REPRESENTATION ZETA FUNCTIONS OF WREATH PRODUCTS WITH FINITE GROUPS

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Abstract. Let G be a group which has a finite number $h_n(G)$ of irreducible linear representations in $GL_n(\mathbf{C})$ for all $n \geq 1$. Let $\zeta(G,s) = \sum_{n=1}^{\infty} h_n(G) n^{-s}$ be its representation zeta function.

First, in case $G = H \wr_X Q$ is a permutational wreath product with respect to a permutation group Q on a finite set X, we establish a formula for $\zeta(G,s)$ in terms of the zeta functions of H and of subgroups of Q, and of the Möbius function associated to the lattice $\Pi_Q(X)$ of partitions of X in orbits under subgroups of Q.

Then, we consider groups $W(Q,k) = (\cdots(Q \wr_X Q) \wr_X Q \cdots) \wr_X Q$ which are iterated wreath products (with k factors Q), and several related infinite groups $W(Q)$, including the profinite group $\varprojlim_k W(Q,k)$, a locally finite group $\lim_k W(Q,k)$, and several finitely generated dense subgroups of $\varprojlim_k W(Q,k)$. Under convenient hypotheses (in particular Q should be perfect), we show that $h_n(W(Q)) < \infty$ for all $n \geq 1$, and we establish that the Dirichlet series $\zeta(W(Q),s)$ has a finite and positive abscissa of convergence $s_0 = s_0(W(Q))$. Moreover, the function $\zeta(W(Q),s)$ satisfies a remarkable functional equation involving $\zeta(W(Q),es)$ for $e \in \{1,\dots,d\}$, where $d = |X|$. As a consequence of this, we exhibit some properties of the function, in particular that $\zeta(W(Q),s)$ has a singularity at s_0, a finite value at s_0, and a Puiseux expansion around s_0.

We finally report some numerical computations for $Q = A_5$ and $Q = PGL_3(\mathbf{F}_2)$.

1. Introduction

Let G be a group. We denote by $\widehat{G}$ the set of equivalence classes of irreducible linear representations of G in complex vector spaces of finite dimension; in case G is a topological group, we assume that the representations are continuous. The set $\widehat{G}$ is called here (and somewhat abusively) the *dual* of G. For $n \geq 1$, let $h_n(G) \in \{0,1,2,\dots,\infty\}$ denote the number of $\pi \in \widehat{G}$ of degree $\deg \pi = n$. The group G is *rigid* if $h_n(G) < \infty$ for all $n \geq 1$.

For example, an infinite cyclic group is not rigid, since $h_1(\mathbf{Z}) = \infty$. Rigid groups include finite groups, compact semisimple Lie groups (as a consequence of Weyl's formula for the dimensions of their irreducible representations), and many arithmetic groups (see [LaLu–08]).

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The *representation zeta function* of a rigid group G is the Dirichlet series

$$\zeta_G(s) \;=\; \sum_{\pi \in \widehat{G}} (\deg \pi)^{-s} \;=\; \sum_{n \geq 1} h_n(G) n^{-s} \ . \tag{1.1}$$

Whenever convenient, we write $\zeta(G, s)$ instead of $\zeta_G(s)$. For example, $\zeta(SU(2), s)$ is the Riemann zeta function since $h_n(SU(2)) = 1$ for any $n \geq 1$.

On the one hand, this function captures a very small part only of the representation theory of G; when G is finite, it is just a way to organise the information contained in the *degree pattern* of G, which is the list of the integers $\deg \pi$, including multiplicities (see [Hupp–98]). On the other hand, $\zeta_G(s)$ happens to be strongly related to several interesting questions, as shown by numerous articles including [Zagi–96], [LiSh–05], and [LaLu–08]. As far as we know, the first appearance of these zeta functions is in [Witt–91], where Formula (4.72) relates the evaluation at $2g-2$ of $\zeta_G(s)$ to the volume of the moduli space of flat connections of G–principal bundles over Σ_g, where G is a compact, simple, simply connected Lie group and Σ_g an orientable closed surface of genus $g \geq 2$.

Some elementary facts on representation zeta functions are collected in Section 9. The final Section 10 contains a remark concerning the "unitary variation" of representation zeta functions and a question about a possible strengthening of Kazhdan's Property (T).

The aim of the work reported here is to exhibit remarkable properties of these zeta functions for groups which are wreath products, and in particular for groups G which are isomorphic to some wreath products with themselves, namely of the form $G \cong G \wr_X Q$. Our main result is to show that there exists a group G for which $\zeta_G(s)$ has a *root singularity* at its abscissa of convergence (see Theorem 3), in sharp contrast to what happens for compact Lie groups (see Theorem 20).

2. Wreath products with a finite permutation group

Let H be a topological group and let Q be a finite group acting on a finite set X. Let

$$G \;=\; H \wr_X Q \;=\; B \rtimes Q, \qquad \text{with } \; B \doteqdot H^X, \tag{2.1}$$

denote the corresponding *wreath product*. Here, H^X denote the group of all applications from X to H, with pointwise multiplication, and $\rtimes$ indicates the semi–direct product with respect to the natural action of Q on B, given by $(qb)(x) = b(q^{-1}x)$ for $q \in Q$, $b \in B$, and $x \in X$. The topology on $B \rtimes Q$ is that for which $B = H^X$ has the product topology and is an open subgroup.

Our first goal is to revisit part of the *representation theory of wreath products.* This is a classical subject: see among others [Spec–33], [Kerb–71] (which contains historical comments on wreath products and their representations), and [Hupp–98, § 25]; let us also mention that the theory of iterated wreath products (see Section 3) goes back to Kaloujnine [Kalo–45, Kalo–48] and his students. As a consequence, an important part of what follows consists of variations on standard themes. This holds in particular for our first result, before the statement of which we need to define the appropriate Möbius function.

Any subgroup S of Q gives rise to a partition of X in S–orbits; we denote by $\Pi_Q(X)$ the *lattice* of all partitions of X of this kind, where, for $P, P' \in \Pi_Q(X)$, we have $P \leq P'$ if P is a refinement of P' (namely if every *block* of P is contained in a block of P'). For a partition $P = (P_1, \ldots, P_\ell)$ in $\Pi_Q(X)$, its *stability subgroup* is the corresponding subgroup

$$Q_P \;=\; \{q \in Q \mid q(P_1) = P_1, \ldots, q(P_\ell) = P_\ell\} \tag{2.2}$$

of Q. The smallest partition $\widehat{0}$, that for which all blocks are singletons, has stability subgroup the subgroup of Q of trivially acting elements (the one–element subgroup $\{1\}$ if the action is faithful), and the largest partition $\widehat{1}$, that for which the blocks are the Q–orbits, has stability subgroup Q (so that $\widehat{1}$ has just one block if Q acts transitively on X). We denote by μ_X the Möbius function of the lattice $\Pi_Q(X)$; recall that the domain of μ_X is $\Pi_Q(X)^2$, that $\mu_X(P, P'') = 0$ unless $P \leq P''$, that $\mu_X(P, P) = 1$ for all $P \in \Pi_X(Q)$, and that

$$\mu(P, P'') \;=\; - \sum_{\substack{P' \in \Pi_Q(X) \\ P \leq P' < P''}} \mu(P, P') \qquad \text{if } \; P < P''. \tag{2.3}$$

We refer to [Stan–97]; see in particular his Example 3.10.4 for the lattice of all partitions of X, which is our $\Pi_Q(X)$ for Q the group of all permutations of X. For an easier example, see the proof of (4.6) below.

Our first result is the following consequence of Clifford's theory, proven in Section 4.

1. Theorem. *Let H be a topological group, let Q be a finite group acting on a finite set X, and let $H \wr_X Q$ be the corresponding permutational wreath product, as above. Then $H \wr_X Q$ is a rigid if and only if H is rigid. Moreover, when this is the case, we have*

$$\begin{aligned} \zeta(H \wr_X Q, s) \;=\; & \sum_{P \in \Pi_Q(X)} [Q : Q_P]^{-1-s} \zeta_{Q_P}(s) \\ & \qquad \sum_{P' = (P'_1, \dots, P'_\ell) \geq P} \mu_X(P, P') \; \zeta_H(|P'_1|s) \cdots \zeta_H(|P'_\ell|s) \end{aligned} \tag{2.4}$$

(see also Formula (8.1) below).

For example, when Q is the permutation group of three objects acting on $\{1, 2, 3\}$, we have

$$\begin{aligned} \zeta(H \wr_3 S_3, s) \;=\; & 6^{-1-s} \big(\zeta_H(s)^3 - 3\zeta_H(s)\zeta_H(2s) + 2\zeta_H(3s)\big) \\ & + \, 3 \times 3^{-1-s} \times 2 \left(\zeta_H(s)\zeta_H(2s) - \zeta_H(3s)\right) \\ & + \, \big(2 + 2^{-s}\big) \, \zeta_H(3s) \\ = \; & 6^{-1-s}\zeta_H(s)^3 + \big(- \, 6^{-s}/2 + 2 \times 3^{-s}\big) \, \zeta_H(2s)\zeta_H(s) \\ & + \, \big(6^{-s}/3 - 2 \times 3^{-s} + 2^{-s} + 2\big) \, \zeta_H(3s). \end{aligned} \tag{2.5}$$

In particular, for the Weyl group $C_2 \wr_3 S_3$ of type B_3 and of order 48, Formula (2.5) specialises to

$$\zeta(C_2 \wr_3 S_3, s) \;=\; 4 + 2 \times 2^{-s} + 4 \times 3^{-3}, \tag{2.6}$$

so that we recover the well–known degree pattern $1, 1, 1, 1, 2, 2, 3, 3, 3, 3$. For other specialisations, see Sections 4 and 8.

3. Iterated wreath products

A finite group Q acting on a finite set X gives rise to a tower of iterated wreath products

$$W(Q,0) = \{1\}, \quad W(Q,1) = Q, \quad \ldots, \quad W(Q,k+1) = W(Q,k) \wr_X Q, \quad \ldots; \tag{3.1}$$

observe that

$$|W(Q,k)| = |Q|^{(d^k-1)/(d-1)}. \tag{3.2}$$

Recall the associativity of wreath products: given two finite groups Q, R acting respectively on two finite sets X, Y, there is a natural action of $Q \wr_Y R$ on $X \times Y$ and a natural isomorphism

$$\big(H \wr_X Q\big) \wr_Y R = H \wr_{(X\times Y)} \big(Q \wr_Y R\big). \tag{3.3}$$

In particular, we also have wreath product decompositions

$$W(Q,k+1) = Q \wr_{X^k} W(Q,k) \tag{3.4}$$

and corresponding natural split epimorphisms

$$W(Q,k+1) \longrightarrow W(Q,k), \tag{3.5}$$

which give rise to a profinite group

$$W^{prof}(Q) = \varprojlim_k W(Q,k). \tag{3.6}$$

For example, if $Q = S_d$ is the symmetric group acting in a standard way on $\{1,\ldots,d\}$, then $W^{prof}(S_d)$ is the full automorphism group of the infinite d–ary regular rooted tree (more on these trees below in Section 7). Groups of this kind are far from being rigid in general: for example,

$$h_n(W^{prof}(C_p)) = \begin{cases} \infty & \text{if } n = p^e \text{ for some } e \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

and $h_n(W^{prof}(S_d)) \geq h_n(W^{prof}(C_2))$ for all $d \geq 2$ and $n \geq 1$.

The situation is radically different if Q is perfect. More precisely, and this is *the main purpose of this paper,* our goal is to show that the representation Dirichlet series of the group $W^{prof}(Q)$ has an abscissa of convergence $s_0 = s_0(W^{prof}(Q))\} > 0$, and that the resulting function, holomorphic in the half–plane $\{\mathrm{Re}(s) > s_0\}$, has remarkable properties.

2. Theorem. *Let Q be a finite group acting transitively on a finite set X of size $d \geq 2$. The following properties are equivalent:*

(i) *the finite group Q is perfect,*
(ii) *the profinite group $W^{prof}(Q)$ is rigid.*

3. Theorem. *Let Q be as in the previous theorem and assume that Properties (i) and (ii) hold. Then the abscissa of convergence s_0 of $\zeta(W^{prof}(Q), s)$ is a finite positive number, and the function $\zeta(W^{prof}(Q), s)$ has a singularity at s_0 with a Puiseux expansion of the form*

$$\zeta(W^{prof}(Q), s) \;=\; \sum_{n=0}^{\infty} a_n (s - s_0)^{\frac{n}{e}} \tag{3.8}$$

for some integer e with $2 \leq e \leq |X|$.

Proofs are given in Section 6. In the situation of Theorem 3, additional information is contained in Proposition 17.

Moreover, we can replace $W^{prof}(Q)$ by various groups having the same representation zeta function, in particular a locally finite group $W^{locfin}(Q)$ and a finitely generated group $W^{fingen}(Q)$; see Section 7.

By contrast, in the situation of a semisimple compact connected Lie group G, the series $\zeta_G(s)$ has completely different properties. Some are given in Theorem 20, which puts together results of Weyl and Mahler, from the 1920's.

Let us particularise the situation of Theorems 2 and 3 to the smallest nontrivial perfect finite group, namely to the alternating group A_5 of order 60.

