OPERATOR ALGEBRAS, FREE GROUPS AND OTHER GROUPS

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Abstract. The operator algebras associated to non commutative free groups have received a lot of attention, by F.J. Murray and J. von Neumann and by later workers. We review some properties of these algebras, both for free groups and for other groups such as lattices in Lie groups and Gromov hyperbolic groups. Our guideline is the following list of results for the free group F_n over $n \geq 2$ generators.

(1) $W^*_\lambda(F_n)$ is a full factor (Murray and von Neumann, 1943, rephrased by Connes).
(2) There exists a continuous interpolating family $(L(F_r))_{1<r\leq\infty}$ which satisfies $L(F_n) = W^*_\lambda(F_n)$ and the free product relation $L(F_r) \star L(F_{r'}) = L(F_{r+r'})$ (Voiculescu, Dykema, Radulescu, 1990's).
(3) $C^*_\lambda(F_n)$ is a simple C^*-algebra with a unique trace (Powers, 1975).
(4) $C^*_\lambda(F_n)$ is not nuclear (Takesaki, 1964).
(5) $C^*(F_n)$ is not exact (S. Wassermann, 1976).
(6) $\mathbb{C}[F_n]$ has no zero divisor (G. Higman, 1940).
(7) $C^*_\lambda(F_n)$ has no non trivial idempotent (Pimsner and Voiculescu, 1982).
(8) The norm in $C^*_\lambda(F_n)$ of the sum of the generators of F_n and of their inverses is $\frac{1}{n}\sqrt{2n-1}$ (Kesten, 1959).
(9) $C^*_\lambda(F_n)$ contains $H^\infty(F_n)$(Haagerup, 1979, rephrased by Jolissaint).

We have collected a list of open problems; most of them are standard.

1. *INTRODUCTION.*

Let Γ be a group. We denote by $\mathbb{C}[\Gamma]$ the group algebra of complex linear combinations of elements of Γ, given together with the involution

$$X = \sum_{\gamma\in\Gamma} z_\gamma \gamma \mapsto X^* = \sum_{\gamma\in\Gamma} \overline{z_\gamma}\gamma^{-1} \ .$$

The **operator algebras** of interest here are various completions of $\mathbb{C}[\Gamma]$. Non abelian **free groups** are among the most studied examples of groups in this context. We denote by F_n the non abelian free group on n generators, where n is either an integer, $n \geq 2$, or $n = \infty$, meaning an infinite countable number of generators.

Our guiding principle is that the special case of free groups indicates typical behaviours which hold in many other cases of geometrical interest. This has suggested the three main aspects of the report below :

a survey of some properties of operator algebras associated to the F_n 's,

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an exploration of "geometric" groups giving rise to algebras with similar properties,

a list of open problems (some of them are numbered, from 1 to 19, and others appear in the text).

We shall concentrate on groups Γ which are lattices in semi-simple Lie groups ([Rag], [Mas]) or hyperbolic [Gr1], and on algebras which are either von Neumann algebras or C^*-algebras. But we shall mention on occasions other groups and other algebras. Unless explicitely stated otherwise, Γ denotes a **countable** group and operator algebras are **separable** in the appropriate sense.

Many important developments are left untouched. In particular, we say very little on K-theory and KK-theory related to group C^*-algebras, and nothing at all on the Novikov conjecture.

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2. *THE VON NEUMANN ALGEBRA* $W^*_\lambda(\Gamma)$.

2.1. *Generalities.*

For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ the involutive algebra of bounded operators on $\mathcal{H}$ and by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. Any **unitary representation** $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ of a group Γ gives rise to a morphism of involutive algebras $\mathbb{C}[\Gamma] \to \mathcal{L}(\mathcal{H})$ which is again denoted by π, and defined by

$$\pi\left(\sum_{\gamma\in\Gamma} z_\gamma \gamma\right) = \sum_{\gamma\in\Gamma} z_\gamma \pi(\gamma) \ .$$

We denote by $W^*_\pi(\Gamma)$ the weak closure of $\pi\left(\mathbb{C}[\Gamma]\right)$ in $\mathcal{L}(\mathcal{H})$.

Consider in particular the space $l^2(\Gamma)$ of square summable complex valued functions on Γ and the left regular representation

$$\lambda : \Gamma \to \mathcal{U}\left(l^2(\Gamma)\right)$$

where $\left(\lambda(\gamma)\xi\right)(x) = \xi(\gamma^{-1}x)$ for all $\gamma, x \in \Gamma$ and for all $\xi \in l^2(\Gamma)$. The weak closure $W^*_\lambda(\Gamma)$ of $\lambda(\mathbb{C}[\Gamma])$ is the **von Neumann algebra** of Γ.

There is a **finite normal trace** $\tau : W^*_\lambda(\Gamma) \to \mathbb{C}$ which extends the map $\mathbb{C}[\Gamma] \to \mathbb{C}$ given by $\sum_{\gamma\in\Gamma} z_\gamma\gamma \mapsto z_1$, and this trace is faithful. Thus the von Neumann algebra $W^*_\lambda(\Gamma)$ is finite, of the form $W_I \oplus W_{II} = (\oplus_{i=1}^{\infty} W_i) \oplus W_{II}$ with each W_i of type I_i, say with unit e_i, and with W_{II} of type II_1, say with unit e.

One has $e = 0$ if and only if Γ contains an abelian group of finite index. Let Γ_f denote the subgroup of Γ of elements with finite conjugacy classes and let $D\Gamma_f$ denote its commutator subgroup; then one has $e = 1$ if and only if either $[\Gamma : \Gamma_f] = \infty$ or $[\Gamma : \Gamma_f] < \infty$ and $|D\Gamma_f| = \infty$. See [Kan], [Sm1] and [Tho]. In case Γ is finitely generated, one has either $e = 0$ or $e = 1$. (This appears in [Ka2], but it is also a straightforward consequence of [Kan]. Indeed $e \neq 1$ implies $[\Gamma : \Gamma_f] < \infty$ by [Kan, Satz 1]; as Γ_f is also finitely generated in this case, the centre of Γ_f is of finite index in Γ_f [Tom, Corollary 1.5], and thus also in Γ; consequently $e = 0$.) But there are already in [Kap] examples, due to B.H. Neumann, which show that one may have $0 \neq e \neq 1$. Here is one of these examples : for each $i \in \mathbb{N}$, denote by D_i a copy of the dihedral group of order 8 and by C_i its center, which is of order 2 and which is also its derived group; let B be the direct sum of the B_i 's and let C be the subgroup of elements $(c_i)_{i\in\mathbb{N}} \in B$ such that $c_i \in C_i$ for each $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} c_i = 1$; this B.H. Neumann example is the quotient A/B; its von Neumann algebra is the direct product of $\mathbb{C}$ (with $\tau(e_1) = 1/2$) and of a factor of type II_1 (with $\tau(e) = 1/2$).

For a group Γ, Kaplansky has observed that $\tau(e_1)$ is the inverse of the order of the derived group of Γ [Kap, Theorem 1]. There are formulas giving $1 - e$ [Fo2]. The sum $1-e = \sum e_i$ is finite, and indeed $e_i = 0$ whenever $i^2 > |\Gamma/\Gamma_f|$ [Sm2]. When $e \neq 1$, one has $W_I \approx W^*_\lambda(\Gamma/\Gamma_0)$, where the von Neumann kernel Γ_0 of Γ is defined as $\bigcap_\pi Ker(\pi : \Gamma \to U(n))$, the intersection being over all finite dimensional representations of Γ [Sch, Satz 1].

Let Γ be a group such that $0 \neq e \neq 1$; I do not know whether there exists a group Γ_{II} naturally associated to Γ and such that $W_{II} \approx W^*_\lambda(\Gamma_{II})$. Here is a similar question : let Γ be a group such that $W^*_\lambda(\Gamma)$ is *not* a II_1-factor but contains a central projection c such that $cW^*_\lambda(\Gamma)$ is a II_1-factor; does there exist a group Γ_c naturally associated to Γ and such that $cW^*_\lambda(\Gamma) \approx W^*_\lambda(\Gamma_c)$?

Observe that, in case Γ is a hyperbolic group, Γ_f is precisely the so-called *virtual center* of Γ, denoted by $Z_{virt}(\Gamma)$ in [Cha].

2.2. *Free groups.*

Historically, the first examples of factors of type II_1 are given by Murray and von Neumann in [MNI], as crossed products which involve abelian groups (indeed subgroups of $\mathbb{R}$) acting ergodically on appropriate spaces. Several years later, they give a new construction which is "considerably simpler than our previous procedures, but it is clearly related to them" [MNIV, Introduction, §5]. Among other things, they show the following results. Recall that a group Γ has **infinite conjugacy classes,** or in short is icc, if *all* its conjugacy classes distinct from $\{1\}$ are infinite; for example, F_n is icc for all $n \geq 2$.

Theorem 1 (Murray and von Neumann).
(i) Let Γ be a group. Then $W^*_\lambda(\Gamma)$ is a factor if and only if Γ is icc.
(ii) For each $n \geq 2$ the factor $W^*_\lambda(F_n)$ does not possess Property Gamma.

This is shown in [MNIV] : see Lemma 5.3.4 for (i), Definition 6.1.1 for Property Gamma and §6.2 for (ii) when $n = 2$; moreover Lemma 6.3.1 shows that $W^*_\lambda(\Gamma_1 \star \Gamma_2)$ is a factor which does not possess Property Gamma whenever Γ_1 [respectively Γ_2] is a group containing at least two [resp. three] elements (the star denotes a free product). About the meaning of (ii), let us recall that a von Neumann algebra M does *not* have Property Gamma if and only if it is **full**, namely if and only if the group $Int(M)$ of its inner automorphisms is closed in the group $Aut(M)$ of all its automorphisms (see [Co74, Corollary 3.8] and [Co76, Theorem 2.1]).

Though we do *not* consider twisted crossed products in this report, let us at least mention that many of the results discussed here have "twisted formulations". For example, for claim (i) of Theorem 1 above, see [Pac, Proposition 1.3].

Claim (ii) suggests immediately the following, which is Problem 4.4.44 in [Sak].

Problem 1. *Does it happen that $W^*_\lambda(F_n) \approx W^*_\lambda(F_{n'})$ for $n \not\approx n'$?*

Though Problem 1 is still open, progress has been obtained recently, using Voiculescu's theory of freeness in noncommutative probability spaces (see among others [Vo2], [VDN] and [Sk2]). For example, one must have

either $W^*_\lambda(F_n) \approx W^*_\lambda(F_{n'})$ for all n, n' such that $2 \leq n, n' \leq \infty$
or $W^*_\lambda(F_n) \not\approx W^*_\lambda(F_{n'})$ for all n, n' such that $2 \leq n < n' \leq \infty$.

