IMRN International Mathematics Research Notices
1994, No. 7

Simplicity of the Reduced C^*-Algebra of $PSL(n, \mathbb{Z})$

M. Bekka, M. Cowling, and P. de la Harpe

Let Γ be a discrete group. We denote by λ_Γ the left regular representation of Γ on the Hilbert space $\ell^2(\Gamma)$ and by $C_r^*(\Gamma)$ the *reduced* C^*-*algebra* of Γ, generated by $\lambda_\Gamma(\Gamma)$ in the C^*-algebra of all bounded operators on $\ell^2(\Gamma)$. In the case where Γ is a nonabelian free group, R. Powers [Pow] has shown that $C_r^*(\Gamma)$ is a simple algebra. Equivalently (see, e.g., [Dix], 3.4.4 and 18.1.4), any unitary representation π of Γ which is weakly contained in λ_Γ is weakly equivalent to λ_Γ. After Powers's result, the simplicity of $C_r^*(\Gamma)$ was shown in various cases, such as appropriate free products [PaS], Fuchsian groups [Ake], and Gromov-hyperbolic groups (see the review in [Har]). Our main result solves a well-known conjecture.

Theorem. Let G be a connected real semisimple Lie group without compact factors and with trivial centre. Let Γ be a Zariski-dense subgroup of G. If Γ is viewed as a discrete group, then the reduced C^*-algebra $C_r^*(\Gamma)$ is simple.

The Borel density theorem shows that this applies to a lattice Γ of G. The purpose of this paper is to give a proof of this result when $G = PSL(n, \mathbb{R})$ and $\Gamma = PSL(n, \mathbb{Z})$ for some $n \geq 2$. The proof of the general case as well as other results may be found in [BCH].

Given a locally compact group G, a closed subgroup H of G with compact quotient $B = G/H$, and a discrete subgroup Γ of G, we introduce the following combinatorial-geometric property.

Definition 1. The action of Γ on B is said to have Property (P_{cg}) if, for any finite subset F of $\Gamma \setminus \{1\}$, there exist $y_0 \in \Gamma$ and Borel subsets $V, B_1, \dots, B_n$ of B such that

(i) $B = V \cup B_1 \cup \cdots \cup B_n$,

(ii) $xV \cap V = \emptyset$ for all $x \in F$,

(iii) $y_0^{-j} B_s \cap B_s = \emptyset$ for all integers $j \geq 1$ and $s \in \{1, \dots, n\}$.

Received 19 May 1994.

Proposition 1. The canonical action of $\Gamma = PSL(n, \mathbb{Z})$ on the homogeneous space $B = \mathbb{P}^{n-1}(\mathbb{R})$ of the group $G = PSL(n, \mathbb{R})$ has Property (P_{cg}).

Proof. Let F be a finite subset of $\Gamma \setminus \{1\}$. It is easy to find $y_1' \in \Gamma$ with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. Let $b_1', b_2', \ldots, b_n' \in B$ be the corresponding eigendirections. Let us check that there exists a conjugate y_1 of y_1' in Γ with eigendirections $b_1, \ldots, b_2 \in B$ such that

$$x\{b_1, \ldots, b_n\} \cap \{b_1, \ldots, b_n\} = \emptyset$$

for all $x \in F$.

Given $s, t \in \{1, \ldots, n\}$ and $x \in F$, the set $G_{x,s,t}$ of those $y \in G$ such that $yxy^{-1}b_s' \neq b_t'$ is clearly Zariski-open in G. Suppose, by contradiction, that it is empty. Then $xb_s' = b_t'$, so $xy^{-1}b_s' = y^{-1}xb_s'$ for all $y \in G$; thus b_s' and b_t' have the same stabilizer in G. As stabilizers in G of points in B are their own normalizers, this implies that $s = t$. Therefore, $xy^{-1}b_s' = y^{-1}b_s'$ for all $y \in G$. This means that x fixes every point in B . Hence $x = 1$, a contradiction.

Now the intersection of the finite collection of the Zariski-dense sets $G_{x,s,t}$ is still Zariski-dense. We thus find $y \in \Gamma$ such that $xy^{-1}b_s' \neq y^{-1}b_t'$ for all $x \in F$ and $s, t \in \{1, \ldots, n\}$. Finally, we set $y_1 = y^{-1}y_1'y \in \Gamma$.