4. Example. *For $Q = A_5$ acting in the canonical way on a set of $d = 5$ elements, the representation zeta function $\zeta(s) = \zeta(W^{prof}(A_5), s)$ satisfies the functional equation*

$$\begin{aligned}\zeta(s) \;=\; & 60^{-1-s}\Big(\zeta(s)^5 - 10\zeta(s)^2\zeta(3s) - 15\zeta(s)\zeta(2s)^2 \\ & \qquad + 30\zeta(2s)\zeta(3s) + 30\zeta(s)\zeta(4s) - 36\zeta(5s)\Big) \\ & + 15 \times 30^{-1-s} \times 2 \times \Big(\zeta(s)\zeta(2s)^2 - 2\zeta(2s)\zeta(3s) - \zeta(s)\zeta(4s) + 2\zeta(5s)\Big) \\ & + 10 \times 20^{-1-s} \times 3 \times \Big(\zeta(s)^2\zeta(3s) - \zeta(2s)\zeta(3s) - 2\zeta(s)\zeta(4s) + 2\zeta(5s)\Big) \\ & + 10 \times 10^{-1-s}\left(2 + 2^{-s}\right)\Big(\zeta(2s)\zeta(3s) - \zeta(5s)\Big) \\ & + 5 \times 5^{-1-s}\left(3 + 3^{-s}\right)\Big(\zeta(s)\zeta(4s) - \zeta(5s)\Big) \\ & + \left(1 + 2 \times 3^{-s} + 4^{-s} + 5^{-s}\right)\zeta(5s).\end{aligned} \tag{3.9}$$

Then numerical computations show that

$$\begin{aligned}\zeta(s) \;\approx\; & 4.186576086287 - 6.740797357\,(s - s_0)^{\frac{1}{2}} \\ & + 5.6535295(s - s_0) - 1.421\,(s - s_0)^{\frac{3}{2}} + \cdots\end{aligned} \tag{3.10}$$

near

$$s_0 \approx 1.1783485957546400082. \tag{3.11}$$

Formula (3.9) makes it easy to obtain the first "few" terms of $\zeta(W^{prof}(A_5), s)$ from a computer:

$$\begin{aligned}\zeta(s) \;=\; & 1 \,+\, 2 \times 3^{-s} \,+\, 4^{-s} \,+\, 5^{-s} \,+\, 6 \times 15^{-s} \,+\, 3 \times 20^{-s} \,+\, 3 \times 25^{-s} \\ & + 2 \times 45^{-s} \,+\, 60^{-s} \,+\, 19 \times 75^{-s} \,+\, 4 \times 90^{-s} \,+\, 9 \times 100^{-s} \,+\, \cdots\end{aligned} \tag{3.12}$$

Indeed, if we agree that

$$[1,1],[3,2],[4,1],[5,1],[15,6],[20,3],[25,3],[45,2],[60,1],[75,19],[90,4],[100,9]$$

is a shorthand for the 12 first terms of the right–hand side of (3.12), the non–zero terms $[n, h_n]$ of $\zeta(s)$ for $n \leq 10^4$ are:

$[1,1],[3,2],[4,1],[5,1],[15,6],[20,3],[25,3],[45,2],[60,1],[75,19],$
$[90,4],[100,9],[125,9],[160,2],[180,5],[225,12],[240,6],[243,2],[250,2],[270,4],$
$[300,12],[320,1],[375,60],[400,3],[405,6],[450,12],[500,28],[540,2],[625,27],[640,2],$
$[675,2],[729,4],[800,6],[810,4],[900,52],[972,2],[1024,1],[1080,4],[1125,55],[1200,54],$
$[1215,16],[1250,8],[1280,4],[1350,20],[1440,4],[1500,81],[1600,12],[1620,12],[1875,189],[2000,27],$
$[2025,24],[2160,7],[2250,52],[2400,4],[2430,6],[2500,94],[2700,30],[3000,2],[3072,2],[3125,85],$
$[3200,6],[3375,18],[3600,20],[3645,16],[3750,2],[3840,13],[4000,24],[4050,40],[4096,1],[4320,8],$
$[4500,339],[4800,4],[4860,10],[5120,4],[5400,48],[5625,225],[5760,4],[6000,333],[6075,92],[6250,30],$
$[6400,18],[6480,8],[6750,92],[7200,36],[7500,442],[8000,75],[8100,106],[8640,3],[9000,12],[9375,603],$
$[9600,20],[9720,2],[10000,165]$

There are 2752 non-trivial coefficients of degree $\leq 10^{12}$, and it would be easy to extend the computations further.

In Section 8, we explain our numerical computations.

4. Proof of Theorem 1, and examples involving small groups Q

The proof of Theorem 1 is a simple application of Arthur Clifford's theory [Clif–37] which, in particular, provides a description of the dual $\widehat{G}$ of a group G given as an extension

$$1 \longrightarrow B \longrightarrow G \longrightarrow Q \longrightarrow 1 \tag{4.1}$$

with finite quotient Q. Let us recall this description in our case, which is simpler than the general case in two respects: the sequence splits, $G = B \rtimes Q$, and the representations of the subgroups of Q which are involved are linear (rather than, more generally, projective). Clifford's description of an irreducible representation of $G = H \wr_X Q$ has two ingredients.

The first ingredient is a Q–orbit in $\widehat{B}$, represented by some irreducible representation ρ. Since $B = H^X$, the dual $\widehat{B}$ can be identified with $(\widehat{H})^X$, and ρ can be written uniquely as an outer tensor product $\boxtimes^{x \in X} \rho_x$, with $\rho_x : H \longrightarrow GL(V_x)$ in $\widehat{H}$ for each $x \in X$. The *stability subgroup* of ρ is the subgroup

$$Q_\rho \,=\, \{q \in Q \mid q\rho \sim \rho\}\,, \tag{4.2}$$

where $\sim$ indicates equivalence of representations. We have

$$Q_\rho \,=\, Q_{P_\rho}, \tag{4.3}$$

where the right–hand side is defined by (2.2), and where the partition P_ρ is that for which $x, y \in X$ are in the same block if and only if $\rho_x \sim \rho_y$. Moreover, since the action of Q on $\widehat{B} = (\widehat{H})^X$ is induced by the action of Q on X, the representation $\rho = \boxtimes^{x \in X} \rho_x$ of B in the vector space $\otimes^{x\in X} V_x$ extends to a representation ρ' of $B \rtimes Q_\rho$ in the same space, defined by $\rho'(b,s) = \rho(b)$ for all $b \in B$ and $s \in Q_\rho$.

The second ingredient is an irreducible representation $\sigma \in \widehat{Q_\rho}$. We view it as an irreducible representation σ' of $B \rtimes Q_\rho$, of which the group Q_ρ is a quotient, so that $\rho' \otimes \sigma'$ is also an irreducible representation of $B \rtimes Q_\rho$. Denote by

$$\pi_{\rho,\sigma} \;=\; \mathrm{Ind}_{B\rtimes Q_\rho}^{G} \, (\rho' \otimes \sigma') \tag{4.4}$$

the induced representation, and observe that

$$\deg \pi_{\rho,\sigma} \;=\; [G : B \rtimes Q_\rho] \; \deg \rho \; \deg \sigma \; . \tag{4.5}$$

5. Proposition (Clifford). *With the notation above, the representation $\pi_{\rho,\sigma}$ of $G = H \wr_X Q$ is irreducible, and any irreducible representation of G is of this form, in a unique way.*

In other words, the dual $\widehat{G}$ is fibred (as a set) over the orbit space $Q\backslash\widehat{B}$; the fibre over an orbit represented by $\rho \in \widehat{B}$ is the dual $\widehat{Q_\rho}$.

In particular, G is rigid if and only if H is rigid.

[The general case of a group extension is more complicated, since ρ' is a projective representation of $G_\rho \doteqdot \{q \in Q \mid q\rho \sim \rho\}$ which need not be linear. One has to choose σ as a projective representation of Q_ρ such that the class of ρ' in $H^2(G_\rho, \mathbf{C}^*)$ is minus the pull–back of the class of σ in $H^2(Q_\rho, \mathbf{C}^*)$. In our particular case, $G_\rho = B \rtimes Q_\rho$. Yet there is a formulation of the proposition which carries over to the general case of (4.1).]

We are now ready to check the formula of Theorem 1.

Consider a partition $P = (P_1, \ldots, P_j)$ in $\Pi_Q(X)$. Let $\widehat{B}_P^{\leq}$ be the subset of $\widehat{B}$ consisting of those $\boxtimes^{x\in X}\rho_x$ for which $\rho_x \sim \rho_y$ as soon as x, y are in the same block of P. We have a product formula

$$\sum_{\rho=(\rho_x)_{x\in X}\in\widehat{B}_P^{\leq}} \deg\left(\boxtimes^{x\in X}\rho_x\right)^{-s} \;=\; \zeta_H(|P_1|s)\cdots\zeta_H(|P_j|s).$$

Let $\widehat{B}_P^{=}$ be the subset of $\widehat{B}_P^{\leq}$ consisting of those $\boxtimes^{x\in X}\rho_x$ for which $\rho_x \sim \rho_y$ if and only if x, y are in the same block of P. By the defining property of the Möbius function of $\Pi_Q(X)$, we have

$$\sum_{\rho=(\rho_x)_{x\in X}\in\widehat{B}_P^{=}} \deg\left(\boxtimes^{x\in X}\rho_x\right)^{-s} \;=\; \sum_{P'=(P'_1,\ldots,P'_\ell)\geq P} \mu_X(P,P')\zeta_H(|P'_1|s)\cdots\zeta_H(|P'_\ell|s).$$

It follows that the contribution to $\zeta_G(s)$ of the representations $\pi_{\rho,\sigma}$, with $\rho \in \widehat{B}_P^{=}$ and $\sigma \in \widehat{Q_P}$ (recall that Q_P has been defined in (2.2)), is

$$\sum_{\rho\in Q_P\backslash\widehat{B}_P^{=},\sigma\in\widehat{Q_\rho}} \deg\pi_{\rho,\sigma}^{-s} \;=\; [Q:Q_P]^{-1-s}\zeta_{Q_P}(s)\sum_{P'\geq P}\mu_X(P,P')\zeta_H(|P'_1|s)\cdots\zeta_H(|P'_\ell|s),$$

where $P' = (P'_1, \ldots, P'_\ell)$. The factor $[Q : Q_P]^{-s}$ is due to the induction from $B \rtimes Q_P$ to G, which multiplies the degrees of representations by $[Q : Q_P]$; as we have to count only one

ρ by Q–orbit in $\widehat{B}$, or more precisely here one ρ by Q_P–orbit in $\widehat{B}^{=}_{\overline{P}}$, there is an extra factor $[Q : Q_P]^{-1}$ on the right–hand side. A summation over $P \in \Pi_Q(X)$ gives rise to the formula of Theorem 1.

In general, computing the Möbius function of a lattice, for example of $\Pi_Q(X)$, is a tedious problem; but special cases can be worked out. For example, when the finite group Q is abelian and acts on itself ($X = Q$) by multiplications, the computation of μ_X goes back to [Dels–48].

Let us consider here some easy specialisations of Theorem 1. For an integer $d \geq 2$, we denote by I_d the finite set $\{1, \dots, d\}$, by C_d the cyclic group of order d acting on I_d by cyclic permutations, by S_d the symmetric group of I_d, and by $H \wr_d \cdots$ the corresponding wreath products. We denote by μ the usual Möbius function of elementary number theory.

6. Proposition. *With the notation above we have for an integer $d \geq 2$:*

$$\zeta(H \wr_d C_d, s) \;=\; \sum_{e|d} \left(\frac{d}{e}\right)^{-1-s} e \sum_{f \text{with } e|f|d} \mu\left(\frac{f}{e}\right) \zeta_H(fs)^{m/f} \; . \tag{4.6}$$

In particular, if $m = p$ is prime, the summation has three terms:

$$\zeta(H \wr_p C_p, s) \;=\; p^{-1-s}\Big(\zeta_H(s)^p - \zeta_H(ps)\Big) \;+\; p\zeta_H(ps) \; . \tag{4.7}$$

If $m = 4$, the summation has five terms

$$\begin{aligned}\zeta(H \wr_4 C_4, s) \;=\;& 4^{-1-s}\Big(\zeta_H(s)^4 - \zeta_H(2s)^2\Big) \\ &+\; 2^{-s}\Big(\zeta_H(2s)^2 - \zeta_H(4s)\Big) \;+\; 4\zeta_H(4s) \; .\end{aligned} \tag{4.8}$$

For the permutation group on three objects, we have

$$\begin{aligned}\zeta(H \wr_3 S_3, s) \;=\;& 6^{-1-s}\zeta_H(s)^3 + \big(-\, 6^{-s}/2 + 2 \times 3^{-s}\big)\, \zeta_H(2s)\zeta_H(s) \\ &+\; \big(6^{-s}/3 - 2 \times 3^{-s} + 2^{-s} + 2\big)\, \zeta_H(3s).\end{aligned} \tag{4.9 = 2.5}$$

For the permutation group on four objects, we have

$$\begin{aligned}&\zeta(H \wr_4 S_4, s) \;=\; \\ &\quad 24^{-1-s}\left(\zeta_H(s)^4 - 6\zeta_H(s)^2\zeta_H(2s) + 8\zeta_H(s)\zeta_H(3s) + 3\zeta_H(2s)^2 - 6\zeta_H(4s)\right) \\ &\quad +\; 6 \times 12^{-1-s} \times 2 \times \left(\zeta_H(s)^2\zeta_H(2s) - 2\zeta_H(s)\zeta_H(3s) - \zeta_H(2s)^2 + 2\zeta_H(4s)\right) \\ &\quad +\; 4 \times 4^{-1-s}(2 + 2^{-s})\;\; (\zeta_H(s)\zeta_H(3s) - \zeta_H(4s)) \\ &\quad +\; 3 \times 6^{-1-s} \times 4\;\; \left(\zeta_H(2s)^2 - \zeta_H(4s)\right) \\ &\quad +\; \left(2 + 2^{-s} + 2 \times 3^{-s}\right) \zeta_H(4s).\end{aligned} \tag{4.10}$$

Remark. The reader will have no problem to guess the formula for $\zeta(H \wr_5 A_5, s)$ from Formula (3.9); see (8.3).

Proof. For an integer $d \geq 2$, the subgroups of the cyclic group C_d of order d are in one–to–one correspondence with the positive divisors of d. If C_d acts on itself by multiplication,

$X = C_d$, the orbits of a subgroup C_e of C_d, with $e|d$, are of the form $(j, j+d/e, j+2d/e, ...)$. If e, f are positive divisors of d, the partition P into orbits of the subgroup C_e is a refinement of the Partition P' in the orbits of the subgroup C_f if and only if $e|f$, or equivalently if and only if $C_e \leq C_f \leq C_d$. When this is the case, $\mu_X(P, P') = \mu(f/e)$, with μ the standard Möbius function. Thus Formula (4.6) for $\zeta(H \wr_X C_d, s)$ is indeed a particular case of the formula of Theorem 1.