This has been first proved for $n, n' < \infty$, independently by K. Dykema and F. Radulescu; moreover, this holds for $n, n' \leq \infty$ by [Ra5, Corollary 4.7]. Let us also mention that

$$W^*_\lambda(\bigstar_{n=1}^{\infty}\Gamma_n) \approx W^*_\lambda(F_\infty)$$

whenever Γ_n is a nontrivial amenable group for all $n \geq 1$ (see [Vo2, Corollary 3.5] and [Dy2, Corollary 5.4]), and that

$$W^*_\lambda(\Gamma \star \Gamma') \approx W^*_\lambda(F_2)$$

when Γ , Γ' are infinite amenable groups [Dy2, particular case of Corollary 5.3]. See also [HaVo].

One of the novelties connected with the results above is the discovery, due independently to K. Dykema [Dy1] and F. Radulescu [Ra4], of a continuous family of II_1-factors $L(F_r)$ interpolating the free group factors. In the next theorem, we denote by $M_1 \star M_2$ the II_1-factor which is the reduced free product of two finite factors M_1, M_2; this is a crucial notion in Voiculescu's approach [Vo2].

Theorem 2 (Dykema, Radulescu, Voiculescu). *For each extended real number r such that $1 < r \leq \infty$, there exists a II_1-factor $L(F_r)$ such that*

$L(F_r) \star L(F_{r'}) \approx L(F_{r+r'})$ for all $r, r' \in]1, \infty]$,
$p\,(L(F_r) \otimes M_n(\mathbb{C})) \approx L(F_{1+\gamma^{-2}(r-1)})$ for any $r \in]1, \infty]$ and any projection $p \in L(F_r) \otimes M_n(\mathbb{C})$ of trace $\gamma \in]0, \infty[$ (where n is large enough),
$L(F_n) \approx W^*_\lambda(F_n)$ for all $n \in \{2, 3, ..., \infty\}$,
$L(F_r) \otimes M \approx L(F_{r'}) \otimes M$ for all $r, r' \in]1, \infty[$ whenever M is either $\mathcal{L}(\mathcal{H})$, or R, or $W^*_\lambda(F_\infty)$,
the isomorphism class of $L(F_r) \otimes L(F_{r'})$ depends only on $(r-1)(r'-1)$, for all $r, r' \in]1, \infty]$.

In the theorem, $\mathcal{L}(\mathcal{H})$ denotes the factor of type I_∞ and R denotes the hyperfinite factor of type II_1; moreover p is of trace γ for the trace of value 1 on the unity of $L(F_r)$. The first result quoted after Problem 1 is in fact

either $L(F_r) \approx L(F_{r'})$ for all r, r' such that $1 < r, r' \leq \infty$
or $L(F_r) \not\approx L(F_{r'})$ for all r, r' such that $1 < r < r' \leq \infty$.

Let us mention that some attention has been paid to free groups on uncountably many generators : if $F_{!!}$ denotes such a free group, then $W^*_\lambda(F_{!!})$ hasn't any "regular MASA"; also (we anticipate here on Section 3) the reduced C^*-algebra $C^*_\lambda(F_{!!})$, which clearly is not separable, has only separable abelian *-subalgebras [Po1, Section 6].

2.3. *Other groups.*

Considerable effort has been devoted to understand whether various factors of the form $W^*_\lambda(\Gamma)$ are or are not isomorphic to each other. The oldest result of this kind follows from Claim (ii) of Theorem 1 above on one hand and from the consideration of locally finite groups which are icc on the other hand; this result, which is the existence of two non isomorphic factors of type II_1, is recorded as the achievement of Chapters V and VI in [MNIV, Theorem XVI]. Later, the same construction $\Gamma \mapsto W^*_\lambda(\Gamma)$ has been used by D. McDuff to show that there are uncountably many pairwise non isomorphic type II_1 factors (see [McD] or [Sak, 4.3.10]).

We shall review now how some properties of an icc group are reflected in properties of the corresponding factor.

2.4. *Injectivity.*

For an icc group Γ, A. Connes has shown that $W^*_\lambda(\Gamma)$ is the *unique* injective factor of type II_1 if and only if Γ is amenable [Co76, in particular Corollary 7.2].

There is a very large number of pairwise non isomorphic icc amenable countable groups. Let us mention icc locally finite groups, as in [MNIV, Lemma 5.6.1], and icc solvable groups, such as the group

$$\Gamma_{\mathbb{K}} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{K}) \ : \ a \in \mathbb{K}^* \ , \ b \in \mathbb{K} \right\}$$

where $\mathbb{K}$ is a countable infinite field (the so-called $ax+b$ group associated to $\mathbb{K}$). It is known that there exist uncountably many pairwise nonisomorphic groups which are locally finite and icc (indeed simple) [KeW, Corollary 6.12]. It is also easy to check that two groups $\Gamma_{\mathbb{K}}$ and $\Gamma_{\mathbb{K}'}$ as above are isomorphic if and only if the fields $\mathbb{K}$ and $\mathbb{K}'$ are isomorphic, and there are uncountably many pairwise nonisomorphic countable fields (examples : the fields $\mathbb{K}_S = \mathbb{Q}\left((\sqrt{p})_{p\in S}\right)$ where S is a set of prime numbers; $\mathbb{K}_S \approx \mathbb{K}_{S'}$ if and only if $S = S'$, as it follows from Kummer's theory [Bou, V, p. 85, Théorème 4]; I am grateful to M. Ojanguren for explanations on this). There are many other ways to construct uncountable families of icc amenable groups; the way suggested in [Wat] provides groups with pairwise nonisomorphic C^*-algebras.

Yet all these groups provide the same factor.

2.5. *Fullness.*

The proof in [MNIV] of Theorem 1.ii above uses arguments which go much beyond free products. Indeed, one has the following, for which we refer to Effros [Ef1] and to [BdH]. As the terminology is unfortunately *not* uniform (compare [Pat, page 84]), let us recall that, *here*, a group is **inner amenable** if there exists a finitely additive measure $\mu : \mathcal{P}(\Gamma - \{1\}) \to [0,1]$ defined on all subsets of $\Gamma - \{1\}$, which is normalized by $\mu(\Gamma - \{1\}) = 1$ and which satisfies $\mu(\gamma D \gamma^{-1}) = \mu(D)$ for all $\gamma \in \Gamma$ and $D \subset \Gamma - \{1\}$. A group which has a finite conjugacy class distinct from $\{1\}$ is inner amenable.

Proposition 1 (Effros). *If Γ is a group which is not inner amenable, the algebra $W^*_\lambda(\Gamma)$ is a full factor.*

Problem 2. *Does there exist an icc group Γ which is inner amenable and such that the von Neumann algebra $W^*_\lambda(\Gamma)$ is a full factor ?*

There are many examples of families of groups which are known to be not inner amenable, and thus to give rise to full factors. Here are some of them

Let G be a connected semi-simple real Lie group without centre and without compact factor. It is a simple corollary of Borel density theorem that a lattice Γ in G is an icc group. (See e.g. [BkH, Proposition 2], which proves a slight strengthening of this; for the density theorem, see [Bo1] or [Zim, Theorem 3.2.5].) Also Γ is not amenable (indeed, C.C. Moore has shown this for any Zariski-dense subgroup of G [Zim, 4.1.11 and 4.1.15]). It can be shown that such a lattice Γ is never inner amenable, so that $W^*_\lambda(\Gamma)$ is a full factor; this carries over to lattices in adjoint semi-simple Lie groups over local fields [HS3].

In case G is moreover a simple real Lie group of rank one, it is also known that an icc subgroup of G which is discrete (not necessarily a lattice) and which is not amenable cannot be inner amenable (Georges Skandalis, private communication of December, 1992, and [Sk1]).

Let now Γ be a group which is hyperbolic and non elementary (a hyperbolic group is said to be elementary if it contains a cyclic subgroup of finite index). Such a group is not necessarily icc, for example because there may exist a subgroup Γ_0 of Γ such that Γ is the direct product of Γ_0 and of a non trivial finite subgroup. Consider however

$$\Gamma_f = \{\gamma \in \Gamma : \text{ the centralizer of } \gamma \text{ in } \Gamma \text{ is of finite index in } \Gamma\}.$$

Then Γ_f is a finite normal subgroup in Γ and the quotient $\Gamma_{icc} = \Gamma/\Gamma_f$ is icc [Cha, cor. 2.2.2]. If Γ is moreover torsion free (this implies $\Gamma_f = \{1\}$), then Γ is icc and is not inner amenable [Har4], so that $W^*_\lambda(\Gamma)$ is a full factor. I do not know if this holds under the more general condition $\Gamma_f = \{1\}$.

Let B_n denote the Artin braid group on n strings, let C_n denote its centre (which is isomorphic to $\mathbb{Z}$) and let DB_n denote is commutator group. It has been shown in [GiH] that $W^*_\lambda(B_n/C_n)$ and $W^*_\lambda(DB_n)$ are full factors for all $n \geq 3$. From the same paper, we repeat here the following.

Problem 3. *Let $K \subset \mathbb{S}^3$ be a piecewise linear knot which is not a torus knot and let Γ_K denote the fundamental group of the knot complement $\mathbb{S}^3 - K$. Show that Γ_K is icc and not inner amenable.*

One may of course repeat for the groups above the question of Problem 1 : in particular, if K and K' are two such knots, when are $W^*_\lambda(\Gamma_K)$ and $W^*_\lambda(\Gamma_{K'})$ isomorphic ? (Compare with Problems 4 and 6 below.) One may also formulate similar problems for other classes of groups appearing in geometry, such as mapping class groups, or infinite irreducible Coxeter groups which are neither finite nor affine. (The latter have free subgroups [Har3]; for many examples, see the references quoted in [Har5, nos 78-81].) Ditto for various notions of generic or random groups [Cha], [Gr2, § 9].

2.6. *Fundamental groups.*

The **fundamental group** of a factor M of type II_1 with trace τ is the group of positive real numbers

$$\begin{aligned}\mathcal{F}(M) = \Big\{ t \in \mathbb{R}_+^* \ : \ & \text{there exist a projection } e \in M \otimes \mathcal{L}(\mathcal{H}) \\ & \text{such that } \tau(e) = t \ \text{ and } \ e\big(M \otimes \mathcal{L}(\mathcal{H})\big)e \approx M \Big\} \\ = \Big\{ t \in \mathbb{R}_+^* \ : \ & \text{there exists } \alpha \in Aut\big(M \otimes \mathcal{L}(\mathcal{H})\big) \\ & \text{such that } \tau(\alpha(x)) = t\tau(x) \ \text{ for all } \ x \in M \otimes \mathcal{L}(\mathcal{H}) \Big\}\end{aligned}$$

defined by Murray and von Neumann. Chapter V of [MNIV] ends with the disappointing observation that "as to $\mathcal{F}(M)$, we know nothing beyond Theorem XV" (which shows that the fundamental group of the injective II_1 factor is $\mathbb{R}_+^*$). Today, we know at least two more things

if Γ has Kazhdan Property (T), then $\mathcal{F}\left(W_\lambda^*(\Gamma)\right)$ is a countable subgroup of $\mathbb{R}_+^*$ (see [Co80], and [Po5] for a generalization),

the fundamental group of $W_\lambda^*(F_\infty)$ is $\mathbb{R}_+^*$ [Ra1]; more precisely there exists a one parameter group $(\alpha_t)_{0<t<\infty}$ of automorphisms of $M = W_\lambda^*(F_\infty) \otimes \mathcal{L}(\mathcal{H})$ such that $\tau(\alpha_t(x)) = t\tau(x)$ for all $t \in \mathbb{R}_+^*$ and $x \in M$, where τ denotes the canonical trace on M [Ra2].

