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Acyclic groups of automorphisms

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1. Introduction

A discrete group Γ is said to be acyclic if its Eilenberg–MacLane homology groups $H_i(\Gamma)$ with coefficients in the trivial Γ-module $\mathbf{Z}$ are zero for all $i>0$. In this paper we show that certain groups, such as the group $GL(V)$ of all continuous linear automorphisms of an infinite dimensional Hilbert space V, are acyclic. This is a folk theorem which was surely known long ago to experts in the field such as Quillen and Segal. However it seems worthwhile to publish a proof in view of the recent interest shown in such questions. For example, Herman pointed out in [He] that the group of diffeomorphisms of a compact manifold admits a canonical representation in $GL(V)$. Therefore, if $GL(V)$ had carried non-trivial cohomology, one might have been able to define non-trivial characteristic classes for groups of diffeomorphisms. See also section 2.6 in [Ma] and the concluding remark of [H2].

We will consider the following groups.

1. The group $\Sigma(X)$ of all permutations of an infinite set X.
2. The group $\mathcal{A}(\Omega)$ of measure preserving automorphisms of a Lebesgue measure space $(\Omega, \mathcal{B}, \mu)$ where μ is infinite and non-atomic. (As usual one identifies two automorphisms which agree μ-a.e.)
3. The group $GL(W)$ of all linear automorphisms of an infinite dimensional vector space W.
4. The group $GL(V)$ of all continuous linear automorphisms of an infinite dimensional Hilbert space V over the real, complexes or quaternions, as well as the group $U(V)$ of invertible isometries of V.
5. The group $GL(M)$ of invertible elements in a properly infinite von Neumann algebra M, and the subgroup $U(M)$ of unitary elements.

THEOREM. *The groups defined above are acyclic.*

The above list is by no means complete. One could add many "classical

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groups" in the sense of [H3], and also the group of continuous linear automorphisms of an infinite dimensional topological vector space E for suitable E. The Banach spaces c_0 and l_p, $1 \leq p < \infty$, are possible candidates: see proposition 2.a.2 in [LT]. However Douady [D] constructs a Banach space E for which the group of connected components of $GL(E)$ is isomorphic to $\mathbf{Z}$. It follows that $GL(E)$ is not perfect and hence not acyclic. Therefore the above theorem does not hold for $GL(E)$ where E is an arbitrary Banach space. See also [St]. For acyclic groups of a quite different nature from those of our list, see [BDH] and [BDM].

Here is one consequence of the theorem.

COROLLARY. *Let G be one of the groups above and let A be a finitely generated abelian group. Then any extension*

$$0 \to A \to \tilde{G} \to G \to 1$$

is trivial.

Proof. Any non-trivial normal subgroup of G is of uncountable index. (See Appendix 1.) In particular any homomorphism from G to Aut (A) is trivial and so G acts trivially on A in the above extension. Our main theorem implies that $H^2(G; A)$ is zero. Hence the extension is a semi-direct product. Again using the fact that the action of G on A is trivial, we see that the product is direct. ∎

A notable feature of the groups in 2, 4 and 5 is that they are contractible when given their natural topologies. (See [Ke] for $\mathcal{A}(\Omega)$, [DD] for $U(V)$ and $U(M)$ with the strong topology, [Ku] for $GL(V)$ and $U(V)$ with the uniform topology, and [BW] for $GL(M)$ and $U(M)$ with the uniform topology.) There are many other contractible groups of automorphisms which are acyclic when considered as discrete groups: for example, the group of compactly supported homeomorphisms of $\mathbf{R}^n$ [M], and the group of diffeomorphisms of $\mathbf{R}^n$ which are the identity near the origin [Se]. On the other hand, Sah pointed out that the universal cover $\widetilde{SL(2, \mathbf{R})}$ of $SL(2, \mathbf{R})$ is contractible as a topological group but is not acyclic as a discrete group [SW]. The main tool which we use in proving acyclicity is the infinite repetition argument of Mather [M] and Wagoner [W]. (See also [BDH] §4 and [Be] ch. 3.) There are several contractible groups which are more "flexible" than $\widetilde{SL(2, \mathbf{R})}$, but are still not large enough for this argument to be used. We have in mind groups such as $\mathcal{A}(\Omega)$, where Ω has finite measure, or the group of compactly supported homeomorphisms of $\mathbf{R}^n$ which preserve Lesbegue measure, for $n > 2$. These groups are known to be perfect [F1], [F2], and it would be interesting to know if they are acyclic. One could also consider much bigger groups such as the group of all homeomorphisms of a Hilbert cube or a Hilbert space. These are shown to be contractible in [Re]. The groups $GL(M)$ and $U(M)$, where M is a finite continuous von Neumann factor, are not contractible. They are discussed further in section 4.

The theorem is not hard to prove. We first show that the subgroup G_F of elements in G which are the identity on an appropriately defined "flag" F is acyclic. Then we show, using a technique due to Segal (§2 in [Se]), that this forces the whole group G to be acyclic. The first of these two steps uses the infinite repetition argument of [M] and [W] and, in the general case, an elegant algebraic trick due to Quillen [Q2]. The second step works essentially because the Tits building (or partially ordered set) formed by the flags is contractible. We give the proof for $GL(V)$ in full detail, and in section 4 sketch the modifications needed for the other groups.

We discuss in Appendix 1 the results about normal subgroups of G needed for the corollary above. Though these are old results, we indicate for $GL(W)$ and $GL(V)$ a proof much shorter than the originally published ones. Doing this, we again show that G is perfect, namely that $H_1(G)$ is trivial. This is what our main result and proof reduce to when cleared from homological machinery.

Finally Appendix 2 describes a result due to Quillen according to which the monoids (or semi-groups) related to our groups are contractible and hence acyclic.

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2. Subgroups of $GL(V)$

In this section and the next one, V denotes an infinite dimensional Hilbert space. Let F be a *flag* in V: we mean by this that F is a nested sequence $S_1 \supset S_2 \supset S_3 \supset \cdots$ of closed subspaces of $V = S_0$ such that S_{i-1}/S_i is isomorphic to V for each $i \geq 1$. Define

$$G_i = \{g \in GL(V) \mid g = \text{id on } S_i\}$$

and

$$G_i' = \{g \in G_i \mid g(S_i^\perp) = S_i^\perp\}$$

for each $i \geq 0$. Define also G_∞ to be the union of the G_i's and G_∞' that of the G_i''s. Then

$$\begin{array}{ccccccccc} 1 = G_0 & \subset & G_1 & \subset \cdots \subset & G_i & \subset \cdots \subset & G_\infty \\ \| & & \cup & & \cup & & \cup \\ G_0' & \subset & G_1' & \subset \cdots \subset & G_i' & \subset \cdots \subset & G_\infty'. \end{array}$$

For $g \in G_\infty$, observe that $g = \mathrm{id}$ on $S_\infty = \bigcap S_i$. For notational convenience, we assume $S_\infty = \{0\}$. (But proposition 1 as well as its consequences in section 3 and the variations of section 4 would obviously hold without this assumption.) The result of this section is:

PROPOSITION 1. *The groups G'_∞ and G_∞ are acyclic.*

We shall recall the following facts from §2 in [W]. A *flabby* group is a group Γ such that there exist homomorphisms

$$\amalg : \Gamma \times \Gamma \to \Gamma \quad \text{(direct sum)}$$
$$\tau : \Gamma \to \Gamma \quad \text{(infinite repetition)}$$

with the following properties: For any finite subset $\Phi \subset \Gamma$, there are elements a, b, c in Γ satisfying

(1) $g \amalg 1 = aga^{-1}$, $1 \amalg g = bgb^{-1}$ where 1 is the identity element in Γ,

(2) $g \amalg \tau(g) = c\tau(g)c^{-1}$

for all $g \in \Phi$.