If $m = p$ is prime, observe that the summation in (4.6) has only three terms:

$$(d, f) \;=\; (1, 1),\;\; (1, p),\;\; (p, p),$$

and (4.7) follows.

If $m = 4$, there are five terms:

$$(d, f) \;=\; (1, 1),\;\; (1, 2),\;\; (2, 2),\;\; (2, 4),\;\; (4, 4),$$

and (4.8) follows; the pair $(d, f) = (1, 4)$ does not contribute, because $\mu(4) = 0$.

For $Q = S_3$ acting on $I_3 = \{1, 2, 3\}$, the lattice $\Pi_Q(I_3)$ consists of five partitions: the partition $\widehat{0}$ in singletons, three partitions $P^{(j)}$ in the singleton $\{j\}$ and a block of size two, $j = 1, 2, 3$, and the partition $\widehat{1}$ in one block. The values of the Möbius function μ_X are given by the following table

$$\begin{array}{llllll} \mu(\widehat{0},\widehat{0}) & = 1 & \mu(\widehat{0},P^{(j)}) & = -1 & \mu(\widehat{0},\widehat{1}) & = \;\; 2 \\ & & \mu(P^{(j)},P^{(j)}) & = \;\; 1 & \mu(P^{(j)},\widehat{1}) & = -1 \\ & & & & \mu(\widehat{1},\widehat{1}) & = \;\; 1. \end{array}$$

[Observe that the subgroups of S_3 of the form $(S_3)_P$ for some $P \in \Pi_{S_3}(I_3)$ are the subgroups of order 1, 2, and 6, but not the subgroup of order 3; indeed, the stability subgroup of the orbits of the subgroup of order 3 is the whole group S_3.] The left–hand side of Formula (2.4) specialises to

$$\begin{aligned} & 6^{-1-s}\Big(\zeta_H(s)^3 - 3\zeta_H(s)\zeta_H(2s) + 2\zeta_H(3s)\Big) \\ + \;\; & 3 \times 3^{-1-s} \times 2\Big(\zeta_H(s)\zeta_H(2s) - \zeta_H(3s)\Big) \\ + \;\; & (2 + 2^{-s})\zeta_H(3s), \end{aligned}$$

namely to (4.9) after minor reorganisation.

Let us finally check Formula (4.10). The lattice $\Pi_{S_4}(I_4)$ has

— 1 partition with blocks of size $1, 1, 1, 1$,
— 6 partitions with blocks of size $2, 1, 1$,
— 4 partitions with blocks of size $3, 1$,
— 3 partitions with blocks of size $2, 2$,
— 1 partition with one block of size 4,

namely altogether 15 partitions. We leave it to the reader to compute the Möbius function. The summation in (4.10) has $5 + 4 + 2 + 2 + 1 = 14$ terms, more precisely:

Terms with $(S_4)_P = \{1\}$, and therefore with $\zeta_{(S_4)_P}(s) = 1$, contribute

$$24^{-1-s}\left(\zeta_H(s)^4 - 6\zeta_H(s)^2\zeta_H(2s) + 8\zeta_H(s)\zeta_H(3s) + 3\zeta_H(2s)^2 - 6\zeta_H(4s)\right).$$

Terms with $(S_4)_P \cong S_2$ (fixing two of the four points of I_4), and therefore with $\zeta_{(S_4)_P}(s) = 2$, contribute

$$6 \times 12^{-1-s} \times 2 \times \left(\zeta_H(s)^2\zeta_H(2s) - 2\zeta_H(s)\zeta_H(3s) - \zeta_H(2s)^2 + 2\zeta_H(4s)\right).$$

Terms with $(S_4)_P \cong S_3$, and therefore with $\zeta_{(S_4)_P}(s) = 2 + 2^{-s}$, contribute

$$4 \times 4^{-1-s}(2 + 2^{-s}) \ \left(\zeta_H(s)\zeta_H(3s) - \zeta_H(4s)\right).$$

Terms with $(S_4)_P \cong S_2 \times S_2$ contribute

$$3 \times 6^{-1-s} \times 4 \ \left(\zeta_H(2s)^2 - \zeta_H(4s)\right).$$

The term with $(S_4)_P = S_4$ contributes

$$\left(2 + 2^{-s} + 2 \times 3^{-s}\right)\zeta_H(4s).$$

□

5. Reminder on representations of profinite groups

Concerning representations of profinite groups, Claim (i) in the following proposition[1] is well–known, but we did not find any convenient reference. Claim (ii) is a straightforward consequence of the definitions (and is a particular case of, for example, [Wils–98, Proposition 1.2.1]).

If G is a profinite group (or more generally a compact group, for example a finite group!), recall that any representation of G in a Hilbert space is unitarisable. Recall also that, in this paper, representations are continuous, unless explicitly stated otherwise.

7. Proposition. *Let G be a profinite group.*

(i) Let $\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ be an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$. Then π factors through a finite quotient of G.

(ii) Assume that $G = \varprojlim F_n$ is an inverse limit of a system $\left(p_{n,n-1} : F_n \longrightarrow F_{n-1}\right)_{n\geq 1}$ of finite groups and epimorphisms (here indexed by integers). If F is a finite group and $p : G \longrightarrow F$ a continuous epimorphism, then there exists an integer n such that p is the composition of the canonical epimorphism $p_n : G \longrightarrow F_n$ and some epimorphism $F_n \longrightarrow F$.

Proof. Recall that a *profinite group* is a compact topological group in which every neighbourhood of 1 contains an open normal subgroup (a subgroup of a compact group which is open is necessarily of finite index). Equivalently, a profinite group is an inverse limit of finite groups.

(i) Since irreducible representations of compact groups are finite dimensional (see for example [Robe–83, Corollary 5.8]), $\mathcal{H}$ is finite dimensional; we can assume that $\mathcal{H} \neq \{0\}$.

Consider a nonzero vector $\xi \in \mathcal{H}$. Since π is continuous at 1, there exists a neighbourhood U of 1 in G such that

$$\|\pi(g)\xi - \xi\| < 1 \quad \text{for all } g \in U.$$

[1] We are grateful to Bachir Bekka for showing to us his personal notes on this.

Since G is profinite, U contains a normal subgroup N of finite index. Thus

$$\|\pi(g)\xi - \xi\| < 1 \quad \text{for all } g \in N.$$

This implies that $\eta \doteqdot \int_N \pi(n)\xi dn$ (with dn the Haar measure on N of mass one) is a non–zero N–invariant vector, and in particular that the space $\mathcal{H}^N$ of N–invariant vectors is not $\{0\}$.

Since N is normal, the space $\mathcal{H}^N$ is $\pi(G)$–invariant. Since π is irreducible, $\mathcal{H}^N = \mathcal{H}$; in other words, π factors through G/N.

(ii) Since the kernel of p is closed and of finite index, it is also open. It follows from the definition of the topology of $\varprojlim F_n$ that $\ker p$ contains $\ker p_n$ for some n, and the claim follows. □

8. Corollary. *Let Q be a finite group acting on a finite set X, as in Section 2. We have*

$$h_n(W(Q,k+1)) \ge h_n(W(Q,k)) \tag{5.1}$$

for all $n \ge 1$, $k \ge 0$, and

$$h_n(W^{prof}(Q)) = \lim_{k\to\infty} h_n(W(Q,k)) \in \{0,1,2,\dots,\infty\} \tag{5.2}$$

for all $n \ge 1$.

Moreover, $h_n(W^{prof}(Q)) = 0$ unless n is of the form $p_1^{e_1} \dots p_r^{e_r}$, where $p_1, \dots, p_r$ are the prime factors of the order of Q; and the same holds for $h_n(W(Q,k))$ for all $k \ge 0$.

Proof. Inequalities (5.1) follow from (3.5) and Equality (5.2) follows from Proposition 7. The last statement follows from (3.2) and from the general fact according to which the degrees of the irreducible representations of a finite group divide the order of this group. □

9. Particular case. *For $d \ge 2$, any irreducible representation of the group $W^{prof}(S_d)$ of automorphisms of the d–ary tree factors through $W(S_d,k)$, for some $k \ge 0$.*

Similarly, any irreducible representation of the group $W^{prof}(C_d)$ of d-adic automorphisms of the d–ary tree factors through $W(C_d,k)$, for some $k \ge 0$.

In the case of $Q = C_p$ cyclic of prime order p, we can be more specific. For a finite p–group H, let $p^{\delta_{\max}(H)}$ denote the maximum of the degrees of the irreducible representations of H. A first rather straightforward consequence of Formula (4.7) is

$$\delta_{\max}(W(C_p,k)) = \begin{cases} 2^{k-2} + 2^{k-3} - 1 & \text{if } p = 2 \text{ and } k \ge 3, \\ 0 \ [\text{respectively } 1] & \text{if } p = 2 \text{ and } k = 1 \ [\text{respectively } k = 2], \\ 1 + p + \cdots + p^{k-2} & \text{if } p \ge 3 \text{ and } k \ge 2. \end{cases} \tag{5.3}$$

A second set of consequences of (4.7), using slightly more calculus, is that

$$\lim_{k\to\infty} h_{p^j}(W(C_p,k)) = \infty \quad \text{for all } j \ge 0 \tag{5.4}$$

and that the degree set of $W(C_p,k)$ is

$$cd(W(C_p,k)) = \left\{p^j \mid 0 \le j \le \delta_{\max}(W(C_p,k))\right\}. \tag{5.5}$$

Formula (3.7) for $h_n(W^{prof}(C_p))$ follows.

Recall that the *degree set* of a finite group G is defined by $cd(G) = \{n \in \mathbf{N} \mid h_n(G) > 0\}$. About (5.5), let us recall that the possible degree sets of finite p–groups are known; indeed, by a theorem of Isaacs, any finite subset of $\mathbf{N}$ of the form

$$\{p^{e_j} \mid 0 \le j \le m\} \quad \text{with } e_0 = 0 < e_1 < \cdots < e_m$$

is such a set [Hupp–98, p. 352]. Much less seems to be known about the degree sets of more general finite groups; see in particular [Hupp–98, Remarks 24.5 and § 27]. Possible degree patterns (namely possible representation zeta functions of finite groups) are even more mysterious [Hupp–98, § 6].

Observe that $W^{prof}(C_d)$ also has *non–continuous* unitary representations. For example, it can be seen that the abelianisation of $W^{prof}(C_d)$ is isomorphic to the direct product $\prod_{k\ge1} C_d$ of infinitely many copies of C_d. Given any free ultrafilter ω on $\mathbf{N}$ and a character $\chi \ne 1$ of C_d, the composition of the abelianisation $W^{prof}(C_d) \longrightarrow \prod_{k\ge1} C_d$ with the ω–limit $\prod_{k\ge1} C_d \longrightarrow \mathbf{C}^*$, $(c_k)_{k\ge1} \longmapsto \lim_\omega \chi(c_k)$ is a discontinuous character $W^{prof}(C_d) \longrightarrow \mathbf{C}^*$.

6. Proof of Theorems 2 and 3

Let Q be a finite group acting on a finite set X of size $d \ge 2$. In this long section, we shall denote by $W(Q)$ the profinite group denoted by $W^{prof}(Q)$ in Section 3. We will first prove Theorem 2:

— $W(Q)$ is rigid if and only if Q is perfect;

and then Theorem 3, namely, in case Q is perfect, that:

— the Dirichlet series $\zeta(W(Q), s)$ has a finite abscissa of convergence, say $s_0 = s_0(W(Q))$, see Proposition 15,
— $s_0 > 0$, see Proposition 16,
— $\zeta(W(Q), s)$ has near s_0 a Puiseux expansion of the form $\sum_{n=0}^{\infty} a_n(s - s_0)^{n/e}$, for some $e \le d$, see Proposition 17.

10. Observation. *(i) For two integers $n \ge 2$ and $d \ge 2$, we have*

$$\left[\sqrt[d]{n}\right] \le \frac{n}{2}. \tag{6.1}$$

(ii) For a pair of integers e, f such that $0 \le f \le e$ and a prime p, we have

$$\frac{2+e}{(2+e-f)p^f} \begin{cases} = 1 & \text{if } f = 0, \\ \le \dfrac{3}{4} & \text{if } f \ge 1. \end{cases} \tag{6.2}$$

Proof. (i) If $n \ge 3$ and $d \ge 3$, the inequality follows from $2^d \le 8 \times 2^{d-3} < 9 \times n^{d-3} \le n^{d-1}$. If $n \ge 4$ and $d = 2$, ditto from $\sqrt{n} \le \frac{n}{2}$. If $n = 3$ and $d = 2$, then $[\sqrt{n}\,] = 1 < 1.5 = \frac{n}{2}$. If $n = 2$ and $d \ge 2$, then $[\sqrt[d]{n}\,] = 1 = \frac{n}{2}$.

(ii) If $f = 0$, the equality is obvious. If $f = 1$ (so that $e \ge 1$), we have

$$\frac{2+e}{(1+e)p} \le \frac{\frac{3}{2}(1+e)}{(1+e)p} \le \frac{3}{4}.$$

If $f = 2$ (so that $e \geq 2$), we have

$$\frac{2+e}{ep^2} \leq \frac{2}{p^2} \leq \frac{1}{2}.$$

If $f \geq 3$ and $f \leq \frac{3}{4}e$, we have

$$\frac{2+e}{(2+e-f)p^f} \leq \frac{2+e}{(2+\frac{1}{4}e)8} \leq \frac{1}{2}.$$

If $f \geq 3$ and $f \geq \frac{3}{4}e$, we have

$$\frac{2+e}{(2+e-f)p^f} \leq \frac{2e}{2 \times 2^{3e/4}} \leq \frac{3}{4}.$$

□

For all positive $\nu \in \mathbf{R}$ and $k \in \mathbf{N}$, set

$$h_{\nu,k} = \begin{cases} h_n(W(Q,k)) & \text{if } \nu = n \in \mathbf{N}, \\ 0 & \text{if } \nu \notin \mathbf{N}, \end{cases} \qquad \text{and} \qquad h_\nu = \begin{cases} h_n(W(Q)) & \text{if } \nu = n \in \mathbf{N}, \\ 0 & \text{if } \nu \notin \mathbf{N}. \end{cases}$$

(Values for $\nu \notin \mathbf{N}$ will only occur in Lemma 14.) We write d for $|X|$.