In particular $W_\lambda^*(\Gamma) \not\approx W_\lambda^*(\Gamma')$ if Γ has Property (T) and if Γ' is amenable or is F_∞.

Problem 4. *Does one have $\mathcal{F}\left(W_\lambda^*(\Gamma)\right) = \mathcal{F}\left(W_\lambda^*(F_2)\right)$ whenever Γ is one of the following groups ?*
a non elementary Fuchsian group,
a lattice in $SO^0(1,n)$ or in $PSU(1,n)$ for some $n \geq 2$,
a quotient B_n/C_n of a braid group by its centre ($n \geq 3$).

Dykema and Radulescu have shown that

either $\mathcal{F}\left(W_\lambda^*(F_n)\right) = \{1\}$ for all n such that $2 \leq n < \infty$, and then $W_\lambda^*(F_n) \not\approx W_\lambda^*(F_{n'})$ for all $n, n' \in \{2, 3, ...\}$ such that $n \neq n'$,
or $\mathcal{F}\left(W_\lambda^*(F_n)\right) = \mathbb{R}_+^*$ for all n such that $2 \leq n < \infty$, and then $W_\lambda^*(F_n) \approx W_\lambda^*(F_{n'})$ for all $n, n' \in \{2, 3, ...\}$.

Then Radulescu has shown that the second possibility implies $W_\lambda^*(F_n) \approx W_\lambda^*(F_\infty)$ for all $n \geq 2$. For Fuchsian groups, see [HaVo].

2.7. *Jones' invariants.*

Let M be a separable factor of type II_1. V. Jones has defined the invariant

$$\mathcal{I}(M) = \{\ r \in [1, \infty] \ : \ \text{there exists a } II_1\text{-subfactor } N \text{ of } M \text{ with index } r \ \}$$

and has shown that it satisfies the following properties :

$r_1, r_2 \in \mathcal{I}(M) \Longrightarrow r_1 r_2 \in \mathcal{I}(M)$,
$\left\{n^2\right\}_{n=1,2,...} \cup \{\infty\} \subset \mathcal{I}(M)$,

for each $x \in \mathcal{F}(M)$ one has $x + 2 + x^{-1} \in \mathcal{I}(M)$,

$\mathcal{I}(M) \subset \left\{4\cos^2 \frac{\pi}{n}\right\}_{n=3,4,\dots} \cup [4,\infty] = \mathcal{I}(R)$,

where R denotes the injective II_1-factor. Moreover, if Γ is an icc group

$\mathcal{I}(\Gamma) \subset \mathcal{I}\left(W^*_\lambda(\Gamma)\right)$

where $\mathcal{I}(\Gamma) = \{n \in \mathbb{N} \ : \ \text{there exists a subgroup of } \Gamma \text{ of index } n\}$. For all this, see [Jo83]. It is also known that

$\mathcal{I}(M)$ is countable if M has Property (T),

for example if $M = W^*_\lambda(\Gamma)$ for an icc group Γ with Property (T) (see [PiP], and a generalization in [Po5]).

A recent computation of Radulescu shows that

$\mathcal{I}\left(W^*_\lambda(F_\infty)\right) = \mathcal{I}(R)$.

Moreover, let $s \in \mathcal{I}(R)$ be the index of a subfactor of R with trivial relative commutant obtained by iteration of Jones' "basic construction" from a commuting square; then $s \in \mathcal{I}\left(W^*_\lambda(F_n)\right)$ for each $n \geq 2$; in particular $\mathcal{I}\left(W^*_\lambda(F_n)\right) \cap [1,4] = \mathcal{I}(R) \cap [1,4]$ for each $n \geq 2$. For all this, see [Ra3], [Ra5].

Problem 5. *Let Γ be an icc group which has Kazhdan Property (T). Compute $\mathcal{F}\left(W^*_\lambda(\Gamma)\right)$ and $\mathcal{I}\left(W^*_\lambda(\Gamma)\right)$.*

The question about $\mathcal{F}\left(W^*_\lambda(\Gamma)\right)$ appears in [Co90, section 3.10, problème 3] and [Co93, Section V.11]. One could of course add a (probably even more difficult) problem about the invariant $\mathcal{C}(M) = \{r \in [1,\infty] \ : \ \text{there exists a } II_1\text{-subfactor } N \text{ of } M \text{ with index } r \text{ and with trivial relative commutant}\}$.

2.8. *The constants of Cowling and Haagerup.*

In their work on completely bounded multipliers, M. Cowling and U. Haagerup have defined constants $\Lambda(\Gamma)$, $\Lambda(G)$, $\Lambda(M) \in [1,\infty]$, associated respectively to a discrete group Γ, a second countable locally compact group G and a finite von Neumann algebra M. Moreover :

$\Lambda(G) = 1$ if G is amenable,

$\Lambda(\Gamma) = \Lambda(G)$ if Γ is a lattice in G,

$\Lambda(\Gamma) = \Lambda\left(W^*_\lambda(\Gamma)\right)$,

$\Lambda(M_1) \leq \Lambda(M_2)$ if M_1 is a subalgebra of the finite algebra M_2,

$\Lambda(G) = 1$ if G is locally isomorphic to one of $SO(1,n)$ or $SU(1,n)$ for some $n \geq 2$,

$\Lambda(G) = 2n - 1$ if G is locally isomorphic to $Sp(1,n)$ for some $n \geq 2$,

$\Lambda(G) = \infty$ if G is a connected simple real Lie group with finite centre which is non compact and which is of real rank at least 2.

This provides many examples of pairs of lattices $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ such that $W^*_\lambda(\Gamma_1)$ is not isomorphic to any subalgebra of $W^*_\lambda(\Gamma_2)$. For all this, see [Haa3] and [CoH], as well as [LHa] for the universal covering of $SU(1,n)$; there is also a nice review by Cowling [Cw1]. Let us finally mention that $\Lambda(G) = 1$ for a locally compact group acting properly on a locally finite simplicial tree (a result of Szwarc, see [Va3, Proposition 6]) or for various free products of amenable groups amalgamated onver a common open compact subgroup

[BoP], and that $\Lambda(\Gamma) = 1$ for a Coxeter group Γ in the so-called "right-angled" class [Va4]; it is believed that $\Lambda(\Gamma) = 1$ for any Coxeter group Γ.

Superrigidity à la Margulis suggests the following problem, again due to A. Connes. See [Co90, section 3.10, problème 2], [Co93, Section V.11], and also the last question in [CoH] about lattices in $Sp(1, 11)$ and in $F_{4(-20)}$.

Problem 6. *Let Γ_1, Γ_2 be two icc groups which have Kazhdan Property (T). Show that $W^*_\lambda(\Gamma_1)$ and $W^*_\lambda(\Gamma_2)$ are isomorphic if and only if Γ_1 and Γ_2 are isomorphic.*

Here is a related problem concerning rigid groups. I believe it is also due to A. Connes.

Problem 7. *Find an icc group Γ such that any automorphism of the factor $W^*_\lambda(\Gamma)$ is inner.*

If such a group Γ exists, it has to be perfect and any automorphism of Γ itself has to be inner [Beh1, Theorems 5.1 and 5.2], [Ka1, Remark 2.3].

There is a notion of Property (T) for von Neumann algebras [CoJ] which is well adapted to the groups we discuss here : if a group Γ is icc, or more generally if the subgroup Γ_f of these elements of Γ which have a finite conjugacy class is finite, then $W^*_\lambda(\Gamma)$ has Property (T) if and only if Γ has Property (T). But Jolissaint has observed that, if Γ is a group which has Property (T) and which is such that Γ_f is infinite, then $W^*_\lambda(\Gamma)$ does *not* have Property (T) of [CoJ]. (See [Jo3], which gives also a characterization in terms of $W^*_\lambda(\Gamma)$ of Property (T) for an arbitrary group Γ; for examples, due to Serre, of groups Γ which have Property (T) and which have infinite centres, see [HaVa, § 3.d].)

It is known that a II_1-factor with Property (T) cannot be isomorphic to a subfactor of $W^*_\lambda(F_2)$ [CoJ, Corollary 4]. Moreover, if Γ is a group which has Property (T), any homomorphism from Γ to the unitary group of $W^*_\lambda(F_2)$ has an image whose strong closure is a compact subgroup of $\mathcal{U}(W^*_\lambda(F_2))$; this is a particular case of a result in [Rob].

One may ask whether there exists a sequence $(T_1) = (T)$, (T_2) , ... of strengthenings of Property (T) such that, if Γ or $W^*_\lambda(\Gamma)$ has and if Γ' or $W^*_\lambda(\Gamma')$ has not Property (T_n), then $W^*_\lambda(\Gamma)$ cannot be a subfactor of $W^*_\lambda(\Gamma')$. (This is a suggestion of M. Gromov.)

Let us finally repeat here an old problem which is still open (see e.g. [Po4, § 4.3]).

Problem 8. *Let M be a factor of type II_1 which is not injective. Does there exist a subfactor of M isomorphic to $W^*_\lambda(F_2)$?*

2.9. *The χ invariant and other invariants.*

For any factor M with separable predual, A. Connes has defined an abelian Borel group $\chi(M)$, which is the centre of the image in the outer automorphism group $Out(M) = Aut(M)/Int(M)$ of the group $\overline{Int}(M)$ of approximately inner automorphisms of M. He has shown that $\chi(M) = \{1\}$ if M is the injective factor of type II_1 or if $M = W^*_\lambda(F_n)$ for some integer $n \geq 2$. The invariant is meant (among other things) as an obstruction to a factorization as a tensor product of a full factor and a hyperfinite factor. A. Connes has constructed examples M such that $\chi(M) \neq \{1\}$ (some of these examples can be realized as group factors, but this is not used for the computation of their χ). He has also used the invariant χ to show that there exist factors of type II_1 which are *not* anti-isomorphic to themselves, and in particular *not* of the form $W^*_\lambda(\Gamma)$. See [Co75], as well as [C76b, Section 3.10], [Jo79], [Jo80] and [Kaw]. It seems appropriate to formulate explicitely the following question.

Problem 9. *What can be said about $\chi(W^*_\lambda(\Gamma))$ for other icc groups Γ ?*

In [Po2], S. Popa has made a detailed study of the maximal injective von Neumann subalgebras of $W^*_\lambda(F_n)$. He has shown in particular that each free generator of F_n generates such a maximal injective subalgebra, abelian and isomorphic to $W^*_\lambda(\mathbb{Z}) \approx L^\infty(\mathbb{S}^1)$. He asks moreover the following problem.

