MANIFOLDS WITHOUT MIDDLE DIMENSIONAL HANDLES

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Let M^n be a closed CAT-manifold of dimension n (CAT = PL or DIFF). Consider M as a handlebody : $M = \bigcup_{i=0}^{n}$ (i-handles). The middle dimensional handles are those of index k if $n = 2k$ and those of index k and $k + 1$ if $n = 2k + 1$.

A large amount of work on manifolds (related for instance to surgery) consists of the study of the middle dimensional structure of a manifold, i.e. the way in which middle dimensional handles fit together. This gives rise to invariants like intersection or linking pairings, signature , etc.

In this paper, we take a somewhat complementary approach, in which we study manifolds admitting a handle decomposition without middle dimensional handles. Such manifolds are called (WMDH)-manifolds (the initials WMDH stand for "without middle dimensional handles").

We first introduce some fundamental constructions producing (WMDH)-manifolds. Call a compact manifold V^{n-1} a stable thickening if it admits a handle decomposition with handles of index $\leqslant k - 1$ ($n = 2k$ or $2k + 1$). Let $\beta : \partial(V \times I) \longrightarrow \partial(V \times I)$ be a CAT-homeomorphism. The stable twisted double $STD(V \times I,\beta) = (V \times I) \cup_{\beta} (V \times I)$ is a (WMDH)-manifold. If $\beta \mid V \times 0 \cup \partial V \times I = id$, the manifold $STD(V \times I,\beta)$ is an open book (in the sense of Winkelkemper [W1]) with page V and monodromy $\alpha = \beta | V \times I$. Such a (WMDH)-manifold is called a stable open book and we write it SOB(V,α) . Finally, if $\alpha = \mathrm{id}$, the manifold SOB(V,id) is just the boundary of the "super stable thickening" $V \times D^2$.

Our first results concern the classification up to homeomorphism and up to h-cobordism of the stable open books and of the boundaries of super stable thickenings (these two classes are closed under h-cobordism). The results are analogous to the Tietze theorem for group presentations, i.e. they tell us that SOB(V,α) is homeomorphic (or h-cobordant) to SOB(V',α') if and only if (V',α') can be obtained from (V,α) by some transformations. The author was not able to find such a classification for the class of stable twisted doubles. One may observe that the latter is not closed under h-cobordism (see Theorem (12.7)).

Having classified the stable open books and the boundary of super stable thickenings, we next define invariants for a (WMDH)-manifold M which tell us wether or not M belongs to these simpler classes. For instance, the torsion invariant $t(M) \in H^{n-1}(Z_2; Wh(M))$ which vanishes if and only if M is a stable open book (Wh(M) is the Whitehead group of M). Using this invariant, we construct (WMDH)-manifolds homotopy equivalent to SOB(V,id) which are not stable open books. We also show the existence of a (WMDH)-manifold which is not h-cobordant to a stable twisted double.

Other invariants for a (WMDH)-manifold M are the virtual monodromy and the virtual detached page $\mathscr{V}(M)$, which carry information on the tangential homotopy type of M. The virtual detached page is a homeomorphism class of interiors of stable thickenings (its name comes from the fact that $\mathscr{V}(\mathrm{SOB}(V,\alpha)) = [\mathrm{int}V]$).

In Part IV, we study the set $\mathscr{M}_{CAT}(\mathring{V})$ of h-cobordism classes of (WMDH)-manifold with virtual detached page equal to the open manifold $\mathring{V}$. If V is a stable thickening with $\mathrm{int}V \simeq \mathring{V}$, the set $\mathscr{M}_{CAT}(V)$ is shown to be the quotient of an abelian group $\Gamma_{CAT}(V)$ which is trivial if and only if any (WMDH)-manifold with virtual detached page $\mathring{V}$ is the boundary of a super stable thickening. We then construct a space $\mathscr{C}_{CAT}(V)$ such that $\pi_1(\mathscr{C}_{CAT}(V)) \cong \Gamma_{CAT}(V \times D^1)$. The space $\mathscr{C}_{CAT}(V)$ is involved in a pull-back diagram of fibrations, which permits us to put the group $\Gamma_{CAT}(V)$ in relation with some classical gadgets of surgery, homotopy and Whitehead theory. As an application, we prove that $\Gamma_{CAT}(V)$ is finitely generated when $\pi_1(V)$ is finite. A last section is devoted to some computations of $\Gamma_{CAT}(V)$ is particular cases.

Some ideas of Part IV were already present in the author's thesis [Ha 1]. The theory of (WMDH)-manifolds has had application in knot theory [Ha 2 and 3]. In a subsequent paper, the author will give a solution of the problem of characterizing (WMDH)-manifolds in terms of homotopy invariants (to be a (WMDH)-manifold is a condition on the homotopy type of the manifold; see Proposition (2.3)).

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I. GENERALITIES

1. Notations and definitions :

The abreviation "CAT" stands for PL or DIFF C^∞ . The words "manifold" and "homeomorphism" mean always "CAT-manifold" and "CAT-homeomorphism". The proofs given here usually work for CAT = PL . We leave to the reader the standard arguments to make them valid in the smooth category. The classifying space BCAT is understood as usual as to be BO for CAT = DIFF and BPL otherwise.