LEMMA 2 (Wagoner). *A flabby group is acyclic.*

Sketch of proof. Any inner automorphism of Γ acts trivially on homology. By (1), this implies first that $\amalg$ induces a (non associative) ring structure $\amalg_* : H_*(\Gamma) \otimes H_*(\Gamma) \to H_*(\Gamma)$ on homology, with two-sided unit the number 1 in $H_0(\Gamma) = \mathbf{Z}$. By (2), this implies also that $\amalg(\mathrm{id} \times \tau)\Delta$ and τ act the same way on homology, where $\Delta : \Gamma \to \Gamma \times \Gamma$ is the diagonal map.

Let i be an integer, $i > 0$, and assume inductively that $H_n(\Gamma)$ is trivial for $0 < n < i$ (this holds trivially if $i = 1$). Choose $z \in H_i(\Gamma)$. By the Künneth formula

$$\Delta_*(z) = z \otimes 1 + 1 \otimes z \in H_i(\Gamma) \otimes H_0(\Gamma) + H_0(\Gamma) \otimes H_i(\Gamma) = H_i(\Gamma \times \Gamma)$$

so that

$$(\amalg(\mathrm{id} \times \tau)\Delta)_*(z) = \amalg_*(z \otimes 1 + 1 \otimes \tau_*(z)) = z + \tau_*(z) \in H_i(\Gamma).$$

As this must coincide with $\tau_*(z)$ one has $z = 0$. Hence $H_i(\Gamma)$ is trivial. ■

LEMMA 3. *The group G'_∞ is flabby.*

Proof. Let T_0^0 be a Hilbert space isomorphic to V. For any pair (j, k) of positive integers, let T_j^k be a copy of T_0^0. We identify V and $T = \bigoplus_k \bigoplus_j T_j^k$ in such a way that

$$S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$$

(where $\bigoplus_k$ means $\bigoplus_{k=0}^{\infty}$). For each $j \geqslant 0$ define an isometry ρ_j from $\bigoplus_k T_j^k$ onto T_j^0 and an isometry (shift) σ_j from $\bigoplus_k T_j^k$ onto $\bigoplus_{k=1}^{\infty} T_j^k$ with $\sigma_j(T_j^k) = T_j^{k+1}$ for all $k \geqslant 0$. Denote by ρ the isometry $\bigoplus_j \rho_j$ from T onto $\bigoplus_j T_j^0$ and by σ the shift $\bigoplus_j \sigma_j$. Define the maps

$$\amalg : \begin{cases} GL(T) \times GL(T) \to GL(T) \\ (g, h) \to \rho g \rho^* + \sigma h \sigma^* \end{cases}$$

and

$$\tau : \begin{cases} GL(T) \to GL(T) \\ g \to \sum_k \sigma^k \rho g \rho^* \sigma^{*k} \end{cases}$$

(The series converges strongly, and ρ^* is the adjoint of ρ; in view of section 4, it is appropriate to define ρ^* by $\rho^*(\xi) = \eta$ if $\eta = \rho(\xi) \in \operatorname{Im}(\rho)$ and $\rho^*(\xi) = 0$ if $\xi \perp \operatorname{Im}(\rho)$.)

It is easy to check that $\amalg$ and τ are homomorphism because ρ and σ are isometries with orthogonal complementary ranges. Similarly $\amalg(\mathrm{id} \times \tau)\Delta = \tau$. For each $i \geqslant 0$ one has $\amalg(G_i' \times G_i') \subset G_i'$ and $\tau(G_i') \subset G_i'$ because $\rho_j \rho_j^* + \sigma_j \sigma_j^*$ coincides with the identity on $\bigoplus_k T_j^k$ for $j \geqslant i$. It follows that $\amalg$ and τ induce homomorphisms $G'_\infty \times G'_\infty \to G'_\infty$ and $G'_\infty \to G'_\infty$, denoted below by $\amalg$ and ρ again. Requirement (2) in the definition of a flabby group obviously holds (with $c = 1$).

Consider some integer $i \geqslant 0$. Let a_i be an invertible isometry of T which acts as $\bigoplus_{j=0}^{i-1} \rho_j$ on $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$, as the identity on $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$, and (thus) maps in some way $\bigoplus_k T_i^k$ onto

$$\left(\bigoplus_{k=1}^{\infty} \bigoplus_{j=0}^{i-1} T_j^k \right) \oplus \left(\bigoplus_k T_i^k \right).$$

One has $a_i \in G'_{i+1} \subset G'_\infty$ and $a_i g a_i^* = g \amalg 1$ for all $g \in G_i'$. Similarly, let b_i be an invertible isometry of T which acts as $\bigoplus_{j=0}^{i-1} \sigma_j$ on $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ and as the identity on $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$. Then $b_i \in G'_{i+1}$ and $b_i g b_i^* = 1 \amalg g$ for all $g \in G_i'$. It follows that requirement (1) above holds. ■

We know thus that G'_∞ is acyclic. The reader who is interested in $U(V)$ and not in $GL(V)$ may skip the end of this section since $G_\infty \cap U(V) = G'_\infty \cap U(V)$.

Let us now recall what we need from a result due to Quillen (theorem 1′ of [Q2]). Let A be a $\mathbf{Q}$-algebra with unit, let Γ be the group of invertible (2×2)-matrices over A which have the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, let Γ' be the subgroup of Γ consisting of diagonal matrices and let $\pi : \Gamma \to \Gamma'$ be the homomorphism defined by

$$\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

If R is a $\mathbf{Z}[\Gamma]$-module, we denote by $H_i(\Gamma, R)$ the i^{th} Eilenberg–MacLane homology group of Γ with coefficients in R; moreover R is assumed to have the trivial $\mathbf{Z}[\Gamma]$-structure if there is no strong reason for any other one (such as $R = H_t(N; \mathbf{K})$ below).

LEMMA 4 (Quillen). *Let* $\mathbf{K}$ *be a field which is either finite or the rationals. Then* π *induces an isomorphism on* $H_*(-; \mathbf{K})$.

Proof. Let N be the subgroup of Γ consisting of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, which is isomorphic to the additive group of the algebra A. As N is torsion-free and abelian, $H_*(N; \mathbf{Z})$ is isomorphic to the additive group $\bigwedge_{\mathbf{Z}} N$. (This holds for finitely generated free abelian groups, as one checks knowing homology of compact tori; this holds in general because N and the inductive limit of finitely generated subgroups of N have the same homology.) It follows that $H_*(N; \mathbf{K}) \approx (\bigwedge_{\mathbf{Z}} N) \otimes_{\mathbf{Z}} \mathbf{K}$ for any field $\mathbf{K}$. In particular $H_*(N; \mathbf{K}) = H_0(N; \mathbf{K}) = \mathbf{K}$ if $\mathbf{K}$ is finite (because N is divisible) and $H_*(N; \mathbf{Q}) = \bigwedge_{\mathbf{Q}} A$. (This is a highly degenerate form of the results described in §8 of [Q2].)