Denote by $[\xi_1 : \cdots : \xi_n]$ homogeneous coordinates on $B = \mathbb{P}^{n-1}(\mathbb{R})$ such that $b_s = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$, with 1 in the sth place $(1 \leq s \leq n)$. For $\varepsilon > 0$, set

$$V_s = \left\{[\xi_1 : \cdots : \xi_n] \in B : \xi_s \neq 0 \text{ and } \left|\frac{\xi_t}{\xi_s}\right| < \varepsilon \text{ when } t \neq s\right\}$$

for $s \in 1, \ldots, n$, and let $V = V_1 \cup \cdots \cup V_n$. Then, for ε small enough, the following holds:

(i) $B = \cup_{j=0}^{\infty} y_1^{-j} V$,

(ii) $xV_s \cap V_t = \emptyset$ for all $x \in F$ and $s, t \in \{1, \ldots, n\}$,

(iii) if $s \in \{1, \ldots, n\}$, $j \geq 1$ and $b \in V_s$ are such that $y_1^j \notin V_s$, then $y_1^{j+1} \notin V_s$.

For $s \in \{1, \ldots, n\}$ and $j \geq 1$, set

$$B_{s,j} = y_1^{-j} V_s \setminus \bigcup_{k=0}^{j-1} y_1^{-k} V_s.$$

By the compactness of B, there exists an integer $Q > 0$ such that

$$B \setminus V \subset \bigcup_{j=1}^{Q} \bigcup_{s=1}^{n} B_{s,j}.$$

Property (P_{cg}) holds with $y_0 = y_1^Q$, with V as above, and with $B_s = \cup_{j=1}^{Q} B_{s,j}$ for all s, $1 \leq s \leq n$. ∎

Proposition 1 holds in the more general situation of the theorem above, with the Furstenberg boundary of G for B (see [BCH]).

Next, we define the following analytical property of a group Γ. Let $\mathbb{Z}^+$ denote the set of strictly positive integers, $\|\cdot\|_2$ the Hilbert norm in $\ell^2(\mathbb{Z}^+)$, and $\|\cdot\|$ the norm of operators on $\ell^2(\Gamma)$.

Definition 2. A group Γ is said to have Property (P_a) if, for any finite subset F of $\Gamma \setminus \{1\}$, there exist $y_0 \in \Gamma$ and a constant $C > 0$ such that

$$\left\| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| \leq C \, \|a\|_2$$

for all sequences $a = (a_j)_{j\geq 1} \in \ell^2(\mathbb{Z}^+)$ and $x \in F$.

Proposition 2. If the action of Γ on B has Property (P_{cg}) of Definition 1, then Γ has Property (P_a).

Proof. Fixing a quasi-invariant measure on B, we consider the corresponding space $L^2(B)$, with Hilbert norm denoted by $\|\cdot\|$, and the quasi-regular representation ρ of G on $L^2(B)$. Let F be a finite subset of $\Gamma \setminus \{1\}$. Let $y_0 \in \Gamma$ and $V, B_1, \ldots, B_n \subset B$ satisfy the conditions of Definition 1. We claim first that

$$\left\| \sum_{j=1}^{\infty} a_j \rho(y_0^{-j} x y_0^j) \right\| \leq 2n \, \|a\|_2$$

for all $a \in \ell^2(\mathbb{Z}^+)$.

Indeed, denoting by χ_U the characteristic function of a subset U of B, one has for all $f, g \in L^2(B)$ and $x \in F$

$$\begin{aligned}
\left|\langle \rho(x) f, g\rangle\right| &\leq |\langle \rho(x)\chi_V f, g\rangle| + |\langle \rho(x)\chi_{B\setminus V} f, g\rangle| \\
&= |\langle \chi_{xV}\rho(x) f, g\rangle| + |\langle \rho(x)\chi_{B\setminus V} f, g\rangle| \\
&= |\langle \chi_{xV}\rho(x) f, \chi_{B\setminus V} g\rangle| + |\langle \rho(x)\chi_{B\setminus V} f, g\rangle| \\
&\leq \|f\| \left\|\chi_{B\setminus V} g\right\| + \left\|\chi_{B\setminus V} f\right\| \|g\| \\
&\leq \sum_{s=1}^{n} \left(\|f\| \left\|\chi_{B_s} g\right\| + \left\|\chi_{B_s} f\right\| \|g\| \right)
\end{aligned}$$