In all what follows, we assume that $d \geq 2$, and that Q acts transitively on X. (The transitivity hypothesis could most likely be weakened, however arbitrary actions – and in particular points of X fixed by Q – would introduce unnecessary complications.)

11. Lemma. *Assume that the group Q is perfect and acts transitively on a set X of size $d \geq 2$. For any $n \geq 1$, we have*

$$h_n = h_{n,\ell} = h_{n,k} \quad \text{for all } \ell \geq k \text{ and } k > \log_2 n. \tag{6.3}$$

Proof. Step one. If Q is perfect, then $W(Q,k)$ is also perfect, so that $h_{1,k} = 1$ for all $k \geq 1$. Hence, the lemma holds for $n = 1$.

Step two. From Clifford's theory applied to $W(Q,k+1) = W(Q,k) \wr_X Q$, we see that any irreducible representation π of $W(Q,k+1)$ of degree n is of the form.

$$\pi = \pi_{\rho_1,\dots,\rho_d;\sigma} = \mathrm{Ind}_{W(Q,k)^X \rtimes Q_P}^{W(Q,k)^X \rtimes Q = W(Q,k+1)} (\rho_1 \boxtimes \cdots \boxtimes \rho_d \boxtimes \sigma), \tag{6.4}$$

where $(\rho_1, \dots, \rho_d)$ is a d–uple of irreducible representations of $W(Q,k)$, where P is the partition of X for which $\rho_i \sim \rho_j$ if and only if i,j are in the same block of P, and where σ is an irreducible representation of Q_P (compare with (4.4)). Set

$$f = [W(Q,k+1) : W(Q,k)^X \rtimes Q_P] \deg\sigma = [Q : Q_P] \deg\sigma$$

and distinguish two cases.

In the first case, $f = 1$. Thus, on the one hand $[Q : Q_P] = 1$, and therefore $\rho_1 = \cdots = \rho_d$, and on the other hand $\deg \sigma = 1$, and therefore $\sigma = 1$. Hence

$$n \;=\; \deg \pi_{\rho_1,\dots,\rho_1;1} \;=\; (\deg \rho_1)^d \quad \text{or} \quad \deg \rho_1 \;=\; \sqrt[d]{\deg \pi} \;=\; \sqrt[d]{n}. \tag{6.5}$$

In the second case, $f \geq 2$. Hence

$$n \;=\; \deg \pi_{\rho_1,\dots,\rho_d;\sigma} \;\geq\; 2\deg(\rho_1)\cdots\deg(\rho_d) \;\geq\; 2\deg(\rho_i)$$

namely

$$\deg \rho_i \;\leq\; \frac{n}{2} \qquad \forall i \in \{1,\dots,d\}. \tag{6.6}$$

By (6.1),

$$\deg \rho_i \;\leq\; \left[\frac{n}{2}\right] \qquad \forall i \in \{1,\dots,d\}$$

in the two cases.

Step three. Let us show that the lemma holds for $n < 2^m$, by induction on m. Since the case $m = 1$ is covered by Step one, we can assume that $m > 1$ and that the lemma holds up to $m - 1$. Step two shows that there exists a formula of the type

$$h_{n,\ell} \;=\; F\left(h_{1,\ell-1},\dots,h_{[n/2],\ell-1}\right) \tag{6.7}$$

where F is an expression independent of ℓ. Since $[n/2] < 2^{m-1}$, we have

$$h_{1,\ell-1} \;=\; h_{1,k-1}, \;\; \dots, \;\; h_{[n/2],\ell-1} = h_{[n/2],k-1}$$

for $\ell - 1 \geq k - 1$ and $k - 1 > m - 1$, by the induction hypothesis. It follows from (6.7) that

$$h_{n,\ell} \;=\; h_{n,k} \qquad \text{for all } \;\ell \geq k\; \text{ and } \;k > \log_2 n$$

whenever $n < 2^m$, and this completes the induction step. □

12. Proof of Theorem 2.

If Q is perfect, then $W(Q)$ is rigid by the previous lemma.

Assume Q is not perfect. It is known that the abelianisation of $W(Q, k)$ is isomorphic to the direct sum of k copies of the abelianisation of Q. (See [BORT–96, § 4.4, p. 145] for a more general result, since there the action of Q on X need not be transitive.) In particular,

$$h_1(W(Q)) \;\geq\; h_1\Big(\bigoplus_{j=1}^{\infty}(Q/[Q,Q])_j\Big) \;=\; \infty$$

(where each $(Q/[Q,Q])_j$ denotes a copy of $Q/[Q,Q]$), and $W(Q)$ is not rigid. □

We now proceed to prove Theorem 3.

13. Lemma. *There exists a constant $t_0 \geq 0$ with the following property. For any pair of integers $d \geq 1$ and $g \geq 0$, we have*

$$\sum_{\substack{g_1,\dots,g_d \geq 0 \\ \sum_{i=1}^d g_i = g}} \left(\frac{1+\frac{g}{2}}{(1+\frac{g_1}{2})\cdots(1+\frac{g_d}{2})} \right)^t \leq 4^{d-1} \qquad \text{for all } t \geq t_0. \tag{6.8}$$

More precisely, any t_0 such that

$$1 + 2\left(\frac{5}{6}\right)^{t_0} + \sum_{h=3}^{\infty} \left(\frac{2}{1+\frac{h}{2}}\right)^{t_0} \leq 2 \tag{6.9}$$

is suitable.

Proof. For $d = 1$, the lemma holds with $t_0 = 0$. We assume from now on that $d \geq 2$, and we proceed by induction on d, assuming that the lemma holds up to $d-1$. For $g = 0$ and $g = 1$, the inequality reduces respectively to $1 \leq 4^{d-1}$ and $d \leq 4^{d-1}$, so that $t_0 = 0$ is again suitable; we can assume therefore that $g \geq 2$.

The left–hand side of (6.8) can be written as

$$\sum_{g_1=0}^{g} \left(\frac{1+\frac{g}{2}}{(1+\frac{g_1}{2})(1+\frac{g-g_1}{2})} \right)^t \sum_{\substack{g_2,\dots,g_d \geq 0 \\ \sum_{i=2}^d g_i = g-g_1}} \left(\frac{1+\frac{g-g_1}{2}}{(1+\frac{g_2}{2})\cdots(1+\frac{g_d}{2})} \right)^t, \tag{6.10}$$

where the second summation is bounded by 4^{d-2}, by the induction hypothesis. If $g = 2$, the first sum is $2 + \left(\frac{8}{9}\right)^t$, and in particular is bounded by 4, so that we can assume now that $g \geq 3$.

The summation on g_1 from 0 to g in (6.10) is bounded by twice the summation on g_1 from 0 to $[g/2]$, namely by

$$2\left\{ 1 + \left(\frac{1+\frac{g}{2}}{\frac{3}{2}(1+\frac{g-1}{2})} \right)^t + \left(\frac{1+\frac{g}{2}}{2(1+\frac{g-2}{2})} \right)^t + \sum_{g_1=3}^{[g/2]} \left(\frac{1+\frac{g}{2}}{(1+\frac{g_1}{2})(1+\frac{g-g_1}{2})} \right)^t \right\}.$$

Since $g \geq 3$, we have firstly

$$1 + \frac{g}{2} \leq \frac{5}{8} + \frac{g}{8} + \frac{g}{2} = \frac{5}{6} \times \frac{3}{2}\left(\frac{1}{2} + \frac{g}{2}\right) \qquad \text{and therefore} \qquad \frac{1+\frac{g}{2}}{\frac{3}{2}(1+\frac{g-1}{2})} \leq \frac{5}{6}.$$

We have secondly

$$\frac{1+\frac{g}{2}}{2(1+\frac{g-2}{2})} = \frac{1}{g} + \frac{1}{2} \leq \frac{5}{6}.$$

And we have thirdly (recall that $g_1 \leq g/2$)

$$1 + \frac{g}{2} \leq 1 + (g - g_1) < 2\left(1 + \frac{g-g_1}{2}\right) \qquad \text{so that} \qquad \frac{1+\frac{g}{2}}{(1+\frac{g_1}{2})(1+\frac{g-g_1}{2})} < \frac{2}{1+\frac{g_1}{2}}.$$

Hence, the left–hand side of (6.8) is bounded by

$$2 \times \left\{ 1 + 2\left(\frac{5}{6}\right)^t + \sum_{h=3}^{\infty} \left(\frac{2}{1+\frac{h}{2}}\right)^t \right\} \times 4^{d-2},$$

namely by $2 \times 2 \times 4^{d-2} = 4^{d-1}$ if t is large enough, as was to be shown. □

Here as in Corollary 8, we denote by $p_1, \ldots, p_r$ the prime factors of $|Q|$. Observe that they also include the prime factors of the degree of any irreducible representation of one of the groups $W(Q,k)$ or $W(Q)$.

14. Lemma. *There exists a constant $t_1 \geq 0$ with the following property. For any $n \geq 1$, with prime decomposition $n = p_1^{e_1} \cdots p_r^{e_r}$ $(e_1, \ldots, e_r \geq 0)$, we have*

$$h_{n,k} \leq \left(\frac{n}{(1+\frac{e_1}{2})\cdots(1+\frac{e_r}{2})}\right)^t \quad \text{for all } t \geq t_1 \text{ and } k \geq 0. \tag{6.11}$$

More precisely, any t_1 such that

$$|Q|\,|X|^{|X|}\,4^{r(d-1)}|Q|\left(\frac{3}{4}\right)^{t_1} \leq 1 \quad \text{and } t_1 \geq t_0 \tag{6.12}$$

(with t_0 as in Lemma 13) is suitable.

Proof. Step one. For an integer $n \geq 1$ and $e_1, \ldots, e_r$ as above, set

$$\overline{h}_n = \left(\frac{n}{(1+\frac{e_1}{2})\cdots(1+\frac{e_r}{2})}\right)^t,$$

so that we wish to show that $h_{n,k} \leq \overline{h}_n$ for all $k \geq 0$. For $\nu \in \mathbf{R}$, $\nu > 0$, $\nu \notin \mathbf{N}$, it is convenient to set $\overline{h}_\nu = 0$, so that we have obviously $h_{\nu,k} = \overline{h}_\nu$ $\forall k \geq 0$.

If $k = 0$, we distinguish two cases: if $n = 1$, then $h_{1,0} = 1$ and $\overline{h}_1 = 1$ (the last equality for all t), so that $h_{n,0} = \overline{h}_n$; if $n \geq 2$, then $h_{n,0} = 0$ and again $h_{n,0} \leq \overline{h}_n$. We assume from now on that the lemma is proven for some $k \geq 0$, and we will show by induction that it holds also for $k+1$.

The irreducible representations of $W(Q, k+1)$ of the first type, see (6.5) in the proof of Lemma 11, contribute to $h_{n,k+1}$ by

$$h^{(i)}_{n,k+1} = h_{n^{1/d},k}$$

(recall that $h_{n^{1/d},k} = 0$ if $n^{1/d}$ is not an integer). The representations of the second type, see (6.6) in the proof of Lemma 11, contribute to $h_{n,k+1}$ by

$$h^{(ii)}_{n,k+1} \leq \Big(\sum_{P \in \Pi_Q(X)} \sum_{\sigma \in \widehat{Q_P}}\Big)' \sum_{\substack{n_1,\ldots,n_d \geq 1, \\ \prod_{i=1}^d n_i = \frac{n}{(\deg \sigma)[Q:Q_P]}}} h_{n_1,k} \cdots h_{n_d,k}$$

where the prime in $(\sum_P \sum_\sigma)'$ indicates that the pair $(P,\sigma) = (\hat{1},1)$ does not occur. Observe that $\deg\sigma$ is a divisor of $|Q_P|$, so that $[Q : Q_P]\deg\sigma$ is a divisor of $|Q|$. If we introduce the constant

$$K = \text{number of pairs } (P,\sigma), \text{ with } P \in \Pi_Q(X),\ \sigma \in \widehat{Q_P}, \text{ and } (P,\sigma) \neq (\hat{1},1),$$

it follows that

$$h_{n,k+1} = h^{(i)}_{n,k+1} + h^{(ii)}_{n,k+1} \le h_{n^{1/d},k} + K \sum_{\substack{2\le f\le |Q| \\ f|n}} \sum_{\substack{n_1,\dots,n_d\ge 1 \\ \prod_{i=1}^d n_i = n/f}} h_{n_1,k}\cdots h_{n_d,k}. \tag{6.13}$$

Observe that the number of choices for P is strictly bounded by $|X|^{|X|}$ and that the number of choices for σ is bounded by $|Q|$, so that $K+1 \le |Q||X|^{|X|}$.

