Erratum to our paper
"A simplification problem in manifold theory"

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September 6 2022

As was pointed out to us by Oscar Randal-Williams, the proof of our Proposition 4.7 in [4] is not correct. As a consequence, this proposition and its corollaries 4.9 and 4.10 are currently unsettled. We recall here the statement of the incriminated proposition.

Proposition 4.7. *(Unsettled) Let M be a smooth connected closed manifold of dimension $n \geq 5$. Let $\sigma \in \mathrm{Wh}(M)$ such that $\sigma = (-1)^n\bar{\sigma}$. Then $\sigma \in I(M)$.*

First of all, the dimension restriction should be $n \geq 6$ (see below). In the proof, the manifold M is decomposed as $M = A \cup B$ ($\partial A = \partial B$), where A is a regular neighborhood of an embedded finite 2-complex K with $\pi_1(K) \approx \pi_1(M)$. Whitehead torsions are measured in $\mathrm{Wh}(\pi_1(K))$. An h-cobordism (W, M, M') with $\tau(W, M) = \sigma$ may thus be constructed as $W = V \cup (B \times I)$, where (V, A, A') is an h-cobordism with $\tau(V, A) = \sigma$.

We then consider an h-cobordism $(T, \partial A, X)$ with $\tau(T, \partial A) = \sigma$. The mistake in our argument occurs with the claim

$$\textit{the condition } \sigma = (-1)^n\bar{\sigma} \textit{ means that } T^{-1} = \bar{T} \quad .$$

This is wrong because T^{-1} is the inverse of T in the cobordism category Cob (see Section 2.2), meaning that $T \circ T^{-1}$ and $T^{-1} \circ T$ are both diffeomorphic to $\partial A \times I$ relative to both ends. But this cannot be controlled by Whitehead torsion alone. All we are allowed to say is

$$\textit{the condition } \sigma = (-1)^n\bar{\sigma} \textit{ means that } T \circ \bar{T} \textit{ is an s-cobordism}$$

Given that, the last paragraph of our tentative proof may be modified to get at least some information on the manifold M'. By the s-cobordism theorem (since $\dim \partial A \geq 5$), there exists a diffeomorphism $\Phi \colon \partial A \times I \xrightarrow{\approx} T \circ \bar{T}$ such that $\Phi(x, 0) = x$, restricting to a diffeomorphism $\varphi \colon \partial A' = \partial A \times \{1\} \xrightarrow{\approx} \partial A$. Form the manifold

$$R = V \cup_{(\partial A \times I)} C_\Phi \,,$$

where C_Φ is the mapping cylinder of Φ (see [4, Section 2.3]). This manifold may be seen as an h-cobordism from $C = A \circ T$ to $C' = A' \cup_\varphi T$, relative boundary.

Using the chain of inclusions $A \subset A \circ T \subset R$, one sees that (R, C, C') is an s-cobordism. Therefore, there exists a diffeomorphisms $C \xrightarrow{\approx} C'$ (rel boundary) which can be extended to a diffeomorphisms $A \circ T \circ T^{-1} \xrightarrow{\approx} A \cup_\varphi T \circ T^{-1}$ (rel boundary). Identifying $T \circ T^{-1}$ with a collar neighborhood inside B, this proves the following

Proposition 4.7. *(New statement) Let M be a smooth connected closed manifold of dimension $n \geq 6$. Let $\sigma \in \mathrm{Wh}(M)$ such that $\sigma = (-1)^n \bar{\sigma}$. Suppose that M is decomposed as $M = A \cup B$ as above. Then there exists a diffeomorphism $\varphi\colon \partial A \xrightarrow{\approx} \partial A$ such that $M' \approx_{\mathrm{diff}} A \cup_\varphi B$.*

Remarks. 1. There are many other possible ways of chosing the decomposition $M = A \cup B$. The essential property is that the inclusion $\partial A = \partial B \subset M$ induces an isomorphism on π_1.

2. Now write $\partial A = \partial B = N$. By the s-cobordism theorem, two trivializations $\Phi_1, \Phi_2\colon N \times I \xrightarrow{\approx} T \circ \bar{T}$ of $T \circ \bar{T}$ differ by a concordance relative to N. Thus, the concordance class of φ in the above proposition depends only on σ. One can see that the correspondence $\sigma \mapsto [\varphi]$ defines a homomorphism

$$\delta_N : S_n(M) = \{\sigma \in Wh(M) | \sigma = (-1)^n \bar{\sigma}\} \to \mathrm{Diff}^c(N)\,,$$

where $\mathrm{Diff}^c(N)$ is the group of diffeomorphisms modulo concordances (δ_N actually coincides with the homomorphism Δ^A_{Diff} introduced in [3, p. 15]). Moreover, one can show that δ_N factors through

$$S_n(N)/\{\tau + (.1)^n \bar{\tau}\} = \hat{H}^n(\mathbb{Z}/2; Wh(N)) \xrightarrow{\rho_N} \mathcal{S}^s(N \times I) \to \mathrm{Diff}^c(N)\,,$$

where ρ_N is the map in the Rothenberg sequence for structure sets (compare [3, Proposition 5.2]). A consequence is the following weaker form of Corollary 4.9 of [4]

Corollary 4.9. *(Weaker) Let M be a smooth connected closed manifold of even dimension $n \geq 6$, and suppose $\pi = \pi_1(M)$ is finite of odd order. Then $S_n(M) \subset I(M)$. If, in addition, M is orientable and π is abelian, then $I(M) = Wh(M)$.*

Proof. In a decomposition $M = A \cup_N B$ as above, the dimension of N is odd. But then $\delta_N = 0$, since ρ_N factors through $L^s_{odd}(\pi) = 0$ [1]. The last assertion follows from the fact that if π is finite abelian, then the involution on the Whitehead group is trivial [2]. □

References

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[4] JC. Hausmann and B. Jahren. A simplification problem in manifold theory. *Enseign. Math.*, 64(1-2):207–248, 2018.