If V is a manifold, ∂V denotes its boundary. If a manifold M bounds two manifold A and B, we denote by $A \cup_\partial B$ the manifold constructed by gluing A to B along their common boundary M, with the identity as gluing map.

2. Stable pre-open book decompositions :

A pre-open book decomposition of a closed manifold M^n is a homeomorphism $g : V \times I \cup_\partial W \longrightarrow M$ where V is a compact manifold with non-empty boundary and $(M, V \times 0, V \times 1)$ is an h-cobordism. If V is a stable thickening (i.e. V has a handle decomposition with handles of index $\leqslant k - 1$ ($n = 2k$ or $2k + 1$)) we talk about a stable pre-open book decomposition .

(2.1) Proposition : If $n \geqslant 6$, a (WMDH)-manifold M^n admits a stable pre-open book decomposition.

Proof : Let $Z \subset M$ be the union of the handles of index $\leqslant k - 1$ of a (WMDH)-handle decomposition of M. Thus one has an embedding of a $(k - 1)$-dimensional complex K into Z which is a simple homotopy equivalence. By general position, this embedding is isotopic to an embedding $K \subset \partial Z$. Let V be a regular neighborhood of K in ∂Z. We check easily that $(Z, V, Z - \mathrm{int}V)$ is an s-cobordism. As $n \geqslant 6$, the s-cobordism theorem implies that the pair (Z, V) is homeomorphic to $(V \times I, V \times 0)$.

The manifold $W = M - \mathrm{int}Z$ has also a handle decomposition with handles of index $\leqslant k - 1$ (the dual handle decomposition to the handle decomposition of the pair (M, Z)). Consider the following commutative diagram of inclusions :

$$\begin{array}{ccc} V & \xrightarrow{\ i\ } & W \\ \simeq \downarrow & & \downarrow i_1 \\ Z & \xrightarrow{\ i_2\ } & M \end{array}$$

The homomorphisms $\pi_j(i_1)$ and $\pi_j(i_2)$ are isomorphisms for $j \leqslant n - k \geqslant k$. Thus i is a k-connected map between (k-1)-dimensional complexes, hence a homotopy equivalence. The same argument holds as well for the inclusion $\partial Z - \text{int}V \subset W$. Therefore $(W, V \times 0, V \times 1)$ is an h-cobordism.

Remark : In order to avoid complicate notations, the homeomorphism $g : V \times I \cup_\partial W \longrightarrow M$ will often be ommited in the definition of a pre-open book decomposition and we shall write $M = V \times I \cup_\partial W$. We can do that easily, even when several pre-open book decompositions of the same manifold are considered, as in the following proposition :

(2.2) Proposition : Let $M = (V \times I)\cup_\partial W = (V' \times I)\cup_\partial W'$ be two stable pre-open book decompositions of M^n, $n \geqslant 5$. Then, there exists an ambient isotopy $H_t : M \longrightarrow M$ with $H_0 = \text{id}$, such that :

1) $H_1(V \times 0) \subset \text{int}(V' \times 0)$
2) $H_1(x,t) = (H_1(x,0),t)$ for $(x,t) \in V \times I$.
3) $V' \times 0 - \text{int}(H_1(V \times 0))$ is an h-cobordism from $\partial V' \times 0$ to $H_1(\partial V \times 0)$.

Proof : Consider $V \times 0$ as a regular neighborhood in $\partial(V \times I)$ of a (k-1)-dimensional complex K(dimM = 2k or 2k + 1). By general position in M, the inclusion $K \hookrightarrow M$ is isotopic to an inclusion of K $\text{int}(V' \times I)$. By general position again in $V' \times I$ and in $\partial(V' \times I)$, the inclusion $K \hookrightarrow V' \times I$ is isotopic to an inclusion $K \hookrightarrow \text{int}(V' \times 0)$. Let $U \times 0$ be a regular neighborhood of K in $\text{int}(V' \times 0)$. By extension of isotopies and unicity of regular neighborhood there exists an ambiant isotopy G_t of M with $G_0 = \text{id}$ such that $G_1(V \times I) = U \times I \subset \text{int}V' \times I$. By the same kind of argument, one can assume that $G_1(V \times 0) = U \times 0$. Now, $G_1 | V \times I$ and the embedding $(x,t) \longrightarrow (G_1(x,0)t)$ are two homeomorphisms from $V \times I$ onto $U \times I$ which coincide on $V \times 0$. Therefore, there are connected by an isotopy relative to $V \times 0$, which can be covered by an ambient $\overline{G}_t(\overline{G}_0 = \text{id})$. The ambient isotopy $H_t = \overline{G}_t \circ G_t$ satisfies Properties 1) and 2) by construction.

As in the proof of (2.1), one shows by general position that the inclusion $H_1(V \times 0) \hookrightarrow V' \times 0$ is a homotopy equivalence. By excision, one deduces that $H_1(\partial V \times 0) \hookrightarrow W = V' \times 0 - \text{int}(H_1(V \times 0))$ is also a homotopy equivalence. One checks by general position and Van-Kampen Theorem that the inclusion $\partial V' \times 0 \hookrightarrow W$ induces an isomorphism on the fundamental groups. (One uses here the fact that $n \geqslant 5$). Thus, by Poincaré duality, the map $\partial V' \times 0 \hookrightarrow W$ is also a homotopy equivalence and $(W, H_1(\partial V \times 0), \partial V' \times 0)$ is an h-cobordism.

