)^k$-Hermitian and sesquillineraty properties can checked by hand, while the statements on μ can be found in [CLMa, Lemma 4.11]. This concludes our proof sketch. □

Summarizing, for every k-connected degree one normal map $f\colon M^{2k} \to X$, we have associated a triple $(K_k(M), \lambda, \mu)$. Next, we review the algebra necessary to describe such objects.

2.2.2 Symmetric and quadratic forms

The goal of this short subsection is to collect some terminology on quadratic forms. References include [Ran81, Wal70, KL05].

Rings are understood to be associative and with a unit and all modules are assumed to be finitely generated. A *ring with involution* is a ring R together with an involution $x \mapsto \overline{x}$ that satisfies $\overline{ab} = \overline{b}\overline{a}, \overline{a+b} = \overline{a} + \overline{b}$ and $\overline{\overline{x}} = x$. Recall that an R-module is *projective* if there exists an R-module Q such that $P \oplus Q$ is free.

In practice, we shall mostly work with free modules. However, as the next remark shows, it is convenient to have a class of modules that contains stably free modules.

Example 2.2.11. Stably free modules are projective (but the converse does not hold) and in particular, if $f\colon M^{2k} \to X$ is a k-connected degree one normal map, then $K_k(M)$ is projective since Proposition 2.2.10 shows that it is stably free over $R = \mathbb{Z}[\pi_1(M)]$.

Set $\varepsilon = \pm 1$. Given a projective left R-module P, a pairing $\lambda\colon P \times P \to R$ is *sesquilinear* if it satisfies $\lambda(x_1, ax_2 + by_2) = \overline{a}\lambda(x_1,x_2) + \overline{b}\lambda(x_1,y_2)$ and $\lambda(ax_1 + by_1, x_2) = \lambda(x_1,x_2)a + \lambda(y_1,x_2)b$ for all $x_1, x_2, y_2, y_2 \in P$ and all $a, b \in R$. A sesquilinear pairing is *ε-Hermitian* if $\lambda(y,x) = \varepsilon\overline{\lambda(x,y)}$ for all $x, y \in P$. A *ε-symmetric form* consists of a pair (P, λ) where P is a projective left R-module and λ is a ε-Hermitian form on P.

Example 2.2.12. If $f\colon M^{2k} \to X$ is a k-connected degree one normal map, then Proposition 2.2.10 shows that $(K_k(M), \lambda)$ is a $(-1)^k$-symmetric form.

In order to discuss non-singularity, we start with some remarks on module structures.

[3] The meaning of the overline will be explained in Subsection 2.2.2 below.

Remark 2.2.13. If P is a left R-module, then $\mathrm{Hom}_{\text{left-}R}(P,R)$ is endowed with a *right* R-module structure via the action $\varphi \cdot r(x) := \varphi(x)r$. In order to obtain a left R-module, we use the following construction: for a right R-module M, we use $\overline{M}$ to denote the left R-module with the same underlying abelian group structure as M but with action $r \cdot m := m\overline{r}$. From now, we write P^* for left R-module $\overline{\mathrm{Hom}_{\text{left-}R}(P,R)}$ endowed with the action $\varphi \cdot r(x) = \varphi(x)\overline{r}$.

A sesquilinear form determines an R-linear map $\lambda^\bullet \colon P \to P^*$ by setting $\lambda^\bullet(x)(y) = \lambda(x,y)$. A symmetric form (P,λ) is *non-degenerate* if $\lambda^\bullet$ is injective and *nonsingular* if $\lambda^\bullet$ is an isomorphism.

These remarks lead to a common reformulation of the notion of a symmetric form.

Exercise 2.2.14. Consider the map $\mathrm{ev} \colon P \to (P^*)^*, p \mapsto (\varphi \mapsto \varphi(p))$. Show that a sesquilinear pairing λ is Hermitian if and only $\varepsilon\lambda^\bullet$ coincides with the following composition:

$$P \xrightarrow{\mathrm{ev}} (P^*)^* \xrightarrow{\phi^*} P^*.$$

Show furthermore that if $\phi \colon P \to P^*$ is R-linear and satisfies $\epsilon\phi = \phi^* \circ \mathrm{ev}$, then the assignment $\lambda(x,y) := \phi(x)(y)$ defines an ε-Hermitian form on P.

Next, we describe a class of symmetric form which occurs frequently in surgery theory.

Example 2.2.15. We describe the *standard hyperbolic symmetric form* $H^\varepsilon(R^u)$. This form has underlying module R^{2u}, a basis $e_1,\dots,e_u,f_1,\dots,f_u$ and the symmetric form is described by $\lambda(e_i,e_j) = 0 = \lambda(f_i,f_j)$ and $\lambda(e_i,f_j) = \delta_{ij}$.

Returning to topology, given a k-connected degree one normal map $f \colon M^{2k} \to X$, we know that the pair $(K_k(M),\lambda)$ is a non-singular symmetric form. We now introduce the terminology needed to describe the self-intersection number μ. First, set

$$Q_\varepsilon(R) = R/\{r - \varepsilon\overline{r} \mid r \in R\}.$$

Next, abstracting the properties of self-intersections, we make the following definition.

Definition 19. A *quadratic refinement* of an ε-symmetric form (P,λ) is a map $\mu \colon P \to Q_\varepsilon(R)$ such that the following properties hold for all $r \in R$ and all $p, p_1, p_2 \in P$:

1. $\mu(rp) = r\mu(p)\overline{r}$,
2. $\lambda(p,p) = \mu(p) + \varepsilon\overline{\mu}(p)$, [4]
3. If we use $\mathrm{pr} \colon R \to Q_\varepsilon(R)$ to denote the canonical projection, then

$$\mu(p_1+p_2) - \mu(p_1) - \mu(p_2) = \mathrm{pr}(\lambda(p_1,p_2)).$$

An *ε-quadratic form* is a triple (P,λ,μ) consisting of a symmetric form (P,λ) together with a quadratic refinement μ of λ.

We now define the quadratic analogue of Example 2.2.15.

Example 2.2.16. We describe the *standard hyperbolic quadratic form* $H_\varepsilon(R^u)$. This form has underlying module R^{2u}, a basis $e_1,\dots,e_u,f_1,\dots,f_u$, the symmetric form is described by $\lambda(e_i,e_j) = 0 = \lambda(f_i,f_j)$ and $\lambda(e_i,f_j) = \delta_{ij}$, the quadratic form is $\mu(e_i) = 0 = \mu(f_i)$ for all i. We will sometimes call such a basis a *hyperbolic basis.*

Next, we define sums and isomorphisms. Two symmetric forms (P_1,λ_1) and (P_2,λ_2) are *isometric* if there is an R-linear isomorphism $f \colon P_1 \to P_2$ such that $\lambda_2(f(x),f(y)) = \lambda_1(x,y)$ for every $x,y \in P_1$. Two quadratic forms (P_1,λ_1,μ_1) and (P_2,λ_2,μ_2) are *isometric* if there is

[4] This means that $\mu(p) + \varepsilon\overline{\mu}(p) \in R$ is the image of $\mu(p)$ by the map $Q_\varepsilon(R) \to R, [r] \mapsto r + \varepsilon\overline{r}$.

an isomorphism $f\colon P_1 \to P_2$ such that $\lambda_2(f(x), f(y)) = \lambda_1(x, y)$ and $\mu_2(f(x)) = \mu_1(x)$ for every $x \in P_1$. The *sum* of the symmetric forms (P_1, λ_1) and (P_2, λ_2) is the symmetric form $(P_1 \oplus P_2, \lambda_1 \oplus \lambda_2)$, where $\lambda_1 \oplus \lambda_2((x_1, x_2), (y_1, y_2)) = \lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)$. For quadratic refinements, we set $\mu_1 \oplus \mu_2(x, y) = \mu_1(x) + \mu_2(y)$.

We conclude this short subsection with some additional terminology.

Definition 20. A quadratic form is *hyperbolic* if it is isomorphic to a standard hyperbolic form. Two quadratic forms (P_1, λ_1, μ_1) and (P_2, λ_2, μ_2) are *stably isomorphic* if $(P_1, \lambda_1, \mu_1) \oplus H_\varepsilon(R)^u$ and $(P_2, \lambda_2, \mu_2) \oplus H_\varepsilon(R)^v$ are isomorphic for some non-negative integers u and v.