Step two. In the last sum of Inequality (6.13), any n_i which occurs, namely any n_i such that $h_{n_i,k} \neq 0$, is a product of the p_j's. We repeat the definition of the exponents e_j, and we define exponents $e_{i,j}$, f_j by

$$n = \prod_{1\le j\le r} p_j^{e_j}, \qquad n_i = \prod_{1\le j\le r} p_j^{e_{i,j}} \ \text{ for } i=1,\dots,d, \qquad f = \prod_{1\le j\le r} p_j^{f_j}. \tag{6.14}$$

We use the induction hypothesis to bound the first term of (6.13):

$$h_{n^{1/d},k} \le \overline{h}_{n^{1/d}} \cdots \overline{h}_{n^{1/d}}\, \overline{h}_{n^{1/d}/p_j},$$

where we have $d-1$ factors $\overline{h}_{n^{1/d}}$ and one factor in which j is such that p_j divides n (in case there does not exist any such j, we have $h_{n^{1/d},k} = 0$). We use again the induction hypothesis to bound the second term of (6.13), and we collect terms to obtain

$$h_{n,k+1} \le (K+1) \sum_{\substack{2\le f\le |Q| \\ f|n}} \sum_{\substack{n_1,\dots,n_d\ge 1 \\ f\prod_{i=1}^d n_i = n}} \overline{h}_{n_1}\cdots\overline{h}_{n_d}.$$

Using the definition of the $\overline{h}_{n_i}$ and reordering the terms, we have

$$\begin{aligned}
h_{n,k+1} &\le (K+1) \sum_{\substack{2\le f\le |Q| \\ f|n}} \sum_{\substack{n_1,\dots,n_d\ge 1 \\ f\prod_{i=1}^d n_i = n}} \\
&\qquad \left(\frac{n_1}{(1+\frac{e_{1,1}}{2})\cdots(1+\frac{e_{1,r}}{2})}\right)^t \cdots \left(\frac{n_d}{(1+\frac{e_{d,1}}{2})\cdots(1+\frac{e_{d,r}}{2})}\right)^t \\
&= (K+1) \sum_{\substack{2\le f\le |Q| \\ f|n}} \sum_{\substack{n_1,\dots,n_d\ge 1 \\ f\prod_{i=1}^d n_i = n}} \prod_{i=1}^d \left(n_i\right)^t \prod_{j=1}^r \left(\frac{1}{1+\frac{e_j}{2}}\right)^t \\
&\qquad \left(\frac{1+\frac{e_1}{2}}{(1+\frac{e_{1,1}}{2})\cdots(1+\frac{e_{d,1}}{2})}\right)^t \cdots \left(\frac{1+\frac{e_r}{2}}{(1+\frac{e_{1,r}}{2})\cdots(1+\frac{e_{d,r}}{2})}\right)^t \\
&= (K+1) \left(\frac{n}{\prod_{j=1}^r (1+\frac{e_j}{2})}\right)^t \sum_{\substack{2\le f\le |Q| \\ f|n}} \frac{1}{f^t} \sum_{\substack{n_1,\dots,n_d\ge 1 \\ f\prod_{i=1}^d n_i = n}} \\
&\qquad \left(\frac{1+\frac{e_1}{2}}{(1+\frac{e_{1,1}}{2})\cdots(1+\frac{e_{d,1}}{2})}\right)^t \cdots \left(\frac{1+\frac{e_r}{2}}{(1+\frac{e_{1,r}}{2})\cdots(1+\frac{e_{d,r}}{2})}\right)^t
\end{aligned}$$

where the term $\left(\frac{n}{\prod_{j=1}^{r}(1+\frac{e_j}{2})}\right)^t$ is precisely $\overline{h}_n$. We replace now a sum of products by a product of sums, and we obtain

$$\begin{aligned} h_{n,k+1} \leq (K+1)\,\overline{h}_n \sum_{\substack{2\leq f\leq |Q| \\ f|n}} \frac{1}{f^t} \times & \left(\sum_{\substack{e_{1,1},\dots,e_{d,1}\geq 0 \\ \sum_{i=1}^d e_{i,1}=e_1-f_1}} \left(\frac{1+\frac{e_1}{2}}{(1+\frac{e_{1,1}}{2})\cdots(1+\frac{e_{d,1}}{2})} \right)^t \right) \times \\ & \cdots \times \left(\sum_{\substack{e_{1,r},\dots,e_{d,r}\geq 0 \\ \sum_{i=1}^d e_{i,r}=e_r-f_r}} \left(\frac{1+\frac{e_r}{2}}{(1+\frac{e_{1,r}}{2})\cdots(1+\frac{e_{d,r}}{2})} \right)^t \right) \\ = (K+1)\,\overline{h}_n \sum_{\substack{2\leq f\leq |Q| \\ f|n}} & \left(\frac{1+\frac{e_1}{2}}{\left(1+\frac{e_1-f_1}{2}\right)p_1^{f_1}} \right)^t \cdots \left(\frac{1+\frac{e_r}{2}}{\left(1+\frac{e_r-f_r}{2}\right)p_r^{f_r}} \right)^t \times \\ & \left(\sum_{\substack{e_{1,1},\dots,e_{d,1}\geq 0 \\ \sum_{i=1}^d e_{i,1}=e_1-f_1}} \left(\frac{1+\frac{e_1-f_1}{2}}{(1+\frac{e_{1,1}}{2})\cdots(1+\frac{e_{d,1}}{2})} \right)^t \right) \times \\ & \cdots \times \left(\sum_{\substack{e_{1,r},\dots,e_{d,r}\geq 0 \\ \sum_{i=1}^d e_{i,r}=e_r-f_r}} \left(\frac{1+\frac{e_r-f_r}{2}}{(1+\frac{e_{1,r}}{2})\cdots(1+\frac{e_{d,r}}{2})} \right)^t \right). \end{aligned}$$

Each of the r terms $\frac{1+\frac{e_1}{2}}{\left(1+\frac{e_1-f_1}{2}\right)p_1^{f_1}}$, ... , $\frac{1+\frac{e_r}{2}}{\left(1+\frac{e_r-f_r}{2}\right)p_r^{f_r}}$ above is bounded by 1, and at least one of them is bounded by $\frac{3}{4}$ (see Observation 10.ii). Each of the next r sums over d–uples of $e_{\star,\star}$'s is bounded by 4^{d-1}, by Lemma 13. It follows that

$$\begin{aligned} h_{n,k+1} &\leq \overline{h}_n\,(K+1)\,4^{r(d-1)} \sum_{\substack{2\leq f\leq |Q| \\ f|n}} \left(\frac{3}{4}\right)^t \\ &\leq \overline{h}_n\,(K+1)\,4^{r(d-1)}|Q|\,\left(\frac{3}{4}\right)^t. \end{aligned}$$

For t large enough, this shows that

$$h_{n,k+1} \leq \overline{h}_n$$

and ends the induction argument. □

15. Proposition. *Let Q be a perfect finite group acting transitively on a finite set X with at least two points. Then the representation zeta function $\zeta(W(Q),s)$ has a finite abscissa of convergence, say $s_0(W(Q))$.*

Moreover, the function $\zeta_G(s)$, which is holomorphic in the half–plane defined by $\mathrm{Re}(s) > s_0(W(Q))$*, has a singularity at* $s_0(W(Q))$.

Proof. It is elementary to check that $h(W(Q)) = \sum_{n=1}^{\infty} h_n(W(Q)) = \infty$ (if necessary, see the first step of the proof of the next proposition, which is independent of the present proof).

For any rigid group G such that $h(G) = \infty$, namely such that the *Dirichlet series* $\zeta_G(s)$ diverges at $s = 0$, it is a classical result (see e.g. Theorems 7 and 10 in [HaRi–15]) that the abscissa of convergence of $\zeta_G(s)$ is given by

$$s_0(G) \;=\; \limsup_{n\to\infty} \frac{\ln\left(\sum_{j=1}^{n} h_j(G)\right)}{\ln n} \tag{6.15}$$

and that $s_0(G)$ is a singular point of the *function* $\zeta_G(s)$. Hence, by Lemma 14, and using the notation $\overline{h}_n$ of the proof of Lemma 14:

$$\begin{aligned}
s_0(W(Q)) \;&\le\; \limsup_{n\to\infty} \frac{\ln\left(\sum_{j=1}^{n} \overline{h}_j\right)}{\ln n} \\
&\le\; \limsup_{n\to\infty} \frac{1}{\ln n} \ln\left(\sum_{d_1=0}^{e_1} \cdots \sum_{d_r=0}^{e_r} \left(\frac{n}{(1+\frac{d_1}{2})\cdots(1+\frac{d_r}{2})} \right)^t \right) \\
&\le\; \limsup_{n\to\infty} \frac{1}{\ln n} \ln\left(\sum_{d \mid n} n^t \right) \;\le\; \limsup_{n\to\infty} \frac{1}{\ln n} \ln\left(n \times n^t\right) \\
&\le\; t+1
\end{aligned}$$

where $e_1, \dots, e_r$ are defined in terms of n as in Lemma 14. Observe that, at this stage, the only clear bounds are $0 \le s_0(W(Q)) < \infty$. □

16. Proposition. *In the situation of the previous proposition (Q perfect acting transitively on X, with $d = |X| \ge 2$), we have*

$$s_0(W(Q)) \;>\; 0. \tag{6.16}$$

Proof. Step one: there exist infinitely many values of n such that $h_n(W(Q)) \ge 1$.

Indeed, since $Q = W(Q,1)$ is perfect and not reduced to $\{1\}$, there exists $n \ge 2$ such that $h_n(Q) \ge 1$. Let $k \ge 1$ be such that $h_n(W(Q,k)) \ge 1$ and let ρ_1 be an irreducible representation of $W(Q,k)$ of dimension n. The n^d–dimensional representation $\pi_{\rho_1,\dots,\rho_1;1}$, as in (6.4), contributes to $h_{n^d}(W(Q,k+1)) \ge 1$. Continue with n^d in lieu of n.

Step two: there exist infinitely many values of n for which $h_n(W(Q)) \ge 2$.

Let $N \ge 2$ be an integer. By Step one, there exist $k \ge 1$ and irreducible representations $\rho_1, \dots, \rho_d$ of $W(Q,k)$ of pairwise distinct degrees, all at least N. Since Q is perfect, and therefore strictly contained in the symmetric group of X, the action of Q is not d times transitive. Hence there exists a permutation τ of X such that

$$\rho_1, \dots, \rho_d \quad \text{and} \quad \rho_{\tau(1)}, \dots, \rho_{\tau(d)}$$

are *not* in the same orbit of Q acting on $\left(\widehat{W(Q,k)}\right)^d$. It follows that the irreducible representations

$$\mathrm{Ind}_{W(Q,k)^X}^{W(Q,k+1)}\left(\rho_1 \boxtimes \cdots \boxtimes \rho_d\right) \quad \text{and} \quad \mathrm{Ind}_{W(Q,k)^X}^{W(Q,k+1)}\left(\rho_{\tau(1)} \boxtimes \cdots \boxtimes \rho_{\tau(d)}\right)$$

of $W(Q,k+1)$, which are both of degree $|Q|\prod_{i=1}^d \deg\rho_i$ (a degree $> N^d$), are not equivalent. The claim of Step two follows.

Step three: For any positive integer B, there exist infinitely many values of n for which $h_n(W(Q)) \geq B$.

We proceed by induction on B (see Step two for $B = 2$). Suppose that Step three has been shown for some value $B_0 \geq 2$. Let $N \geq 2$ be an integer. By the induction hypothesis, there exist integers $k \geq 1$ and $n_1, \ldots, n_d$ such that $N < n_1 < \cdots < n_d$ and $h_{n_i}(W(Q,k)) \geq B_0$, $i = 1, \ldots, d$. For each i, choose $h_{n_i}(W(Q,k))$ pairwise inequivalent irreducible representations $\rho_{i,j}$ of $W(Q,k)$ of degree n_i. The irreducible representations

$$\mathrm{Ind}_{W(Q,k)^X}^{W(Q,k+1)}\left(\rho_{1,j_1} \boxtimes \cdots \boxtimes \rho_{d,j_d}\right), \qquad 1 \leq j_i \leq h_{n_i}(W(Q,k)) \quad (i = 1, \ldots d)$$

of $W(Q,k+1)$ are pairwise inequivalent, all of the same degree, which is $|Q|\prod_{i=1}^d n_i > N^d$, and there are $\prod_{i=1}^d h_{n_i}(W(Q,k)) \geq (B_0)^d$ of them.

Step four: Set $K = \sqrt[d]{\frac{1}{d!\,2^d}}$. Let $n \geq 1$ be such that $h_n(W(Q)) \geq \min\{2d, \frac{2}{K}\}$. Then

$$h_{|Q|n^d}(W(Q)) \;\geq\; \Big(K\,h_n(W(Q))\Big)^d.$$

Observe that the existence of the integer n involved in the claim of Step four follows from Step three. Let $k \geq 1$ be such that there exist pairwise inequivalent irreducible representations $\rho_1, \ldots, \rho_{h(n)}$ of $W(Q,k)$ of dimension n, where we have written $h(n)$ for $h_n(W(Q))$. Any choice of d distinct representations $\rho_{j_1}, \ldots, \rho_{j_d}$ among $\rho_1, \ldots, \rho_{h(n)}$ provides an irreducible representation

$$\mathrm{Ind}_{W(Q,k)^X}^{W(Q,k+1)}\left(\rho_{j_1} \boxtimes \cdots \boxtimes \rho_{j_d}\right)$$

of $W(Q,k+1)$ of dimension $|Q|n^d$. Different choices provide nonequivalent representations, and there are $\binom{h(n)}{d} \geq \frac{(h(n)/2)^d}{d!} = (Kh(n))^d$ such choices.

Step five: end of proof.