(where we used conditions (i) and (ii) of Definition 1). Hence, for all $j \geq 1$,

$$\left|\left\langle \rho(y_0^{-j} x y_0^j) f, g \right\rangle\right| \leq \sum_{s=1}^{n} \left(\|f\| \left\|\chi_{y_0^{-j} B_s} g\right\| + \left\|\chi_{y_0^{-j} B_s} f\right\| \|g\| \right)$$

and, for all $a = (a_j)_{j\geq 1} \in \ell^2(\mathbb{Z}^+)$,

$$\begin{aligned} \left|\left\langle \sum_{j=1}^{\infty} a_j \rho(y_0^{-j} x y_0^j) f, g \right\rangle\right| &\leq \sum_{j=1}^{\infty} \sum_{s=1}^{n} |a_j| \left(\|f\| \left\|\chi_{y_0^{-j}B_s} g\right\| + \left\|\chi_{y_0^{-j}B_s} f\right\| \|g\| \right) \\ &\leq \sum_{s=1}^{n} \|a\|_2 \left\{ \|f\| \left(\sum_{j=1}^{\infty} \left\|\chi_{y_0^{-j}B_s} g\right\|^2 \right)^{1/2} + \left(\sum_{j=1}^{\infty} \left\|\chi_{y_0^{-j}B_s} f\right\|^2 \right)^{1/2} \|g\| \right\} \\ &\leq \sum_{s=1}^{n} \|a\|_2 \{2 \|f\| \|g\|\} = 2n \|a\|_2 \|f\| \|g\| \end{aligned}$$

(where we used the Cauchy-Schwarz inequality and condition (iii) of Definition 1). This proves the claim.

Now, Herz's principle of majoration states that $\|\lambda_G(\mu)\| \leq \|\rho(\mu)\|$ for any bounded positive measure μ on G (see, e.g., [EyL], page 186). As Γ is closed in G, one has $\|\lambda_\Gamma(\mu)\| = \|\lambda_G(\mu)\|$ for any bounded measure μ supported on Γ. Hence, if $a = (a_j)_{j\geq 1} \in \ell^2(\mathbb{Z}^+)$ has finite support, the above majoration, applied to the measure μ which assigns mass $|a_j|$ to each point $y_0^{-j} x y_0^j$, implies that

$$\left\| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| \leq \left\| \sum_{j=1}^{\infty} |a_j| \, \rho(y_0^{-j} x y_0^j) \right\| \leq 2n \|a\|_2 .$$

This inequality extends to any $a \in \ell^2(\mathbb{Z}^+)$, so that Γ has Property (P_a). ■

Proposition 3. If the group Γ has Property (P_a), then the C^*-algebra $C_r^*(\Gamma)$ is simple.

Proof. Let J be a two-sided ideal in $C_r^*(\Gamma)$ different from$\{0\}$. Choose $Y = \sum_{y\in\Gamma} z'_y \lambda_\Gamma(y) \in J$, $Y \neq 0$. Set

$$X = \left(\sum_{y\in\Gamma} |z'_y|^2 \right)^{-1} Y^* Y = 1 + \sum_{x\in\Gamma\setminus\{1\}} z_x \lambda_\Gamma(x) \in J,$$

and choose a finite subset $F \subset \Gamma \setminus \{1\}$ such that the element

$$X' = 1 + \sum_{x\in F} z_x \lambda_\Gamma(x) \in C_r^*(\Gamma)$$

satisfies $\|X' - X\| \leq 1/3$. Using Property (P_a), we find $y_0 \in \Gamma$ and $C > 0$ such that

$$\left\| \frac{1}{N} \sum_{j=1}^{N} \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| \leq C \frac{1}{\sqrt{N}}$$

for all $x \in F$ and $N \geq 1$. For all $N \geq 1$, set

$$X_N = \frac{1}{N} \sum_{j=1}^{N} \lambda_\Gamma(y_0^{-j}) X \lambda_\Gamma(y_0^j) \in J$$

$$X'_N = \frac{1}{N} \sum_{j=1}^{N} \lambda_\Gamma(y_0^{-j}) X' \lambda_\Gamma(y_0^j) \in C_r^*(\Gamma).$$

Then $\|X'_N - X_N\| \leq \|X' - X\| \leq 1/3$ and

$$\|X'_N - 1\| \leq \sum_{x \in F} \left\| \frac{z_x}{N} \sum_{j=1}^{N} \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| \leq \left(\sum_{x \in F} |z_x| \right) C \frac{1}{\sqrt{N}} \leq \frac{1}{3},$$

where the last inequality holds for N large enough. Hence, for N large enough,

$$\|X_N - 1\| \leq \|X_N - X'_N\| + \|X'_N - 1\| \leq \frac{2}{3} < 1.$$

and $X_N \in J$ is invertible. It follows that $J = C_r^*(\Gamma)$. ■

In order to deal with nondiscrete subgroups of G, such as $PSL(n, \mathbb{Q})$, one has to use slightly more refined methods. As a consequence, one obtains the following result, which is related to those in [HoR].