Consider the Hochschild–Serre spectral sequence

$$E^2_{s,t} = H_s(\Gamma'; H_t(N; \mathbf{K})) \Rightarrow H_{s+t}(\Gamma; \mathbf{K})$$

corresponding to the extension

$$0 \to N \to \Gamma \to \Gamma' \to 1.$$

If $\mathbf{K}$ is a finite field, one has $H_t(N; \mathbf{K}) = 0$ for $t > 0$ and $H_0(N; \mathbf{K}) = \mathbf{K}$. The spectral sequence therefore degenerates, giving the desired result.

Suppose $\mathbf{K}=\mathbf{Q}$. Make $\mathbf{Q}^*$ act on Γ by

$$\lambda \cdot \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & \lambda b \\ 0 & 1 \end{pmatrix}.$$

Thus $\lambda \in \mathbf{Q}^*$ acts on the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N & \to & \Gamma & \to & \Gamma' & \to & 1 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mathrm{id}} & & \\ 0 & \to & N & \to & \Gamma & \to & \Gamma' & \to & 1 \end{array}$$

and consequently also on the spectral sequence. As $\lambda \in \mathbf{Q}^*$ acts on $H_t(N; \mathbf{Q}) = \bigwedge_{\mathbf{Q}}^t (N \otimes_{\mathbf{Z}} \mathbf{Q})$ by multiplying by λ^t, and acts trivially on Γ', it follows that λ acts on $E^2_{s,t}$ by multiplying by λ^t. Assume $\lambda \neq \pm 1$; as the differentials commute with the $\mathbf{Q}^*$-action and as $\lambda^t \neq \lambda^{t'}$ for $t \neq t'$, all differentials are zero. It follows that

$$E^2_{s,t} = E^\infty_{s,t} \quad \text{for all} \quad s, t \geqslant 0.$$

Now $\bigoplus_{s+t=n} E^\infty_{s,t}$ is the graded object associated to the natural filtration of $H_n(\Gamma; \mathbf{Q})$ for each integer $n \geqslant 1$. Since $\mathbf{Q}^*$ acts on Γ by inner automorphisms, the induced action on $H_n(\Gamma; \mathbf{Q})$ is trivial; thus $\mathbf{Q}^*$ acts trivially on each $E^\infty_{s,t}$. Hence $E^\infty_{s,t} = 0$ for any (s, t) with $s \geqslant 0$ and $t > 0$. This shows that $H_s(\Gamma'; \mathbf{Q}) = H_s(\Gamma; \mathbf{Q})$ for any $s \geqslant 0$. ■

COROLLARY 5 (a universal coefficient argument). *The homomorphism* $\pi : \Gamma \to \Gamma'$ *induces an isomorphism on* $H_*(-) = H_*(-; \mathbf{Z})$.

Proof. We know that π induces an isomorphism for $H_*(-; R)$ if R is the additive group of a finite field. Using direct products and extensions of the coefficients, one checks the same holds for R a finite abelian group. As homology commutes with inductive limits of coefficients, this holds also when $R = \mathbf{Q}/\mathbf{Z}$. Using the sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$$

and the fact that π_* is an isomorphism for $R = \mathbf{Q}$ and $R = \mathbf{Q}/\mathbf{Z}$, one proves the claim. ■

The proof of Proposition 1. We use again the notations defined earlier in this section, and we denote by $L(V)$ the algebra of all bounded operators on V. For each $i>0$ the spaces $S_i^{\perp}$ and S_i are both isomorphic to V. It follows that G_i is isomorphic to

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \ a, b \in L(V) \text{ with } a \text{ invertible}\right\}$$

and that G_i' consists of matrices in G_i with $b=0$. Quillen's argument shows that the inclusion of G_i' in G_i induces an isomorphism $H_*(G_i') \approx H_*(G_i)$. It follows that the inclusion of G_∞' in G_∞ induces also an isomorphism $H_*(G') \approx H_*(G)$, so that the proof of proposition 1 is complete. ■

Let us end this section by two observations. First the groups of our main theorem are not flabby. Consider for example $G=U(V)$ with V an infinite dimensional separable complex Hilbert space, and suppose there exists a "direct sum" homomorphism $\perp\!\!\perp : G \times G \to G$ with property (1) preceding lemma 2; we shall reach a contradiction.

Choose an orthonormal basis $(e_j)_{j\in N}$ of V and a sequence $(\lambda_j)_{j\in N}$ of pairwise distinct numbers in the interval $]-\pi, \pi[$. Define $r\in G$ by $r(e_j)=\exp(i\lambda_j)e_j$ for $j\in N$. The centralizer of r in G is the abelian group T of unitary operators which are diagonal with respect to the chosen basis.

Consider the homomorphism $\perp\!\!\perp_1 : G \to G$ given by $g \mapsto \perp\!\!\perp(g, 1)$. By hypothesis $\perp\!\!\perp_1(g)$ is conjugate to g. Therefore, $\perp\!\!\perp_1$ is injective and, because its image commutes with $\perp\!\!\perp(1, r)$, the centralizer of $\perp\!\!\perp(1, r)$ is not abelian. But there exists $b\in G$ with $\perp\!\!\perp(1, r)=brb^{-1}$. Therefore the centralizer of $\perp\!\!\perp(1, r)$ is the abelian group bTb^{-1}. This contradiction shows that G is not flabby.

The second observation is that there are plenty of (non trivial) G-modules R with non trivial $H_*(G, R)$ or $H^*(G, R)$. Consider for example a subgroup G_1 of G and a G_1-module R_1. Let $R=\mathrm{Hom}_{\mathbf{Z}G_1}(\mathbf{Z}G, R_1)$, where $\mathbf{Z}G$ is considered as a left $\mathbf{Z}G_1$-module and as a right $\mathbf{Z}G$-module; then R is naturally a G-module (namely a left $\mathbf{Z}G$-module). A standard result known as Shapiro's lemma states that $H^n(G_1, R_1)$ is naturally isomorphic with $H^n(G, R)$ for all $n\geq 0$; see for example §34.2 in [Bab]. Choose in particular a finite cyclic subgroup G_1 of G and let R_1 be a trivial G_1-module isomorphic to G_1 as abelian group. Then $H^n(G, R)\neq 0$ for all $n>0$.

This is quite a general construction. Indeed, let Γ be any group with more than one element. One shows by induction from a (possibly infinite) cyclic subgroup of Γ that there exists a Γ-module M and an integer $n>0$ with $H^n(\Gamma; M)\neq 0$.

3. The set of flags

Let Gr be the set of those closed subspaces S of V which are isomorphic to V/S. (Thus Gr is the set of points in a Grassmannian space.)