2.2.3 Surgery on hyperbolic surgery kernels

The aim of this subsection is to provide a sufficient condition for a k-connected degree one normal map to be normal bordant to a homotopy equivalence.

Let $f\colon M \to X$ be a normal map. Recall that $I_r(M)$ denotes the set of based regular homotopy classes of based immersions $S^r \looparrowright M$. Recall furthermore that $I_r(M)$ is a $\mathbb{Z}[\pi_1(M)]$-module (a fact we only mentioned in passing, recall Remark 2.2.7) and that the map $\alpha\colon \pi_{r+1}(f) \to I_r(M)$ of Lemma 2.2.9 is a $\mathbb{Z}[\pi_1(M)]$-homomorphism. The zero element of $I_r(M)$ is represented by the standard embedding $S^r \to \mathbb{R}^{r+1} \to \mathbb{R}^r \subset M$, where the last map is any embedding. Therefore we can always do surgery on the zero element of $\pi_{r+1}(f)$.

The next lemma describes the effect of such a surgery on the surgery kernel.

Lemma 2.2.17. *Let $k \geq 2$ and let $f\colon M \to X$ be a k-connected degree one normal map. If we do surgery on the zero element of $K_{k-1}(M) \cong \pi_k(f)$, then the effect on the surgery kernels is to replace $(K_k(M), \lambda, \mu)$ by $(K_k(M), \lambda, \mu) \oplus H_{(-1)^k}(\mathbb{Z}[\pi_1(M)])$.*

Proof. We claim that the result of this surgery is to replace M by $M\#S^k \times S^k$. By definition of the 0-element of $I_k(M)$ (which we recalled above the statement of the lemma), we can assume that the standardly embedded sphere S^{k-1} on which we are doing surgery is contained in an S^{2k} summand of a connected sum decomposition $M \cong M\#S^{2k}$. It is therefore enough to show that the effect of surgery along a standardly embedded (framed) $S^{k-1} \subset S^{2k}$ replaces S^{2k} by $S^k \times S^k$. Since this embedding is framed, we can write $S^{2k} = S^{k-1} \times D^{k+1} \cup D^k \times S^k$. Performing the surgery replaces this S^{2k} by an $S^k \times S^k$, as claimed.

Next, we study the effect of this surgery on $K_k(M)$ Set $M' := M\#S^k \times S^k$ and $\pi := \pi_1(M)$. Pull back the universal of X to M'. Since $S^k \times S^k$ is simply connected, the restriction of this cover to $S^k \times S^k$ is the trivial π-cover. It follows that we get a direct sum decomposition $H_k(\widetilde{M'}; \mathbb{Z})$ as $H_k(\widetilde{M}; \mathbb{Z}) \oplus \mathbb{Z}[\pi][S^k \times \{\text{pt}\}] \oplus \mathbb{Z}[\pi][\{\text{pt}\} \times S^k]$. Since the degree one normal map on the $S^k \times S^k$ summand of M' was obtained from a degree one normal map on S^{2k}, we deduce that it is zero on the $\mathbb{Z}[\pi]^2$-summand of $H_k(\widetilde{M'}; \mathbb{Z})$. Passing to surgery kernels, we obtain the direct sum decomposition $K_k(M') = K_k(M) \oplus \mathbb{Z}[\pi][S^k \times \{\text{pt}\}] \oplus \mathbb{Z}[\pi][\{\text{pt}\} \times S^k]$. Using the definition of (self)-intersection numbers, we deduce that this surgery adds a hyperbolic form to $K_k(M)$, concluding the proof of the lemma. □

The next theorem provides a sufficient condition for a highly connected degree one normal map to be normal bordant to a homotopy equivalence.

Theorem 2.2.18. *Let $f\colon M^{2k} \to X$ be a k-connected degree one normal map. If $k \geq 3$ and $(K_k(M), \lambda, \mu)$ is stably hyperbolic, then f is normal bordant to a homotopy equivalence.*

Proof. Without loss of generality, we can assume that the kernel form $(K_k(M), \lambda, \mu)$ is hyperbolic, i.e. isometric to some $H_{(-1)^k}(\mathbb{Z}[\pi]^v)$: Lemma 2.2.17 shows that performing surgery on the trivial element of $\pi_k(f) \cong K_{k-1}(M)$ adds a hyperbolic summand to $(K_k(M), \lambda, \mu)$. Since we are working up to normal bordism, we can indeed assume that $(K_k(M), \lambda, \mu) \cong H_{(-1)^k}(\mathbb{Z}[\pi]^v)$.

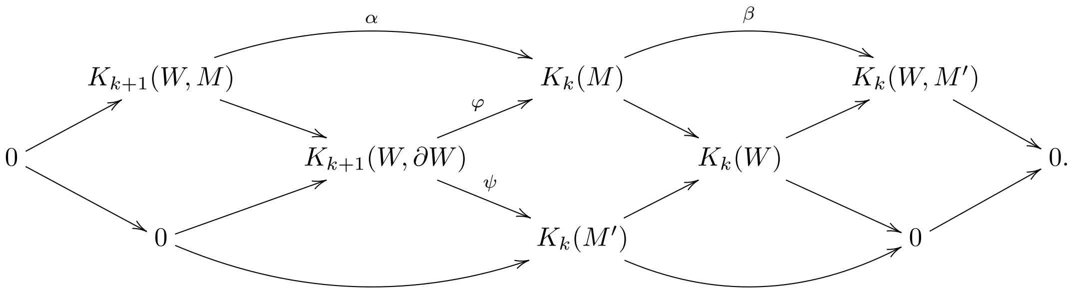
Endow $(K_k(M), \lambda, \mu)$ with a hyperbolic basis $(b_1, \dots, b_v, c_1, \dots, c_v)$ (recall Example 2.2.16), and note that f is a homotopy equivalence if and only if $v = 0$. We must therefore show that f is normal bordant to a degree one normal map $f' \colon M' \to X$ such that $K_k(M')$ has rank $v - 1$ and $(K_k(M'), \lambda', \mu')$ is still hyperbolic. The conclusion will then follow by induction.

We do surgery on the basis element $b_v \in K_k(M) \cong \pi_{k+1}(f)$. Note that this is possible since we assumed $k \geq 3$: since $\mu(b_v) = 0$, the Wall embedding theorem (recall Theorem 2.2.8) ensures that the regular homotopy class determined by b_v contains a framed embedding. The normal map f extends to the trace W of this surgery and the result is a map

$$(F, f, f') \colon (W, M, M') \to (X \times [0,1], X \times \{0\}, X \times \{1\}).$$

This triple gives rise to surgery kernels $K_i(W), K_i(M), K_i(M')$. These kernels admit natural generalisations to pairs. For instance $K_i(W, M) := \ker(H_j(\widetilde{W}, \widetilde{M}) \to H_j(\widetilde{X} \times [0,1], \widetilde{X} \times \{0\})$, and $K_i(W, M')$ is defined similarly. As in the proof of Proposition 2.2.2, similar definitions can be made in cohomology (using cokernels instead of kernels), and the $\mathbb{Z}[\pi_1(M)]$-modules $K_i(W, \partial W), K_i(W, M), K_i(W, M')$ satisfy Poincaré duality and fit into long exact sequences.

