CORRECTIONS AND UPDATES III
(AUGUST 2004 – MAY 2005)

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A first set of "Corrections and updates" has appeared in the 2003 printing of my book [Harpe–00], as well as in the 2003 list of Geneva's preprints [Harpe–03]. A second set has appeared in the 2004 list of Geneva's preprints [Harpe–04].

Here is a third set. We wish to single out a first item in Section 1 below on growth (related to VI.19 in [Harpe–00]) and a second item in Section 2 on fundamental groups of non-positively curved manifolds (see VII.6 and VII.31). Other items, in Section 3, are ordered according to numbers to which they correspond in [Harpe–00].

1. Testing polynomial growth of a finitely generated group on a subsequence of a sphere sequence

Consider a finitely-generated group Γ, a finite set S of generators of Γ, and the resulting word length ℓ_S. For each integer $k \geq 0$, denote by $B(\Gamma, S; k)$ the ball of those $\gamma \in \Gamma$ such that $\ell_S(\gamma) \leq k$. Recall that the *growth function* of the pair (Γ, S) assigns to an integer $k \geq 0$ the size $\beta(\Gamma, S; k)$ of $B(\Gamma, S; k)$, and that the *spherical growth function* is defined by $\sigma(\Gamma, S; k) = \beta(\Gamma, S; k) - \beta(\Gamma, S; k-1)$, with $\sigma(\Gamma, S; 0) = 1$.

Question VI.19 in [Harpe–00] asks wether it is true that an infinite group Γ contains a cyclic group of finite index as soon as there exists an infinite subsequence $(k_i)_{i=0}^{\infty}$ such that $\sup_{i \geq 0} \sigma(\Gamma, S; k_i) < \infty$. A positive answer was given by A. Erschler[1] and recorded in [Harpe–03]. Following again her indications, we show below that, more generally, if $\sigma(\Gamma, S; k)$ is polynomial on an infinite subsequence, then the group Γ itself has polynomial growth. More precisely:

Proposition (A. Erschler). *The notation being as above, assume that there exist an infinite increasing sequence $(k_i)_{i=0}^{\infty}$ and constants $C, d \geq 0$ such that*

$$\sigma(\Gamma, S; k_i) \leq C(k_i)^d \qquad \textit{for all} \quad i \geq 0.$$

Then the group Γ has polynomial growth.

Observe that the conclusion could be stated in a stronger form since, for an infinite group Γ of polynomial growth, there is $d \geq 1$, necessarily an integer, such that the limit $\lim_{k\to\infty} \beta(k)/k^d$ exists ([Pansu–83], and Item VII.33).

[1]The problem has been solved again by Ádám Timár [Timar], together with problems on cutsets in infinite Cayley graphs from a paper by Babson and Benjamini [BabBe–99].

Proof. The two main ingredients of the proof are

(i) the version of Van den Dries and Wilkie of Gromov's theorem on groups of polynomial growth [VdDW–84a],

(ii) an inequality of Coulhon and Saloff-Coste (see [CouSC–93], as well as Section E_+ of Chapter 6 in [GroLP–99]).

We write $B(k)$, $\beta(k)$, $S(k)$, and $\sigma(k)$ for $B(\Gamma, S; k)$, ..., and $\sigma(\Gamma, S; k)$. We proceed by contradiction, assuming that Γ is infinite and not of polynomial growth.

It follows from [VdDW–84a] that, for any pair of constants $C', d' > 0$, we have $\beta(k_i) \geq C'(k_i)^{d'}$ for infinitely many values of i. In particular, we have $\beta(k_i) \geq C(k_i)^{d+2}$, so that

$$(1) \qquad \frac{|S(k_i)|}{|B(k_i)|} \leq \frac{C(k_i)^d}{C(k_i)^{d+2}} = \frac{1}{k_i^2} \qquad \text{for infinitely many values of } i.$$

In particular, $(B(k_i))_{i=0}^{\infty}$ is a Følner sequence, so that Γ is amenable.