LEMMA 6. *Let $\{S_1, \ldots, S_p\}$ be a finite subset of Gr. There exist $S'_1, \ldots S'_p \in Gr$ with $S'_m \subset S_m$ $(1 \leq m \leq p)$ and $S'_m \perp S'_n$ $(1 \leq m < n \leq p)$.*

Proof. Any subspace of V whose codimension is strictly smaller than the dimension of V intersects non trivially any element of Gr. One may thus choose unit vectors as follows

$$v_{1,1} \in S_1,\ v_{2,1} \in S_2 \cap \{v_{1,1}\}^{\perp}, \ldots, v_{p,1} \in S_p \cap \{v_{1,1}, \ldots, v_{p-1,1}\}^{\perp}$$

and in general

$$\begin{aligned} &v_{1,i} \in S_1 \cap \{v_{1,1}, \ldots, v_{p,1}, \ldots, v_{1,i-1}, \ldots, v_{p,i-1}\}^{\perp}, \\ &\ldots, \\ &v_{p,i} \in S_p \cap \{v_{1,1}, \ldots, v_{p,1}, \ldots, v_{1,i}, \ldots, v_{p-1,i}\}^{\perp}. \end{aligned}$$

(The index i runs over $\mathbf{N}^*$ if V is separable and over some suitable infinite set if V is "larger".) Define S'_m to be the closed linear span of the $v_{m,i}$'s. Then $S'_1, \ldots, S'_p$ have the desired properties. ■

LEMMA 7. *Let $S_1, \ldots, S_p \in Gr$ and let $h_1, \ldots, h_p \in GL(V)$. There exist $S'_1, \ldots, S'_p \in Gr$ with $S'_m \subset S_m$ $(1 \leq m \leq p)$, $S'_m \perp S'_n$ and $h_m(S'_m) \perp h_n(S'_n)$ $(1 \leq m < n \leq p)$.*

Proof. By Lemma 6 there exist $S''_1, \ldots, S''_p \in Gr$ with $S''_m \subset S_m$ $(1 \leq m \leq p)$ and $S''_m \perp S''_n$ $(1 \leq m < n \leq p)$. Define $T_m = h_m(S''_m)$ $(1 \leq m \leq p)$. There exist also $T'_1, \ldots, T'_p \in Gr$ with $T'_m \subset T_m$ $(1 \leq m \leq p)$ and $T'_m \perp T'_n$ $(1 \leq m < n \leq p)$. Define $S'_m = h_m^{-1}(T'_m)$ $(1 \leq m \leq p)$. ■

Now consider the set $\mathfrak{F}$ of flags $F = \{S_1 \supset S_2 \supset \cdots\}$ with $\bigcap S_i = \{0\}$ as defined in section 2. Let $F = \{S_1 \supset S_2 \supset \cdots\}$, $F' = \{S'_1 \supset S'_2 \supset \cdots\}$ and $h \in GL(V)$. We write $F' \leq F$ if $S'_i \subset S_i$ for all i. If $S'_i \perp S_i$, we write $F' \perp F$. If in addition $S_1 \oplus S'_1 \in Gr$, the spaces $S_1 \oplus S'_1 \supset S_2 \oplus S'_2 \supset \cdots$ form a flag which we call $F' \oplus F$. Finally the flag $\{h(S_1) \supset h(S_2) \supset \cdots\}$ is called $h(F)$.

We may reformulate lemma 7 for flags.

LEMMA 8. *Let* $F_1, \ldots, F_p \in \mathfrak{F}$ *and let* $h_1, \ldots, h_p \in GL(V)$. *There exist* $F'_1, \ldots, F'_p \in \mathfrak{F}$ *with* $F'_m \leqslant F_m$ $(1 \leqslant m \leqslant p)$, $F'_m \perp F'_n$ *and* $h_m(F'_m) \perp h_n(F'_n)$ $(1 \leqslant m < n \leqslant p)$.

Proof. Let $F_m = \{S_{m,1} \supset S_{m,2} \supset \cdots\}$ and write $T_{m,i} = S_{m,i}^{\perp} \cap S_{m,i-1}$ where $S_{m,0} = V$ $(1 \leqslant m \leqslant p$ and $i \geqslant 1)$. Then $S_{m,i} = \bigoplus_{j=i+1}^{\infty} T_{m,j}$. The result now follows by applying lemma 7 to the spaces $T_{1,j}, \ldots, T_{p,j}$ for each $j \geqslant 1$. ■

We review now the Milnor construction for classifying space (see e.g. [Hu], chap. 4, §11). Given any (discrete) group Γ, let $E\Gamma$ be the simplicial complex whose p-simplices are the ordered subsets $(\gamma_0, \ldots, \gamma_p)$ of Γ. We denote by $|E\Gamma|$ the topological space obtained by realizing $E\Gamma$. It is well-known and easy to see that $|E\Gamma|$ is contractible (compare the proof of lemma 10 below). Moreover the group Γ acts freely on $|E\Gamma|$ by multiplication on the left. Thus the quotient space $B\Gamma = \Gamma \backslash |E\Gamma|$ is a model (the "infinite join" model) for the classifying space of the group Γ. In particular this means that the groups $H_i(\Gamma)$ $(i \in \mathbf{N})$ are just the integral homology groups of the space $B\Gamma$.

For the rest of this section, we will write G for $GL(V)$, E for $EGL(V)$ and B for $BGL(V)$. For each flag $F = \{S_1 \supset S_2 \supset \cdots\}$ in $\mathfrak{F}$, let G_F be the subgroup of G containing those operators which agree with the identity on S_i for i large enough, and let E_F be the subcomplex of E defined as follows: a k-simplex $(g_0, \ldots, g_k)$ of E is in E_F if $g_0, \ldots, g_k$ agree on S_i for i large enough. (For short, we will say that $g_0, \ldots, g_k$ agree on F.) Let $F, F' \in \mathfrak{F}$. If $F' \leqslant F$, observe that $G_F \subset G_{F'}$ and that E_F is a subcomplex of $E_{F'}$. If $F \perp F'$ and if $F \oplus F' \in \mathfrak{F}$, then $G_{F \oplus F'} = G_F \cap G_{F'}$.

LEMMA 9. *For any* $F \in \mathfrak{F}$, *the complex* E_F *is* G-*invariant and the quotient* $G \backslash |E_F|$ *is naturally isomorphic to* BG_F.

Proof. "Naturally" means that, if $F, F' \in \mathfrak{F}$ with $F' \leqslant F$, then the map $BG_F \to BG_{F'}$ induced by $G_F \hookrightarrow G_{F'}$ is just the inclusion of BG_F in $BG_{F'}$ (both are subspaces of B).

The space $|E_F|$ is not connected. Indeed two 0-simplices (g) and (g') define points lying in the same connected component if and only if there is a sequence of 1-simplices in E_F of the form

$$(g, g_1), (g_1, g_2), \ldots, (g_m, g').$$

This holds if and only if g and g' agree on F, namely if and only if g and g' belong to the same right coset of G_F in G. It follows that connected components of $|E_F|$

are parametrized by G/G_F. The coset G_F corresponds to $|E'_F|$, where E'_F is the subcomplex of E_F consisting of simplices $(g_0, \ldots, g_k)$ where $g_0, \ldots, g_k$ agree with the identity on F.

It is clear that E_F is G-invariant. It follows from the discussion above that $G \backslash |E_F|$ may be identified with $G_F \backslash |E'_F|$, which is nothing but the infinite join model BG_F for the classifying space of G_F. ■

Let E_* be the union of the E_F's over $F \in \mathfrak{F}$; it is a subcomplex of E which is invariant by G. Let $B_* = G \backslash |E_*|$; it is a subspace of B which is the union of the $G \backslash |E_F|$'s over F in $\mathfrak{F}$.

