CORRECTIONS AND UPDATES IV
(SEPTEMBER 2005 – DECEMBER 2006)

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24th of January, 2007

A first set of "Corrections and updates" for my book [Harpe–00] has appeared, in the 2003 printing as well as in the 2003 list of Geneva's preprints [Harpe–03]. A second set [Harpe–04] and a third set [Harpe–05] have appeared later. Here is a fourth set.

II.31, misprint in the exercise.

Read $k, l \geq 1$ instead of $k, l \geq 0$.

II.37, free groups of rotations

Let g be a rotation of the 2–sphere $\mathbf{S}^2$ of angle $\frac{2\pi}{p}$, and let h be a rotation of $\mathbf{S}^2$ of angle $\frac{2\pi}{q}$, where p, q are integers, at least 2; assume that the axis of g and h define an angle of $2\pi\frac{n}{m}$, where $m \geq 3$ and n, m are coprime. The structure of the subgroup $G_{n/m}(p, q)$ of $SO(3)$ generated by g and h has been worked out in [RadSu–99]. For example, if m, n are both odd, then $\langle g, h \rangle$ is isomorphic to the free product $(\mathbf{Z}/p\mathbf{Z}) * (\mathbf{Z}/q\mathbf{Z})$.

II.41, and dense free subgroups of various groups.

See [Bhatt–95], which applies among other situations to profinite completions.
NB.: there are previous updates concerning II.41 in [Harpe–03, 04, 05].

III.4, the group $SL(n, \mathbb{K})$ is either finite or not finitely generated.

For *any* infinite field (the point is that the hypothesis of characteristic zero, stated in III.4, is redundant), the group $SL(n, \mathbb{K})$ is not finitely generated. One of the indicated ingredients, the fact that finitely generated subgroups of $GL(n, \mathbb{K})$ are residually finite, does indeed hold for *any* field; see for example Theorem 7.11, page 143, in the book by Lyndon and Schupp.

V.A.23, on a bound for $H_2(\Gamma, \mathbb{Z})$.

Read $m - (n - \dim_{\mathbb{Q}}(H_1(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}))$ instead of $m - (n - \dim_{\mathbb{Q}}(H_1(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q})$.

III.18.x, on residual finiteness of quotients by finite normal subgroups.

The part starting with "There are examples of a different kind ..." makes no sense in the present Item.

III.24, maximal *discrete* subgroups

There are various results concerning subgroups which are maximal *in a given class* of subgroups. For example, $SL_n(\mathbf{Z})$ is maximal as a *discrete* subgroup of $SL_n(\mathbf{R})$. This is due for $n = 2$ to Hecke (unpublished) and can be found for all n in [Raman–64]. The same result holds for $Sp_n(\mathbf{Z})$ in $Sp_n(\mathbf{R})$, and many other arithmetic subgroups of the classical groups.

III.36, further uses of Neumann's construction.

See [Pyber–03], mainly Section 5. See also [BauMi].

V.25, on an analogue of Higman's universal finitely-presented group for quasi-isometric embeddings, and on the lack of a Schröder–Bernstein theorem for quasi–isometric embeddings.

A question to Denis Osin. Does there exists a finitely-presented group V in which every finitely-presented group can be quasi-isometrically embedded?

Osin's asnwer (November 7, 2005). Yes, such a group does exist. It follows from two theorems proved by Ol'shanskii. These results provide a "quasi-isometric version" of the Higman embedding. The first theorem can be found in [Ol'sh–97] and the second one can be extracted from [Ol'sh–99]; "extracted" means that it does not follow immediately (you have to show that the the group R in Theorem 2 is recursively presented, but this is easy if you just look at the proofs and take into account the fact that P is recursively presented).

Theorem 1. *Let R be a finitely generated recursively presented group with word length l_R on R corresponding to some finite generating set of R. Then there exists a finitely presented group Q, containing R as a subgroup, and a finite generating set of Q inducing the word length l_Q on Q such that $(R, l_R) \subset (Q, l_Q)$ is a quasi–isometric embedding.*

Theorem 2. *Let $G_1, G_2, ...$ be the set of all finitely presented (or even recursively presented) groups. Let $P = G_1 * G_2 * \cdots$ be the free product of all these groups. Then P can be embedded into a 2–generated recursively presented group R such that, for each i, the embedding $G_i \longrightarrow R$ is a quasi–isometry.*

Theorems 1 and 2 show that the answer to the question above is affirmative. Moreover, the groups $V * Z$ and $V \times Z$ show that there cannot exist any Cantor–Bernstein theorem for quasi–isometric embeddings.

VII.B and V.31 as in [Harpe–04], on positive deficiency and uniformly exponential growth.

Jack Oliver Button has pointed out that this Item is not formulated precisely enough, and that it does not give enough credit to the work of Romanovskii, *Free subgroups of fintiely presented groups*, which is an essential ingredient of Wilson's argument.

VII.33, on growth of nilpotent groups.

Let Γ be a finitely generated nilpotent group with polynomial growth of degree d; choose a finite generating set of Γ and let $\beta(\cdot)$ denote the corresponding growth function. We have quoted in [Harpe–05] "an unpublished result" according to which there exists a constant $C > 0$ such that

$$\beta(k) \; = \; Ck^d + O(k^{d-\frac{1}{2}}). \qquad (???)$$

Since there is no proof available to back this claim, it is appropriate to quote an announcement of E. Breuillard which implies that the limit

$$\lim_{k\to\infty} \frac{\beta(k)}{k^d}$$

exists and is strictly positive (at the time of writing, I only know of this announcement by what I've heard from Y. de Cornulier ...!).

VIII.85, solution to the problem.

Yes, the Grigorchuk Γ is a quotient of the fundamental group $\pi_1(\Sigma_2)$ of a surface of genus two.

Indeed, this is an easy exercise, as observed independently by several people, including Yu. S. Semenov and Jack Oliver Button. Consider the presentations

$$\begin{aligned}\pi_1(\Sigma_2) \;&=\; \langle A, B, C, D \mid ABCDA^{-1}B^{-1}C^{-1}D^{-1} = 1\rangle \\ &=\; \langle S, T, U, V \;\mid STS^{-1}T^{-1}UVU^{-1}V^{-1} = 1\rangle.\end{aligned}$$

Let first the generators A, B, C, D be mapped respectively onto the generators a, b, c, d of Γ (Button's solution). Since $ABCDA^{-1}B^{-1}C^{-1}D^{-1}$ is mapped onto $abcdabcd = ad^2ad^2 = 1$, this defines a homomorphism $\pi_1(\Sigma_2) \longrightarrow \Gamma$ which is clearly onto.

Consider then the free product

$$(\mathbf{Z}/2\mathbf{Z}) * \mathbf{V} \;=\; \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1\rangle$$

of a group of order 2 and a Vierergruppe (Semenov's solution). Let the generators S, T, U, V be mapped respectively onto the elements b, da, b, cac of $(\mathbf{Z}/2\mathbf{Z}) * \mathbf{V}$. Since $STS^{-1}T^{-1}UVU^{-1}V^{-1}$ is mapped onto $bdabadbcacbcac = ca^2c = 1$, this defines a homomorphism $\pi_1(\Sigma_2) \longrightarrow (\mathbf{Z}/2\mathbf{Z}) * \mathbf{V}$; its image contains b, $bda = ca$, $cacca = c$ as well as a and d, so that this homomorphism is onto. As Γ is a quotient of $(\mathbf{Z}/2\mathbf{Z}) * \mathbf{V}$, this also solves Problem VIII.85.

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