We already know that $K_i(M)$ is non-zero if and only if $i = k$. Recall that up to homotopy equivalence, W is obtained by adding a $(k+1)$-cell to M. As a consequence, using Proposition 2.1.4, we know that $K_i(W) = \pi_{i+1}(F) = \pi_{i+1}(f) = K_i(M)$ vanishes for $i \leq k - 1$. We also know from Proposition 2.1.4 that $K_i(M') = K_i(M) = 0$ for $i \leq k - 1$, or put differently, that $f' \colon M' \to X$ is also a k-connected degree one normal map. Arguing as in Proposition 2.2.2, we deduce that $K_i(M') = 0$ for $i \neq k$. It follows that $K_i(\partial W) = 0$ for $i \neq k$ and that $K_i(W, M), K_i(W, M')$ and $K_i(W, \partial W)$ all vanish for $i \leq k - 1$. Since f is k-connected, we also deduce that $K_i(W, M) = 0$ for $i = k$. Duality and some homological algebra imply that $K_i(W, M) = 0$ for all $i \neq k + 1$ and $K_i(W, M') = 0$ for all $i \neq k$. The long exact sequence for (W, M') implies that $K_i(W) = 0$ for $i \neq k$. It follows that $K_i(W, \partial W) = 0$ for $i \neq k + 1$. We therefore get the following braid of four interlocking exact sequences:



The long exact sequences that come into play are those of the pairs $(W, M), (W, M')$ and of the triples $(W, \partial W, M), (W, \partial W, M')$. Set $\pi := \pi_1(M)$ and note that $K_{k+1}(W, M) \cong \mathbb{Z}[\pi]$ is freely generated by the core of the $(k+1)$-handle ϕ^{k+1} that we attached to $M \times [0,1]$: recall that up to homotopy equivalence, W is obtained from M by adding a $(k+1)$-cell; therefore we have $H_k(W, M) = \mathbb{Z}[\pi][\phi^{k+1}]$, and the observation follows. A similar argument shows that $K_k(W, M') \cong \mathbb{Z}[\pi]$ is freely generated by the core of the dual k-handle ψ^k. Next, note that the homomorphism $\beta \colon K_k(M) \to K_k(W, M')$ is given by $x \mapsto \lambda(b_v, x)\psi^k$: briefly, this can be seen by noting that $\alpha(\phi^{k+1}) = b_v$, using that λ can be defined by composing Poincaré duality with the evaluation map (a fact we chose not to mention in Subsection 2.2.1), and considering the following commutative diagram:

$$\begin{array}{ccccc}
K_k(M) & \xrightarrow{\text{PD},\cong} & K^k(M) & \xrightarrow{\text{ev},\cong} & \overline{\operatorname{Hom}_{\mathbb{Z}[\pi]}(K_k(M), \mathbb{Z}[\pi])} \\
\big\downarrow \beta & & \big\downarrow \alpha^* & & \big\downarrow \alpha^* \\
K_k(W, M') & \xrightarrow{\text{PD},\cong} & K^{k+1}(W, M) & \xrightarrow{\text{ev},\cong} & \overline{\operatorname{Hom}_{\mathbb{Z}[\pi]}(K_{k+1}(W, M), \mathbb{Z}[\pi])}.
\end{array}$$

We claim that there exists a basis $(b'_1, \dots, b'_v, c'_1, \dots c'_{v-1})$ for $K_{k+1}(W, \partial W)$ such that $\varphi(b'_i) = b_i$ and $\varphi(c'_i) = c_i$ for each i. Observe that by exactness of the braid at the parts involving φ and β, we obtain $K_{k+1}(W, \partial W) \cong \operatorname{im}(\varphi) = \ker(\beta)$. Since we chose a hyperbolic basis for $K_k(M)$, the previously mentioned description of β implies that $\beta(c_v) = \lambda(b_v, c_v) = 1$ and $\beta(c_i) = \lambda(b_v, c_i) = 0$ for $i \neq v$ and $\beta(b_i) = \lambda(b_v, b_i) = 0$ for all i. The claim follows.

We now turn to the $\mathbb{Z}[\pi]$-module $K_k(M')$. By exactness of the diagonal involving the map ψ, we know that this $\mathbb{Z}[\pi]$-module is isomorphic to $\mathbb{Z}[\pi]^{2(v-1)}$ and is freely generated by the $\psi(b_i')$ and $\psi(c_i')$ for $i = 1, \ldots, v-1$. We outline why these elements still form a hyperbolic basis. Since $(b_1, \ldots, b_v, c_1, \ldots, c_v)$ for a hyperbolic basis and since $k \geq 3$, we can use the Whitney trick to arrange that the spheres $b_1, \ldots, b_{v-1}, c_1, \ldots, c_{v-1}$ do not intersect b_v [Ran02, Corollary 7.30]. As a consequence, the surgery on b_v will not affect the other intersection numbers. [5] This concludes the proof of the theorem. □

Given a degree one normal map $f \colon M \to X$, our goal is to define a "surgery obstruction" $\sigma(f)$ that only depends on $f \colon M \to X$ and not on an intermediate highly connected degree one normal map. The first step is to describe the group to which this surgery obstruction will belong.

2.2.4 The even quadratic L-groups

A first glance at Theorem 2.2.18 suggests that given a degree one normal map $f \colon M^{2k} \to X$, the required "surgery obstruction" should involve the kernel form $(K_k(f'), \lambda', \mu')$ of a highly connected f' that is normal bordant to f. A second glance at Theorem 2.2.18 suggests that this surgery obstruction ought to belong to a group of "symmetric forms modulo stably hyperbolic forms". The aim of this subsection is define such a group. Reference include [Ran81, Wal70, KL05].

Let R be a ring. We start with two important conventions.

Remark 2.2.19. In this subsection, all R-modules are understood to be **free** left R-modules and all quadratic forms are assumed to be nonsingular.

The main definition of this subsection is the following.

Definition 21. The $2k$*-th quadratic L-group* $L_{2k}(R)$ is the abelian group of stable isomorphism classes of $(-1)^k$-quadratic forms; the sum is given by the direct sum operation, the zero element is represented by the class of $H_{(-1)^k}(R)^u$ for any $u \geq 0$, and the inverse of $[(H, \lambda, \mu)]$ is $[(H, -\lambda, -\mu)]$.

More explicitly, $L_0(R)$ (which is equal to $L_{4n}(R)$ for all n) consists of stable isomorphism classes of quadratic forms, while $L_2(R)$ (which is equal to $L_{4n+2}(R)$ for all n) consists of stable isomorphism classes of (-1)-quadratic forms (i.e. the form λ is skew-Hermitian).

The goal of the remainder of this subsection to show that $L_{2k}(R)$ is indeed an abelian group. First, note that stable isomorphism is clearly an equivalence relations which is compatible with the sum operation. Associativity and commutativity of the direct sum are also readily verified. By definition of the equivalence relation, the class of the hyperbolic form is indeed the zero element. Consequently, it remains to show that the inverse of $[(H, \lambda, \mu)]$ is represented by $(H, -\lambda, -\mu)$.

This result will follow readily from the following useful proposition.

Proposition 2.2.20. *Let (H, λ, μ) be an ε-quadratic form, where H is of rank $2n$. If there exists a free half rank summand $G \subset H$ such that $\lambda|_{G \times G} = 0$ and $\mu|_G = 0$, then (H, λ, μ) is isomorphic to $H_\varepsilon(R^n)$.*

Proof. Choose a basis $x_1, \ldots, x_n$ of G. Since λ is nonsingular, we have an isomorphism $H \to H^*$. Since G is a summand of H, we can extend the basis $x_1, \ldots, x_n$ of G to a basis $x_1, \ldots, x_{2n}$ of H. For $i = 1, \ldots, n$, consider the linear map $\delta_i \colon H \to R$ given by $\delta_i(x_j) = \delta_{ij}$. These δ_i define elements in H^*. Since λ is non-singular, its adjoint $\lambda^\bullet \colon H \to H^*$ is an isomorphism and therefore

[5] the key underlying point is: "in high dimensions, the Whitney trick can be used to realize algebraic intersections geometrically."

there exists elements $y_1, \dots, y_n$ of H such that $\lambda(x, y_j) = \delta_i(x)$ for all $x \in H$. In particular, we have $\lambda(x_i, y_j) = \delta_{ij}$ for $i = \dots, n$ and therefore $(x_1, \dots, x_n, y_1, \dots, y_n)$ forms a basis of H.