LEMMA 10. *The space E_* is contractible.*

Proof. Let $\sigma_1, \ldots, \sigma_p$ be simplices in E_*. Choose

$$F_1 = \{S_{1,1} \supset S_{1,2} \supset \cdots\}, \ldots, F_p = \{S_{p,1} \supset S_{p,2} \supset \cdots\}$$

in $\mathfrak{F}$ with $\sigma_m \in E_{F_m}$. There is an integer k such that the vertices in σ_m agree on $S_{m,k}$; denote by h_m their common restriction on $S_{m,k}$ $(1 \leq m \leq p)$. Let $F'_1, \ldots, F'_p$ be as in lemma 8: one has $\sigma_m \in E_{F'_m} (1 \leq m \leq p)$. Then the cone on $\sigma_1 \cup \cdots \cup \sigma_p$ with vertex h_0 is in E_*.

It follows that, for any finite subcomplex K of E_*, there exists a subcomplex L of E_* containing K and contracting to a point. Hence $|E_*|$ itself is c ·tractible (see e.g. corollary 7.6.24 in [Sp]). ■

LEMMA 11. *The inclusion $B_* = \bigcup_{F \in \mathfrak{F}} BG_F \to B = BG$ is a homotopy equivalence.*

Proof. Since the quotient maps $|E| \to B$ and $|E_*| \to B_*$ are covering maps, this follows immediately from the two previous lemmas. ■

The following lemma holds for $p = 1$ by section 2.

LEMMA 12. *Let $F_1, \ldots, F_p \in \mathfrak{F}$. Then $BG_{F_1} \cup \cdots \cup BG_{F_p}$ is contained in an acyclic subspace of B_*.*

Proof. Choose any flag $F_0 \in \mathfrak{F}$. By Lemma 8 there exist $F'_0, F'_1, \ldots. F'_p \in \mathfrak{F}$ with $F'_m \leq F_m$ $(0 \leq m \leq p)$ and $F'_m \perp F'_n$ $(0 \leq m < n \leq p)$; in particular $F'_1 \oplus \cdots \oplus F'_p$ is a flag in $\mathfrak{F}$. As $BG_{F_m} \subset BG_{F'_m}$ $(1 \leq m \leq p)$, it suffices to check that $BG_{F'_1} \cup \cdots \cup BG_{F'_p}$ is acyclic. Hence we may assume without loss of generality that $F_m \perp F_n$ $(1 \leq m < n \leq p)$ and that $F_1 \oplus \cdots \oplus F_p \in \mathfrak{F}$.

Let us assume as induction hypothesis that, in this situation, both

$$BG_{F_1} \cup \cdots \cup BG_{F_{p-1}} \quad \text{and} \quad BG_{F_1 \oplus F_{p-1}} \cup \cdots \cup BG_{F_{p-2} \oplus F_{p-1}}$$

are acyclic. (When $p = 2$, the former works by proposition 1 and the latter is vacuous.)

Consider first the Mayer–Vietoris homology sequence of the subcomplexes

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-2} \oplus F_p} \quad \text{and} \quad BG_{F_{p-1} \oplus F_p}$$

of B_* with intersection

$$BG_{F_1 \oplus (F_{p-1} \oplus F_p)} \cup \cdots \cup BG_{F_{p-2} \oplus (F_{p-1} \oplus F_p)}.$$

By the induction hypothesis, two of any three consecutive terms in this sequence vanish. Hence all terms vanish and

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-1} \oplus F_p}$$

is acyclic.

Consider now the Mayer–Vietoris sequence of the subcomplexes

$$BG_{F_1} \cup \cdots \cup BG_{F_{p-1}} \quad \text{and} \quad BG_{F_p}$$

of B_* with intersection

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-1} \oplus F_p}.$$

From the previous step and from the induction hypothesis it follows that

$$BG_{F_1} \cup \cdots \cup BG_{F_p}$$

is acyclic. ■

THEOREM 13. *The group G is acyclic.*

Proof. The homology of a complex is generated by that of its finite subcomplexes. Thus lemma 12 implies that B_* is an acyclic space, and lemma 11 that G is acyclic. ■

4. Variations

Unitary group $U(V)$ of an infinite dimensional Hilbert space V.

The proof that $U(V)$ is acyclic is much simpler than for $GL(V)$ since section 2 may be reduced to Lemmas 2 and 3. Section 3 is unchanged.

Symmetric group $\Sigma(X)$ of an infinite set X

Here a flag is a nested sequence $\{S_1 \supset S_2 \supset \cdots\}$ of subsets of $X = S_0$ such that $S_{i-1} - S_i$ is equipotent with X for each $i \geqslant 1$ and such that $\bigcap S_i = \emptyset$. Define

$$\Sigma_i = \{g \in \Sigma(X) \mid g = \text{id on } S_i\}$$

for each $i \geqslant 0$ (no distinction here between Σ_i' and Σ_i) and $\Sigma_\infty = \bigcup_{i=0}^{\infty} \Sigma_i$. The argument of Lemma 3 shows that Σ_∞ is a flabby group. Read "disjoint union" instead of "direct sum", "injection" instead of "isometry". The adjoint ρ^* of an injection ρ is defined only on the image of ρ by $\rho^*\rho = \text{id}$; then a formula like $\rho g \rho^* + \sigma h \sigma^*$ is clear because $\rho g \rho^*$ is a permutation of some subset of X and $\sigma h \sigma^*$ is a permutation of its complement. The group Σ_∞ is consequently acyclic.

Let Gr be the set of those subsets S of X equipotent with their complements $S^\perp = X - S$. For two subsets S_1, S_2 of X, read $S_1 \cap S_2 = \emptyset$ for $S_1 \perp S_2$. Lemmas 7 and 8 may then be repeated without change and all of section 3 with minor changes only. It follows that $\Sigma(X)$ is acyclic.

Automorphism group $\mathscr{A}(\Omega)$ of a Lebesgue space $(\Omega, \mathscr{B}, \mu)$

Let $(\Omega, \mathscr{B}, \mu)$ be a Lebesgue space where the measure μ is infinite and non atomic. A flag is now a nested sequence $F = \{S_1 \supset S_2 \supset \cdots\}$ of measurable subsets of $\Omega = S_0$ such that $S_{i-1} - S_i$ has infinite measure for each $i \geqslant 1$ and such that $\bigcap S_i$ has measure zero. Comments for $\sum(X)$ above apply to $\mathscr{A}(\Omega)$, with the understanding that everything in view is now measurable. Therefore $\mathscr{A}(\Omega)$ is also acyclic.

Let $(\tilde{\Omega}, \mathscr{B}, \mu)$ be a Lebesgue measure space. Let X be the set of atoms in $\tilde{\Omega}$, let $X = \coprod_j X_j$ be the partition of X according to the masses of the atoms, and let $\Omega = \tilde{\Omega} - X$. Then the sequence

$$1 \to \mathscr{A}(\Omega) \to \mathscr{A}(\tilde{\Omega}) \to \prod_j \sum (X_j) \to 1$$

is exact (and splits). Suppose $\mu(\Omega) = \infty$, and suppose that X is not empty. Then $\mathscr{A}(\tilde{\Omega})$ is clearly acyclic if and only if each X_j is either one point or an infinite set.