Recall that the *Følner functions* $F_{\Gamma,S}$ and $\text{Føl}_{\Gamma,S}$ are defined for $\epsilon > 0$ and $n \geq 1$ by

$$F_{\Gamma,S}(\epsilon) = \min\left\{ N \;\middle|\; A \subset \Gamma, \quad |A| = N, \quad \text{and} \quad |\partial A| \,/\, |A| < \epsilon \right\}$$

$$\text{Føl}_{\Gamma,S}(n) = F_{\Gamma,S}\left(\frac{1}{n}\right).$$

From the inequalities (1) and the definition of the function $\text{Føl}_{\Gamma,S}$, we have

$$(2) \qquad \text{Føl}_{\Gamma,S}(k_i^2) \leq |B(k_i)| = \beta(k_i)$$

for infinitely many values of i.

By [CouSC–93] and the "translation" below, there exist constants $K, K' > 0$ such that

$$(3) \qquad \text{Føl}_{\Gamma,S}(n) \geq K\beta(K'n) \qquad \text{for all} \quad n \geq 1.$$

From (2) and (3), it follows that

$$(4) \qquad \beta(k_i) \geq K\beta(K'k_i^2)$$

for infinitely many values of i.

In any infinite group, there exists for i large enough an arbitrarily large number of disjoint translates of $B(k_i)$ inside the ball $B(K'k_i^2)$. Hence

$$(5) \qquad K\beta(K'k_i^2) \geq 2\beta(k_i)$$

for i large enough. The two previous inequalities lead to a contradiction, and this ends the proof. □

A convenient form of the Coulhon–Saloff-Coste inequality. Let us explain how an inequality established in [CouSC–93] can be translated as (3).

Choose an integer $n \geq 1$. By definition of the function $\mathrm{Føl}_{\Gamma,S}$, there exists a finite subset A of Γ such that

$$\mathrm{Føl}_{\Gamma,S}(n) \,=\, |A| \qquad \text{and} \qquad \frac{|\partial A|}{|A|} \,<\, \frac{1}{n}. \tag{6}$$

The growth function β gives rise to the function Φ defined by

$$\Phi(\lambda) \,=\, \inf\{k \geq 1 \;\mid\; \beta(k) > \lambda\}.$$

Observe that, if $\Phi(\lambda) > k'$, then $\beta(k') \leq \lambda$. Now, by Theorem 1 of [CouSC–93]:

$$\frac{|\partial A|}{|A|} \,\geq\, \frac{1}{4\,|S|\,\Phi(2\,|A|)}.$$

Together with (6), this implies

$$\frac{1}{n} \,>\, \frac{1}{4\,|S|\,\Phi(2\,|A|)}.$$

Hence $\frac{n}{4|S|} < \Phi(2\,|A|)$ and, by the observation above,

$$\beta\left(\frac{n}{4\,|S|}\right) \,<\, 2\,|A| \,=\, 2\mathrm{Føl}_{\Gamma,S}(n),$$

as used in the previous proof. □

2. Growth of the fundamental group of negatively curved (VII.6) and non-negatively curved (VII.31) manifolds

I am grateful to Patrick Ghaanat for indications which made possible the present update. Let M be a connected compact Riemannian manifold and let Γ denote its fundamental group.

Theorem VII.6 is Theorem 2 in [Milno–68]: if M has negative sectional curvature, then Γ has exponential growth. Just after the proof of this theorem, Milnor asks "Perhaps the hypothesis of negative definite mean curvature would already suffice ?" ("mean curvature" is nowadays more often called "Ricci curvature"). The answer is no: any manifold of dimension ≥ 3 (compact or not) admits a complete metric of negative Ricci curvature [Lohka–94]. In particular, spheres of dimensions $n \geq 3$ underly examples of Riemannian manifolds of negative Ricci curvature with $\Gamma = \{1\}$; for $n = 3$, this was known earlier from [GaoYa–86] and [Brook–89].