The existence of a pre-open book decomposition for a (WMDH)-manifold can be used to prove the following result :

(2.3) Proposition : A manifold of dimension $n \geqslant 6$ which is homotopy equivalent to a (WMDH)-manifold is itself a (WMDH)-manifold.

Proof : Let $f : M \longrightarrow M'$ be a homotopy equivalence, where M' is a (WMDH)-manifold. Let $M' = V' \times I \cup_\partial W'$ be a stable pre-open book decomposition for M'. By [Ca, Theorem 1, (ii)], there is a decomposition $M = W_1 \cup_\partial W_2$ such that f is homotopic to a homotopy equivalence of 4-ads $\bar{f} : (M, W_1, W_2, \partial W_1) \longrightarrow (M', V' \times I, W', \partial W')$. Thus the pairs $(W_i, \partial W_i)$ are (n-k)-connected. By the classical procedure for eliminating handles [Ke], W_i can be constructed from ∂W_i by adding handles of index $\geq n - k + 1$. Therefore $W_1 = \bigcup_{i \leq k-1}$ (i-handles). This proves that M is a (WMDH)-manifold.

3. <u>Stable thickening and their doubles</u>:

A compact manifold V^n is a <u>stable thickening</u> if it admits a handle decomposition with handles of index $\leq \frac{m-1}{2}$. It is a super-stable thickening if it admits a handle decomposition with handles of index $\leq \frac{m-3}{2}$.

Let W^n be a compact manifold with non-empty boundary. Let $\beta : \partial W \longrightarrow \partial W$ be a CAT-homeomorphism. We define the <u>twisted-double</u> $TD(W,\beta)$ as the closed CAT-manifold :

$$TD(W,\beta) = W \cup_\beta W, \text{ i.e. : } TD(W,\beta) = W_1 \amalg W_2 / \{\beta(x_2) = x_1 \text{ for } x_1 \in \partial W_1\},$$

where the W_i's are copies of W. When $W = V \times I$ with V a stable thickening, we talk about a <u>stable twisted double</u> and write $STD(W,\beta)$.

(3.1) <u>Lemma</u> : A stable twisted double is a (WMDH)-manifold.

<u>Proof</u> : Construct a handle decomposition for $STD(W^m,\beta)$ by taking a handle decomposition of W_1 of index $\leq \frac{n-2}{2}$ and the corresponding dual handle decomposition for W_2. This gives a (WMDH)-handle decomposition of $STD(W,\beta)$.

Let V be a stable thickening and let $\alpha : V \longrightarrow V$ be a CAT-homeomorphism such that the restriction of α to a neighborhood ∂V is equal to the identity. The <u>stable open book</u> with page V and monodromy α is defined as :

$$SOB(V,\alpha) = STD(V \times I, \beta_\alpha)$$

where $\beta_\alpha : (V \times I) \longrightarrow (V \times I)$ is defined by :

$$\beta_\alpha(x,t) = \begin{cases} (x,t) & \text{if } t \neq 1 \\ (\alpha(x),1) & \text{if } t = 1 . \end{cases}$$

This definition coincides with the usual definition of an open book : $OB(V,\alpha) = M(\alpha) \cup_\partial V \times D^2$ (union over $\partial M(\alpha) = \partial V \times S^1$), where $M(\alpha)$ is the mapping torus of α(see [Ha 4])). The manifold ∂V is the binding of the open book and intV is the detached page. Being a stable twisted double, a stable open book is a (WMDH)-manifold by (3.1).

When $\alpha = id$, one gets $SOB(V,id) = \partial(V \times D^2)$. Observe that $V \times D^2$ is a super stable thickening. Thus, the stable open books with monodromy equal to the identity are exactly the boundaries of super stable thickenings.

(3.2) <u>Remark</u> : A stable open book decomposition $M \cong SOB(V,\alpha)$ of a (WMDH)-manifold M produces a stable pre-open book decomposition $M = V \times I \cup_\partial W$ of M with $(W, V \times 0) \cong (V \times I, V \times 0)$.

II. CLASSIFICATION OF STABLE OPEN BOOKS AND OF BOUNDARIES OF SUPER-STABLE THICKENING

4. Classification of stable thickenings :

Let us consider pairs (K,γ) where K is a finite complex and $\gamma: K \longrightarrow BCAT$ is a continuous map. Two such pairs (K_1,γ_1) are called equivalent if there exists simple homotopy equivalence $g : K_1 \longrightarrow K_2$ such that $\gamma_2 \circ g$ is homotopic to γ_1. One denotes by $\mathcal{T}_{CAT}(m)$ the set of equivalence classes of pairs (K,γ) as above with $\dim K \leqslant \frac{m-1}{2}$.