Automorphisms of an infinite dimensional vector space W over a (possibly skew) field $\mathbf{F}$

Case (i): Char $\mathbf{F}=0$.

A flag is in this case a nested sequence $\{S_1 \supset S_2 \supset \cdots\}$ of subspaces of $W=S_0$ such that S_{i-1}/S_i is isomorphic to W for each $i \geq 1$ and such that $\bigcap S_i = \{0\}$. As in Lemma 3 we may identify W with $\bigoplus_k \bigoplus_j T_j^k$, where each $T_j^k \cong W$, in such a way that $S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$ for all i. Then the subspace $R_i = \bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ complements S_i.

Define

$$G_i^W = \{g \in GL(W) \mid g = \text{id on } S_i\},$$
$$G_i^{W'} = \{g \in G_i^W \mid g(R_i) = R_i\}.$$

One checks as in Lemma 3 that $G_\infty^{W'}$ is flabby. When Char $\mathbf{F}=0$, Lemma 4 and 5 show that G_∞^W is acyclic.

In Lemmas 6 to 8, understand $S'_m \perp S'_n$ as $S'_m \cap S'_n = \{0\}$, and $v \in S \cap \{v_1, \ldots, v_m\}^\perp$ as $v \in S$ with v not in the linear span of $\{v_1, \ldots, v_m\}$. Then section 3 holds for $GL(W)$, which is consequently an acyclic group. All our arguments allow the field $\mathbf{F}$ to be non-commutative.

Case (ii): Char $\mathbf{F}=p>0$.

The arguments of section 2 show that $\tilde{H}_*(G_\infty^W; \mathbf{K})=0$ if Char $\mathbf{K} \neq$ Char $\mathbf{F}$ (where $\tilde{H}_*$ denotes reduced homology). It follows that $\tilde{H}_*(GL(W); \mathbf{K})=0$ when Char $\mathbf{K} \neq$ Char $\mathbf{F}$. Therefore, in order to show that $GL(W)$ is acyclic, it will suffice to prove that $\tilde{H}_*(GL(W); \mathbf{K})=0$ when $\mathbf{K}$ is the algebraic closure $\bar{\mathbf{k}}$ of the finite field $\mathbf{k}$ with p elements. To do this we need

LEMMA 14. *For each flag F and integer $d>0$ there is a subgroup G_F^d of $GL(W)$ which contains G_F and is such that $H_j(G_F^d; \bar{\mathbf{k}})=0$ for $0<j<d$.*

Proof. Quillen proves the following lemma in [Q2] §9.

LEMMA. *Let $\bar{\mathbf{k}}$ be an algebraically closed field and d an integer >0. Then there exists an order D in a number field of degree d over $\mathbf{Q}$ with the following properties: Given any D-module N, let the group of units D^* act on it by multiplication, and let the group homology $H_*(N, \bar{\mathbf{k}})$ be endowed with the induced action of D^*. Then for each t, $H_t(N, \bar{\mathbf{k}})$ is a direct sum of one-dimensional representations of D^* over $\bar{\mathbf{k}}$. Furthermore, $H_t(N, \bar{\mathbf{k}})$ does not contain the trivial representation for $0<t<d$.*

Let D be as in this lemma. The choice of a basis over $\mathbf{Z}$ for D gives rise to a ring homomorphism

$$\rho_0 : D \to M_d(\mathbf{Z}) \to M_d(\mathbf{F})$$

where $M_d(A)$ is the ring of d-by-d matrices over A and where $M_d(\mathbf{Z}) \to M_d(\mathbf{F})$ is reduction mod p. Let F be the flag $\{S_1 \supset S_2 \supset \cdots\}$. For each pair (j, k) of positive integers, let now T_j^k be a copy of $\mathbf{F}^d$. We identify W and $T = \oplus_k \oplus_j T_j^k$ in such a way that $S_i = \oplus_k \oplus_{j=i}^{\infty} T_j^k$, and we denote by R_i "the" complement $\oplus_k \oplus_{j=0}^{i-1} T_j^k$ of S_i. Define a ring homomorphism $\rho_i : D \to GL(W)$ by setting

$$\rho_i(\lambda) = \begin{cases} \rho_0(\lambda) & \text{in } T_j^k \text{ for } j \geq i, \text{ all } k \\ \text{id} & \text{in the other } T_j^k. \end{cases}$$

Now put

$$G_i^d = \{g \in GL(W) \mid g = \rho_i(\lambda) \text{ in } S_i \text{ for some } \lambda \in D^*\}$$

and let $G_F^d = \bigcup_{i \geq 1} G_i^d$. Clearly $G_F \subset G_F^d$. We must show that $H_j(G_F^d; \bar{\mathbf{k}}) = 0$ for $0 < j < d$.

Let

$$G_i^{d'} = \{g \in G_i^d \mid g(R_i) = R_i\}.$$

and consider the induced D^*-action on the spectral sequence of the extension $0 \to N \to G_i^d \to G_i^{d'} \to 1$. It follows from the lemma that each E_{st}^r, $2 \leq r \leq \infty$, breaks up into a sum of one dimensional representations preserved by the differentials. Since D^* acts trivially on the abutment, the subspaces on which D^* acts trivially form a spectral sequence which converges to $H_*(G_i^d; \bar{\mathbf{k}})$. By the lemma, the terms E_{st}^2 of this sequence vanish when $0 < t < d$. Hence $H_j(G_i^d; \bar{\mathbf{k}}) \cong H_j(G_i^{d'}; \bar{\mathbf{k}})$ for $0 < j < d$.

Now note that $G_i^{d'}$ is the product of G_i' with $\rho_i(D^*)$. But $\rho_i(D^*)$ is isomorphic to a subgroup of the group of units of $D/pD \cong \mathbf{k}_d$, where $\mathbf{k}_d$ is the field of order p^d. Hence $\rho_i(D^*)$ has order prime to p. Therefore $\tilde{H}_*(\rho_i(D^*); \bar{\mathbf{k}}) = 0$ which implies that $H_*(G_i^{d'}; \bar{\mathbf{k}}) \cong H_*(G_i'; \bar{\mathbf{k}})$. Now consider the diagram

$$\begin{array}{ccc} G_i^d & \xrightarrow{\alpha_3} & G_{i+1}^d \\ {\scriptstyle\alpha_2}\uparrow & & \nearrow \\ G_i^{d'} & {\scriptstyle\alpha_4} & \\ {\scriptstyle\alpha_1}\uparrow & \nearrow & \\ G_i' & & \end{array}$$

We have seen that the inclusions α_1 and α_2 induce an isomorphism on $H_j(-;\bar{\mathbf{k}})$, $0<j<d$. Since α_4 factors through a group isomorphic to G'_∞, it induces the zero map on $\tilde{H}_j(-;\bar{\mathbf{k}})$. Hence α_3 must induce the zero map on $H_j(-;\bar{\mathbf{k}})$, $0<j<d$. This implies that

$$H_j(G^d_F;\bar{\mathbf{k}}) = \lim_i H_j(G^d_i;\bar{\mathbf{k}}) = 0, \qquad 0<j<d. \quad \blacksquare$$

To finish the proof of the theorem we must find an appropriate substitute for Lemma 12. If $F_1, \ldots, F_n$ are disjoint flags such that $F_1 \oplus \cdots \oplus F_n$ is also a flag, choose groups $G^d_{F_i}$ as above and, for each subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, set

$$G^d_{F_{i_1}\oplus\cdots\oplus F_{i_k}} = G^d_{F_{i_1}} \cap \cdots \cap G^d_{F_{i_k}}.$$

The proof of Lemma 14 shows that these groups G^d_F, for $F = F_{i_1} \oplus \cdots \oplus F_{i_k}$, are acyclic. The inductive argument of Lemma 12 then readily shows that

$$H_j(BG^d_{F_1} \cup \cdots \cup BG^d_{F_n};\bar{\mathbf{k}}) = 0 \qquad 0<j<d-2n.$$

Clearly, this suffices to show that the inclusion $B_* \hookrightarrow B$ annihilates $\tilde{H}_*(-;\bar{\mathbf{k}})$.