Problem 10. *Classify up to isomorphism the maximal injective von Neumann subalgebras of the II_1-factors.*

Let us define a factor M of type II_1 to be **tensorially indecomposable** if it cannot be written as any tensor product $A \otimes B$ of two factors A, B of type II_1. In [Po1, Corollary 6.6], Popa shows that the nonseparable factor $W^*_\lambda(F_{!!})$ of a free group $F_{!!}$ on uncountably many generators is tensorially indecomposable. In [Po6], he asks for examples of separable II_1-factors which have this property, and asks in particular the following.

Problem 11. *Are the $W^*_\lambda(F_n)$'s tensorially indecomposable ?*

Among other invariants of factors of the form $M = W^*_\lambda(\Gamma)$ which should be investigated, obvious candidates are cohomology spaces, in particular $H^k_c(M, M)$ and $H^k_c(M, M_*)$, where the subscript c indicates cohomology with norm-continuous cochains and where M_* denotes the predual of M. For example, one has $H^k_c(M, M) = \{0\}$ for all $k \geq 1$ if M is hyperfinite, and more generally if M is isomorphic to its tensor product with the hyperfinite II_1-factor; does this hold in general ? (it does for $k = 1$). See [Rin], [Co78]. Recently, A. Sinclair and co-workers have observed that the Gromov bounded cohomology group $H^k_b(\Gamma)$ injects in $H^k_c\left(l^1(\Gamma), l^1(\Gamma)\right)$ for all $k \geq 2$, so that one has for example $H^2_c\left(l^1(F_n), l^1(F_n)\right) \neq \{0\}$. There is now some effort to try and produce an example with

$$H_c^{2 \ or \ 4}\left(W^*_\lambda(\Gamma), W^*_\lambda(\Gamma)\right) \overset{?}{\neq} \{0\}$$

but all this is quite conjectural at the time of writing. (Moreover specialists don't all agree about which way to conjecture; see for example [Po6].)

Here is another open problem, stated in [FaH], slightly related to cohomology considerations : if γ is a free generator of F_n, does there exist $X, Y \in W^*_\lambda(F_n)$ such that $\lambda(\gamma) = XY - YX$?

A more exotic project would be to study homotopy groups of the unitary group (with the strong topology) of II_1-factors of the form $M = W^*_\lambda(\Gamma)$; such a unitary group is contractible if M is hyperfinite or if $\Gamma = F_\infty$ [PoT]. The homotopy groups of the unitary group of a II_1-factor M with respect to the *norm* topology are known; firstly $\Pi_1\left(\mathcal{U}(M)_{\text{norm}}\right) = \mathbb{R}$ by [ASS]; secondly $\Pi_{2k+1}\left(\mathcal{U}(M)_{\text{norm}}\right) = \Pi_1\left(\mathcal{U}(M)_{\text{norm}}\right) = \mathbb{R}$ and $\Pi_{2k}\left(\mathcal{U}(M)_{\text{norm}}\right) = \{0\}$ for all $k \geq 1$ by [Sc1]; see also [Sc2].

Finally, we would like at least to mention the impressive entropy computations of E. Størmer for automorphisms of the factor $W^*_\lambda(F_\infty)$ [Sto].

2.10. *Other representations.*

Let Γ be an irreducible lattice in a connected semi-simple real Lie group G without centre and without compact factor, let ρ be an irreducible unitary representation of G and let $\rho|\Gamma$ denote the restriction of ρ to Γ. If ρ is *not* in the discrete series of G, then $\rho|\Gamma$ is irreducible [CoS, Proposition 2.5], so that $W^*_{\rho|\Gamma}(\Gamma) = \mathcal{L}(\mathcal{H}_\rho)$ by Shur's Lemma. If ρ is in the discrete series, then $W^*_{\rho|\Gamma}(\Gamma) \approx W^*_\lambda(\Gamma)$ by [GHJ, Section 3.3.c].

Let Γ be an infinite group such that $C^*_\lambda(\Gamma)$ is simple (see below) and let M be an infinite hyperfinite factor (e.g. a Powers factor R_λ for some $\lambda \in]0,1[$). It follows from a result of O. Maréchal that there exists a representation π of Γ which is weakly equivalent to the regular representation and such that $W^*_\pi(\Gamma) \approx M$ [Mar]. It is also known that *any* properly infinite von Neumann algebra is of the form $W^*_\pi(PSL_2(\mathbb{Z}))$ [Beh2], and that the same holds for large classes of finite von Neumann algebras [Beh3].

In case $\Gamma = F_n$, finite factors of the form $W^*_\pi(F_n)$ are precisely the II_1-factors which can be generated by n unitaries. In particular, any icc group Γ given together with a set of n generators provides a factor $M = W^*_\lambda(\Gamma)$ and a representation $\pi : F_n \to \Gamma \to \mathcal{U}\left(l^2(\Gamma)\right)$ such that $W^*_\pi(F_n) = M$. For example, for each $k \geq 2$, the group $PGL_k(\mathbb{Z})$ can be generated by 2 elements [CxM, Chapter 7]; thus there exists a representation π of F_2 such that $W^*_\pi(F_2) \approx W^*_\lambda(PGL_k(\mathbb{Z}))$. I don't know of any II_1-factor which could not be generated by two unitaries (see also the end of 4.1 below); nor do I know of any II_1-factor without Cartan subalgebras. (If a II_1-factor with separable predual has a Cartan subalgebra, then it is generated by two unitaries [Po3, Theorem 3.4]; however, given such a factor M, the existence of a Cartan subalgebra of M is "in general" an open problem.)

Given a group Γ and a class of von Neumann algebras, an ambitious project is to classify the representations π of Γ such that $W^*_\pi(\Gamma)$ is in the given class. The appropriate kind of classification is up to quasi-equivalence : π and π' are quasi-equivalent if there exists an isomorphism Φ from $W^*_\pi(\Gamma)$ onto $W^*_{\pi'}(\Gamma)$ such that $\Phi(\pi(\gamma)) = \pi'(\gamma)$ for all $\gamma \in \Gamma$. For the class of finite factors, such a classification can be rephrased in terms of normalized characters of finite type, namely of functions of positive type $\xi : \Gamma \to \mathbb{C}$ which are normalized ($\xi(1) = 1$), central ($\xi(\gamma\gamma') = \xi(\gamma'\gamma)$) and indecomposable ($\xi = a\xi' + (1-a)\xi''$ with $a \in]0,1[$ and ξ', ξ'' normalized central of positive type implies $\xi = \xi' = \xi''$); see [DC*, corollaire 6.7.4 and proposition 17.3.5].

In case Γ is locally finite, factors of the form $W^*_\pi(\Gamma)$ are either the injective II_1-factor or finite dimensional factors. Characters have been classified for a few groups such as the group $S(\infty)$ of permutations with finite supports of an infinite countable set, and related groups. These results are due to Thoma, Vershik-Kerov and Nazarov [Naz].

3. *THE REDUCED C^*-ALGEBRA $C^*_\lambda(\Gamma)$, AND SIMPLICITY.*

3.1. *Generalities.*

Notations being as in the beginning of Chapter 2, the norm closure of $\pi : \mathbb{C}[\Gamma] \to \mathcal{L}(\mathcal{H})$ is denoted by $C^*_\pi(\Gamma)$. In particular, if π is the left regular representation λ, one obtains the **reduced C^*-algebra** $C^*_\lambda(\Gamma)$ of Γ. One may also choose the universal representation π_{un} of Γ (say here that π_{un} is the direct sum of all cyclic representations of Γ, up to equivalence), and one obtains the **full C^*-algebra** $C^*(\Gamma)$ of Γ. For any representation π, one has a natural morphism from $C^*(\Gamma)$ onto $C^*_\pi(\Gamma)$ which is again denoted by π. One has in particular a morphism $C^*(\Gamma) \to C^*_\lambda(\Gamma)$, and this is an isomorphism if and only if Γ is amenable, by a theorem of Hulanicki and Reiter [Ped, Th. 7.3.9].

In the classical case $\Gamma = \mathbb{Z}$, the algebra $C^*_\lambda(\Gamma) \approx C^*(\Gamma)$ is isomorphic via Fourier transform to the algebra $\mathcal{C}(\mathbb{T})$ of continuous functions on the one-dimensional torus $\mathbb{T}$. This carries over to any discrete abelian group Γ and its compact Pontryagin dual : $C^*_\lambda(\Gamma) \approx \mathcal{C}(\hat{\Gamma})$.

The oldest published reference I know involving reduced C^*-algebras of locally compact groups is [Seg].

3.2. *Free groups.*

The following result was published in 1975 [Pow], seven years after it was found [Va2, page 489]. Recall first that a normalized trace on a C^*-algebra A with unit is a linear map $\tau : A \to \mathbb{C}$ such that $\tau(1) = 1$, $\tau(a^*a) \geq 0$ and $\tau(ab - ba) = 0$ for all $a, b \in A$; it follows that $|\tau(a)| \leq ||a||$ for all $a \in A$ [DC*, Proposition 2.1.4]. The **canonical trace** on the reduced C^*-algebra of a group Γ is the extension to $C^*_\lambda(\Gamma)$ of the map $\mathbb{C}[\Gamma] \to \mathbb{C}$ which applies $\sum_{\gamma \in \Gamma} z_\gamma \gamma$ to z_1, as in 2.1 above.

Theorem 3 (Powers). *The reduced C^*-algebra $C^*_\lambda(F_n)$ of a free group on $n \geq 2$ generators is simple and has a unique normalized trace.*

There is a proof in Appendix 2 below. Another formulation of this theorem is that any unitary representation π of F_n which is weakly contained in the regular representation λ of F_n is in fact weakly equivalent to λ, and in particular is such that $C^*_\pi(F_n) \approx C^*_\lambda(F_n)$; for more of this point of view, see [BkH]; it is moreover true that π and λ as above are approximately equivalent in the sense of [Vo1, see in particular Corollary 1.4].

There is no analogue to Problem 1 here because computations of K_1-groups show that the algebras $C^*_\lambda(F_n)$ are pairwise nonisomorphic [PiV, Corollary 3.7]. Concerning pairs Γ_1, Γ_2 of non-isomorphic groups such that $C^*_\lambda(\Gamma_1) \approx C^*_\lambda(\Gamma_2)$, there are several known examples but apparently no systematic study. Let us describe two classes of such examples, the algebras being commutative in one case and simple in the other.

Let first Γ be any infinite countable abelian torsion group. Its Pontryagin dual $\hat{\Gamma}$ is compact, metrisable, totally disconnected and without isolated point [because the locally compact abelian group Γ is respectively discrete, countable, torsion, and infinite]. Consequently $\hat{\Gamma}$, viewed as a topological space, is homeomorphic to the triadic Cantor set K. Hence the C^*-algebra of Γ is isomorphic to the C^*-algebra $\mathcal{C}(K)$ of continuous functions on K, and thus does not depend on the detailed structure of Γ.