I wish I had stated more precisely some consequences of Milnor's theorem VII.31: "If M has nonnegative Ricci curvature, then Γ has polynomial growth of degree bounded by the dimension of M".

For example: consider the manifold $M = G/\Gamma$ where $G = \begin{pmatrix} 1 & 0 & 0 \\ \mathbb{R} & 1 & 0 \\ \mathbb{R} & \mathbb{R} & 1 \end{pmatrix}$ is the real Heisenberg group and where $G = \begin{pmatrix} 1 & 0 & 0 \\ \mathbb{Z} & 1 & 0 \\ \mathbb{Z} & \mathbb{Z} & 1 \end{pmatrix}$ is the Heisenberg group of Items VII.21 and VII.22; since the degree of polynomial growth of $\Gamma = \pi_1(M)$, which is 4, is strictly larger than the dimension of M, which is 3, there does not exist any Riemannian metric on M with non-negative Ricci curvature.

Observation. Let M be a compact manifold; assume that the degree of polynomial growth of $\Gamma = \pi_1(M)$ is d; then the degree of polynomial growth of the abelianization $H_1(M, \mathbb{Z})$ of Γ is bounded by d, so that the first Betti number[2] of M is at most d. Thus, if the first Betti number of M is strictly larger than the dimension of M, there cannot exist on M a Riemannian manifold of non-negative Ricci curvature.

Let M be a compact manifold of dimension n which admits a Riemannian metric of nonnegative Ricci curvature. Cheeger and Gromoll [CheGr–71] have shown that the fundamental group $\Gamma = \pi_1(M)$ contains a finite normal subgroup N such that Γ/N is a crystallographic group, namely a cocompact discrete subgroup of the isometry group $\mathbb{R}^d \rtimes O(d)$ of an Euclidean space $\mathbb{R}^d$ with $0 \leq d \leq n$, and thus a group which fits in a short exact sequence

$$0 \longrightarrow \mathbb{Z}^d \longrightarrow \Gamma \longrightarrow F \longrightarrow 1$$

with finite quotient F; in particular, Γ has polynomial growth of degree exactly d. Moreover, if $d = n$, then M is flat.

The following characterization is due to Wilking. For an abstract group Γ, the following conditions are equivalent:

(i) Γ is isomorphic to the fundamental group of a compact manifold of nonnegative Ricci curvature;
(ii) Γ is isomorphic to the fundamental group of a compact manifold of nonnegative sectional curvature;
(iii) there is a normal subgroup of finite index in Γ which is finitely generated free abelian;
(iv) Γ is isomorphic to a discrete cocompact subgroup of a semi-direct product $\mathbb{R}^d \rtimes_\beta F$ where F is a finite group and $\beta : F \longrightarrow GL(d, \mathbb{R})$ a homomorphism.

In [Wilki–00], there are also theorems giving equivalent geometrical and algebraic conditions for finitely generated fundamental groups of complete Riemannian manifolds of nonnegative Ricci curvature, and for virtually polycyclic groups. Several of the other results quoted above carry over in some form to *complete* Riemannian manifolds with non-negative Ricci curvature.

[2]Recall one definition of the first Betti number: it is the dimension of the rational vector space $H_1(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The following is a standard open problem (stated explicitely in [Milno–68]). Let M be a complete Riemannian manifold with non-negative Ricci curvature. Is the fundamental group $\pi_1(M)$ necessarily finitely generated ? Wilking has reduced the problem to manifolds with abelian fundamental groups [Wilki–00].

There is an exposition of some of the relations between the properties of the fundamental group of a Riemannian manifold and appropriate conditions on the curvature, in Chapter 10 of [Eberl–96].

3. Various Items

II.B, and free subgroups of groups of units in tensor algebras.