Properly infinite von Neumann algebras

Let M be a properly infinite von Neumann algebra, faithfully represented in $L(V)$ for some complex Hilbert space V. A flag is a nested sequence $\{S_1 \supset S_2 \supset \cdots\}$ of closed subspaces of $V = S_0$ with $\bigcap S_i = \{0\}$ such that the orthogonal projection P_i from V onto S_i is in M and such that $P_{i-1} - P_i$ is equivalent to the identity for each $i \geq 1$. It is easy to choose every operator appearing in sections 2 and 3 in the algebra M. Therefore the appropriately defined groups G'_∞ and G_∞ are acyclic, as well as $U(M)$ and $GL(M)$.

It is likely that the argument applies to a large class of infinite C^*-algebras. Let B be such an algebra, let $M(B)$ be its multiplier algebra, let $U(B)$ be the subgroup of the unitary group $U(M(B))$ consisting of those elements g for which $g-1 \in B$, and let $U(B)_0$ be the connected component of $U(B)$ with respect to the norm topology. There are many cases in which $U(B)_0$ is known to be contractible for the norm topology [Mi]; in these cases, $U(B)_0$ and the similarly defined "general linear group" $GL(B)_0$ should "often" be acyclic.

Finite von Neumann algebras

Let M be a finite continuous factor, and let $U(M)$ be the group of unitaries in M. When given the norm topology, $U(M)$ has a fundamental group isomorphic to the additive group of the real numbers: this was first proved in [AS], but it follows also essentially from Bott periodicity as formulated in theorem 1.11 of chapter III of [Ka]. Indeed

$$\pi_i(U(M)_{\text{norm}}) \approx \begin{cases} \mathbf{R} \text{ if } i \text{ is odd, } i \geqslant 0 \\ 0 \text{ if } i \text{ is even, } i > 0 \end{cases}$$

(See III.7.7 in [Ka], or theorem 5 in [Br]; both state the analogous "stable fact", but the isomorphism holds also as above.) Let

$$0 \to \mathbf{R} \to \tilde{U}(M) \to U(M) \to 1$$

be the (topological) universal covering of $U(M)$. It is known that $U(M)$ is perfect (indeed simple up to centre [FH]). One may conjecture that $\tilde{U}(M)$ is also perfect, namely that the short exact sequence above is still a covering in the algebraic sense of [Ker], and thus that there exists a surjective homomorphism of $H_2(U(M))$ onto $\mathbf{R}$. In any event it seems very unlikely that the group $U(M)$ is acyclic.

Appendix 1. About normal subgroups

If X is an infinite countable set, $\Sigma(X)$ has exactly two non trivial normal subgroups: the group $\Sigma_f(X)$ of permutations of X with finite support and its derived group $A_f(X)$ of even permutations [SU]. If X is any infinite set, normal subgroups of $\Sigma(X)$ which are neither trivial nor $A_f(X)$ are in bijection (via supports) with infinite cardinals smaller than the cardinal of X [B].

If $(\Omega, \mathcal{B}, \mu)$ is a Lebesgue measure space with μ infinite and non atomic, $\mathcal{A}(\Omega)$ has exactly one non trivial normal subgroup consisting of those bi-measurable transformations α with support $\{\omega \in \Omega \mid \alpha(\omega) \neq \omega\}$ of finite measure [F1], [Ei].

If W is an infinite dimensional vector space over a field $\mathbf{F}$, normal subgroups of $GL(W)$ have been studied in [R]; we present hereafter part of these results with different proofs inspired by [And], [Ep] and [Hi].

LEMMA A1. *The group $GL(W)$ is perfect.*

Proof. If I is a set and if $(W_i)_{i \in I}$ is a family of copies of W, we write any element in $GL(\bigoplus W_i)$ as an $(I \times I)$-matrix with coefficients in $\operatorname{End}(W)$. If I is countable, we may identify $\bigoplus W_i$ and W.

In $GL(W \oplus W \oplus W)$ one has

$$\begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

for each $x \in \mathrm{End}\,(W)$. It follows that any element of the form $\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$ in $GL(W \oplus W)$ is a product of two commutators. In $GL(\bigoplus_{i\in N} W_i)$, one may apply the infinite repetition argument used in section 2. We write $\gamma_1 \sim \gamma_2$ if two elements γ_1, γ_2 in a group Γ are conjugate. For any $x \in GL(W)$ one has

$$\begin{pmatrix}x & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots\end{pmatrix}\begin{pmatrix}1 & & & & \\ & x & & & \\ & & 1 & & \\ & & & x & \\ & & & & \ddots\end{pmatrix}\sim\begin{pmatrix}1 & & & & \\ & x & & & \\ & & 1 & & \\ & & & x & \\ & & & & \ddots\end{pmatrix}$$

in $GL(\bigoplus_{i\in N} W_i)$. It follows that any element of the form $\begin{pmatrix}* & 0\\ 0 & 1\end{pmatrix}$ in $GL(W \oplus W)$ is a commutator.

Let $g \in GL(W)$. Choose sequences (u_i) and (v_i) of vectors in W as follows:

$$u_1 \in W-\{0\} \qquad u_1' = g(u_1) \qquad v_1 \in W-\mathrm{span}\,(u_1, u_1')$$

and in general

$$u_{i+1} \in W-\mathrm{span}\begin{pmatrix}u_1 & v_1 & g^{-1}(v_1)\\ \cdot & \cdot & \cdot\\ u_i & v_i & g^{-1}(v_i)\end{pmatrix} \qquad u_{i+1}' = g(u_{i+1})$$

$$v_{i+1} \in W-\mathrm{span}\begin{pmatrix}u_1 & u_1' & v_1\\ \cdot & \cdot & \cdot\\ u_i & u_i' & v_i\\ u_{i+1} & & u_{i+1}'\end{pmatrix}.$$

(The index i runs over N^* if the dimension of W is countable and over some suitable set otherwise.) Define

$$U = \text{span}\,(u_1, u_2, \ldots) \qquad V_1 = \text{span}\,(v_1, v_3, \ldots)$$
$$V_2 = \text{span}\,(v_2, v_4, \ldots) \qquad V = V_1 \oplus V_2.$$

It is easy to check that $U \cap V = \{0\}$ and $g(U) \cap V = \{0\}$. Thus there exists $t \in GL(W)$ with $tu_i' = u_i$ and $tv_{2i} = v_{2i}$ for each i. As $t = \text{id}$ on V_2 one has $t \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$; as $tg = \text{id}$ on U one has $tg \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$. It follows from the beginning of the proof that g is a product of commutators in $GL(W)$. ■

The proof above shows also the following *fragmentation lemma*: any element in $GL(W)$ may be written as a product of finitely many elements similar to $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$ in $GL(W \oplus W)$. Indeed, it remains to be checked that $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ has this property, and this is clear if one looks at

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $GL(W \oplus W \oplus W)$.