To a stable thickening V^m, one can associate a pair (K,γ) representing a class in $\mathcal{T}_{CAT}(m)$ in the following way : K is a complex over which V collapses and $\gamma : K \longrightarrow BCAT$ is a characteristic map of the stable tangent bundle. One checks easily that the class of (K,γ) depends only on the CAT-homeomorphism class of V. The classification of CAT-stable thickenings goes as follows :

(4.1) <u>Proposition</u> : If $m \geqslant 6$, the above correspondance is a bijection between the CAT-homeomorphism classes of stable thickenings and the set $\mathcal{T}_{CAT}(m)$.

<u>Proof</u> : This correspondance is surjective by [WA 1, Proposition 5.1]. To prove the injectivity, let V_1 and V_2 be two stable thickening with equivalent characteristic pairs $(K_1,\gamma_1) \sim (K_2,\gamma_2)$. By definition, there exists a simple homotopy equivalence $g : K_1 \longrightarrow K_2$ such that $\gamma_2 \circ g$ is homotopic to γ_1. The composition

$K_1 \xrightarrow{g} K_2 \subset V_2$ is homotopic to an embedding by general position and a regular neighborhood E of K_1 in V_2 is a stable thickening of K_1. As $\gamma_2 \circ g = \gamma_1$, the manifold E is CAT-homeomorphic to V_1 by [Wa 1, Proposition 5.1]. By the excision principle for Whitehead torsion [Co, (23.1)], the cobordism $(V_2 - \text{int}E, \partial E, \partial V_2)$ is an s-cobordism. As $m \geq 6$, it follows from the s-cobordism theorem that V_2 is CAT-homeomorphic to V_1.

The following lemma will be useful. It is an easy consequence of the general position principle and of the s-cobordism theorem.

(4.2) Lemma : A stable thickening V^m of dimension $m \geq 6$ is homeomorphic to a product $V \cong V_0 \times I$.

5. The homomorphism Δ^V_{CAT}

Let V be a CAT-manifold. Denote, as usual, by Wh(V) the Whitehead group of V([Co] or [Mi]) endowed with the involution $\sigma \longmapsto \overline{\sigma}$ depending on the orientation character of V ([Mi, p. 398]. One denotes by $S_m(Wh(V))$ the subgroup of the elements $\sigma \in Wh(V)$ satisfying $\sigma = (-1)^m \overline{\sigma}$.

If $\partial V \neq \emptyset$, define $\text{Aut}_{CAT}(V \text{ rel}\partial)$ as the group of CAT-homeomorphisms $\alpha : V \longrightarrow V$ such that $\alpha\ |(\text{neighborhood of } \partial V) = \text{id}$. The group of concordance classes of elements of $\text{Aut}_{CAT}(V \text{ rel}\partial)$ is denoted by $\text{Aut}^c_{CAT}(V \text{ rel}\partial)$ (concordances relative to V). If $n - 1 = \dim V$, we will define a homomorphism

$$\Delta^V_{CAT} : S_n(Wh(V)) \longrightarrow \text{Aut}^c_{CAT}(V \text{ rel}\partial) .$$

Let W^n be a compact CAT-manifold such that $\partial W = \partial(V \times I)$ and such that $(W, V \times 0, V \times 1)$ is a s-cobordism. Let $\sigma \in S_n(Wh(V))$ and let (R_0, W, W') be an h-cobordism such that $\tau(R_0, W) = i_{V,R_0}(\sigma)$. As $\sigma \in S_n(Wh(V))$ the duality formula [Mi, p. 394] implies that $(W', V \times 0, V \times 1)$ is a s-cobordism (see [Ha 4, proof of (5.3)). The s-cobordism theorem then provides two CAT-homeomorphisms :

$$h : V \times I \longrightarrow W$$
$$g : V \times I \longrightarrow W'$$

such that $h, g|V \times 0 \cup \partial V \times I \cup (\text{neighborhood of } \partial V \times 1) = \text{id}$. Then $g^{-1} \circ h|V \times 1$ determines a class in $\text{Aut}^c_{CAT}(V \text{ rel}\partial)$ which we take as a definition for $\Delta^V_{CAT}(\sigma)$. One checks easily that Δ^V_{CAT} is thus well defined.

(5.1) <u>Lemma</u> : Δ^V_{CAT} is a homomorphism of groups.

<u>Proof</u> : Let U_1 and $U_2 \subset \text{int } V$ be two compact codimension 0 CAT-submanifolds such that $\pi_1(U_i) \longrightarrow \pi(V)$ is an isomorphism and $U_1 \cap U_2 = \emptyset$. If σ_1 and $\sigma_2 \in S_n(Wh(V))$, the h-cobordisms (R^i_0, W, W') and (R_0, W, W') used in the definition of $\Delta^V_{CAT}(\sigma_i)$ and $\Delta^V_{CAT}(\sigma_1 + \sigma_2)$ can be chosen so that :

$$R^i_0 = [W - \text{int } h(U_i \times I)] \times I \cup T_i$$

$$R_0 = [W - \text{int } h((U_1 \cup U_2) \times I)] \times I \cup T_1 \cup T_2$$

($h : V \times I \longrightarrow W$ is a CAT-homeomorphism), where $(T_i, h(U_i \times I), L_i)$ are suitable h-cobordisms. They give rise to CAT-homeomorphisms $g_i : V \times I \longrightarrow W'_i$ and $g : V \times I \longrightarrow W'$ such that $g_i \,|(V - \text{int } U_i) \times I = h$ and $g|V - \text{int}(U_1 \cup U_2) = h$. By our definition of Δ^V_{CAT}, this implies that $\Delta^V_{CAT}(\sigma_1 + \sigma_2)$ is the composition of $\Delta^V_{CAT}(\sigma_1)$ with $\Delta^V_{CAT}(\sigma_2)$.