Let X be a set and $\mathbb{K}$ a commutative ring with unit, and with $1 \neq 0$. Denote by $A(X) = \oplus_{n=0}^{\infty} A^n(X)$ the tensor algebra of the free $\mathbb{K}$-module $\mathbb{K}[X]$, with its standard graduation. The *Magnus algebra of the set* X is the direct product module $\hat{A}(X) = \prod_{n=0}^{\infty} A^n(X)$, with the product topology of the discrete topologies on the $A^n(X)$'s; this has a filtration with order function $\omega : \hat{A}(X) \longrightarrow \mathbb{N} \cup \{\infty\}$ defined by $\omega((a_n)_{n \geq 0}) = m$ if $a_n = 0$ for $n < m$ and $a_n \neq 0$, and $\omega(0) = \infty$. The *Magnus group of the set* X is the group $\Gamma(X)$ of those invertible elements in $\hat{A}(X)$ which are of the form $1 + x$, with $\omega(x) \geq 1$.

The subgroup of $\Gamma(X)$ generated by $(1+x)_{x \in X}$ is isomorphic to the free group $F(X)$ on X (this can also be seen as a subgroup of the group of units in $A(X)$).

Let us assume that $\mathbb{K}$ is a field of characteristic zero; then, similarly, the subgroup of $\Gamma(X)$ generated by $(\exp(x) = 1 + \sum_{n=1}^{\infty} x^n/n!)_{x \in X}$ is isomorphic to $F(X)$.

These are particular cases of Theorem 1 in [Bourb–72, Chap. II, § 5, No 3].

III.5, and infinite finitely generated simple groups with exactly two conjugacy classes.

A very extreme way for a group to be simple is to have exactly two conjugacy classes, namely $\{e\}$ and another one. For a long time, it has been an open problem to know whether there exists such a group which would be finitely generated and not of order 2; this is recorded as Problem 9.10, due to V. Guba, in the 1984 Kourovka Notebook. A positve answer has been announced by [IvaOl–91], but these authors did not provide a full proof, and the problem was later again considered as open.

Denis Osin has shown that such groups do exist; indeed, any countable torsion-free group can be embedded into a 2-generated group with exactly two conjugacy classes (such a group is by necessity torsion-free), and it follows that there exist uncountably many pairsise non-isomorphic torsion-free 2-generated groups with exactly two conjugacy classes [Osin–S].

In the same paper, Osin shows that there exist uncountably many pairwise non-isomorphic 2-generated divisible groups (indeed verbally complete groups).

III.14, and free products with amalgamation.

The construction is interesting outside the world of finitely generated groups. For a field $\mathbb{K}$, the group of automorphisms of the affine plane $\mathbb{K}^2$ viewed as an affine algebraic variety, the so-called *affine Cremona group*, has a decomposition

$$\mathrm{Aut}(\mathbb{K}^2) \,=\, \mathrm{Aff}(\mathbb{K}^2) *_T \mathcal{J}(\mathbb{K}^2)$$

where $\mathrm{Aff}(\mathbb{K}^2)$ is the affine groupe, containing elements $x \longmapsto ax + b$ with $a \in GL_2(\mathbb{K})$ and $b \in \mathbb{K}^2$, where $\mathcal{J}(\mathbb{K}^2)$ is the *de Jonquières group*, consisting of the automorphisms of the form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 + P(x_1) \end{pmatrix}$, with $P \in \mathbb{K}[X]$, and where $T = \mathrm{Aff}(\mathbb{K}^2) \cap \mathcal{J}(\mathbb{K}^2)$ is the lower triangular group of matrices of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. See [Kulk–53] as well as references in [Lamy–02].

Appendix I, Item concerning III.B (Ulam's problem).