The second example is due to G. Skandalis. Let F_1, F_2 be two non isomorphic finite abelian groups of the same order, say n, for example $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. We identify both $C^*(F_1)$ and $C^*(F_2)$ to the same algebra A, isomorphic to $\mathbb{C}^n$. For $i \in \{1, 2\}$, set $\Gamma_i = F_i \star \mathbb{Z}$. The full C^*-algebra of Γ_i is canonically isomorphic to a free product with amalgamation over $\mathbb{C}$, and one has more precisely $C^*(\Gamma_i) \approx A \underset{\mathbb{C}}{\star} C^*(\mathbb{Z})$ in the sense of [Bro]. Moreover, the canonical trace $\tau_i : C^*(\Gamma_i) \to \mathbb{C}$ is independent on i when viewed as a map $A \underset{\mathbb{C}}{\star} C^*(\mathbb{Z}) \to \mathbb{C}$. Consequently the ideal $\mathcal{K}_i = \{x \in C^*(\Gamma_i) \mid \tau_i(x^*x) = 0\}$ and the quotient $C^*_\lambda(\Gamma_i) = C^*(\Gamma_i)/\mathcal{K}_i$ are both independent on i. On the other hand, Proposition 2 below shows that $C^*_\lambda(\Gamma_i)$ is a simple C^*-algebra with unique trace.

There exists however an uncountably infinite family $(\Gamma_\iota)_{\iota \in I}$ of countable groups such that the C^*-algebras $C^*_\lambda(\Gamma_\iota)$ are pairwise non-isomorphic, each being simple with a unique normalized trace; this follows easily from McDuff's result quoted in 2.3 above [AkL, Corollary 9].

It is known that some group Banach algebras determine the group Γ. (Examples : $L^1(\Gamma)$, $A(\Gamma)$ and $B(\Gamma)$; this holds indeed for a locally compact group; see [Lep], [Wa1], [Wa2] and [Wen].) But we do not discuss these algebras further here.

3.3. *Other groups with simple reduced C^*-algebras.*

Theorem 3 has begotten many generalizations : see among others [Ake], [AkL], [Be1], [Be2], [BCH], [BN1], [BN2], [Har2], [HSk], [HoR], [PaS], [Ros]. Let us indicate some of these results.

Proposition 2. *Let Γ be a group which admits at least one of the following descriptions :*

(a) a free product $\Gamma_1 \star \Gamma_2$ where $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$,

(b) a Zariski-dense subgroup in a semi-simple connected real Lie group without centre and without compact factor,

(c) a group $PSL(n, K)$ for some integer $n \geq 2$ and for some field K which is either of characteristic zero, or of characteristic p and not algebraic over $\mathbb{F}_p$.

(d) a group of K-rational points $\mathbb{G}(K)$ for some field K of characteristic zero and for some connected semi-simple algebraic group $\mathbb{G}$ defined over K,

(e) a torsionfree hyperbolic group which is not elementary.

Then $C^*_\lambda(\Gamma)$ is simple with unique normalized trace.

See [PaS] for (a), [BCH] for (b) and (d), [HoR] and [Ros] for (c), and [Har4] for (e). In (b), the subgroup is not supposed to be discrete in the ambient Lie group, but it is viewed as a discrete (possibly uncountable) group in the statement about $C^*_\lambda(\Gamma)$. There is also in [BCH] a proof of (c) for the particular case of a field K of characteristic zero.

The following repeats [Har2, Section 2, Question (2)]. (Question (3) of the same reference, same section, has been answered in [Be1].)

Problem 12. *Does there exist a group Γ such that $C^*_\lambda(\Gamma)$ is simple but has several traces? or such that $C^*_\lambda(\Gamma)$ has a unique trace but is not simple ?*

Here are two observations resulting from a conversation with A. Valette.

Assume firstly Γ such that $C^*_\lambda(\Gamma)$ has a unique trace; then any normal subgroup N of Γ distinct from $\{1\}$ is icc and not amenable. Indeed, $C^*_\lambda(N)$ has a unique trace for any normal subgroup N of Γ, because any trace on $C^*_\lambda(N)$, composed with the canonical conditional expectation $C^*_\lambda(\Gamma) \to C^*_\lambda(N)$, provides a trace on $C^*_\lambda(\Gamma)$; the conclusion follows.

Assume secondly Γ such that $C^*_\lambda(\Gamma)$ is simple; then Γ is an icc group. Indeed, as Γ_f is an amenable group [Tom, Corollary 1.5], the identity representation of Γ_f is weakly contained in the regular representation of Γ_f, and one sees by induction from Γ_f to Γ that the regular representation of Γ/Γ_f (viewed as a representation of Γ) is weakly contained in the regular representation of Γ. In other words, one has a morphism of C^*-algebras π from $C^*_\lambda(\Gamma)$ onto $C^*_\lambda(\Gamma/\Gamma_f)$. Now π is an isomorphism if $C^*_\lambda(\Gamma)$ is simple, so that the natural map $\mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma/\Gamma_f]$ is injective, and thus $\Gamma_f = \{1\}$ as claimed above.

All this being said, problems of simplicity of reduced C^*-algebras of groups should not conceal other problems. In particular, it would be pleasant to know "many" examples of groups Γ which are *not* C^*_λ-simple but for which two-sided ideals of $C^*_\lambda(\Gamma)$ are *classified* in some way (a few examples appear in Theorem 4 of [BCH]). The corresponding program for the Fourier algebra $A(\Gamma)$ of a discrete group Γ is the subject of [For, see in particular Theorem 3.20].

3.4. *Other representations*

The diversity of C^*-algebras of the form $C^*_\pi(F_n)$ has no limit. Indeed, let A be a separable C^*-algebra with unit acting in some Hilbert space $\mathcal{H}$, and assume that A is generated as a C^*-algebra by a finite or infinite sequence $(a_1, a_2, ...)$ of length n. As any element in a C^*-algebra with unit is a linear combination of unitaries [Ped, 1.1.11], there is no loss of generality if we assume that the a_i 's are unitary. Consider now a free set of generators s_i of F_n, and define a representation $\pi : F_n \to \mathcal{U}(\mathcal{H})$ by $\pi(s_i) = a_i$. Then $C^*_\pi(F_n) = A$.

In particular, there are C^*-algebras with unit of the form $C^*_\pi(\Gamma)$ which cannot be reduced C^*-algebras of discrete groups. A specific example is the algebra $\mathbb{C} \oplus \mathcal{K}$ acting on a separable infinite dimensional Hilbert space, which is generated by the identity and by the compact operators : it is $C^*_\pi(F_2)$ for an appropriate π [HRV1, Section A], and it cannot be of the form $C^*_\lambda(\Gamma)$. (Indeed, assume firstly that Γ has an element γ of infinite order. Then γ generates a sub-C*-algebra isomorphic to $C^*\left(\gamma^{\mathbb{Z}}\right) \approx \mathcal{C}(\mathbb{T})$, so that the spectrum of γ is the circle $\mathbb{T}$. But the spectrum of any element in $\mathbb{C} \oplus \mathcal{K}$ is countable, and thus $\mathcal{C}(\mathbb{T})$ cannot be isomorphic to a subalgebra of $\mathbb{C} \oplus \mathcal{K}$. Assume secondly that Γ has an element γ of some finite order $k > 1$. As Γ is infinite, it follows that any kth root of 1 appears in the spectrum of γ with infinite multiplicity. This cannot happen in $\mathbb{C} \oplus \mathcal{K}$.)

It would be interesting to understand better, for a given group Γ,

(i) which are the algebras of the form $C^*_\pi(\Gamma)$,
(ii) what are the automorphism groups of these $C^*_\pi(\Gamma)$,
(iii) for which π these algebras $C^*_\pi(\Gamma)$ are simple.

About (ii), it is known that $\overline{Inn}(A)/Inn(A)$ is uncountable for any separable C^*-algebra A which does not have continuous trace, e.g. for $A = C^*_\lambda(\Gamma)$ whenever Γ is not of type I. (See [Phi, Theorem 3.1], and compare with Problem 7 above.) It is also known that $\overline{Inn}\left(C^*_\lambda(F_2)\right)/\overline{Inn_0}\left(C^*_\lambda(F_2)\right)$ is non trivial, where $\overline{Inn_0}(A)$ denotes the closure of the group of inner automorphisms of A determined by unitaries connected to 1 in the group of automorphisms of the C^*-algebra A, closure for the topology of pointwise convergence [ElR, 4.13].

About (iii), see Proposition 4 of Appendix 2.

Representations of free groups are discussed in [FTP]; see also [FTN], [Sz1] and [Sz3].

In a remarkable paper of the early 50's, Yoshizawa has constructed an irreducible representation π of F_2 which weakly contains *any* irreducible representation of F_2, namely which is such that the natural morphism $C^*(F_2) \to C^*_\pi(F_2)$ is an isomorphism [Yos, § 3]. In other words, the C^*-algebra $C^*(F_2)$ is primitive, namely has a representation which is both irreducible and faithful [Ped, 3.13.7].

Problem 13. *What are the groups with primitive full C^*-algebras ?*

Let Γ be a group given as a discrete subgroup of some Lie group G. A natural way to obtain representations of Γ is to consider a representation ρ of G and its restriction $\rho|\Gamma$ to Γ. If Γ and G are as in 2.10 above, it is a natural question to ask about properties of $C^*_{\rho|\Gamma}(\Gamma)$. Here is a partial and easy answer, from [BkH] : let G be a simple connected real Lie group which is non compact and with centre reduced to $\{1\}$, let ρ be a unitary representation of G in the principal series and let Γ be a lattice in G; then $C^*_{\rho|\Gamma}(\Gamma) \approx C^*_\lambda(\Gamma)$.

Another natural representation of Γ to consider is $\rho_{un}|\Gamma$, where ρ_{un} denotes the universal representation of the Lie group G; if this is again a non compact simple connected real Lie group without centre, it is conjectured that $C^*(\Gamma) \to C^*_{\rho_{un}|\Gamma}(\Gamma)$ is *never* an isomorphism, and this has been proved in many cases (e.g. if G has Property (T)) in [BeV].

4. *THE C^*-ALGEBRAS $C^*_\lambda(\Gamma)$ AND $C^*(\Gamma)$: SOME OTHER PROPERTIES.*

4.1. *Nuclearity and other finiteness conditions.*

Given two C^*-algebras A and B, there are in general several ways to complete the algebraic tensor product of A and B to obtain a C^*-tensor product. The algebra A is said to be **nuclear** if these ways coincide, for any B. For more on this, see [La2] and [Ta2]. Nuclearity for $C^*_\lambda(\Gamma)$ is settled by the following result (of which (ii) has the remarkable property that the "only if" part does *not* extend to the locally compact case).

Theorem 4 (Takesaki, 1964, and Lance, 1973).
(i) The algebra $C^*_\lambda(F_2)$ is not nuclear.
(ii) Let Γ be a group. Then $C^*_\lambda(\Gamma)$ is nuclear if and only if Γ is amenable.