Besides its simple geometric definition, the homomorphism Δ^V_{CAT} has another description involving surgery ingredients. Let $\mathscr{S}^s_{CAT}(V \times I \text{ rel}\partial)$ be the set of homotopy CAT-structures on $V \times I$ [Wa 2, Chapter 10] , i.e. the set of concordance classes of maps $f : X \longrightarrow V \times I$, where X is a compact CAT-manifold, f is a simple homotopy equivalence and $f|\partial X : \partial X \longrightarrow (V \times I)$ is a CAT-homeomorphism. The Wall surgery obstruction group $L^s_{n+1}(\pi_1(V),\omega)$ ($\omega : \pi_1(V) \longrightarrow \mathbb{Z}_2$ is the orientation character of V) acts on $\mathscr{S}^s_{CAT}(V \times I \text{ rel}\partial)$ [Wa 2, Theorem 10.5] and so we get a map

$$\gamma : L^s_{n+1}(\pi_1(V),\omega) \longrightarrow \mathscr{S}^s_{CAT}(V \times I \text{ rel}\partial)$$

by restricting this action to $id_{V\times I}$. Now define a map $\delta : \mathscr{S}^s_{CAT}(V \times I \text{ rel}\partial) \longrightarrow \text{Aut}^c_{CAT}(V, \text{rel } \partial)$ in the following way. Let $f : (X,\partial X) \longrightarrow (V \times I, \partial(V \times I))$ represent a class in $\mathscr{S}^s_{CAT}(V \times I \text{ rel}\partial)$ thus $f|\partial X : \partial X \longrightarrow \partial(V \times I)$ is a CAT-homeomorphism. The manifold X is then a s-cobordism from $f^{-1}(V \times 0)$ to $f^{-1}(V \times I)$. By the s-cobordism theorem, there exists a CAT-homeomorphism $h : V \times I \longrightarrow X$ such that $h|V \times 0 \cup \partial V \times I \cup \text{neighborhood } (\partial V \times 1) = f^{-1}$. One defines $\delta([f])$ to be the concordance class of $f \circ h|V \times 1$.

Finally, recall that the elements of the form $x + (-1)^n\overline{x}$ constitute a subgroup $S_n(Wh(V))$ and that the quotient $S_n(Wh(V))/\{x + (-1)^n\overline{x}\}$ is isomorphic to $H^n(\mathbb{Z}_2;Wh(V))$. There is a homomorphism $\varepsilon_n : H^n(\mathbb{Z}_2;Wh(V)) = H^{n+2}(\mathbb{Z}_2;Wh(V)) \longrightarrow L^s_{n+1}(\pi_1(V),\omega)$ which takes place in the Rothenberg exact sequence [Sh, §4] .

(5.2) <u>Proposition</u> : The map Δ^V_{CAT} is equal to the composition

$$S_n(Wh(V)) \xrightarrow{\rho_n} H^n(\mathbb{Z}_2;Wh(V)) \xrightarrow{\varepsilon_n} L^s_{n+1}(\pi_1(V),\omega)$$
$$\xrightarrow{\gamma} \mathscr{S}^s_{CAT}(V \times I \text{ rel}\partial) \xrightarrow{\delta} \text{Aut}^c_{CAT}(V \text{ rel}\partial)$$

where ε_n, γ and δ are the maps defined above.

<u>Proof</u> : This follows immediately from our various definitions, using the geometric interpretation of the homomorphism ε_n (see [Sh, p. 315-316] ; for more details, see [Ha 1, § 4.2 and 4.3]).

6. Transformations of open book decompositions by h-cobordisms

First we complete the notations of § 5 : if $\gamma \in Aut_{CAT}(V\ rel\partial)$, we denote by γ^c its class in $Aut^c_{CAT}(V\ rel\partial)$. The Whitehead torsion of a homotopy equivalence $f : X \longrightarrow Y$ is denoted by $\tau(f)$ if it is measured in $Wh(Y)$ or by $\tau^0(f)$ if it is defined as an element of $Wh(X)$ (with the relation $\tau(f) = f_*(\tau^0(f))$. If $X \subset Y$ is an inclusion, we denote by $i_{X,Y} : Wh(X) \longrightarrow Wh(Y)$ the induced homomorphism.

(6.1) Theorem : Let $M^n = OB(V,\alpha)$ with $n \geq 6$. Let $\sigma \in S_n(Wh(V))$ and (R,M,M') be an h-cobordism with $\tau^0(W,M) = i_{V,M}(\sigma)$. Then M' is CAT-homeomorphic to $OB(V,\alpha \circ d)$ where $d \in Aut_{CAT}(V\ rel\ \partial V)$ satisfies $d^c = \Delta^V_{CAT}(\sigma)$.