We now prove the proposition by induction on n. Suppose $n = 1$, write x, y instead of x_1, y_1 for simplicity, and note that $G = \langle x \rangle$ and $H = \langle x, y \rangle$. By definition of G, we know that $\lambda(x, x) = 0$ and $\mu(x) = 0$, and by construction of y, we have $\lambda(x, y) = 1$. Set $y' = y - ax$ with a a representative of $\mu(y)$. Using the properties of λ and μ (recall Subsection 2.2.2), we check that $\langle x, y' \rangle$ is a hyperbolic basis of H:

$$\begin{aligned} \lambda(x, y') &= \lambda(x, y) - \overline{a}\lambda(x, x) = \lambda(x, y) = 1, \\ \lambda(y', y') &= \lambda(y, y) - a\lambda(x, y) - \overline{a}\lambda(y, x) = \lambda(y, y) - (a + \varepsilon\overline{a}) = 0, \\ \mu(y') &= \mu(y) - \mu(ax) - \lambda(y, ax) = \mu(y) - \varepsilon\overline{a} = a - \overline{a} = 0. \end{aligned}$$

Next, assume inductively that the proposition is true for $i \leq n-1$. If $\lambda(y_i, y_j) = 0$ and $\mu(y_j) = 0$ for all i, j, then the form is already hyperbolic. Otherwise, pick an y_j for which one these equalities does not hold. Without loss of generality, we can assume that $j = 1$ and define $y_1' = y_1 - ax_1$, where a represents $\mu(y_1)$. The computation above shows that $X := \langle x_1, y_1' \rangle$ is hyperbolic. We set $H' := \langle x_1, y_1' \rangle^\perp$ so that $H = X \oplus H'$ with H' free of rank $2(n-1)$. Observe that the summand $G' := \langle x_2, \dots, x_n \rangle$ of H' has half rank. We apply the inductive step to $G' \subset H'$. It follows that H' is hyperbolic and therefore so is $H = X \oplus H'$. This concludes the proof of the proposition. □

We can now prove the desired result:

Proposition 2.2.21. *$L_{2k}(R)$ is an abelian group.*

Proof. As we saw in the discussion following Definition 21, we need only show that the inverse of $[(H, \lambda, \mu)]$ is represented by $(H, -\lambda, -\mu)$. In other words, we must show that the direct sum $(H, \lambda, \mu) \oplus (H, -\lambda, -\mu)$ is hyperbolic. This follows by applying Proposition 2.2.20 to the diagonal

$$G := \{(x, x) \mid x \in H\} \subset H \oplus H.$$

Indeed, we have $(\lambda \oplus -\lambda)|_{G \times G} = 0$ and $(\mu \oplus -\mu)|_G = 0$. This concludes the proof of the proposition. □

It is worth mentioning that a G as in Proposition 2.2.20 is often referred to as a quadratic *lagrangian.*

Next we discuss some examples of L-groups.

Example 2.2.22. We describe the even quadratic L-groups of $\mathbb{Z}$.

1. The signature gives rise to an isomorphism $L_0(\mathbb{Z}) \cong 8\mathbb{Z}$. (see e.g. [CLMb, Theorem 7.93] for a proof). Here recall that the *signature* $\operatorname{sign}(\lambda)$ of symmetric form (H, λ) is obtained by representing λ by a Hermitian matrix A and defining $\operatorname{sign}(\lambda)$ as the number of positive eigenvalues of A minus the number of negative eigenvalues of A. The signature of a quadratic form is defined as the signature of its underlying symmetric form (the fact that λ admits a quadratic refinement ensures that $\operatorname{sign}(\lambda) \in 8\mathbb{Z}$ [vdB59]). It is well known that the signature is independent of the matrix representative of λ.

2. The Arf invariant gives rise to isomorphisms $L_2(\mathbb{Z}) \cong \mathbb{Z}_2$ and $L_2(\mathbb{Z}_2) \cong \mathbb{Z}_2$ (see e.g. [CLMb, Theorem 7.105] for a proof). We briefly describe the Arf invariant of a quadratic form (H, λ, μ). Since the Arf invariant of a quadratic form over $\mathbb{Z}$ is defined by first tensoring by $\mathbb{Z}_2$ (i.e. reducing λ and μ mod 2), we assume that (H, λ, μ) is a quadratic form over $\mathbb{Z}_2$. Since λ is non-singular H must be even-dimensional, say $H \cong \mathbb{Z}_2^{2n}$. All non-singular skew-symmetric forms over a field admit a hyperbolic basis. Pick such a hyperbolic

basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ for (H, λ) and define the Arf invariant as

$$\operatorname{Arf}(H, \lambda, \mu) := \sum_{i=1}^{n} \mu(e_i)\mu(f_i).$$

This definition is well known to be independent of all the choices involved. As an example, we make the easy verification that the Arf invariant of the 2-dimensional hyperbolic form on $\langle e, f \rangle$ vanishes (since the Arf invariant is additive unders direct sums, this shows that the Arf invariant descends to a map on $L_2(\mathbb{Z}_2)$): as $\mu(e) = 0 = \mu(f)$, we conclude that $\operatorname{Arf}(H, \lambda, \mu) = \mu(e)\mu(f) = 0$.

We conclude this subsection with a brief outlook on L-theory.

Remark 2.2.23. Up to now, we have only considered the *even* L-groups and assumed that all modules were free. We list some other flavors of L-groups and refer to [Ran81] for details.

1. There are L-groups $L_k(R)$ when k is odd. Instead of quadratic forms, the definition of $L_k(R)$ involves so-called "quadratic formations". These L-groups are 4-periodic: $L_{k+4}(R) = L_k(R)$ and, for instance, it is known that $L_1(\mathbb{Z}) = 0 = L_3(\mathbb{Z})$.

2. L-groups can also be defined using projective R-modules instead of free R-modules. In this case, the resulting groups $L^p_n(R)$ are different and are related to the $L_n(R)$ by the so-called Ranicki-Rothenberg exact sequence [Ran81, Ran73a, Ran73b]. If R is a PID, then projective modules are known to be free, and so $L^p_n(R) = L_n(R)$. Use $R[t^{\pm 1}]$ to denote the ring of Laurent polynomials with coefficients in a commutative ring R. Shaneson showed that $L_n(R[t^{\pm 1}]) \cong L^p_{n-1}(R) \oplus L_n(R)$ [Sha69]. For instance, this *Shaneson splitting* shows that $L_0(\mathbb{Z}[\mathbb{Z}]) = L_0(\mathbb{Z}) \oplus L_3(\mathbb{Z}) = L_0(\mathbb{Z})$ (since $\mathbb{Z}$ is a PID, we dropped the decorations).

3. Given a multiplicative subset $S \subset R$, there are also relative L-groups of linking forms and linking formations. These groups are frequently denoted $L_n(R, S)$ and fit into a *localisation exact sequence*
$$\ldots \to L_n(R) \to L_n(S^{-1}R) \to L_n(R) \to L_{n-1}(R) \ldots$$

4. There are symmetric L-groups $L^k(R)$ of $(-1)^k$-symmetric forms/formations: one disregards the quadratic refinements (an arbitrary symmetric form need not admit a quadratic refinement!) These groups coincide with the classical "Witt groups" of Hermitian forms.

5. Quadratic and symmetric L-groups have been reformulated using particular types of chain complexes, known as "algebraic Poincaré complexes".

2.2.5 The surgery obstruction in the even-dimensional case

In this subsection, we define the surgery obstruction $\sigma(f)$ of a degree one normal map $f\colon M^{2k} \to X$. Theorem 2.2.24 shows that in high dimensions $\sigma(f)$ provides a necessary and sufficient condition to decide whether f is normal bordant to a homotopy equivalence. References include [LÖ2, Ran02, Wal70, Wal16, CLMa].

Let $f\colon M^{2k} \to X$ be a degree one normal map. Using surgery below the middle dimension (recall Theorem 2.1.13), we can assume that f is normal bordant to a k-connected degree one normal map $f'\colon M' \to X$. We know from Proposition 2.2.10 that $K_k(M')$ is a stably free $\mathbb{Z}[\pi_1(X)]$-module. We can therefore choose a natural number r so that $K_k(M') \oplus \mathbb{Z}[\pi_1(X)]^{2r}$ is free. We deduce that $(K_k(M'), \lambda, \mu) \oplus H_{(-1)^k}(\mathbb{Z}[\pi]^r)$ determines an element in $L_{2k}(\mathbb{Z}[\pi_1(X)])$.

The main definition of this subsection is the following.

Definition 22. The *surgery obstruction* of a degree one normal map $f\colon M^{2k} \to X$ is defined as

$$\sigma(f) := [(K_k(M'), \lambda', \mu') \oplus H_{(-1)^k}(\mathbb{Z}[\pi]^r)] \in L_{2k}(\mathbb{Z}[\pi_1(X)]),$$

where $f' \colon M' \to X$ is a k-connected degree one normal map that is normal bordant to f, and r is any natural number such that $K_k(M') \oplus \mathbb{Z}[\pi_1(X)]^{2r}$ is free.