Choose B such that $\frac{d}{d-1}\ln K + \ln B > 0$. By iteration of the inequality shown in Step four, for some $m \geq 2$, we have

$$h(m) \geq B, \quad h(|Q|m^d) \geq (Kh(m))^d,$$
$$h(|Q|^{1+d}m^{d^2}) \geq \left(K(Kh(m))^d\right)^d = K^{d+d^2}h(m)^{d^2}, \quad \ldots$$

and more generally

$$h\left(|Q|^{(d^t-1)/(d-1)}m^{d^t}\right) \;\geq\; K^{d(d^t-1)/(d-1)}\,B^{d^t} \qquad \text{for any integer } t \geq 1.$$

Hence

$$\begin{aligned} s_0(W(Q)) &= \limsup_{n\to\infty} \frac{\ln\left(\sum_{j=1}^n h(j)\right)}{\ln n} \\ &\geq \limsup_{t\to\infty} \frac{\ln\left(h\left(|Q|^{(d^t-1)/(d-1)} m^{d^t}\right)\right)}{\ln\left(|Q|^{(d^t-1)/(d-1)} m^{d^t}\right)} \\ &\geq \limsup_{t\to\infty} \frac{\ln\left(K^{d(d^t-1)/(d-1)} B^{d^t}\right)}{\ln\left(|Q|^{(d^t-1)/(d-1)} m^{d^t}\right)} \\ &= \limsup_{t\to\infty} \frac{d\frac{d^t-1}{d-1}\ln K + d^t \ln B}{\frac{d^t-1}{d-1}\ln|Q| + d^t \ln m} \\ &= \frac{\frac{d}{d-1}\ln K + \ln B}{\frac{1}{d-1}\ln|Q| + \ln m} \end{aligned}$$

and the last fraction is positive by the choice of B. □

We concentrate briefly on the group $Q = A_5$ to introduce the next proposition. Consider the polynomial $\Psi \in \mathbf{Q}[X_1, X_2, X_3, X_4, X_5, Y_2, Y_3, Y_5]$ defined by

$$\begin{aligned} \Psi(X_1, X_2, X_3, X_4, X_5, Y_2, Y_3, Y_5) &= \\ &\frac{1}{60} Y_2^2 Y_3 Y_5 \Big(X_1^5 - 10X_1^2 X_3 - 15X_1 X_2^2 + 30X_2X_3 + 30X_1X_4 - 36X_5\Big) \\ &+ Y_2Y_3Y_5\Big(X_1X_2^2 - 2X_2X_3 - X_1X_4 + 2X_5\Big) \\ &+ \frac{3}{2}Y_2^2Y_5\Big(X_1^2X_3 - X_2X_3 - 2X_1X_4 + 2X_5\Big) \\ &+ Y_2Y_5\,(2+Y_2)\Big(X_2X_3 - X_5\Big) \\ &+ Y_5\,(3+Y_3)\Big(X_1X_4 - X_5\Big) \\ &+ \big(1 + 2Y_3 + Y_2^2 + Y_5\big)\, X_5. \end{aligned} \tag{6.17}$$

Since $W(A_5) \wr_5 A_5 \cong W(A_5)$, the representation zeta function $\zeta(s) = \zeta(W(A_5), s)$ is a solution of the equation

$$\Psi\left(\zeta(s), \zeta(2s), \zeta(3s), \zeta(4s), \zeta(5s), 2^{-s}, 3^{-s}, 5^{-s}\right) - \zeta(s) = 0 \tag{6.18}$$

(compare with (3.9)).

Let X_* stand for $\{X_2, X_3, X_4, X_5\}$ and Y_* for $\{Y_2, Y_3, Y_5\}$; we can also write

$$\Psi(X_1, X_*, Y_*) = \sum_{i=0}^{5} v_i(X_*, Y_*) X_1^{5-i},$$

with

$$v_0(X_*, Y_*) = \frac{1}{60} Y_2^2 Y_3 Y_5, \qquad v_3(X_*, Y_*) = -\frac{1}{6} X_3 Y_2^2 Y_3 Y_5 + \frac{3}{2} X_3 Y_2^2 Y_5, \qquad \ldots$$

(it happens that $v_1(X_*, Y_*) = v_2(X_*, Y_*) = 0$, but this does not play any role in the argument below). Let R denote the ring of holomorphic functions in the half–plane of inequation $\mathrm{Re}(s) > \frac{1}{2}s_0(W(A_5))$. Since the Dirichlet series for $\zeta(2s), \dots, \zeta(5s)$ converge when $\mathrm{Re}(s) > \frac{1}{2}s_0(W(A_5))$, we can view $v_0, \dots, v_5$ as elements of R and $\Psi(...)$ as a one variable polynomial $\Phi(X_1)$ in the ring $R[X_1]$, so that $\zeta(s)$ is now a root of the polynomial $\Phi(X_1) - X_1$. Near s_0, all coefficients of this polynomial are bounded; moreover, the top coefficient is an entire function which is never zero; if follows that $\zeta(s)$ tends to a finite limit $\zeta(s_0)$ when $s \to s_0$ (say by real values s with $s > s_0$). Thus $\zeta(s)$ is a solution of a polynomial of the form

$$\Theta(X) \;=\; X^5 + \sum_{i=1}^{5} u_i(s) X^{d-i} \tag{6.19}$$

where the coefficients u_i are in the ring R. Let

$$\Delta \doteqdot \mathrm{Disc}(\Theta(X)) \;=\; \prod_{1 \le i < j \le 5} (\alpha_i - \alpha_j)^2 \;\in\; R, \tag{6.20}$$

denote the *discriminant* of Θ, where $\alpha_1, \dots, \alpha_5$ are the roots of Θ in an appropriate extension of the field of fractions of R (for discriminants, see e.g. [Bour–81, chapitre IV, § 6, no 7, proposition 11]). Since Δ is a function which is holomorphic in a neighbourhood of $s_0 \doteqdot s_0(W(A_5))$, we can evaluate Δ at s_0 to find its value $\Delta(s_0) \in \mathbf{C}$.

We return to the general case. For $k \ge 0$, denote by $\mathcal{H}(k)$ the half–plane defined by $\mathrm{Re}(s) > 2^{-k}s_0(W(Q))$ and by $\mathcal{O}(\mathcal{H}(k))$ the ring of holomorphic functions there. Observe that the Dirichlet series defining $\zeta(s)$ converges in $\mathcal{H}(0)$, that

$$\mathcal{H}(0) \subset \mathcal{H}(1) \subset \cdots \subset \mathcal{H}(k) \subset \dots \subset \bigcup_{k=0}^{\infty} \mathcal{H}(k) \;=\; \{s \in \mathbf{C} \mid \mathrm{Re}(s) > 0\}.$$

We write now $\zeta(s)$ for the representation zeta function $\zeta(W(Q), s)$.

From Theorem 1 and as in the particular case above, we deduce that there exists a polynomial

$$\Theta(X) \;=\; X^d + \sum_{i=1}^{d} u_i(s) X^{d-i} \in \mathcal{O}(\mathcal{H}(1))[X] \quad \text{with discriminant} \quad \Delta(s) \in \mathcal{O}(\mathcal{H}(1)) \tag{6.21}$$

such that $\zeta(s)$ is a root of Θ.

17. Proposition. *Let Q be a perfect group acting transitively on a set X with $d \ge 2$ points, as in Propositions 15 and 16, and let $\zeta(s)$, s_0, $\Delta(s)$ be as above.*

(i) We have $\Delta(s_0) = 0$.

(ii) When $s \to s_0$ by real values, with $s > s_0$, the limit $\lim_{s \to s_0} \zeta(s)$ exists and is finite.

(iii) Near s_0, the function $\zeta(s)$ has a Puiseux expansion of the form

$$\zeta(s) \;=\; \sum_{n=0}^{\infty} a_n (s - s_0)^{n/e}, \tag{6.22}$$

where e is an integer such that $2 \leq e \leq d$.

(iv) The function $\zeta(s)$ extends as an analytic function with only root singularities in the half–plane defined by $\mathrm{Re}(s) > 0$.

Proof. (i) If s_0 were not a zero of Δ, we could extend holomorphically $\zeta(s)$ to the left of s_0, in contradiction with the definition of s_0 and the classical result, used in the proof of Proposition 15, that s_0 is a singular point.

(ii) As $\zeta(s)$ near s_0 satisfies a monic polynomial equation with all coefficients bounded, it has a finite limit when $s \to s_0$ with $s > s_0$.

(iii) This is standard in the Newton–Puiseux theory ; see for example [Casa–00, Theorems 1.5.4 and 1.7.2]. Observe that one could not have $e = 1$ in (6.22), since, then, the point s_0 would not be a singularity of $\zeta(s)$.

(iv) We can define inductively for each $k \geq 0$ an analytic extension of $\zeta(s)$ in $\mathcal{H}(k)$, such that $\zeta(s)$ is a root of a polynomial formally identical to (6.21), but now with coefficients analytic in $\mathcal{H}(k+1)$. Claim (iv) follows. □

The proof of Theorem 3 is now complete.

18. Question. *Is the imaginary axis a natural frontier of the function $\zeta(s)$ of Proposition 17?*

More precisely: is it true that, for any curve in the half–plane $\{\mathrm{Re}(s) > 0\}$ avoiding the singularities and ending on the imaginary axis, the function $\zeta(s)$ does not *extend holomorphically along this curve up to the end point?*

Numerical estimates. *A numerical estimate for $s_0 = s_0(W(Q))$ when $Q = A_5$ acts on $X = I_5$ is given in (3.11). The first terms*[2] *of the Puiseux expansion of $\zeta(W(A_5), s)$ near s_0 are as in (3.10); in particular, $e = 2$.*

Thus, in the case of the group A_5, as before Proposition 17, we have $e = 2 < d = 5$. It follows that the polynomial $\Theta(X)$, now regarded with coefficients in the ring of germs of holomorphic functions around s_0, is reducible, with an irreducible factor of degree $e = 2$.

A few numerical experiments suggest a positive answer to Question 18. In the case of $W(A_5)$, it seems that the discriminant vanishes at s_0 and two other places

$$s'_0 \pm it'_0 \approx 0.8973038819 \pm 0.0264098303\, i \tag{6.23}$$

in $\{\mathrm{Re}(s) > s_0/2\}$.

A small program was written to do the following: a small complex step ϵ is chosen, for instance $0.001 + 0.0001i$, and an integer N large enough so that $\mathrm{Re}(N\epsilon) > 2$ is chosen. The values of $\zeta(W(A_5), s)$ at $s = \epsilon, 2\epsilon, \ldots, N\epsilon$ will be approximated by numerical values z_n. These are computed in decreasing order $z_N, z_{N-1}, \ldots, z_1$. If $\mathrm{Re}(n\epsilon) > 2$, then z_n is computed using the first 10^{12} terms of the power series. Otherwise, the program ensures that $z_{2n}, \ldots, z_{5n}$ are computed, and obtains a polynomial functional equation for z_n, with five roots; the root closest to z_{n+1} is chosen as approximation for z_n.

If one plots in 3-space the points $(\mathrm{Re}(\epsilon)n, \mathrm{Re}(z_n), \mathrm{Im}(z_n))$, one sees discrete approximations of continuous curves, that approximate $\zeta(W(A_5), s)$ along radial half–lines $\mathbf{R}_+\epsilon$. The following remarks can be made empirically:

— these curves remain bounded as $\mathrm{Re}(\epsilon) \to 0$;

[2] We are grateful to Don Zagier for having computed these terms for us.

— they are smooth except when ϵ is a real multiple of s_0 or $s_0' \pm it_0'$;
— for ϵ a real multiple of $1+i$, one sees as $\mathrm{Re}(\epsilon)n \to 0$ a spiral with non-vanishing radius and faster and faster winding;
— other random values of ϵ indicate either convergence, or oscillation, or spiralling.

19. Remark. Let G be a connected compact Lie group which is semisimple and simply connected; denote by ℓ its rank and by $\kappa = \frac{1}{2}(\dim(G) - \ell)$ the number of positive roots of the Lie algebra $\mathrm{Lie}(G) \otimes_{\mathbf{R}} \mathbf{C}$. By Hermann Weyl's theory, there exists a polynomial P of degree κ in ℓ variables such that

$$\zeta(G, s) = \sum_{n_1, \dots, n_\ell \geq 0} P(n_1, \dots, n_\ell)^{-s}.$$

For example, if G is simple of rank $\ell = 2$, we have [Hump–72, Page 140]

$$\begin{aligned}
\zeta(SU(3), s) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \Big(\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)\Big)^{-s} \\
&= 1 + 2\cdot 3^{-s} + 2\cdot 6^{-s} + 8^{-s} + 2\cdot 10^{-s} + 4\cdot 15^{-s} + 2\cdot 21^{-s} + 2\cdot 24^{-s} \\
&+ 27^{-s} + 2\cdot 28^{-s} + 2\cdot 35^{-s} + 2\cdot 36^{-s} + 2\cdot 42^{-s} + 2\cdot 45^{-s} + 2\cdot 48^{-s} \\
&+ 2\cdot 55^{-s} + 2\cdot 60^{-s} + 2\cdot 63^{-s} + 64^{-s} + 2\cdot 66^{-s} + \cdots\cdots\cdots \\
\zeta(Spin_5(\mathbf{C}), s) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \Big(\frac{1}{6}(m_1+1)(m_2+1)(m_1+m_2+2)(2m_1+m_2+3)\Big)^{-s} \\
\zeta(G_2, s) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \Big(\frac{1}{120}(m_1+1)(m_2+1)(m_1+m_2+2) \times \\
&\qquad (m_1+2m_2+3)(m_1+3m_2+4)(2m_1+3m_2+5)\Big)^{-s}.
\end{aligned}$$

20. Theorem (Weyl and Mahler). *Let G be a connected compact Lie group which is semisimple and simply connected, and let ℓ, κ be as above.*

The abscissa of convergence of $\zeta(G,s)$ is $\frac{\ell}{\kappa}$ and $\zeta(G,s)$ extends to a meromorphic function in the whole complex plane. Moreover the poles of $\zeta(G,s)$ are all simple and at rational points of the real axis; in particular, the largest singularity $s_0(G) = \frac{\ell}{\kappa}$ is a simple pole of $\zeta(G,s)$.

References. This theorem[3] is a straightforward consequence of several classical results. More precisely, given the formula of Weyl for the dimensions of the irreducible representations of G, it becomes a particular case of Satz II in [Mahl–28], where the more general case of $\sum_{m_1,\dots,m_\ell \geq 0} P_1(m_1,\dots,m_\ell)P_2(m_1,\dots,m_\ell)^{-s}$ is considered.

See also [LaLu–08, Theorem 5.1], as well as other results on Dirichlet series of related form in [Sarg–84]. □

A comparison with Proposition 17 shows how different representation zeta functions can be from each other.