Proof : Write $M = V \times I \cup_\partial W$, using (3.2). As the h-cobordism R is characterized by its Whitehead torsion, one has $R = V \times I \times I \cup R_0$, where (R_0,W,W') is an h-cobordism with $\tau^0(R_0,W) = i_{V,W}(\sigma)$. As in the definition of Δ^V_{CAT} , one proves that $(W',V \times 0 \times 1,\ V \times 1 \times 1)$ is an s-cobordism, which implies $M' = OB(V,\beta)$, for some $\beta \in Aut_{CAT}(V\ rel\partial\ )$. In fact, one has $\alpha = h|V \times 1$ and $\beta = g|V \times 1$, where h and g are the homeomorphisms arising in the definition of $\Delta^V_{CAT}(\sigma)$ (see, for the details of this kind of consideration, [Ha 4, (2.5)]) . Thus $\beta = \alpha \circ (h^{-1} \circ g|V \times 1)$ and, by definition, $(h^{-1} \circ g)|V \times 1$ represents $\Delta^V_{CAT}(\sigma)$.

(6.2) Remark : In the case $V = V_0 \times I$, the group $Aut^c_{CAT}(V\ rel\partial)$ is abelian (by disjunction of supports), thus the order of α

and d^c in the formula of (6.1) is irrelevant.

Another transformation of open book structure by h-cobordism is given in [Ha 4, (3.1)] . For the convenience of the reader, we recall the statement here (Theorem (6.3) below) :

Let V^n be a CAT-manifold, with $\partial V \neq \emptyset$. Let $\sigma \in Wh(\partial V)$. A σ-enlargement of V is a manifold V_σ such that $V \subset int\ V_\sigma$ and $U = V_\sigma - int\ V$ is an h-cobordism from ∂V to ∂V_σ with $\tau^0(U,\partial V) = \sigma$.

For any σ-enlargement V_σ of V, one has a map $e(V_\sigma) : Aut_{CAT}(V\ rel\partial V) \longrightarrow Aut_{CAT}(V_\sigma\ rel\partial V_\sigma)$ defined by

$$e(V_\sigma)(\alpha) = \begin{cases} \alpha & \text{on } V \\ id & \text{elsewhere .} \end{cases}$$

Clearly $e(V_\sigma)$ is a monomorphism of groups. It induces a homomorphism $e^c(V_\sigma) : Aut^c_{CAT}(V\ rel\partial V) \longrightarrow Aut^c_{CAT}(V_\sigma\ rel\partial V_\sigma)$. As $V_\sigma - V$ is contained in a collar along ∂V_σ , the homomorphism $e^c(V_\sigma)$is an isomorphism.

(6.3) Theorem : Let $M^n = OB(V,\alpha)$ with $n \geq 5$. Let (R,M,M') be an h-cobordism with $\tau^0(R,M) = i_{\partial V,M}(\sigma)$ for $\sigma \in Wh(\partial V)$. Then , for any σ-enlargement V_σ of V, the manifold M' has an open book decomposition $OB(V_\sigma,\ e(V_\sigma)(\alpha)) \xrightarrow{\simeq} M'$.

The comparison between Theorems (6.1) and (6.3) gives the following result :

(6.4) <u>Proposition</u> : Let V^{n-1} be a compact CAT-manifold with $n \geq 7$ and $\partial V \neq \emptyset$. Let $\alpha \in \mathrm{Aut}_{CAT}(V\ \mathrm{rel}\partial V)$ and $\sigma \in Wh(\partial V)$ such that $i_{\partial V,V}(\sigma) \in S_n(Wh(V))$. Let $d(\sigma) \in \mathrm{Aut}_{CAT}(V\ \mathrm{rel}\partial V)$ be a representative of $\Delta^V_{CAT}(i_{\partial V,V}(\sigma))$. Then, for any σ-enlargements V_σ of V, one has the following CAT-homeomorphisms :

1) $OB(V_\sigma,\ e(V_\sigma)(\alpha)) \simeq_{CAT} OB(V, \alpha \circ d(\sigma))$.

2) $OB(V,\alpha) \simeq_{CAT} OB(V_\sigma,\ e(V_\sigma)[\alpha \circ d(\sigma)])$.

Putting α = id, one gets in consequence :

1') $OB(V_\sigma,\ id) \simeq_{CAT} OB(V,\ d(\sigma))$.

2') $OB(V,\ id) \simeq_{CAT} OB(V_\sigma,\ e(V_\sigma)(d(\sigma)))$.