Since $L_{2k}(\mathbb{Z}[\pi_1(X)])$ consists of stable isomorphism classes of quadratic forms, we deduce that $\sigma(f)$ is independent of the choice of r. On the other hand, showing that $\sigma(f)$ is independent of f' requires some more work, as we shall see below.

The next result answers the initial question of this chapter by providing a necessary and sufficient condition for a degree one normal map to be normal bordant to a homotopy equivalence.

Theorem 2.2.24. *Let $k \geq 3$ and let $f \colon M^{2k} \to X$ be a degree one normal map.*

1. *The surgery obstruction $\sigma(f) \in L_{2k}(\mathbb{Z}[\pi_1(X)])$ is well defined.*
2. *f is normal bordant to a homotopy equivalence if and only if $\sigma(f) = 0$.*

Proof. We start by proving the second assertion, assuming that the first has been proved. Note that a homotopy equivalence $g \colon M \to X$ induces an isomorphism on all homotopy groups. It follows that all $\pi_j(g)$ vanish and therefore so do all the surgery kernels. In particular, the surgery obstruction of g vanishes. Consequently if f is normal bordant to a homotopy equivalence g, the first assertion implies that $\sigma(f) = \sigma(g) = 0$. Conversely, if $\sigma(f) = 0$, then f is normal bordant to a k-connected degree one normal map $f' \colon M' \to X$ with stably hyperbolic surgery kernel. Theorem 2.2.18 implies that f' is normal bordant to a homotopy equivalence and therefore so is f. This concludes our proof of the second assertion, and we must now prove the first.

Given two normal bordant degree one normal maps $f_0 \colon M_0 \to X$ and $f_1 \colon M_1 \to X$, we must show that $\sigma(f_0) = \sigma(f_1)$. Let $F \colon W^{2k+1} \to X \times [0,1]$ be a degree one normal cobordism between f_0 and f_1. Without loss of generality, we can assume that f_0, f_1, F are k-connected and that the surgery kernels of f_0, f_1 are free. An argument involving "handle substractions" shows that $K_k(W, \partial W)$ can be assumed to be zero [Wal70, CLMa]. Let λ and μ denote intersection and self-intersection numbers on ∂W. Since $\partial W = M_0 \sqcup -M_1$, we deduce that

$$(K_k(\partial W), \lambda, \mu) = (K_k(M_0), \lambda_0, \mu_0) \oplus (K_k(M_1), -\lambda_1, -\mu_1).$$

To show that $\sigma(f_0) = \sigma(f_1)$, it is therefore enough to prove that $K_k(\partial W), \lambda, \mu)$ is hyperbolic. Thanks to Proposition 2.2.20, it is enough to find a lagrangian for $(K_k(\partial W), \lambda, \mu)$, i.e. $G \subset K_k(\partial W)$ such that $G = G^{\perp}$ and $\mu|_G = 0$. Since we arranged that $K_k(W, \partial W) = 0$, we can consider the following portion of the long exact sequence of the pair $(W, \partial W)$ for surgery kernels:

$$K_{k+1}(W, \partial W) \xrightarrow{\partial} K_k(\partial W) \xrightarrow{i} K_k(W) \to 0 \tag{2.6}$$

Set $G := \operatorname{im}(\partial) = \ker(i)$. We first show that $G = G^{\perp}$. This is a standard "half lives half dies" argument. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
K_{k+1}(W, \partial W) & \xrightarrow{\partial} & K_k(\partial W) & \xrightarrow{i} & K_k(W) & \longrightarrow & 0. \\
{\scriptstyle \mathrm{PD}}\downarrow\cong & & {\scriptstyle \mathrm{PD}}\downarrow\cong & & {\scriptstyle \mathrm{PD}}\downarrow\cong & & \\
K^k(W) & \xrightarrow{i^*} & K^k(\partial W) & \xrightarrow{\delta} & K^{k+1}(W, \partial W) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
K_k(W)^* & \xrightarrow{i^*} & K_k(\partial W)^* & \xrightarrow{\partial^*} & K_{k+1}(W, \partial W)^* & &
\end{array} \tag{2.7}$$

As we have alluded to several times in the previous sections, the middle vertical composition coincides with the adjoint $\lambda^\bullet$ of λ. We write $\lambda_r^\bullet$ for the rightmost vertical compositions and λ_r for the resulting pairing. Note that $\lambda_r^\bullet$ is an isomorphism: the bottom right evaluation map is an isomorphism because $K_i(W, \partial W) = 0$ for $i \leq k$.

We first show that $G^{\perp} \subset G$ i.e. that $\lambda(\partial(x), \partial(y)) = 0$ for all $x, y \in G$. The diagram displayed in (2.7) shows that $\lambda(\partial(x), \partial(y)) = \lambda_r(i \circ \partial(x), y)$ for all $x, y \in K_{k+1}(W, \partial W)$. The exactness of the top row of (2.7) implies that $i \circ \partial(x) = 0$, whence $\lambda(\partial(x), \partial(y)) = 0$. We now prove the reverse inclusion, namely $G^\perp \subset G$. Assume that $a \in K_k(\partial W)$ satisfies $\lambda(a, \partial(y)) = 0$ for all $y \in K_{k+1}(W, \partial W)$; we must show that $i(a) = 0$. Using the diagram displayed in (2.7), we deduce that $0 = \lambda(a, \partial(y)) = \lambda_r(i(a), y)$ for all y. The fact that $\lambda_r^\bullet$ is an isomorphism now implies that $i(a) = 0$, as desired. This concludes the proof that $G = G^\perp$.

Next, we must prove that μ vanishes on $G = \operatorname{im}(\partial)$. Given $x \in K_{k+1}(W, \partial W)$, we must show that $\mu(\partial(x)) = 0$. We shall always use the isomorphism $K_k(\partial W) \cong K_k(M_0) \oplus K_k(M_1)$. As f_0 and f_1 are k-connected, there are isomorphisms $K_k(M_0) \cong \pi_{k+1}(f_0)$ and $K_k(M_1) \cong \pi_{k+1}(f_1)$ (these were discussed in the proof of Proposition 2.2.2). Use ∂_h to denote the connecting homomorphism in the long exact sequence of the pair for homotopy groups and consider the following composition:

$$K_k(M_0) \oplus K_k(M_1) \cong \pi_{k+1}(f_0) \oplus \pi_{k+1}(f_1) \xrightarrow{\partial_h \times \partial_h} \pi_k(M_0) \oplus \pi_k(M_1).$$

Using this composition, observe that the class $\partial(x)$ determines two (homotopy classes of) maps $u_0 \colon S^k \to M_0$ and $u_1 \colon S^k \to M_1$. Using Lemma 2.2.9, these can be assumed to be immersions. Referring to [CLMb] for a discussion of signs, it follows that

$$\mu(\partial x) = \mu(u_0) + \mu(u_1).$$

We must therefore show that the self intersection points of the immersions u_0 and u_1 cancel. To achieve this, consider the following diagram:

$$\begin{array}{ccccccc}
K_{k+1}(W, \partial W) & \xrightarrow{\partial} & K_k(\partial W) & \longrightarrow & K_k(W) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \\
 & & \pi_{k+1}(f_0) \oplus \pi_{k+1}(f_1) & \longrightarrow & \pi_{k+1}(F) & & \\
 & & \downarrow{\scriptstyle \partial_h \times \partial_h} & & \downarrow & & \\
 & & \pi_k(M_0) \oplus \pi_k(M_1) & \xrightarrow{i_*} & \pi_k(W) & &
\end{array} \tag{2.8}$$

Since the top row is exact, the definition of u_0, u_1 and the commutativity of (2.8) implies that $i_*(u_0, -u_1)$ vanishes in $\pi_k(W)$. It follows that u_0 and u_1 are homotopic in W. We write this homotopy as

$$H \colon (S^k \times [0,1], S^k \times \{0,1\}) \to (W, \partial W).$$

To pair up the self intersections of u_0 and u_1, we use H. More precisely, the set of double points of $H(S^k \times [0,1])$ with itself inside W is 1-dimensional and consists of circles (these are not relevant to our purposes) and arcs. Each of these arcs pairs up a double point of u_0 and with a double point u_1 of opposite sign. We deduce that $\mu(\partial x) = \mu(u_0) + \mu(u_1) = 0$, concluding the proof that $\mu|_G = 0$, and therefore the proof that G is a Lagrangian of $(K_k(\partial W), \lambda, \mu)$. As we mentioned above, this shows that $\sigma(f_0)$ and $\sigma(f_1)$ agree in $L_{2k}(\mathbb{Z}[\pi_1(X)])$, thus concluding the proof of the theorem. □

Let $\mathscr{N}_n(X)$ denote the set of normal bordism classes of degree one normal maps $M^n \to X$. The first assertion of Theorem 2.2.24 states that the surgery obstruction gives rise to a well defined map $\sigma \colon \mathscr{N}_{2k}(X) \to L_{2k}(\mathbb{Z}[\pi_1(X)])$. In fact, the map σ fits into the so-called surgery exact sequence which plays a crucial role in surgery theory [Wal70].