[3]We are grateful to Olivier Mathieu who first made one of us aware of this.

7. Locally finite groups and finitely generated groups acting on rooted trees

Besides the group $W^{prof}(Q)$ defined in (3.6), we have also for each $k \geq 1$ a natural injection

$$W(Q,k) \longrightarrow W(Q,k+1) \tag{7.1}$$

(which is the natural splitting of the epimorhism (3.5)), and therefore a corresponding locally finite group

$$W^{locfin}(Q) = \bigcup_{k\geq 1} W(Q,k). \tag{7.2}$$

For integers $d \geq 2$ and $k \geq 0$, let $T_d(k)$ denote the *d–ary rooted tree of height k;* its vertices are the finite words in $I_d = \{1,2,\dots,d\}$ of length at most k, and any word w of length between 0 (the empty word) and $k-1$ is adjacent to the k words wx, for $x \in \{1,\dots,d\}$. We denote by T_d the *infinite d–ary rooted tree*, in which the subtree induced by the vertices at distance at most k from the root is precisely $T_d(k)$.

The particular case of (3.1) and (3.6) in which Q is the symmetric group S_d acting in the standard way on the set $I_d \doteqdot \{1,2,\dots,d\}$ provides *the full automorphism groups of these trees:*

$$W(S_d,k) = \mathrm{Aut}\,(T_d(k)) \qquad \text{and} \qquad W^{prof}(S_d) = \mathrm{Aut}\,(T_d)\,. \tag{7.3}$$

The case of $Q = C_d$ acting by cyclic permutations on I_d gives rise to the so–called *group of d–adic automorphisms* of T_d. Observe that, for p a prime, $W(C_p,k)$ is a p–Sylow subgroup of $W(S_p,k)$, and $W(C_p)$ is a p–Sylow subgroup of $W(S_p)$.

For simplicity, we assume from now on that Q acts faithfully on the set X of size $d \geq 2$, identified to I_d. The groups defined by (3.1) and (3.6) are therefore naturally groups of tree automorphisms:

$$W(Q,k) \subset \mathrm{Aut}\,(T_d(k)) \qquad \text{and} \qquad W^{prof}(Q) \subset \mathrm{Aut}\,(T_d)\,. \tag{7.4}$$

The epimorphism $W(Q,k+1) \longrightarrow W(Q,k)$ of (3.5) is the restriction to $T_d(k)$ of automorphisms of $T_d(k+1)$, and the epimorphism $W(Q,k) \longrightarrow Q$ corresponding in an analogous way to (3.1) is the restriction to X of automorphisms of $T_d(k)$, where X is identified with the set of leaves of $T_d(1)$.

Moreover, corresponding to the splitting which appears in (3.5), we may extend "rigidly" automorphisms of $T_d(k)$ to automorphisms of $T_d(\ell)$, $\ell > k$, and of T_d; otherwise written:

$$W(Q,k) \subset W(Q,k+1) \subset \cdots \subset W^{locfin}(Q) \subset \mathrm{Aut}\,(T_d)\,. \tag{7.5}$$

It follows that the group defined by (7.2) is also a group

$$W^{locfin}(Q) \subset \mathrm{Aut}\,(T_d) \tag{7.6}$$

of tree automorphisms.

There are other interesting dense subgroups of $W^{prof}(Q)$, and in particular finitely generated groups which play an important role in various questions (see e.g. [Bart–03b], [Neum–86], and [Wils–04]) as the examples below show. *Assume furthermore that Q is 2–transitive on X,* and consider two elements of X, written 1 and 2 in (7.9) below. Let

$$W^{fingen}(Q) \;=\; \langle Q, \overline{Q}\rangle \;\subset\; \mathrm{Aut}\,(T_d) \tag{7.7}$$

be the group of automorphisms of T_d generated by two copies of Q. An element q in the first copy Q acts on T_d as follows, with $(x_1, \dots, x_k)$ a typical vertex of T_d:

$$q(x_1, x_2, x_3, \dots, x_k) \;=\; (q(x_1), x_2, x_3, \dots, x_k) \tag{7.8}$$

for all $k \geq 0$ and $x_1, \dots, x_k \in X$. An element $\overline{q}$ of the second copy $\overline{Q}$ acts on T_d by

$$\overline{q}(x_1, x_2, \dots, x_k) \;=\; \begin{cases} (x_1, \dots, x_j, q(x_{j+1}), x_{j+2}, \dots, x_k) \\ \qquad\qquad\qquad \text{if} \quad x_1 = \cdots = x_{j-1} = 1,\; x_j = 2, \\ (x_1, \dots, x_k) \quad \text{otherwise,} \end{cases} \tag{7.9}$$

for all k and $x_1, \dots, x_k$ as above.

We now assume furthermore that Q is generated by $\bigcup_{x \neq y \in X} (\mathrm{Stab}_Q(x) \cap \mathrm{Stab}_Q(y))'$, where the prime indicates a commutator subgroup; this holds for example if Q is simple and $\mathrm{Stab}_Q(x) \cap \mathrm{Stab}_Q(y)$ is not Abelian. The case $Q = A_6$ and $X = I_6$ was considered by Peter Neumann in [Neum–86]. We define a finitely generated group

$$W^{PN}(Q) \;=\; \langle \omega(x, q) \mid x \in X,\; q \in \mathrm{Stab}_Q(x)\rangle \;\subset\; \mathrm{Aut}(T_d) \tag{7.10}$$

where the generators $\omega(x, q)$ are defined by

$$\omega(x, q)(x_1, x_2, \dots, x_k) \;=\; \begin{cases} (x_1, \dots, x_j, q(x_{j+1}), x_{j+2}, \dots, x_k) \\ \qquad\qquad\qquad \text{if} \quad x_1 = \cdots = x_j = x,\; x_{j+1} \neq x, \\ (x_1, \dots, x_k) \quad \text{otherwise.} \end{cases} \tag{7.11}$$

21. Proposition. *Let Q be a perfect finite group acting faithfully and transitively on a finite set X of size $d \geq 2$. Let W be one of the groups*

$$W^{prof}(Q), \qquad W^{locfin}(Q), \qquad W^{fingen}(Q), \qquad W^{PN}(Q)$$

(with hypotheses on the action of Q on X as above for the groups $W^{fingen}(Q)$ and $W^{PN}(Q)$).

Then W is perfect, residually finite, and isomorphic to its own permutational wreath product with Q:

$$W \cong W \wr_X Q.$$

Proof. These groups are perfect because they are generated by perfect subgroups, and are residually finite because they are all subgroups of the profinite group $W^{prof}(Q)$.

It remains to establish the isomorphism; it comes from the natural inclusion of $W^{prof}(Q)$ as the first factor in the base of $W^{prof}(Q) = W^{prof}(Q) \wr_X Q$. The isomorphism is clear for the first two examples. For $W^{fingen}(Q)$, see [Bart–03a, Proposition 3.7] or [Bart–03b, Proposition 2.2]. For $W^{PN}(Q)$, we argue as follows.

Consider $q, r \in \mathrm{Stab}_Q(x) \cap \mathrm{Stab}_Q(y)$. Then $[\omega(x, q), \omega(y, r)]$ acts as $[q, r]$ on T_d as in (7.8). By our hypothesis, $W(Q)^{PN}$ contains the whole of Q acting as in (7.8). It then follows that $W(Q)^{PN}$ contains $\omega(x, q)q^{-1}$ (where q^{-1} acts as in (7.8)), which maps to $(1, \dots, \omega(x, q), \dots, 1)1$ through the embedding $W^{PN}(Q) \to W^{PN}(Q) \wr_X Q$. We conclude that this embedding is an isomorphism. This is in essence the argument given by Neumann in [Neum–86, pp. 307 sqq]. □

22. Proposition (P. Neumann). *Let G be a perfect, residually finite group isomorphic to $G \wr_X Q$. Then G is just infinite.*

More precisely, any homomorphism $G \to H$ with non-trivial kernel factors as $G \to W(Q,k) \to H$ for some $k \in \mathbf{N}$.

Reference. This is part of [Neum–86, Theorem 5.1]. The homomorphism $G \longrightarrow W(Q,k)$ is obtained as

$$\begin{aligned} G &\longrightarrow G \wr_X Q \longrightarrow (G \wr_X Q) \wr_X Q) = G \wr_{X^2} W(Q,2) \longrightarrow \cdots \\ &\longrightarrow G \wr_{X^k} W(Q,k) \longrightarrow W(Q,k). \end{aligned}$$

□

23. Proposition. *In the situation of Proposition 22, there exists for any integer $n \geq 1$ an integer $k(n) \geq 0$ with the following property:*

Any finite dimensional representation $G \longrightarrow GL_n(\mathbf{C})$ factors through $W(Q,k(n))$; in particular $h_n(G) < \infty$ for all $n \geq 1$. More precisely,

$$h_n(G) \; = \; \max_{k \in \mathbf{N}} h_n(W(Q,k)) \; < \; \infty.$$

Proof. Let $\rho : G \to GL_n(\mathbf{C})$ be a representation. In view of Proposition 22, it suffices to show that ρ is not faithful.

Since $G \cong G \wr_X Q$, the group G contains for all $\ell \in \mathbf{N}$ a finite subgroup isomorphic to Q^{X^ℓ}, and *a fortiori* a subgroup isomorphic to $(\mathbf{Z}/p\mathbf{Z})^\ell$, where p is a prime which divides the order of G. By the representation theory of finite abelian groups, the image of $(\mathbf{Z}/p\mathbf{Z})^\ell$ by ρ is up to conjugation a diagonal p–subgroup of $GL_n(\mathbf{C})$, so that the order of this image is at most p^n; it follows that ρ has a non–trivial kernel N.

The proof of Proposition 22, see [Neum–86, Lemma 5.2], actually shows that, if N is a normal subgroup of G with non-trivial image in $W(Q,k)$, then N contains the kernel of the map $G \to W(Q,k)$. □

Remark. This gives another proof of (5.2) in Corollary 8 for $W^{prof}(Q)$.

From the previous propositions, we deduce finally:

24. Theorem. *In the situation of Proposition 21, the groups*

$$W^{prof}(Q), \qquad W^{locfin}(Q), \qquad W^{fingen}(Q), \qquad W^{PN}(Q)$$

are rigid and have the same representation zeta function $\zeta(W^{prof}(Q), s)$, that which is the object of Theorem 2, Theorem 3, and Section 6.

8. Comments on numerical computations

The way chosen to write Formula (2.4) in Theorem 1 shows clearly the inclusion–exclusion ingredient of its proof. However, it is not optimal for numerical computations, because evaluations of $\zeta_H(\cdots)$ are much more expensive to compute that any other term, so that the number of occurrences of these $\zeta_H(\cdots)$ should be kept minimal, as in the second way of writing below.

$$
\begin{aligned}
\zeta(H \wr_X Q, s) &= \sum_{P\in\Pi_Q(X)} [Q : Q_P]^{-1-s}\zeta_{Q_P}(s) \\
&\qquad \sum_{P'=(P'_1,\dots,P'_\ell)\geq P} \mu_X(P,P')\, \zeta_H(|P'_1|s)\cdots\zeta_H(|P'_\ell|s) \\
&= \sum_{P'=(P'_1,\dots,P'_\ell)\in\Pi_Q(X)} \zeta_H(|P'_1|s)\cdots\zeta_H(|P'_\ell|s) \\
&\qquad \sum_{P\leq P'} \mu_X(P,P')\, [Q : Q_P]^{-1-s}\zeta_{Q_P}(s).
\end{aligned}
\tag{8.1 = 2.4}
$$

We used GAP [GAP4] to produce specialisations of Formula (8.1), namely to produce Formulas (4.6) to (4.10), as well as (8.3), (8.4), and (8.6) below. For small groups, they also have or could have been produced by hand, by enumerating all subgroups S of Q; computing the partition P of X they induce; computing the stabilizer Q_P of P; and keeping those pairs (S, P) for which $S = Q_P$.

It turned out to be much faster, for the examples we considered, to enumerate all partitions P of X; to compute their stabilizer Q_P; to compute the partition P' of X induced by Q_P; and to keep those (Q_P, P) for which $P = P'$.

All these commands (enumerating partitions, computing stabilizers, comparing groups) are simple instructions in GAP.

It is then straightforward, using the offspring of (8.1), to compute the zeta function $\zeta(W(Q,k), s)$ for small k, as a polynomial in $p_1^{-s}, \dots, p_r^{-s}$.

We describe now how numerical estimates as those in (3.10) and (3.11) were obtained. We concentrate specifically on $Q = A_5$ in this section; the same method works *mutatis mutandis* for other examples.

We stress that our goal is not to prove formal enclosures for the numerical constants $s_0, \zeta(W(Q), s_0), \dots,$ but rather to obtain good approximations of their decimal expansion, so as to check (e.g. in Plouffe's Inverter `http://pi.lacim.uqam.ca`) whether these constants already appeared in mathematics. We were unsuccessful with the examples we considered.

First, a large number N, say 10^{15}, is chosen; and the degree–$(\leq N)$ truncations $P_k(s)$ of $\zeta(W(Q,k), s)$ are computed. By Corollary 8, the sequence $(P_k(s))_{k\geq 0}$ is eventually constant, with limit $P(s)$; and $P(s)$ is the degree–$(\leq N)$ truncation of $\zeta(W(Q), s)$.

We noted experimentally, by varying N between 10^{10} and 10^{15}, that $\zeta(W(Q), s)$ is very well approximated by $P(s)$ when $s > s_0 + 1$; for example, again for $Q = A_5$, the value at $s = 2.1$ of $P(s)$ differs only at its ath digit when comparing $N = 10^a$ and $N = 10^{a+1}$.