Ulam's problem has been solved positively by S. Thomas [Thoma–99]: there exists indeed a non-trivial action of $SO(3)$ on a countable set. More generally, let G be a nontrivial subgroup of $GL_n(\mathbb{K})$ for some field $\mathbb{K}$ of cardinality 2^ω and for some integer $n \geq 1$, where ω denotes the first infinite cardinal. Then there exists a subgroup $H \neq \{1\}$ of G such that the cardinality of G/H is at most ω; in case G is moreover simple, there exists a subgroup H of G such that G/H is precisely of cardinality ω.

Later but independently, similar results have been published in [ErsCh–04]. See also Problem 15.8 in the Kourovka Notebook.

IV.23, on the fundamental observation of Efremovic-Schwarzc-Milnor.

Theorem IV.23 extends readily to proper isometric actions of compactly generated locally compact groups. This has been observed by several people, including the authors of [FarWe].

IV.25.vii, finitely-generated nilpotent groups, and quasi-isometry.

Let Γ, Γ' be two finitely-generated torsion-free nilpotent groups, let N, N' denote their Mal'cev completions, and let $\mathrm{gr}(N), \mathrm{gr}(N')$ denote the corresponding graded nilpotent groups. Pansu has shown that, if the finitely generated groups Γ and Γ' are quasi-isometric, then the graded Lie groups $\mathrm{gr}(N), \mathrm{gr}(N')$ are isomorphic [Pansu–89]. Shalom has shown that the converse does *not* hold [Shalo].

V.10, on the topology on a space of marked groups named "Cayley topology" in [Harpe–00], and various other names including **Chabauty**.

Let X be a topological space. The *Vietoris topology* on the space $K(X)$ of compact subspaces of X has a basis consisting of the sets

$$\{K \in K(X) \;\mid\; K \subset U_0, K \cap U_1 \neq \emptyset, \ldots, K \cap U_n \neq \emptyset\}$$

for $U_0, U_1, \ldots, U_n$ open in X. This topology had been considered by Vietoris (1922), Michael (1951), and others. Some properties: if X is metrisable [respectively separable, completely metrisable, compact metrizable], so is $K(X)$. See [Engel–89], as well as Section 4.F in [Kechr–95].

In case X is a compact metric space, this topology on $K(X)$ is that associated to the *Hausdorff metric*.

Assume now that X is Polish locally compact. The *Fell topology* on the space $F(X)$ of closed subspaces of X has a basis consisting of the sets

$$\{F \in F(X) \mid F \cap K = \emptyset, F \cap U_1 \neq \emptyset, \dots, F \cap U_n \neq \emptyset\}$$

for K compact and $U_1, \dots, U_n$ open in X. (For X compact, this is again the Vietoris topology.) For some properties, see Section 12.C in [Kechr–95].

In case X is a locally compact space, I would like to understand better how this topology fits with the notions of convergence defined by Gromov in Chapter 3 of [GroLP–81].

For a locally compact group G, the Vietoris-Fell topology induces a topology on the space of closed subgroups of G that we like to call the *Chabauty topology* [Chaba–50] ; see the exposition in § 5 of Chapter 8 in [Bourb–63]. In case G is countable and discrete, this topology coincides with the topology induced on subgroups of G by the compact product topology on 2^G.

In particular, for an integer $m \geq 1$, the space $\mathcal{G}_m$ of normal subgroups of the non-abelian free group F_m on a set S_m of m generators, with the Chabauty topology, is a totally disconnected compact metric space. There is a natural bijection between the set of normal subgroups of F_m and the set of *marked groups* with m generators, namely the set of groups given together with an ordered set of m generators (up to isomorphisms of marked groups). Thus $\mathcal{G}_m$ is also known as *the space of marked groups with* m *generators*.

The space of closed subgroups of the locally compact group $\mathbb{R}^2$ has been identifed as a 4-sphere in [HubPo–79].

VI.62, large growth does not imply non-amenability.