Next, we discuss variants and generalizations of Theorem 2.2.24. First, the results of this chapter can be adapted to manifolds with boundary: surgeries are performed in the interior of the manifold; normal maps $(f, \partial f) \colon (M^{2k}, \partial M) \to (X, \partial X)$ are defined as the closed case, but ∂f is additionally required to be a homotopy equivalence (in fact, it is enough to assume that the maps $H_*(\partial \widetilde{M}; \mathbb{Z}) \to H_*(\partial \widetilde{X}; \mathbb{Z})$ are isomorphisms); surgery below the middle dimension is proved as in the closed case; the middle dimensional surgery kernel of a k-connected map still supports a self intersection form μ and an intersection form λ. The assumption that $H_*(\partial \widetilde{M}; \mathbb{Z}) \to H_*(\partial \widetilde{X}; \mathbb{Z})$

is an isomorphism intervenes to ensure that λ is non-singular; the surgery obstruction can then be defined similarly to the closed case (recall Definition 22), and the analogue of Theorem 2.2.24 holds in this setting. We refer to [Wal70, CLMa] for further details.

We conclude by mentioning two additional settings in which surgery theory works.

Remark 2.2.25. We mention two facts whose proofs each occupy several hundreds pages.

1. Due to foundational work of Kirby and Siebenmann, surgery theory works for topological manifolds [KS77]. In particular, there is a way to make sense of normal maps and the surgery obstruction in the topological category.

2. Making use of Freedman's groundbreaking work on topological 4-manifolds, surgery theory works for topological 4-manifolds, provided the group is "good" [FQ90]. For instance, in the next subsection, we will permit ourselves to use Theorem 2.2.24 for topological 4-manifolds, but the reader should keep in mind that this a very difficult result.

2.3 An application of surgery theory to knot theory

We give an application of surgery theory to knot theory: we prove Freedman's result that Alexander polynomial one knots are topologically slice. We assume no prior background in knot theory, but take a "high dimensional approach" to the subject. In particular, we choose not to define the Alexander polynomial but instead provide a convenient reformulation of the Alexander polynomial one condition. References for the proof of Freedman's theorem include [FQ90, FT05], while introductions to knot theory include [Lic97, Rol76].

A *knot* K is a smooth embedding $S^1 \to S^3$. While knot theory studies knots up to ambient isotopy, our application of surgery is concerned with knot concordance.

Definition 23. A knot is *topologically slice* if it bounds a locally flat embedded flat disc in D^4.

We will neither discuss the classical results in knot concordance (i.e. in the study of slice knots), nor the distinction between topologically slice knots and smoothly slice knots. Instead, we describe a result of Freedman which uses surgery theorem to provide a sufficient condition for a knot to be topologically slice.

We do however need some basic homological facts on knot exteriors, and the next exercise is an application of Alexander duality (or the Mayer-Vietoris exact sequence).

Exercise 2.3.1. Let K be a knot with solid torus neighborhood νK, and let $X_K := S^3 \setminus \nu K$ be its exterior. Show that $H_i(X_K;\mathbb{Z}) = \mathbb{Z}$ for $i = 0, 1$ and vanishes otherwise, i.e. X_K is a *homology circle*. Show that if K is sliced by a disc D, then the slice disc exterior $N_D := D^4 \setminus \nu D$ is also a homology circle.

We study the boundary of the slice disc exterior N_D. Observe that ∂N_D is obtained by gluing a solid torus ST to X_K, identifying the meridian of ST with the (0-framed) longitude of K. [6] In other words, ∂N_D obtained by performing surgery on S^3 along K: we remove $\nu K = K \times D^2$ from S^3 and glue $ST = D^2 \times S^1$. We refer to M_K as the 0-*framed surgery along* K.

The next exercise describes the first homology of M_K.

Exercise 2.3.2. Show that $\pi_1(M_K) = \pi_1(X_K)/\langle \lambda_K \rangle$, where $\langle \lambda_K \rangle$ denotes the normal subgroup generated by the longitude of K. Show that $H_1(M_K;\mathbb{Z}) = H_1(X_K;\mathbb{Z}) \cong$ is generated by the meridian μ_K of K.

[6] The longitude λ_K and meridian μ_K of K are two simple closed curves in $\partial(\nu K)$ such that μ_K is non-separating and bounds a disc in νK, and λ_K is homologous to K in νK and nullhomologous in X_K.

We work towards the statement Freedman's theorem which provides a necessary condition for a knot to be topologically slice. Using Exercise 2.3.2, $H_1(M_K;\mathbb{Z}) \cong \mathbb{Z}\langle \mu_K \rangle$, and we consider the map $\pi_1(M_K) \xrightarrow{\text{ab}} H_1(M_K;\mathbb{Z}) \to \mathbb{Z}$ mapping the meridian of K to 1. The cover $\widehat{M}_K$ corresponding to the kernel of this map is endowed with an action of $\mathbb{Z}$, and therefore $H_*(\widehat{M}_K;\mathbb{Z})$ has the structure of a $\mathbb{Z}[\mathbb{Z}]$-module. It is helpful to think of $\mathbb{Z}[\mathbb{Z}]$ as the ring $\mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials with integer coefficients. In what follows, we shall write

$$H_*(M_K;\mathbb{Z}[\mathbb{Z}]) := H_*(\widehat{M}_K;\mathbb{Z}),$$
$$H^*(M_K;\mathbb{Z}[\mathbb{Z}]) := H^*_c(\widehat{M}_K;\mathbb{Z}).$$

The $\mathbb{Z}[\mathbb{Z}]$-module $H_1(M_K;\mathbb{Z}[\mathbb{Z}])$ is called the *Alexander module* of K. We say that K *has Alexander polynomial* 1 if $H_1(M_K;\mathbb{Z}[\mathbb{Z}]) = 0$.

We can state Freedman's theorem.

Theorem 2.3.3. *If K has Alexander polynomial* 1*, then K is topologically slice.*

We sketch the usual definition of the Alexander polynomial $\Delta_K(t)$ for the interested reader who might be flustered by our unconventional definition of Alexander polynomial one knots.

Remark 2.3.4. The Alexander polynomial $\Delta_K(t)$ is usually defined as the order of $H_1(X_K;\mathbb{Z}[\mathbb{Z}])$ (a Mayer-Vietoris exact sequence shows that M_K and X_K can be used interchangeably). $\Delta_K(t)$ is only defined up to multiplication by $\pm t^n$ and it is not hard to show that $H_1(M_K;\mathbb{Z}[\mathbb{Z}]) = 0$ if and only if $\Delta_K(t) = 1$ up to multiplication by $\pm t^n$. This justifies the terminology we used above.

The remainder of this subsection is devoted to the proof of Theorem 2.3.3. Recall from Exercise 2.3.1 that N_D has the homology of a circle. The basic surgery theoretic idea to slice a knot K is to start from a 4-manifold W whose boundary is M_K, and to perform surgery on W in order to get a homology circle.

In the topological category, this is nearly enough to guarantee the sliceness of K.