It follows easily from Claim (ii) that $C^*(\Gamma)$ is nuclear if and only if Γ is amenable. (Indeed, if Γ is amenable, then $C^*(\Gamma)$ is isomorphic to $C^*_\lambda(\Gamma)$, which is nuclear. If $C^*(\Gamma)$ is nuclear, then $C^*_\lambda(\Gamma)$ is nuclear, because any quotient of a nuclear algebra is nuclear by a result of Choi and Effros [La2], so that Γ is amenable by (ii).)

Besides amenability $\Longleftrightarrow$ nuclearity, there are only few known exact translations between properties of Γ and properties of $C^*_\lambda(\Gamma)$ or of $C^*(\Gamma)$. For example, does

$$\Gamma \ \text{ finitely generated } \overset{?}{\Longleftrightarrow} C^*_\lambda(\Gamma) \ \text{ finitely generated}$$

hold ? One may ask what is the smallest number of generators for $C^*_\lambda(\Gamma)$ or $C^*(\Gamma)$. S. Wasserman [Was2, Section 6] has observed that this number is at least 2 for $C^*(F_2)$, because $C^*(F_2)$ has a quotient isomorphic to $\mathcal{C}(\mathbb{T}^2)$ and because the 2-torus $\mathbb{T}^2$ is not planar. On the other hand, it is known that $W^*_\lambda(F_2)$ and the hyperfinite II_1-factor are both singly generated [Sai, Theorem 2.3 and following example].

And does

$$\Gamma \ \text{ locally finite } \overset{?}{\Longleftrightarrow} C^*_\lambda(\Gamma) \ \text{ approximately finite (AF)}$$

hold ? Known examples of groupoid C^*-algebras which are AF for non obvious reasons [Kum] may suggest that $\Leftarrow$ does *not* hold.

What about

$$\Gamma \ \text{ f.g. and of polynomial growth } \overset{?}{\Longleftrightarrow} C^*_\lambda(\Gamma) \ \text{ essentially of polynomial growth [KiV]}$$

(f.g. holds for finitely generated) ?

The answer to the question

$$\Gamma \ \text{ residually finite } \overset{?}{\Longleftrightarrow} C^*(\Gamma) \ \text{ residually finite dimensional [ExL]}$$

is negative, but can one slightly change the question to have an affirmative answer ? The negative answer follows from properties of the group $\Gamma = SL(2, \mathbb{Z}[1/p])$ where p is a prime, as shown to me by M. Bekka. Indeed, one one hand Γ is residually finite because it is both finitely generated and linear [Mal]. On the other hand Γ does not have Kazhdan's Property (T) because Γ is dense in $SL(2, \mathbb{R})$ [HaV, Propositions 1.6 and 3.6], but the unit representation of Γ is isolated in the set of all its finite dimensional unitary representations [LuZ]; these facts imply that finite dimensional representations of $C^*(\Gamma)$ do not separate elements of this C^*-algebra. However, I do not know whether $\Leftarrow$ holds or not.

What are the properties of $C^*_\lambda(\Gamma)$, or of $C^*(\Gamma)$, which are equivalent to the group being finitely presented ? with solvable word problem ? hyperbolic ? small cancellation ? of finite cohomological dimension (say over $\mathbb{Q}$) ? a torsion group ? solvable ? Dually, what are the properties of Γ which are equivalent to $C^*_\lambda(\Gamma)$ being simple ? generated by one element ? to $C^*(\Gamma)$ having Hausdorff spectrum ? (These lists can be extended at will.)

4.2. *Exactness.*

The C^*-algebra A is said to be **exact** if, given any short exact sequence

$$0 \to J \to B \to B/J \to 0$$

of C^*-algebras, the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

is also exact, where $\otimes$ denotes the minimal (or spatial) tensor product. For a proof of the following result, we refer to [Was1], [Ki1], [Ki2], [Ki3] and [HRV2].

Theorem 5 (S. Wassermann, Kirchberg and others).
(i) The algebra $C^*(F_2)$ is not exact.
(ii) Let Γ be a group; assume that Γ is isomorphic to a subgroup of some locally compact group G such that $C^*(G)$ is a nuclear C^*-algebra. Then $C^*(\Gamma)$ is exact if and only if Γ is amenable.

Recall that the C^*-algebra $C^*(G)$ of a locally compact group G is nuclear as soon as G is almost connected [Co76].

In sharp contrast, there is no known example of a group Γ such that $C^*_\lambda(\Gamma)$ is not exact. For example, let Γ be a group and assume that Γ embeds as a discrete subgroup in some second countable locally compact group G having a closed amenable subgroup P with G/P compact (a connected real Lie group G would do); then $C^*_\lambda(\Gamma)$ embeds in the nuclear C^*-algebra $\mathcal{C}(G/P) \underset{r}{\rtimes} \Gamma$, and in particular $C^*_\lambda(\Gamma)$ is exact. (This is an unpublished result of A. Connes which was circulating in the 1980 Kingston's Conference.) Also $C^*_\lambda(\Gamma)$ is exact for any hyperbolic group Γ (unpublished result of Hilsum-Renault-Skandalis).

Problem 14. *(i) If Γ is any group (not necessarily isomorphic to a subgroup of a C^*-nuclear locally compact group) such that $C^*(\Gamma)$ is exact, does it follow that Γ is amenable ?*
(ii) Does there exist a group Γ such that $C^*_\lambda(\Gamma)$ is not exact ?

Question (ii) is the open problem (P1) of [Ki3].

4.3. *Non-existence of idempotents.*

Given a torsionfree group Γ, it is an old question to know whether $\mathbb{C}[\Gamma]$ may have zero divisors, and in particular idempotents distinct from 0 and 1. This is often attributed to Kaplansky : see [Kou, Problem I.3], and also [Far]. The oldest result I know on this is that of Higman [Hig, particular case of Theorem 12].

Theorem 6 (Higman). *The algebra $\mathbb{C}[F_n]$ has no zero divisor.*

Here is a more recent result, which is a particular case of [Fo1, Theorem 9] and [Bas, § 9]: if Γ is a torsionfree finitely generated linear group, then $\mathbb{C}[\Gamma]$ has no idempotent distinct from 0 and 1.

A C^*-algebra distinct from $\mathbb{C}$ has always zero divisors (this is easy to check via functional calculus) but it is a conjecture going back to Kadison and Kaplansky that

$$C^*_\lambda(\Gamma) \text{ has no idempotent, except } 0 \text{ and } 1$$

for any torsionfree group Γ. This would follow from a more general conjecture of P. Baum and A. Connes [BaC] which involves the K-theory groups $K_i(C^*_\lambda(\Gamma))$. For all this, see the discussion in [Va2].

Theorem 7 (Pimsner-Voiculescu). *For each $n \geq 2$, the C^*-algebra $C^*_\lambda(F_n)$ has no idempotent distinct from 0 and 1.*

Theorem 7 has first appeared in [PiV]. There is a very nice proof of it in [Co86, Section I.1], in terms of a Fredholm module over $C^*_\lambda(F_2)$ associated to the standard action of F_2 on the homogeneous tree of degree 4. (Theorem 7 for F_n follows from the result for F_2 because F_n is a subgroup of F_2.) Conne's proof is so nice that minor variations of it have appeared in semi-popularization journals [Ef2]. There is another proof by Cuntz [Cu2], using the easy result that $C^*(F_2)$ has no idempotent [Cho], [Cu1].

It has been shown that $C^*_\lambda(\Gamma)$ has no idempotent distinct from 0 and 1 for Γ a torsionfree discrete subgroup in a connected Lie group whose semi-simple part is locally isomorphic to a product of compact groups, of Lorentz groups $SO(n, 1)$ [Kas], and of groups $SU(n, 1)$ [JuK]. The published proofs use KK-theory.

The following problem suggests another approach which could work for more groups. It appears in [Co90, section 2.5, problème 11]. It is open even if G is one of the groups $SO(n, 1)$ or $SU(n, 1)$ dealt with by Kasparov and Julg. This has been explained to me by P. Julg and A. Valette.

Problem 15. *Let Γ be a torsionfree subgroup of a connected semi-simple real Lie group G. Show that $C^*_\lambda(\Gamma)$ has no idempotent distinct from 0 and 1 by analyzing the appropriate Fredholm module and its Chern character.*

The same problem holds for an arbitrary torsionfree hyperbolic group.

We refer to [Co90] for explanations about the "appropriate" Fredholm module; see also page 77 of the same reference, and [Co93, Section IV.3].

Let us mention that the following algebras have also been shown to be without non trivial idempotent

$C^*_\lambda(\Gamma)$ for Γ abelian torsionfree (this is Pontryagin Theory [Va2, Theorem 2]),

$C^*_\lambda(\Gamma)$ for Γ locally nilpotent torsion free [KaT] (see also [Ji1, Theorem 5.1]),

$C^*_\lambda(\Gamma)$ for Γ a discrete subgroup of a connected simply connected solvable group [BaC],

$C^*_\lambda(\Gamma)$ for various groups Γ acting on trees as in [Pim],

$l^1(\Gamma)$ for Γ torsionfree hyperbolic [Ji2, Theorem 4.2],

$C^*(\Gamma)$ for a group Γ which is free [Coh1], [Cho], or a free product of torsionfree abelian groups [Cu1], or a free product of torsionfree amenable groups (and a few other cases) [JiP].

Note that $C^*(\Gamma)$ *does* have non trivial projections if Γ has Property (T) by [Va1]; see also [Va5].

These no idempotent results have applications on the structure of various spectra : one appears in [Sun]; another one is the observation (suggested to me by L. Guillopé) following

Theorem 8 below. Let Γ be a group given together with a symmetric probability measure, namely with a function $p : \Gamma \to [0,1]$ in $l^1(\Gamma)$ such that $p(\gamma^{-1}) = p(\gamma)$ for all $\gamma \in \Gamma$ and such that $\sum_{\gamma\in\Gamma} p(\gamma) = 1$. To avoid trivialities, assume moreover that the support of p generates Γ. The **Markov operator** of the associated random walk on Γ is the operator $M(p) : l^2(\Gamma) \to l^2(\Gamma)$ of convolution to the right $\xi \mapsto \xi \star p$. It is obvious that $M(p)$ is self-adjoint and that $||M(p)|| \leq 1$, it is easy to check that $||M(p)|| = max\{\lambda \in \mathbb{R} : \lambda$ is in the spectrum of $M(p)\}$ (see [HRV1, Lemma 8]), and it is a result of H. Kesten that $||M(p)|| = 1$ if and only if Γ is amenable [Ke2]. For p equidistributed on a symmetric set S of generators of Γ, this has been reformulated in terms of "cogrowth" by Grigorchuck and Cohen (see [Coh2], [Sz2], [Woe]).