<u>Proof</u> : Let $M = OB(V,\alpha)$ and (R,M,M') be an h-cobordism with $\tau^0(R,M) = i_{\partial V,M}(\sigma)$. The CAT-homeo 1) comes from Theorems (6.1) and (6.3) which give these two open book decompositions of M'. To prove 2), observe that, by the duality formula [Mi, p. 395] , $\tau(R,M') = i_{\partial V,R}(\sigma)$. Then, giving on M' the open book decomposition of (6.1), one gets $M \simeq OB(V_\sigma,\ e(V_\sigma)[\alpha \circ d(\sigma)])$ by Theorem (6.3). This proves 2). If we exchange the roles of (6.1) and (6.3) in this last argument, we get :

2") $OB(V,\alpha) \simeq_{CAT} OB(V_\sigma,\ e(V_\sigma)(\alpha) \circ d_\sigma(\sigma))$

where $d_\sigma(\sigma) \in \mathrm{Aut}_{CAT}(V_\sigma\ \mathrm{rel}\partial V_\sigma)$ is a representative of $\Delta^{V_\sigma}_{CAT}(j_{\partial V,V_\sigma}(\sigma))$. But 2") can also be deduced from the following general relationship between Δ_{CAT} and e( ) which is a direct consequence of the definitions :

(6.5) <u>Lemma</u> : Let V^k be a compact CAT-manifold with $\partial V \neq \emptyset$ and $k \geq 6$. Let $\sigma \in Wh(\partial V)$, $\eta \in S_{k+1}(Wh(V))$ and $\alpha \in \mathrm{Aut}(V\ \mathrm{rel}\partial V)$. Then, for any σ-enlargement V_σ of V, the following equality holds in $\mathrm{Aut}^c_{CAT}(V_\sigma\ \mathrm{rel}\partial V_\sigma)$:

$$e^c(V_\sigma)[\alpha^c \circ \Delta^V_{CAT}(\eta)] = e^c(V_\sigma)(\alpha) \circ \Delta^{V_\sigma}_{CAT}(\eta) .$$

7. Classification of stable open books

(7.1) Theorem : Two stable open books $SOB(V^{n-1},\alpha)$ and $SOB(T^{n-1},\beta)$ are CAT-homeomorphic if and only if there exists a torsion $\sigma \in S_n(Wh(\partial V))$ and a CAT-homeomorphism $h : T \longrightarrow V_\sigma$, where V_σ is a σ-enlargement of V, such that β is CAT-concordant to $h^{-1} \circ [e(V_\sigma)(\alpha \circ d)] \circ h$, for a representative d of $\Delta^V_{CAT}(\sigma)$.

Proof : Suppose that such a torsion σ and such a homeomorphism h exist. Then, $SOB(T,\beta) \cong SOB(V_\sigma, e(V_\sigma)(\alpha \circ d))$ (see [Ha 4, (2.7)]). By Part 2 of Proposition (6.4), one has $SOB(V_\sigma, e(V_\sigma)(\alpha \circ d)) \cong SOB(V,\alpha)$.

Conversely, suppose that there exists a CAT-homeomorphism $g : SOB(T,\beta) \longrightarrow SOB(V,\alpha)$. Write $SOB(T,\beta) = T \times I \cup_\partial W_T$ and $SOB(V,\alpha) = V \times I \cup_\partial W_V$, where $(W_T, T \times 0, T \times I)$ and $(W_V, V \times 0, V \times 1)$ are s-cobordisms. The homeomorphism g produces a stable pre-open book decomposition : $SOB(V,\alpha) \xrightarrow[\cong]{g} T \times I \cup W_T$. By (2.2), there exists an ambient isotopy H_t : of $SOB(V,\alpha)$ with H_0 = id such that :

$$V \times 0 \subset \mathrm{int}H_1(g(T \times 0))$$

$$H_1(g(x,t)) = (H_1(g(x,0),t) \quad \text{if } H_1(G(x,0)) \in V \times 0$$

and $H_1(g(T \times 0))$ is a σ-enlargement of $V \times 0$ for some $\sigma \in Wh(\partial V)$. As W_V and W_T are both s-cobordisms, it follows from Ha 4, (4.3) that $\sigma \in S_n(Wh(\partial V))$. Proposition (6.4) and its proof show then that the monodromy corresponding to the open book decomposition $H_1 g(T \times I)) \cup H_1(g(W_T))$ is concordant to $e(V_\sigma)(\alpha \circ d)$, where d represents $\Delta^V_{CAT}(\sigma)$. Setting $h = H_1 \circ g|T \times 0$, one obtains

Theorem (7.1).

Definition : For $\alpha \in Aut_{CAT}(V\ rel\partial)$, define $\mathrm{int}\alpha = \alpha|\mathrm{int}V$. Denote by $\mathrm{int}\alpha^c$ the concordance class of $\mathrm{int}\alpha$ (concordance with compact support). Finally, we denote by $\mathrm{int}\Delta^V_{CAT}(\sigma)$ the concordance class with compact support of a representative d of $\Delta^V_{CAT}(\sigma)$.

(7.2) Lemma : Let V be a compact manifold of dimension $\geqslant 6$ with $\partial V \neq \emptyset$. Let $\sigma \in Wh(\partial V)$ and $\alpha \in Aut_{CAT}(V\ rel\partial)$. Then, there exists a CAT-homeomorphism $g : \mathrm{int}V_\sigma \longrightarrow \mathrm{int}V$ such that $\mathrm{int}[\,e(V_\sigma)(\alpha)] = g^{-1} \circ \mathrm{int}\alpha \circ g$.