Proposition 2.3.5. *A knot is topologically slice if M_K bounds a topological* 4*-manifold V such that the following conditions hold:*

1. *the inclusion induced map $H_1(M_K;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is an isomorphism;*
2. *$\pi_1(V)$ is normally generated by the image of a meridian of K;*
3. *$H_2(V;\mathbb{Z}) = 0$.*

Proof. Recall that the 0-framed surgery M_K can be obtained from the knot exterior $X_K = S^3 \setminus \nu K$ by attaching a solid torus: $M_K = X_K \cup (D \times S^1)$, where the meridian $\partial D \times \{operatornamept\}$ is identified with the longitude of K, and the longitude $\{0\} \times S^1$ is identified with the meridian μ_K of K. By assumption, $\pi_1(V)$ is normally generated by the meridian of K. Adding a 4-dimensional 2-handle $D^2 \times D^2$ to V along this meridian therefore leads to a simply-connected 4-manifold N whose boundary is S^3. In particular, we have $H_1(N;\mathbb{Z}) = 0$.

We show that $H_i(N,\mathbb{Z}) = 0$ for $i > 0$. We already know that $H_1(N;\mathbb{Z}) = 0$, and $H_4(N;\mathbb{Z}) = 0$ since N has non-empty boundary. The third assumption states that $H_2(V;\mathbb{Z}) = 0$, and a Mayer-Vietoris argument gives $H_2(N;\mathbb{Z}) = H_2(V;\mathbb{Z}) = 0$. Using the first assumption and the long exact sequence of the pair $(V, \partial V)$, we deduce that $H_1(V,\partial V;\mathbb{Z}) = 0$. It then follows from Poincaré duality and the universal coefficient theorem that $H_3(V;\mathbb{Z}) = 0$. A Mayer-Vietoris argument shows that $H_3(N;\mathbb{Z}) = H_3(V;\mathbb{Z}) = 0$.

We have shown that N is a $\mathbb{Z}$-homology 4-ball with boundary S^3. Freedman's work now implies that N is homeomorphic to D^4 [FQ90]. By construction, $D \subset N$ is the desired slice disk for K. This concludes the proof of the proposition. □

Next, we outline the remainder of the proof of Theorem 2.3.3:

1. We construct a space $X \simeq S^1$ such that (X, M_K) is a4-dimensional *Poincaré pair*, i.e. there is a relative homology class $[X, M_K] \in H_4(X, M_K; \mathbb{Z})$ such that the relative cap product with $[X, M_K]$ yields the following isomorphisms:
$$\cap [X, M_K]\colon H^i(X; M_K; \mathbb{Z}[\mathbb{Z}]) \to H_{4-i}(X; \mathbb{Z}[\mathbb{Z}]),$$
$$\cap \partial[X, M_K]\colon H^i(M_K; \mathbb{Z}[\mathbb{Z}]) \to H_{3-i}(M_K; \mathbb{Z}[\mathbb{Z}]),$$
where $\partial\colon H_4(X, M_K; \mathbb{Z}) \to H_3(M_K; \mathbb{Z})$ is the connecting homomorphism in the long exact sequence of the pair (X, M_K).
2. We construct a topological 4-manifold W with boundary M_K and a degree one normal map
$$f\colon (W, M_K) \to (X, M_K).$$
3. We prove Theorem 2.3.3: starting from f, we construct another degree one normal map $f'\colon (W', M_K) \to (X', M_K)$ such that $\partial W' = \partial W = M_K$, and f' has vanishing surgery obstruction. Theorem 2.3.3 (and Remark 2.2.25) then produces manifold V with boundary M_K which satisfies the three properties of Proposition 2.3.5, thus proving that K is slice.

We now carry out this program, step by step.

Proposition 2.3.6. *If a knot K has Alexander polynomial 1, then there is CW-complex X that is homotopy equivalent to S^1 and such that (X, M_K) is a 4-dimensional Poincaré pair.*

Proof. Recall from Exercise 2.3.2 that $H_1(M_K; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the meridian μ_K of K. Consider the map $\varphi\colon H_1(M_K; \mathbb{Z}) \to \mathbb{Z}, \mu_K \to 1$. Some algebraic topology gives rise to an isomorphism $\operatorname{Hom}_{\mathbb{Z}}(H_1(M_K; \mathbb{Z}), \mathbb{Z}) \cong [X, S^1]$ (here $[X, S^1]$ refers to homotopy classes of maps and not to a relative homology class), and we therefore obtain a map $M_K \to S^1$ corresponding to φ. Let X be the mapping cylinder of this map. We have thus obtained an inclusion $\iota\colon M_K \hookrightarrow X$, where X is homotopy equivalent to S^1.

We must now show that (X, M_K) is a 4-dimensional Poincaré pair. Consider the long exact sequence of (X, M_K) with $\mathbb{Z}$ coefficients. Since X is homotopy equivalent to S^1, we know that $H_i(X; \mathbb{Z}) = 0$ for $i \geq 2$. The connecting homomorphism in the long exact sequence therefore induces an isomorphism $\partial\colon H_4(X, M_K; \mathbb{Z}) \xrightarrow{\cong} H_3(M_K; \mathbb{Z})$. Since M_K is closed, $H_3(M_K; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the fundamental class $[M_K]$ of M_K. We set $[X, M_K] := \partial^{-1}([M_K])$ and check that capping with this class gives rise to the required isomorphisms:
$$\begin{aligned} \cap [X, M_K]\colon H^i(X; M_K; \mathbb{Z}[\mathbb{Z}]) &\to H_{4-i}(X; \mathbb{Z}[\mathbb{Z}]), \\ \cap \partial[X, M_K]\colon H^i(M_K; \mathbb{Z}[\mathbb{Z}]) &\to H_{3-i}(M_K; \mathbb{Z}[\mathbb{Z}]). \end{aligned} \tag{2.9}$$
The second isomorphism holds by definition of $[X, M_K]$: by definition, we have $\partial[X, M_K] = [M_K]$ and, since M_K is a manifold, it satisfies Poincaré duality. We must therefore only check the first isomorphism. Since the $\mathbb{Z}$-cover of S^1 is $\mathbb{R}$, we deduce that $H_i(X; \mathbb{Z}[\mathbb{Z}]) = 0$ for $i \neq 0$ and $H^i(X; \mathbb{Z}[\mathbb{Z}]) = 0$ for $i \neq 1$. It follows that the bottom and top horizontal maps in the following diagram are isomorphisms:
$$\begin{array}{ccccccc} \longrightarrow & H^3(M_K; \mathbb{Z}[\mathbb{Z}]) & \xrightarrow{\cong} & H^4(X, M_K; \mathbb{Z}[\mathbb{Z}]) & \longrightarrow \\ & \cong \downarrow \cap[M_K] & & \downarrow \cap[X, M_K] & \\ \longrightarrow & H_0(M_K; \mathbb{Z}[\mathbb{Z}]) & \xrightarrow{\cong} & H_0(X; \mathbb{Z}[\mathbb{Z}]) & \longrightarrow \end{array}$$
The commutativity of this diagram implies that (2.9) is an isomorphism for $i = 4$. We now focus on the cases $i < 4$: here, since $H_{4-i}(X; \mathbb{Z}[\mathbb{Z}]) = 0$, it is enough to show that $H^i(X, M_K; \mathbb{Z}[\mathbb{Z}]) = 0$. Consider the long exact sequence of the pair (X, M_K) in cohomology:
$$\to H^{i-1}(X; \mathbb{Z}[\mathbb{Z}]) \to H^{i-1}(M_K; \mathbb{Z}[\mathbb{Z}]) \to H^i(X, M_K; \mathbb{Z}[\mathbb{Z}]) \to H^i(X; \mathbb{Z}[\mathbb{Z}]) \to H^i(M_K; \mathbb{Z}[\mathbb{Z}]) \to$$

For $i = 0$, we see that $H^0(X;\mathbb{Z}[\mathbb{Z}]) = 0$ implies $H^0(X, M_K;\mathbb{Z}[\mathbb{Z}]) = 0$. For $i = 1$, we use the exactness together with two facts: $H^0(M_K;\mathbb{Z}[\mathbb{Z}]) \cong H_3(M_K;\mathbb{Z}[\mathbb{Z}]) = 0$ (since $\mathbb{Z}$-coverings are non-compact), and $H^1(X;\mathbb{Z}[\mathbb{Z}]) \to H^1(M_K;\mathbb{Z}[\mathbb{Z}])$ is an isomorphism (since $M_K \to X$ induces an isomorphism on the fundamental groups). For $i = 2$, we recall that $H^2(X;\mathbb{Z}[\mathbb{Z}]) = 0$, and use a second time that $H^1(X;\mathbb{Z}[\mathbb{Z}]) \to H^1(M_K;\mathbb{Z}[\mathbb{Z}])$ is an isomorphism. Finally, for $i = 3$, we have $H^3(X;\mathbb{Z}[\mathbb{Z}]) = 0$ and $H^2(M_K;\mathbb{Z}[\mathbb{Z}]) \cong H_1(M_K;\mathbb{Z}[\mathbb{Z}]) = 0$ by duality and the Alexander polynomial one condition. We have therefore shown that (2.9) is an isomorphism for each i, concluding the proof of the proposition. □

We move on to the second step of our program.