Theorem 8 (Kesten). *Let Γ be a group generated by a finite set $S = \{s_1, ..., s_n\}$ with $n \geq 2$. Let $p : \Gamma \to [0,1]$ be defined by $p(\gamma) = \frac{1}{|S\cup S^{-1}|}$ if $\gamma \in S \cup S^{-1}$ and $p(\gamma) = 0$ otherwise. Then :*

(i) one has $||M(p)|| \geq \frac{1}{n}\sqrt{2n-1}$,
(ii) Γ is free on S if and only if $||M(p)|| = \frac{1}{n}\sqrt{2n-1}$,
(iii) if Γ is free on S, then the spectrum of $M(p)$ is the interval $[-\frac{1}{n}\sqrt{2n-1}\ ,\ \frac{1}{n}\sqrt{2n-1}]$.

Suppose now moreover that Γ is such that $C^*_\lambda(\Gamma)$ has no idempotent distinct from 0 and 1, and let $p : \Gamma \to [0,1]$ be a symmetric probability measure as above. It is an immediate corollary of functional calculus that

the spectrum of $M(p)$ is an interval

as it is the case in Theorem 8.iii. About the following problem, see [KaV] and [HRV1].

Problem 16. *Compute the spectrum and the multiplicity function of $M(p)$ for other pairs (Γ, p), for example when Γ is a Fuchsian group; compute also the spectral measure of $M(p)$.*

It is easy to check that the spectrum of $M(p)$ is finite if and only if the group Γ is finite [HRV3, Section 2.1]. One may also consider non-symmetric probability measures; it is then an open problem to know whether there exists a pair (Γ, p) with Γ infinite and the spectrum of $M(p)$ finite [HRV3].

Of course, group C^*-algebras in general do have projections. For an analysis of the case of $C^*_\lambda(\mathbb{Z}/n\mathbb{Z} \star \mathbb{Z}/m\mathbb{Z})$, see [ABH].

5. *RAPIDLY DECREASING FUNCTIONS ON A FINITELY GENERATED GROUP.*

Let Γ be a group given together with a length function $L : \Gamma \to \mathbb{R}_+$. For simplicity, we will moreover assume here that Γ is generated by a finite set S and that, for each $\gamma \in \Gamma$, the length $L(\gamma)$ is the smallest integer n such that $\gamma = s_1...s_n$ with $s_1, ..., s_n \in S \cup S^{-1}$. For each $s \in \mathbb{R}$, define the Sobolev space

$$H^s(\Gamma) = \left\{ \xi : \Gamma \to \mathbb{C} \;\middle|\; \sum_{\gamma\in\Gamma} |\xi(\gamma)|^2 \left(1 + L(\gamma)\right)^{2s} \ < \ \infty \right\}$$

which is a Hilbert space for the obvious scalar product. The space of **rapidly decreasing functions** on Γ is the Fréchet space

$$H^\infty(\Gamma) = \bigcap_{s\in\mathbb{R}} H^s(\Gamma)\ .$$

It is easy to show that the isomorphism classes of these spaces do not depend on the choice of the finite generating set S.

By definition, $H^\infty(\Gamma)$ is a subspace of $l^2(\Gamma)$. But $H^\infty(\Gamma)$ need not be a convolution algebra. (There may exist $\xi, \eta \in H^\infty(\Gamma)$ such that the convolution $\xi \star \eta$, which is always well defined and in $c_0(\Gamma)$, is not in $H^\infty(\Gamma)$: this happens for example if Γ is amenable and not of polynomial growth [Jo1, Proposition B].)

Proposition 3. *Let Γ be a finitely generated group.*

(i) The space $H^\infty(\Gamma)$ is a subspace of $C^*_\lambda(\Gamma)$ if and only if it is a subalgebra of $C^*_\lambda(\Gamma)$.

(ii) If the conditions of (i) hold, then the inclusion $H^\infty(\Gamma) \subset C^*_\lambda(\Gamma)$ induces an isomophism in K-theory.

We refer to [Jo1] and [Jo2] for the proof. Claim (ii) is due to A. Connes, and is an important step in one application of these ideas to differential topology [CoM]. A finitely generated group is said to have **Property (RD)** if it satisfies the conditions of Claim (i). The following appears in [Haa1, Lemma 1.5]; see also [CaH], and the exposition in [Haa2].

Theorem 9 (Haagerup). *Let n be an integer such that $2 \leq n < \infty$; then $H^2(F_n) \subset C^*_\lambda(F_n)$, and in particular $H^\infty(F_n) \subset C^*_\lambda(F_n)$.*

Various groups have been shown to have Property (RD), and it is in particular the case for groups of polynomial growth, for which $H^s(\Gamma) \subset C^*_\lambda(\Gamma)$ for some s depending on the growth, and for hyperbolic groups, for which $H^2(\Gamma) \subset C^*_\lambda(\Gamma)$ (see mainly [Jo1], and also [JoV] and [Har4]). The known proofs are quite different for these two classes, and it is an interesting open problem to find an argument covering both $\mathbb{Z}^n$ (say) and hyperbolic groups. More generally, we formulate the following question (even though the hope for a positive answer is very small).

Problem 17. *Let Γ be a lattice in $G = PSL_n(\mathbb{R})$ such that G/Γ is compact (and $n \geq 3$); does Γ have Property (RD) ?*

The cocompactness hypothesis is crucial, because $PSL_n(\mathbb{Z})$ does not have Property (RD) as soon as $n \geq 3$ [Jo1, Corollary 3.1.9]. The question of Problem 17 is also open for most Coxeter groups.

Other rapid decay algebras have been introduced by Ogle [Ogl].

APPENDIX 1. *ON FREE GROUP.*

It is almost a tautology to say that free groups play a central role in combinatorial group theory, but this should not conceal the important role of free groups in other parts of mathematics. As an example of an old appearance of free groups in traditional subjects, we may quote Schottky groups, which are free subgroups of $PSL_2(\mathbb{C})$ [Kle, page 200].

Here are three criteria for recognizing free groups. For the first one, a convenient reference is the recent book of Serre [Ser, § I.3], though the result itself is quite old : it appears for example in [Rei, Section 4.20], but "it is, of course, very difficult to claim that something is *not* due to Poincaré" [ChM, page 96].

Criterium 1. *A group which operates freely on a tree is a free group.*

Let us also recall the result of Stallings and Swan according to which a torsionfree group which has a free subgroup of finite index is itself a free group [Swa].

We state now the "Table Tennis Lemma", essentially due to F. Klein: see [Mac], [Tit] and [Har1].

Criterium 2. *Let G be a group acting on a set X, let Γ_1, Γ_2 be two subgroups of G and let X_1, X_2 be two subsets of X; assume that $|\Gamma_2| \geq 3$. Assume that*

$$\begin{aligned} \gamma(X_2) \subset X_1 \quad \text{for all} \quad \gamma \in \Gamma_1 \ , \ \gamma \neq 1 \ , \\ \gamma(X_1) \subset X_2 \quad \text{for all} \quad \gamma \in \Gamma_2 \ , \ \gamma \neq 1 \ . \end{aligned}$$

Then the subgroup Γ of G generated by Γ_1 and Γ_2 is isomorphic to the free product $\Gamma_1 \star \Gamma_2$ of Γ_1 and Γ_2. (In particular, if Γ_1 and Γ_2 are free of rank n_1 and n_2 respectively, then Γ is free of rank $n = n_1 + n_2$.)

The next "quasi-geodesic criterium" is due to Gromov [Gr1, 7.2.C]; see also [GhH, § 5.3], or the much shorter proof in [Del].

Criterium 3. *Let Γ be a δ-hyperbolic group and let $\gamma_1, \gamma_2 \in \Gamma$ be such that the word-length relations*

$$\begin{aligned} |\gamma_j^2| \geq |\gamma_j| + 2\delta + 1 \quad & \text{for} \quad j \in \{1, 2\} \\ |\gamma_1^\epsilon \gamma_2^\eta| \geq \max\left(|\gamma_1|, |\gamma_2|\right) + 2\delta + 1 \quad & \text{for} \quad \epsilon, \eta \in \{1, -1\} \end{aligned}$$

hold. Then the subgroup of Γ generated by γ_1, γ_2 is free of rank 2.

This criterium is related to the following fact (see [Gr1, 5.3.B] and [Del]). Let Γ be a hyperbolic group which is torsionfree and non elementary. Then there exists a finite sequence $\Gamma_1, ..., \Gamma_k$ of subgroups of Γ such that any pair of elements of Γ generates a subgroup which is either free of rank two or conjugated to one of the Γ_j's.

For other examples of free subgroups of geometrically significant groups, see among many others [BeL, appendice] [Bo2], [DeS], [Gla], [Har1], [Hau], [MyW] and [Wag].

APPENDIX 2. *PROOFS OF THEOREM 3 AND PROPOSITION 2(e).*

The litterature contains a large number of proofs of Theorem 3 (see e.g. the references quoted before Proposition 2). On one hand, most of these proofs are minor variations of Powers' original proof. On the other hand however, each proof extends to some other groups than free groups. The following proof is convenient for the discussion below of some research activity on this subject between 1975 (Powers' paper) and now (see also [BCH]). Our first lemma is straightforward.

Lemma 1. *Let $X_1, ..., X_k$ be a finite sequence of operators on a Hilbert space such that the image subspaces $Im(X_1), ..., Im(X_k)$ are pairwise orthogonal. Then*

$$||X_1 + ... + X_k|| \leq \sqrt{k} \max_{1 \leq j \leq k} ||X_j|| \ .$$

Proof of Theorem 3.

Write Γ for F_n.

Step one. Let τ_c denote the canonical trace on $C^*_\lambda(\Gamma)$. Let $\mathcal{J}$ be a non-zero two-sided ideal in $C^*_\lambda(\Gamma)$ and choose $V \in \mathcal{J}$, $V \neq 0$. Upon multiplying V by some $z \in \mathbb{C}$, $z \neq 0$ and by some $\gamma \in \Gamma$, we may assume that $V = 1 + W$ where $\tau_c(W) = 0$. We shall show that

$\mathcal{J}$ contains a sum of conjugates of V which is an invertible element, and that any trace on $C^*_\lambda(\Gamma)$ vanishes on W.

Let ϵ be a real number such that $0 < \epsilon \leq \frac{1}{3}$. As $\lambda_\Gamma(\mathbb{C}[\Gamma])$ is dense in $C^*_\lambda(\Gamma)$, we may choose $X \in \lambda_\Gamma(\mathbb{C}[\Gamma])$ such that $||X - W|| \leq \epsilon$ and $\tau_c(X) = 0$. We may write $X = \sum_{x \in F} z_x x$, where F is a finite subset of $\Gamma - \{1\}$ and where the z_x 's are complex numbers.

Step two. Choose a system $\{s_1, ..., s_n\}$ of free generators of Γ. For a large enough number m, the reduced words $s_1^m x s_1^{-m}$ begin and end with a non-zero power of s_1 for all $x \in F$ (this is [Pow, Lemma 4]). Let C be the subset of Γ of reduced words which begin by s_1^{-m} (followed by a non-zero power of some s_j , $j \neq 1$, or by nothing at all) and set $D = \Gamma - C$. Then one has $xC \cap C = \emptyset$ for all $x \in F$. For each integer $j \geq 1$, set $\gamma_j = s_2^j s_1^m$; one has $\gamma_i D \cap \gamma_j D = \emptyset$ whenever $i \neq j$.