Given an integer $n \geq 2$ and $\epsilon > 0$, there exists a group Γ generated by a set S of n generators such that the corresponding exponential growth rate satisfies $\omega(\Gamma, S) > 2n - 1 - \epsilon$ and such that the group Γ is *amenable* [ArGuG]. This answers Problem VI.62. There are examples with Γ an extension of an abelian group by a nilpotent group, and with Γ an extension of a solvable group of solvable length 2 by a finite group.

VII.B, on uniform exponential growth of Burnside groups.

Let $B(m, n)$ be a free Burnside group on $m \geq 2$ generators and of exponent n which is odd and sufficiently large. It is a corollary of a result of D. Osin on "uniform non-amenability" that $B(m, n)$ if of uniformly exponential growth [Osin-U].

VII.29, on Gromov's theorem characterizing groups of polynomial growth.

Let Π be a group (not necessarily finitely-generated). The following properties are equivalent [Wilki–00]:

(i) There is an integer n such that any finitely generated subroup of Π has polynomial growth of order $\leq n$.

(ii) There is a normal subgroup N of Π satisfying
(a) for any finitely generated subgroup Γ of Π, the group $\Gamma \cap N$ is finite;
(b) the factor group Π/N contains a torsion free, nilpotent subgroup of finite index which is an inductive limit of finitely generated, nilpotent groups of fixed rank.

The group N can be chosen as the maximal locally finite normal subgroup of Π.

VII.33, on growth of nilpotent groups.

Let Γ be a finitely generated nilpotent group with polynomial growth of degree d; choose a finite generating set of Γ and let $\beta(\cdot)$ denote the corresponding growth function. It is an unpubished result of F. Grunewald that there exists a constant $C > 0$ such that

$$\beta(k) \,=\, CK^d + O(k^{d-\frac{1}{2}})$$

(cited by Grigorchuk in [Grigo–91]).

VII.34$\frac{1}{2}$, and what is polynomial growth good for. (There is so far one item of what should become a list....)

(i) Let X be a standard Borel space, Γ a countable group acting on X by Borel automorphisms, and E_Γ^X the corresponding equivalence relation.

Let μ be a probability measure on X which is quasi-invariant by Γ. If Γ is amenable, then E_Γ^X is μ-amenable [KecMi–04, Proposition 9.2], and it follows that E_Γ^X is hyperfinite μ-almost everywhere (a result of Connes-Feldman-Weiss [KecMi–04, Theorem 10.1]).

If Γ is finitely generated and has polynomial growth, then E_Γ^X is hyperfinite (a result of Weiss and Jackson-Kechris-Louveau [KecMi–04, Theorem 11.1]). We do not know whether this carries over to amenable groups.

References

ArGuG. G.N. Arzhantseva, V.S. Guba, and L. Guyot, *Growth rates of amenable groups*, Preprint (15 June 2004).

BabBe–99. E. Babson and I. Benjamini, *Cut sets and normed cohomology with applications to percolation*, Proc. Amer. Math. Soc. **127** (1999), 589–597.

Bourb–63. N. Bourbaki, *Intégration, chapitres 7 et 8*, Hermann, 1963.

Bourb–72. N. Bourbaki, *Groupes et algèbres de Lie, chapitres 2 et 3*, Hermann, 1972.

Brook–89. R. Brooks, *A construction of metrics of negative Ricci curvature*, J. Differential Geometry **29** (1989), 85–94.

Chaba–50. C. Chabauty, *Limite d'ensembles et géométrie des nombres*, Bull. Soc. Math. France **78** (1950), 143–151.

CheGr–71. J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), 119–128.

CouSC–93. T. Coulhon and L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés*, Rev. Mathemática Iberoamericana **9** (1993), 293–314.

Eberl–96. P.B. Eberlein, *Geometry of nonpositively curved manifolds*, The University of Chicago Press, 1996.

Engel–89. R. Engelking, *General topology, revised and completed edition*, Heldermann, 1989.

ErsCh–04. Yu.L. Ershov and V.A. Churkin, *One problem of Ulam*, Dokl. Math. **70:3** (2004), 896–898.