Corollary. Let k be a field of characteristic 0, and let $\mathbb{G}$ be a connected, semisimple algebraic group defined over k, with trivial centre. Let Γ be $\mathbb{G}(k)$, the group of the k-rational points of $\mathbb{G}$, equipped with the discrete topology. Then $C_r^*(\Gamma)$ is a simple C^*-algebra.

By way of contrast, we now present a result for which we cannot give a reference, but which is surely known.

Proposition 4. Let G be a connected locally compact group, $G \neq \{1\}$. Then the reduced C^*-algebra of G contains a nontrivial closed two-sided ideal.

The proof is based on the following simple lemma.

Lemma. Let G be a locally compact group containing an amenable closed normal subgroup $N \neq \{1\}$.Then the reduced C^*-algebra of G contains a nontrivial closed two-sided ideal.

Proof. Let $\lambda_{G/N}$ denote the left regular representation of G/N, viewed as a representation of G. Since N is amenable, the trivial representation 1_N is weakly contained in the regular representation λ_N of N. Hence, $\lambda_{G/N} = \mathrm{Ind}_N^G(1_N)$ is weakly contained in $\lambda_G = \mathrm{Ind}_N^G(\lambda_N)$, where Ind denotes the induced representation.

On the other hand, the coefficients of λ_G separate the points of G, while those of $\lambda_{G/N}$ are constant on N. Hence, λ_G is not weakly contained in $\lambda_{G/N}$. It follows that $C^* \operatorname{Ker}(\lambda_{G/N})/C^* \operatorname{Ker}(\lambda_G)$ is a nontrivial closed two-sided ideal of $C^*_r(G)$. ∎

Proof of Proposition 4. In view of the above lemma, we may assume that G contains no nontrivial amenable closed normal subgroup. Using the standard structure theory of connected locally compact groups and of Lie groups, it is easy to see that G is a noncompact semisimple Lie group with trivial centre.

Let π, π' be two nonequivalent irreducible principal series representations of G. Then π and π' are weakly contained in λ_G, since they are induced from an amenable parabolic subgroup of G. Because G is liminal, π' is not weakly contained in π and vice versa (see [Dix], 15.5.6 and 9.1). This implies that λ_G is not weakly contained in π. Hence, $C^* \operatorname{Ker}(\pi)/C^* \operatorname{Ker}(\lambda_G)$ is a nontrivial closed two-sided ideal of $C^*_r(G)$. ∎

Acknowledgments

This research was partially financed by the Australian Research Council, which supported the first two authors as Senior Research Associate at the University of New South Wales and Senior Research Fellow.

References

[Ake] C. A. Akemann, *Operator algebras associated with Fuchsian groups*, Houston J. Math. **7** (1981), 295–301.

[BCH] M. Bekka, M. Cowling, and P. de la Harpe, *Some groups whose reduced* C^**-algebra is simple*, to appear.

[Dix] J. Dixmier, C^**-Algebras*, North-Holland Math. Library **15**, North-Holland, Amsterdam, 1977.

[EyL] P. Eymard and N. Lohoué, *Sur la racine carrée du noyau de Poisson dans les espaces symétriques et une conjecture de E. M. Stein*, Ann. Sci. École Norm. Sup. (4) **8** (1975), 179–188.

[Har] P. de la Harpe, *Operator algebras, free groups, and other groups*, from the proceedings of the ALGOP, Orléans, July 1992, to appear in Astérisque.

[HoR] R. E. Howe and J. Rosenberg, *The unitary representation theory of* $GL(n)$ *of an infinite discrete field*, Israel J. Math. **67** (1989), 67–81.

[PaS] W. Paschke and N. Salinas, C^**-algebras associated with the free products of groups*, Pacific J. Math. **82** (1979), 211–221.

[Pow] R. T. Powers, *Simplicity of the* C^**-algebra associated with the free group on two generators*, Duke Math. J. **42** (1975), 151–156.

Bekka: Département de Mathématiques, Université de Metz, Ile du Saulcy, F-57045 Metz, France

Cowling: School of Mathematics, University of New South Wales, P. O. Box 1, Kensington, NSW 2033, Australia

P. de la Harpe: Section de Mathématiques, Université de Genève, C. P. 240, CH-1211 Genève 24, Switzerland