This is supported by the following heuristic. We know that the series $\zeta(W(Q), s)$ converges at $s_0 + \epsilon$ for all $\epsilon > 0$, so its general term $h_n n^{-s_0-\epsilon}$ is bounded; say, for simplicity, bounded by 1. We also assume, somewhat crudely, that the tail $\sum_{n>N} h_n n^{-s_0-\epsilon}$ is bounded by 1. Therefore, for $s > s_0 + 1$, the error $\zeta(W(Q), s) - P(s)$ is bounded by

$$\sum_{n>N} h_n n^{-s_0-\epsilon-1} < N^{-1} \sum_{n>N} n^{-s_0-\epsilon} \precsim N^{-1}.$$

For our choice of N, we may therefore expect $P(s)$ to approximate $\zeta(W(Q),s)$ accurately to 15 digits.

We will soon see that $s_0 > 1$, and we will only evaluate the Dirichlet polynomial $P(s)$ at es for $s \approx s_0$ and $e \geq 2$. All our estimations will then be accurate to about 15 digits.

Our first goal is to estimate $s_0(W(Q))$. For this, in (6.17) and as in (6.18), we replace $X_2, \dots, X_5$ by $P(2s), \dots, P(5s)$ respectively, and we replace each Y_j by j^{-s}, yielding a degree–d polynomial Φ in $X = X_1$ whose coefficients are Dirichlet polynomials in s.

It is computationally impractical to evaluate algebraically the discriminant $\Delta(s)$, so we resort to another trick: we first find an enclosure $[l_0, u_0]$ for s_0, such that the number of real roots of $\Phi(X) - X$ differs at $s = l_0$ and at $s = u_0$. We then repeatedly compute the number of real roots of $\Phi(X) - X$ at $m_i \doteqdot (u_i + l_i)/2$, and set $[l_{i+1}, u_{i+1}] \doteqdot [l_i, m_i]$ if the number of real roots at m_i equals that at u_i, and set $[l_{i+1}, u_{i+1}] \doteqdot [m_i, u_i]$ otherwise. The enclosure's width is halved at each step, so after 50 or so iterations, for our choice of N, we have found a good approximation of s_0.

Note that, at $s = s_0$, the equation $\Phi(X) - X$ has a multiple root; let $t + 1$ be its multiplicity. We assume for simplicity that $t = 1$, which is the only case we have encountered in our numerical experiments.

We next set $s = s_0$ in $\Phi(X)$, and obtain a_0, the first term in the Puiseux series of $\zeta(W(Q),s)$ near s_0, see (6.22). We then compute for $i = 0, 1, \dots$ the coefficient a_{i+1} as the average of $(\Phi(X) - a_0 - \cdots - a_i(s - s_0)^{i/2})/(s - s_0)^{(i+1)/2}$ over s very near s_0.

We have obtained in this manner a Puiseux expansion

$$\zeta(W(Q),s) \approx a_0 + a_1\sqrt{s - s_0} + a_2(s - s_0) + \cdots \tag{8.2}$$

for $\zeta(W(Q),s)$ near s_0; see (3.10) and (3.11) for the actual numbers, for the case of A_5 acting on I_5. Let us now consider a few other examples.

* * * * * * *

The group A_5 of order 60 is the smallest nontrivial perfect finite group. As above, we denote a wreath product with respect to its natural action on $\{1,2,3,4,5\}$ by $H \wr_5 A_5$. We can also view $A_5 \cong PSL_2(\mathbf{F}_5)$ as acting on the projective line over the Galois field of order five, and we use the notation $\wr_6$ for this wreath product.

The next smallest finite simple group is $PGL_3(\mathbf{F}_2)$, which is equal to $GL_3(\mathbf{F}_2)$ and has order 168. It has a natural action on the projective plane with 7 points. We denote a wreath product with respect to this action by $H \wr_7 PGL_3(\mathbf{F}_2)$.

Let us write down the particular cases of Formula (8.1) corresponding to these actions, and some results of numerical computations.

25. Proposition. *With the notation described above, we have*

(8.3)
$$\begin{aligned}\zeta(H \wr_5 A_5, s) \;=\; & 60^{-1-s}\zeta_H(s)^5 \\ &+ \Big(20^{-1-s}\times 30 - 60^{-1-s}\times 10\Big)\zeta_H(s)^2\zeta_H(3s) \\ &+ \Big(30^{-1-s}\times 30 - 60^{-1-s}\times 15\Big)\zeta_H(s)\zeta_H(2s)^2 \\ &+ \Big(5^{-1-s}\left(3+3^{-s}\right)5 - 20^{-1-s}\times 60 - 30^{-1-s}\times 30 + 60^{-1-s}\times 30\Big)\zeta_H(s)\zeta_H(4s) \\ &+ \Big(10^{-1-s}\left(2+2^{-s}\right)10 - 20^{-1-s}\times 30 - 30^{-1-s}\times 60 + 60^{-1-s}\times 30\Big)\zeta_H(2s)\zeta_H(3s) \\ &+ \Big(1 + 2\times 3^{-s} + 4^{-s} + 5^{-s} - 5^{-1-s}\left(3+3^{-s}\right)5 - 10^{-1-s}\left(2+2^{-s}\right)10 \\ &\qquad + 20^{-1-s}\times 60 + 30^{-1-s}\times 60 - 60^{-1-s}\times 36\Big)\zeta_H(5s)\end{aligned}$$

and

(8.4)
$$\begin{aligned}\zeta(H \wr_6 PSL_2(\mathbf{F}_5), s) \;=\; & 60^{-1-s}\zeta_H(s)^6 \\ &+ \Big(30^{-s} - \frac{60^{-s}}{4}\Big)\zeta_H(s)^2\zeta_H(2s)^2 \\ &+ \Big(2\cdot 6^{-s} + 2\cdot 12^{-s} - 2\cdot 30^{-s} + \frac{2}{5}\cdot 60^{-s}\Big)\zeta_H(s)\zeta_H(5s) \\ &+ \Big(\frac{4}{3}\cdot 15^{-s} - 30^{-s} + \frac{60^{-s}}{6}\Big)\zeta_H(2s)^3 \\ &+ \Big(2\cdot 10^{-s} + 20^{-s} - 2\cdot 30^{-s} + \frac{60^{-s}}{3}\Big)\zeta_H(3s)^2 \\ &+ \Big(1 + 2\cdot 3^{-s} + 4^{-s} + 5^{-s} - 2\cdot 6^{-s} - 2\cdot 10^{-s} \\ &\qquad - 2\cdot 12^{-s} - \frac{4}{3}\cdot 15^{-s} - 20^{-s} + 4\cdot 30^{-s} - \frac{2}{3}\cdot 60^{-s}\Big)\zeta_H(6s).\end{aligned}$$

For comparison with (6.10), the Dirichlet series $\zeta(W(PSL_2(\mathbf{F}_5)), s)$ converges for

$$\mathrm{Re}(s) \;>\; s_0(W(PSL_2(\mathbf{F}_5))) \;\sim\; 1.13333324(7), \tag{8.5}$$

again with a Puiseux expansion in $\sqrt{s - s_0}$.

We have next

(8.6)

$$\zeta(H \wr_7 PGL_3(\mathbf{F}_2), s) = \left(\frac{168^{-s}}{168}\right)\zeta_H(s)^7$$
$$+ \left(\frac{84^{-s}}{2} - \frac{168^{-s}}{8}\right)\zeta_H(s)^3\zeta_H(2s)^2$$
$$+ \left(\frac{2}{3}42^{-s} - \frac{84^{-s}}{2} + \frac{168^{-s}}{12}\right)\zeta_H(s)^3\zeta_H(4s)$$
$$+ \left(\frac{2}{3}42^{-s} - \frac{84^{-s}}{2} + \frac{168^{-s}}{12}\right)\zeta_H(s)\zeta_H(2s)^3$$
$$+ \left(4 \times 21^{-s} - 3 \times 42^{-s} + \frac{84^{-s}}{2}\right)\zeta_H(s)\zeta_H(2s)\zeta_H(4s)$$
$$+ \left(2 \times 28^{-s} + 56^{-s} - 2 \times 84^{-s} + \frac{168^{-s}}{3}\right) \cdot \zeta_H(s)\zeta_H(3s)^2$$
$$+ \left(2 \times 7^{-s} + 14^{-s} - 2 \times 21^{-s} - 2 \times 28^{-s} + \frac{42^{-s}}{3} - 56^{-s} + 2 \times 84^{-s} - \frac{168^{-s}}{3}\right)$$
$$(\zeta_H(s)\zeta_H(6s) + \zeta_H(3s)\zeta_H(4s))$$
$$+ \Big(1 + 2 \times 3^{-s} + 6^{-s} - 3 \times 7^{-s} + 8^{-s} - 2 \times 14^{-s} + 2 \times 28^{-s} + 42^{-s} + 56^{-s}$$
$$- 2 \times 84^{-s} + \frac{2}{7} \times 168^{-s}\Big)\zeta_H(7s).$$

Moreover $\zeta(W(PGL_3(\mathbf{F}_2)), s)$ *converges for*

$$\mathrm{Re}(s) > s_0(W(PGL_3(\mathbf{F}_2))) \sim 1.112156628, \tag{8.7}$$

again with a Puiseux expansion in $\sqrt{s - s_0}$*.*

9. Some special values of representation zeta functions

Let us record a few general facts about representation zeta functions. If G is a rigid group, observe that

$$\zeta(G, 0) = \sum_{n \geq 1} h_n(G) = |\widehat{G}| \doteqdot h(G) \tag{9.1}$$

is the number (possibly infinite) of equivalence classes of irreducible representations of G.

These functions are well adapted to direct products: if G and H are two rigid groups, we have

$$\zeta(G \times H, s) = \zeta(G, s)\zeta(H, s). \tag{9.2}$$

When G is finite, $h(G)$ is the *class number* of G (= its number of conjugacy classes). In this case, observe also that the number of representations of dimension 1 is

$$\lim_{s \to \infty, s > 0} \zeta(G, s) = h_1(G) = |G/[G, G]|, \tag{9.3}$$

and that

$$\zeta(G,-2) \;=\; |G|. \tag{9.4}$$

If G is a finite group, $\zeta(G,s)$ is an entire function: if G is a finite$p$–group, $\zeta(G,s)$ is a polynomial in p^{-s}.

Let G be a finite group such that $\frac{1}{|G|}\sum_{g\in G}\operatorname{trace}(\pi(g^2)) = 1$ for all $\pi \in \widehat{G}$, namely such that all $\pi \in \widehat{G}$ can be realised over $\mathbf{R}$; then we have

$$\zeta(G,-1) \;=\; \sharp\left\{s \in G \mid s^2 = 1\right\}. \tag{9.5}$$

See for example [Serr–98, Section 13, Exercise 3] or [BeZh–98, Chapter 4, Theorems 13 and 14]. Groups for which (9.5) holds include the dihedral groups D_n for which

$$\zeta(D_{2n+1},-1) \;=\; 2n+2 \quad \text{ and } \quad \zeta(D_{2n},-1) \;=\; 2n+2,$$

the symmetric groups S_n (see e.g. [CuRe–62, § 28]), the alternating groups A_n when $n \in \{5,6,10,14\}$ (and for no other $n \geq 3$), the projective linear groups $PSL_2(\mathbf{F}_q)$ when q is either a power of 2 or of the form $4k+1$; see[4] [Feit–83, Theorem 6.1]. Observe that (9.5) cannot hold if G is of non–trivial odd order.

If we set $h(k) \doteqdot h\left(W(C_2,k)\right)$, the sequence $\left(h(k)\right)_{k\geq 0}$ appears as Number A006893 in [S–EIS], where it is described as counting trees of a kind which can be put in bijective correspondence with the conjugacy classes of $W(C_2,k)$:

$$h(0)=1, \quad h(1)=2, \quad h(2)=5, \quad h(3)=20, \quad h(4)=230, \quad h(5)=26\,795, \quad \ldots$$

10. A unitary variation and a question on some strengthening of Property (T)

Let G be a topological group. For all $n \geq 1$, denote by $u_n(G)$ the number (up to equivalence) of irreducible *unitary* representations of G in the Hermitian space $\mathbf{C}^n$. If $u_n(G) < \infty$ for all $n \geq 1$, set

$$\zeta_G^{(u)}(s) \;=\; \sum_{n\geq 1} u_n(G) n^{-s}.$$

Of course, $\zeta_G^{(u)}(s) \;=\; \zeta_G(s)$ if G is compact, and in many other cases, but not always; indeed, there are groups which are not rigid but are such that $u_n(G) < \infty$ for all $n \geq 1$; we show below that

$$\Gamma \;=\; SL_3(\mathbf{Z}[X])$$

is such an example.

Define for any complex number $z \in \mathbf{C}$ the 3–dimensional representation

$$\pi_z \,:\, \Gamma \longrightarrow GL_3(\mathbf{C}), \qquad g \longmapsto g(z)$$

[4] We are grateful to Alexander Zaleskii for information about Schur indices.

where $g(z)$ denotes the result of evaluating X at z. As the restriction of π_z to $SL_3(\mathbf{Z})$ is the tautological 3–dimensional representation of $SL_3(\mathbf{Z})$, the representation π_z is clearly irreducible. By computing the character at a well–chosen element, for example by computing

$$\text{trace}\left(\pi_z\begin{pmatrix}1+X & X & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) = \text{trace}\begin{pmatrix}1+z & z & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = 3+z,$$

we see that the uncountably many representations π_z are pairwise non–equivalent. Hence $h_3(\Gamma) = \infty$, and in particular Γ is not rigid.

Yet Γ has Kazhdan's Property (T), by a recent result of Leonid Vaserstein [Vase]; see also [Shal–06]. It is known that $u_n(\Gamma) < \infty$ for any finitely generated group Γ which has Kazhdan's Property (T); see Proposition IV of [HaRV–93], which builds up on Propositions 2.5 and 2.6 of [Wang–75], equivalently on Corollary 2 of [Wass–91]. In other words, Property (T) implies "unitary rigidity".

Summing up, $\zeta_\Gamma^{(u)}(s)$ is well defined, and $\zeta_\Gamma(s)$ is not. There are several strengthenings of Kazhdan's Property (T), some already existing and probably some more to come. For countable groups, will one of them imply *bona fide* rigidity?

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