FarWe. B. Farb and S. Weinberger, *Isometries, rigidity and universal covers*, Preprint (2005).

GaoYa–86. L.Z. Gao and s.T. Yau, *The existence of negatively Ricci curved metrics on three manifolds*, Invent. Math. **85** (1986), 637–652.

Grigo–91. R.I. Grigorchuk, *On growth in group theory*, Proceedings of the International Congress of Mathematicians, Kyoto, 1990, **I** (Math. Soc. of Japan, 1991), 325–338.

GroLP–81. M. Gromov, J. Lafontaine and P. Pansu, *Structures métriques pour les variétés riemanniennes*, Cedic/F. Nathan, 1981 [see also the next reference].

GroLP–99. M. Gromov, with appendices by M. Katz, P. Pansu, and S. Semmes, *Metric structures for Riemannian and non-Riemannian spaces*, Edited by J. Lafontaine and P. Pansu, English translation by S.M. Bates, Birkhäuser, 1999.

Harpe–00. P. de la Harpe, *Topics in geometric group theory*, The University of Chicago Press, 2000.

Harpe–03. P. de la Harpe, *Corrections and updates to [Harpe–00]*, see the 2002 preprint list in http://www.unige.ch/math/ (1st January, 2003).

Harpe–04. P. de la Harpe, *Corrections and updates (January 2003 – July 2004)*, see the 2004 preprint list in http://www.unige.ch/math/ (1st August, 2004).

HubPo–79. J. Hubbard and I. Pourezza, *The space of closed subgroups of* $\mathbb{R}^2$, Topology **18** (1979), 143–146.

IvaOl–91. S.V. Ivanov and A.Yu Ol'shanskii, *Some applications of graded diagrams in combinatorial group theory*, in "Groups – St. Andrews 1989", Vol. 2, London Math. Soc. Lecture Note Ser. 160 (Cambridge Univ. Press 1991), 258–308.

Kechr–95. A.S. Kechris, *Classical descriptive set theory*, Springer, 1995.

KecMi–04. A.S. Kechris and B.D. Miller, *Topics in orbit equivalence theory*, Springer Lecture Notes in Math. **1852**, 2004.

Kulk–53. W. van der Kulk, *On polynomial rings in two variables*, Nieuw. Arch. Wisk. **1** (1953), 33–41.

Lamy–02. S. Lamy, *Une preuve géométrique du théoème de Jung*, L'Ens. math. **48** (2002), 291–315.

Lohka–94. J. Lohkamp, *Metrics of negative Ricci curvature*, Ann. of Math. **140** (1994), 655–683.

Milno–68. J. Milnor, *A note on curvature and fundamental group*, J.Diff. Geometry **2** (1968), 1–7.

Osin-U. D. Osin, *Uniform non-amenability of free Burnside groups*, Preprint (2004).

Osin-S. S. Osin, *Small cancellations over relatively hyperbolic groups and embedding theorems*, Preprint (2004).

Pansu–83. P. Pansu, *Croissance des boules et des géodésiques fermées dans les nilvariétés*, Ergod. Th. & Dynam. Sys. **3** (1983), 415–445.

Pansu–89. P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Annals of Math. **129** (1989), 1–60.

Shalo. Y. Shalom, *Measurable group theory*, Preprint (2005).

Thoma–99. S. Thomas, *Infinite products of finite simple groups. II*, J. Group Theory **2** (1999), 401–434.

Timar. A. Timár, *Cutsets in infinite graphs*, Preprint.

VdDW–84a. L. van den Dries and A.J. Wilkie, *On Gromov's theorem concerning groups of polynomial growth and elementary logic*, J. of Algebra **89** (1984), 349–374.

Wilki–00. B. Wilking, *On fundamental groups of manifolds of nonnegative curvature*, Differential Geometry and its Applications **13** (2000), 129–165.

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