Let $N_{\max}$ be the normal subgroup of $GL(W)$ containing those elements of the form $\lambda + X$ with λ a homothety and X an endomorphism of W with rank strictly smaller than the dimension of W. Let $g \in GL(W)$ with $g \notin N_{\max}$. Let us check that there exists a subspace V of W with V isomorphic to W/V and with $V \cap g(V) = \{0\}$.

One may choose a sequence (v_i) of vectors in W as follows:

$$v_1 \in W - \{0\} \quad \text{with} \quad g(v_1) \in W - \text{span}\,(v_1)$$

and in general

$$v_{i+1} \in W - \text{span} \begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \end{pmatrix} \quad \text{with} \quad g(v_{i+1}) \in W - \text{span} \begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \\ v_{i+1} & \end{pmatrix}$$

Indeed, suppose one cannot find v_{i+1}. Let

$$F = \operatorname{span}\begin{pmatrix} v_1 \cdots v_i \\ g(v_1) \cdots g(v_i) \end{pmatrix}.$$

Then $v \in W - F$ implies $g(v) \in \operatorname{span}(F, v)$; for any $u \in F$, one has also $g(v+u) \in \operatorname{span}(F, v)$; hence $g(u) \in \operatorname{span}(F, v)$. It follows that F is invariant by g and that g induces a homothety on W/F. But this is ruled out by hypothesis.

Then $V = \operatorname{span}(v_1, v_2, \ldots)$ has the desired properties.

PROPOSITION A2. *Any non trivial normal subgroup of $GL(W)$ is contained in $N_{\max}$.*

Proof. Let N be a normal subgroup of $GL(W)$ and assume that $N \not\subset N_{\max}$. There exist $f \in N$ and a subspace V of W with V isomorphic to W/V and with $f(V) \cap V = \{0\}$. We may thus view N as a normal subgroup of $GL(W \oplus W)$ containing an element f of the form $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$.

By the fragmentation lemma, it is enough to check that N contains any element of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$. Consider $r, s \in GL(W)$ and define $g = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$. As N is normal, N contains $\hat{h} = hfh^{-1}f^{-1}$ and $g\hat{h}g^{-1}\hat{h}^{-1}$. By a straightforward matrix computation, the latter is of the form

$$g\hat{h}g^{-1}\hat{h}^{-1} = \begin{pmatrix} 1 & * \\ 0 & rsr^{-1}s^{-1} \end{pmatrix}.$$

As $GL(W)$ is perfect, it follows that, for any $k \in GL(W)$, there exists $z \in \operatorname{End}(W)$ with $\begin{pmatrix} 1 & z \\ 0 & k \end{pmatrix} \in N$.

Let now $a, b \in GL(W)$ with $a + b = 1$. (One may define a as an infinite direct sum of automorphisms of a vector space of dimension two, each represented by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and similarly for b with $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$.) There exist $x, y \in \operatorname{End}(W)$ with

$$\begin{pmatrix} 1 & x \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & y \\ 0 & b^{-1} \end{pmatrix}$$

in N. Then

$$\begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -xa \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & z(a-1) \\ 0 & 1 \end{pmatrix} \in N$$

and

$$\begin{pmatrix}1 & z(a-1)\\0 & 1\end{pmatrix}\begin{pmatrix}1 & z(b-1)\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & -z\\0 & 1\end{pmatrix}\in N.$$

It follows that

$$\begin{pmatrix}1 & z\\0 & k\end{pmatrix}\begin{pmatrix}1 & -z\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & k\end{pmatrix}\in N$$

the proof is complete. ■

It would be easy to prove by similar arguments all of theorem B (and thus also theorem A) in [R].

Let now V be an infinite dimensional Hilbert space over the reals, complexes or quaternions and $GL(V)$ be as in the introduction. Let $GE(V, C)$ be the normal subgroup of $GL(V)$ containing those elements of the form $\lambda+x$ with λ a homothety and X a compact operator (we assume V to be separable). It is quite easy to check that $GL(V)$ is perfect (see problems 191 and 192 in [Hal]). There is a fragmentation lemma which follows straightforwardly from polar decomposition and spectral theorem. Any $g\in GL(V)$ with $g\notin GE(V, C)$ is similar to an element of the form $\begin{pmatrix}* & *\\ * & 0\end{pmatrix}$ in $GL(V\oplus V)$: this is corollary 3.4 in [BP] or theorem 1 in [AnS]. Hence the proof above applies, and is very much simpler than that of [H1]. The subgroup of $GL(V)$ containing all bijective isometries of V can be handled either as in [H1] or as suggested in [H3], and we have proved the following result.

PROPOSITION A3. *Any non trivial normal subgroup of $GL(V)$ is contained in $GE(V, C)$. Any non trivial normal subgroup of $U(V)$ is contained in $UE(V, C)=U(V)\cap GE(V, C)$.*

For normal subgroups of $GL(M)$ and $U(M)$, when M is a properly infinite von Neumann algebra, see [H3] and papers reviewed there.

COROLLARY A4. *Let G be one of the groups described in the introduction and let N be a non trivial normal subgroup of G. Then N is of uncountable index in G.*

Let G be as above and let $N_{\max}$ be the maximal normal subgroup of G. There are cases for which we have information about the homology of $N_{\max}$: see works by Nakaoka and Priddy [P] if $G = \sum(X)$ and $N_{\max} = \sum_f(X)$ with X infinite countable, the papers on group cohomology in [E] if $G = GL(W)$, or [BHS] if $G = GL(V)$. In each case our main theorem provides corresponding information about the homology of the quotient $G/N_{\max}$.

Appendix 2. About monoids of monomorphisms

Each of the acyclic groups of automorphisms considered above is the group of units in a corresponding monoid (or semigroup) of monomorphisms. For example, $\sum(X)$ is the group of units in the monoid $M(X)$ formed by all injective maps from X to X. One can form the classifying space BM of a monoid in exactly the same way as that of a group; see [Se]. In particular, the Eilenberg–MacLane homology groups $H_i(M; \mathbf{Z})$ are just the integral homology groups of the space BM. Quillen pointed out in an unpublished version of [Q1] that the classifying spaces of monoids such as $M(X)$ are contractible. Of course, this implies that the monoids are acyclic.

Here is a sketch of his argument. Say two homomorphisms $f, g: M \to M$ are semi-conjugate if there is $m \in M$ such that $mf(n) = g(n)m$ for all $n \in M$. The argument is based on the fact that two homomorphisms which are semi-conjugate induce homotopic maps on BM; see [Q1] §1. Choose $p \in M(X)$ so that the image $p(X)$ of X under p is in Gr. Define $f: M(X) \to M(X)$ by $f(n)(x) = pnp^{-1}(x)$ if $x \in p(X)$ and by $f(n)(x) = x$ otherwise. Then f is semi-conjugate both to the identity homomorphism and to the trivial homomorphism which takes every $n \in M(X)$ to the identity element. It follows that $BM(X)$ is contractible.

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