Step three. Choose an integer $k \geq 1$. For each $j \in \{1, ..., k\}$, let P_j denote the orthogonal projection of $l^2(\Gamma)$ onto the subspace $l^2(\gamma_j D)$ of functions $\Gamma \to \mathbb{C}$ with supports inside $\gamma_j D$. As $xC \cap C = \emptyset$ for all $x \in F$, one has $(1 - P_j)\gamma_j X \gamma_j^{-1}(1 - P_j) = 0$. Thus

$$\gamma_j X \gamma_j^{-1} = P_j X_j' + (P_j X_j'')^* \quad \text{with} \quad \begin{cases} X_j' & = \gamma_j X \gamma_j^{-1} \\ X_j'' & = \gamma_j X^* \gamma_j^{-1}(1 - P_j) \end{cases}$$

for each $j \in \{1, ..., k\}$. Set $Y = \frac{1}{k} \sum_{1 \leq j \leq k} \gamma_j X \gamma_j^{-1}$. As $\gamma_i D \cap \gamma_j D = \emptyset$, Lemma 1 implies

$$||Y|| \;\leq\; \frac{1}{k} \parallel \sum_{1 \leq j \leq k} P_j X_j' \parallel + \frac{1}{k} \parallel \sum_{1 \leq j \leq k} P_j X_j'' \parallel \;\leq\; \frac{2}{\sqrt{k}} \parallel X \parallel \;.$$

For k large enough, one has consequently $||Y|| \leq \frac{1}{3}$ and

$$\parallel \frac{1}{k} \sum_{1 \leq j \leq k} \gamma_j W \gamma_j^{-1} \parallel \;\leq\; \parallel W - X \parallel + \parallel Y \parallel \;\leq\; \frac{2}{3} \;<\; 1 \;.$$

It follows that $\frac{1}{k} \sum_{1 \leq j \leq k} \gamma_j V \gamma_j^{-1} = 1 + \frac{1}{k} \sum_{1 \leq j \leq k} \gamma_j W \gamma_j^{-1}$ is invertible. As this element is obviously in $\mathcal{J}$, one has $\mathcal{J} = C^*_\lambda(\Gamma)$. Thus $C^*_\lambda(\Gamma)$ is a simple C^*-algebra.

Step four. Let τ be any normalized trace on $C^*_\lambda(\Gamma)$. One has

$$|\tau(W) - \tau(X)| \leq \parallel W - X \parallel \leq \epsilon$$

and $|\tau(X)| = |\tau(Y)| \leq \frac{2}{\sqrt{k}} \parallel X \parallel$. As this holds for all $\epsilon > 0$ and for all $k \geq 1$, one has $\tau(W) = 0$. Thus $\tau = \tau_c$, and $C^*_\lambda(\Gamma)$ has a unique trace. $\square$

One strategy of proof of Proposition 2 is to extend the validity of the previous proof.

A first attempt has been do define a **Powers group** as a group Γ which possesses the following property :

for each finite subset $F \subset \Gamma - \{1\}$ and for each integer $k \geq 1$
there exist a partition $\Gamma = C \coprod D$ and elements $\gamma_1, ..., \gamma_k$ in Γ
such that $xC \cap C = \emptyset$ for all $x \in F$ and $\gamma_i D \cap \gamma_j D = \emptyset$ for all $i \neq j$ in $\{1, ..., k\}$.

Step two in the previous proof shows precisely that F_n is a Powers group.

Lemma 2. *If* Γ *is a Powers group, the* C^**-algebra* $C^*_\lambda(\Gamma)$ *is simple with unique trace.*

Proof. see steps one, three and four in the previous proof. □

The conclusion of Lemma 2 can be proved with weaker hypothesis. For example, in the definition of "Powers group", one could replace "for all finite subset $F \subset \Gamma - \{1\}$" by "for all finite subset F inside a conjugacy class distinct from $\{1\}$", so that the lemma applies to the so-called "weak Powers groups". (And direct products such as $F_2 \times F_2$ are weak Powers groups which are not Powers groups; see [BN1, Proposition 1.4] and [Pro, Proposition 3.2].) Or one may consider reduced crossed products $A \rtimes_r \Gamma$ where A is a C^*-algebra with unit which does not have any non trivial Γ-invariant ideal nor any Γ-invariant trace [HS2]. Or one may also cope with twisted reduced crossed products $A \rtimes_{c,r} \Gamma$, where $c : \Gamma \times \Gamma \to \mathcal{U}$ is a 2-cocycle with values in the unitary group of the centre of A [BN2], or even in the unitary group of A itself [Be1], [Be2].

Now comes (at least) some geometry. Let Γ be a group acting by homeomorphisms on a compact topological space Ω. Say the action is **strongly faithful** if, for every finite subset $F \subset \Gamma - \{1\}$, there exists $\omega_0 \in \Omega$ such that $x\omega_0 \neq \omega_0$ for all $x \in F$. Recall that the action is **minimal** if every orbit $\Gamma\omega$ is dense in Ω. Say that $\gamma \in \Gamma$ is **hyperbolic** if there exist two fixed points s_γ, r_γ with the following properties : given neighbourhoods S_γ of s_γ and R_γ of r_γ in Ω, there exists an integer $k \geq 1$ such that

$$\gamma^l(\Omega - S_\gamma) \subset R_\gamma \quad \text{and} \quad \gamma^{-l}(\Omega - R_\gamma) \subset S_\gamma$$

for all integers $l \geq k$. Two hyperbolic elements $\gamma, \gamma' \in \Gamma$ are **transverse** if the four points s_γ , r_γ , $s_{\gamma'}$, $r_{\gamma'}$ are distinct. Say finally that the action of Γ on Ω is **strongly hyperbolic** if, for each integer $k \geq 1$, there exist pairwise transverse hyperbolic elements $\gamma_1, ..., \gamma_k$ in Γ.

Lemma 3. *Let* Γ *be a group acting by homeomorphisms on a compact space* Ω*. Assume that the action is strongly faithful, minimal and strongly hyperbolic. Then* Γ *is a Powers group.*

Proof. see [Har2, Lemma 4]. □

PROOF OF PROPOSITION 2(e). It follows easily from the three previous lemmas. See [Har2] and [Har4] for more details. □

Similar arguments may be used to prove Proposition 2(a), or 2(b) in case the ambient Lie group has real rank 1. It is probably possible to extend this proof to any hyperbolic group Γ with virtual centre Γ_f reduced to $\{1\}$. For this it should be checked that the action of such a group Γ on its Gromov boundary $\partial\Gamma$ is strongly faithful.

However it does not seem possible to extend the proof above to cover, for example, groups such as $PSL_n(\mathbb{Z})$ and $PSL_n(\mathbb{R})$ when $n \geq 3$. (There is an argument in [Ha2] for $n = 3$, but it does not work when $n \geq 4$.) This suggest the following problem.

Problem 18. *Given a group* Γ *and an element* $\gamma \in \Gamma$ *of infinite order, describe obstructions to the existence of a compact space* Ω *on which* Γ *acts by homeomorphisms in such a way that* γ *is hyperbolic. Particular problem :* $\Gamma = PSL_n(\mathbb{Z})$ *with* $n \geq 3$*. (These obstructions vanish if* Γ *is hyperbolic; see e.g.* [GhH, § 8.2]*.)*

A second attempt to extend the validity of the proof of Theorem 3 has been in terms of the following notions. Say that a group Γ is **naively permissive** if, for any finite subset $F \subset \Gamma - \{1\}$, there exists an element $y \in \Gamma$ of infinite order such that the canonical morphism $< x, y > \to < x > \star < y >$ is an isomorphism for each $x \in F$ (where $< x, y >$ [respectively $< x >, < y >$] denotes the subgroup of Γ generated by $\{x, y\}$ [resp. x, y]). One may observe that Lemma 6.3.2 or [MNIV] says that a naively permissive group is an icc group. We leave it to the reader to check that a torsionfree non elementary hyperbolic group is naively permissive.

One may show on one hand that $C^*_\lambda(\Gamma)$ has a unique normalized trace and that it is a simple C^*-algebra if Γ is naively permissive. One may show on the other hand that some of the groups of Proposition 2 are naively permissive [BCH]. However the following is still open.

Problem 19. *If Γ is as in Proposition 3, is Γ naively permissive ? In particular, is $PSL_n(\mathbb{Z})$ naively permissive for all $n \geq 2$?*

A third attempt to extend the validity of the proof of Theorem 3 is in term of other *permissive* properties of groups. For this and for the proof of Proposition 2, we refer to [BCH].

Finally, in connection with Section 3.4, we state and prove the following, due to M. Bekka.

Proposition 4. *Let G be a non compact simple connected real Lie group without centre, let ρ be an irreducible representation of G distinct from the trivial representation of G in $\mathbb{C}$, let Γ be a lattice in G and let $\rho|\Gamma$ denote the restriction of ρ to Γ. Then $C^*_{\rho|\Gamma}(\Gamma)$ does not have any non trivial two sided ideal of finite codimension.*

Proof. Let π be a representation of Γ such that $C^*_\pi(\Gamma)$ has a non trivial two-sided ideal $\mathcal{J}$ of finite codimension. The closure of $\mathcal{J}$ is non trivial (because $C^*_\pi(\Gamma)$ has a unit) and self-adjoint [DC*, proposition 1.8.2]; denote by A the C^*-algebra quotient $C^*_\pi(\Gamma)/\overline{\mathcal{J}}$. There exists an integer $n \geq 1$ and a quotient of A isomorphic to $M_n(\mathbb{C})$. The resulting C^*-morphism $C^*(\Gamma) \to M_n(\mathbb{C})$ defines a finite dimensional representation $\sigma : \Gamma \to U(n)$ which is weakly contained in π, and we write this $\sigma \prec \pi$.

Let now ρ be as in the statement to be proved. There exists a real number p such that ρ is of class L^p. (See [Cw2, théorème 2.5.2, lemmes 2.2.5 et 3.1.2] if the Lie group is of real rank 1; see [Cw2, théorème 2.4.2] if the Lie group is of real rank at least 2.) This implies that there exists an integer $k \geq 1$ such that the tensor product $\rho^{\otimes k}$ is weakly contained in the regular representation λ_G of G. (See [Cw1], [CHH] and [How, pages 288 and 285].)

Suppose now ab absurdo that $\pi = \rho|\Gamma$. Then

$$\sigma^{\otimes k} \prec \pi^{\otimes k} = \left(\rho^{\otimes k}\right)|\Gamma \prec \lambda_G|\Gamma \prec \lambda_\Gamma$$

where the last weak containement $\lambda_G|\Gamma \prec \lambda_\Gamma$ follows from [DC*, proposition 18.3.5]. As $\sigma^{\otimes k}$ is finite dimensional, this implies that Γ is amenable, which is absurd. □

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