Proposition 2.3.7. *If a knot K has Alexander polynomial 1 and (X, M_K) is the Poincaré pair constructed in Proposition 2.3.6, then there exists a degree one normal map of pairs*

$$f\colon (W, M_K) \to (X, M_K).$$

Proof. Our goal is to find a 4-manifold W with $\partial W = M_K$, a map $f\colon W \to X$ that extends the map $\iota\colon M_K \to X$ and a stable trivialisation of $TW \oplus f^*(\xi)$, where ξ is a vector bundle over X. Let ξ' be a trivial bundle over S^1. Since $X \simeq S^1$, this bundle pulls back to a trivial vector bundle ξ over X. As a consequence, the difficulty is to find a pair (W^4, f) with $\partial W = M_K$, a map f that extends $\iota\colon M_K \to X$ and a stable trivialization of TW.

Given a space Y and an integer n, there is a so-called *framed bordism group* $\Omega_n^{\mathrm{fr}}(Y)$ whose elements are equivalence classes of triples (M, θ, f), where M is an n-manifold, θ is a stable trivialisation of TM, and $f\colon M \to Y$ is a map. A triple (M, θ, f) represents zero in $\Omega_n^{\mathrm{fr}}(Y)$ if M bounds an $(n+1)$-manifold W, the map f extends to W, and the stable trivialisation θ extends to a stable trivialisation of TW. If we disregard the space Y and the maps to it, then we get a group Ω_n^{fr}. Since we already have a map $\iota\colon M_K \to X$, the idea of the proof is to define a stable trivialisation θ of TM_K such that $[M_K, \theta, \iota]$ is zero in $\Omega_3^{\mathrm{fr}}(X)$: by definition of this group and recalling the previous paragraph, this will immediately yield a normal nullbordism $W \to X$.

We first endow M_K with a stable trivialisation of its tangent bundle. Since M_K is an orientable 3-manifold, its tangent bundle is trivial, and we can choose (stable) framing θ' of TM_K. The pair (M_K, θ') therefore represents an element in Ω_3^{fr}. Some work shows that θ' can be modified, in order to produce a (stable) framing θ so that (M_K, θ) represents zero in Ω_3^{fr} [FQ90, proof of Lemma 11.6.B], as well as [CP14, proof of Lemma 3.3] (note that a nullbordant stable framing of M_K can also be constructed explicitly). We must now argue that (M_K, θ, ι) represents the zero element in $\Omega_3^{\mathrm{fr}}(X)$.

Recalling that X is homotopy equivalent to S^1, and using the Atiyah-Hirzebruch spectral sequence [CF64], it is known that there is an isomorphism $\Omega_3^{\mathrm{fr}}(S^1) \xrightarrow{\cong} \Omega_3^{\mathrm{fr}} \oplus \Omega_2^{\mathrm{fr}}$. The image of $[M_K, \theta, \iota]$ in the first summand is $[M_K, \theta]$, while its image in $\Omega_2^{\mathrm{fr}} \cong \mathbb{Z}_2$ can be shown to be the so-called "Arf invariant" of K:

$$\begin{aligned} \Omega_3^{\mathrm{fr}}(X) &\xrightarrow{\cong} \Omega_3^{\mathrm{fr}} \oplus \Omega_2^{\mathrm{fr}} \\ [M_K, \theta, \iota] &\mapsto ([M_K, \theta], \mathrm{Arf}(K)). \end{aligned}$$

It is a fairly well known fact in knot theory that the value of $\mathrm{Arf}(K)$ is determined mod 8 by the value of the Alexander polynomial at -1 [Lic97, Theorem 10.7]. In particular, if K has Alexander polynomial 1, then K has vanishing Arf invariant. We have therefore established that (M_K, θ, ι) represents zero in $\Omega_3^{\mathrm{fr}}(X)$ and, as we explained above, this is enough to guarantee the existence of the desired normal nullbordism $f\colon W \to X$.

It only remains to prove that $f\colon W \to X$ has degree one. We must show that f maps the fundamental class $[W, M_K]$ to the class $[X, M_K]$ defined in Proposition 2.3.6. Since the map

$f\colon (W, M_K) \to (X, M_K)$ extends $\iota\colon M_K \to X$, the following diagram commutes:

$$\begin{array}{ccc} H_4(W, M_K) & \xrightarrow{\partial} & H_3(M_K) \\ \downarrow f_* & & \downarrow = \\ H_4(X, M_K) & \xrightarrow{\partial} & H_3(M_K). \end{array}$$

We established in Proposition 2.3.6 that the bottom map is an isomorphism, and we defined the class $[X, M_K]$ as $\partial^{-1}[M_K]$. This shows that f has degree one and concludes the proof of the proposition. □

We can now conclude the proof of Theorem 2.3.3.

Proof of Theorem 2.3.3. Let $f\colon (W, M_K) \to (X, M_K)$ be the degree one normal map constructed in Proposition 2.3.7. Using surgery below the middle dimension, we can arrange that $f\colon W \to X$ is 2-connected. We saw in Theorem 2.2.24 (and Remark 2.2.25) that since $\mathbb{Z}$ is a good group, f is normal bordant to a homotopy equivalence if and only if the surgery obstruction $\sigma(f) \in L_4(\mathbb{Z}[\mathbb{Z}])$ is zero. Using Remark 2.2.23, we know that $L_4(\mathbb{Z}[\mathbb{Z}])$ is isomorphic to $L_4(\mathbb{Z}) = 8\mathbb{Z}$ and is detected by the usual signature.

Since we are in the topological category, Freedman proved that there is a simply connected closed manifold E_8 whose intersection form is the E_8 form. In fact, there is a degree one normal map $E_8 \to S^4$ whose surgery obstruction generates $L_4(\mathbb{Z}) \cong 8\mathbb{Z}$. Taking the connected sum of the degree one normal map $f\colon W \to X$ with $|\sigma(f)|$ copies of $\pm E_8 \to S^4$, we obtain a new degree one normal map $f'\colon W' \to X'$, where W' still has boundary M_K (since E_8 is closed), but f' has vanishing surgery obstruction. We deduce that $(W', M_K) \to (X', M_K)$ is normal bordant to a homotopy equivalence $(V, M_K) \to (X', M_K)$.

Since $M_K = \partial V$, in order to conclude the proof of the theorem, it only remains to show that V satisfies the homological properties of Theorem 2.3.5. Since V is homotopy equivalent to X', it has the same homology and homotopy groups as $X' = X \# S^4$, where $X \simeq S^1$. This immediately implies that $H_2(V; \mathbb{Z}) = 0$. Since, by construction, the map $\iota\colon M_K \to X \simeq S^1$ induces the isomorphism $H_1(M_K; \mathbb{Z}) \to H_1(S^1; \mathbb{Z}) = \pi_1(S^1), \mu_K \to 1$ on homology, we deduce that the group $\pi_1(V) = \pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$ is (normally) generated by the meridian of K and that $H_1(M_K; \mathbb{Z}) \to H_1(V; \mathbb{Z})$ is an isomorphism. We have therefore proved that M_K bounds a topological 4-manifold V which satisfies the three properties listed in Proposition 2.3.5. Applying this proposition, we deduce that K is topologically slice, concluding the proof of the theorem. □

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