Proof : Let $C \subset V$ be a collar neighborhood of V (i.e. $C \simeq V \times I$) such that $\alpha|C = id$. Express the σ-enlargement V_σ as $V_\sigma = V \cup U$. By [Ke, p. 41], there exists a CAT-homeomorphism $g : \mathrm{int}V_\sigma \longrightarrow \mathrm{int}V$ such that $g|cl(V - C) = id$. Such an homeomorphism g satisfies the requierement of our Lemma (7.2).

An analogous argument, left to the reader, permits us to prove the following lemma :

(7.3) Lemma : Let V and T be two compact manifold of dimension $\geqslant 6$, with non-empty boundary. Let $\alpha \in Aut_{CAT}(V\ rel\partial)$ and $\beta \in Aut_{CAT}(T\ rel\partial)$. Suppose that there exists a CAT-homeomorphism $h : \mathrm{int}T \longrightarrow \mathrm{int}V$ such that $h^{-1} \circ \mathrm{int}\alpha \circ h$ is concordant with compact support to $\mathrm{int}\beta$. Then, there exists $\sigma \in Wh(\partial V)$ and a CAT-homeomorphism $H : V_\sigma \longrightarrow T$ such that

$$e(V_\sigma)(\alpha)^c = (H^{-1} \circ \beta \circ H)^c \text{ in } Aut^c_{CAT}(V_\sigma\ rel\partial).$$

We are now prepared to prove the following criterium for two stable open book to be h-cobordant :

(7.4) <u>Theorem</u> : Two stable open book $SOB(V^{n-1},\alpha)$ and $SOB(T^{n-1},\beta)$ are h-cobordant if and only if there is a homeomorphism $h : intT \longrightarrow intV$ such that $int\beta$ is concordant with compact support to $h^{-1} \circ int\alpha \circ intd \circ h$, where d is a representative $\Delta^V_{CAT}(\sigma)$, for some $\sigma \in S_n(Wh(V))$. $(n \geq 6)$.

<u>Proof</u> : Suppose that $SOB(V,\alpha)$ and $SOB(T,\beta)$ are h-cobordant. By [Ha 4, (3.1)], the manifold $SOB(V,\alpha)$ is CAT-homeomorphic to $SOB(T_\sigma, e(T_\sigma)(\beta))$, for some $\sigma \in Wh(T)$. The condition on $int\alpha$ and $int\beta$ then follows from Theorem (7.1) and Lemma (7.2).

Conversely, suppose that $int\beta$ is concordant with compact support to $h^{-1} \circ int\alpha \circ intd \circ h$. Let $\eta \in Wh(\partial V)$ and $H : V_\eta \longrightarrow T$ be as in Lemma (7.3), so that $e(V_\eta)(\alpha)^c =$ $= (H^{-1} \circ \beta \circ H)^c$. By Theorem (7.1), $SOB(V_\eta, e(V_\eta)(\alpha))$ and $SOB(T,\beta)$ are CAT-homeomorphic and by [Ha 4, (3.1)], $SOB(V_\eta, e(V_\eta)(\alpha))$ is h-cobordant to $SOB(V,\alpha)$.

(7.5) <u>Corollary</u> : A stable open book $SOB(V^{n-1}, \alpha)(n \geq 6)$ is CAT-homeomorphic to the boundary of a super stable thickening if and only if $\alpha^c \in Im\Delta^V_{CAT}$.

<u>Proof</u> : If $SOB(V,\alpha)$ is the boundary of a super stable thickening, then $SOB(V,\alpha)$ is CAT-homeomorphic to $SOB(T,id)$ for some stable thickening T. The condition on α^c then follows from (7.1) or (7.4). On the other hand, if $\alpha^c = \Delta^V_{CAT}(\sigma)$ for some $\sigma \in Wh(V)$, then $SOB(V,\alpha)$ is CAT-homeomorphic to $SOB(T_\sigma, id)$ by Part 2, of (6.4).

8. <u>Classification of boundaries of super stable thickenings</u>.

Recall that a stable thickening T^m is characterized by a pair (K,β), where K is a finite complex of dimension $\leq \frac{m-1}{2}$ over which T collapses and $\beta : K \longrightarrow BCAT$ classifies the stable tangent bundle of T (see § 4) .

(8.1) <u>Proposition</u> : Let $M^n_i = \partial T_i$, where T_i is a super stable thickening with characteristic pair (K_i,β_i) $(n \geq 5,\ i = 1,2)$. Then, M_1 and M_2 are h-cobordant if and only if there exists a homotopy equivalence $f : K_1 \longrightarrow K_2$ such that $\beta_2 \circ f$ is homotopic to β_1 .

<u>Proof</u> : Write $M_i = SOB(V_i, id)$, with $V_i \times I = T_i$. If M_1 and M_2 are h-cobordant, the detached pages V $intV_1$ and $intV_2$ are homeomorphic by (7.4), and any homeomorphism will produce a homotopy equivalence f as required.

Conversely, suppose that there exists a homotopy equivalence $f : K_1 \longrightarrow K_2$ satisfying $\beta_2 \circ f \sim \beta_1$. By [Wa 1, Proposition 5.1], there exists a CAT-embedding $F : T_1 \longrightarrow T_2$ such that $F|K_1$ is homotopic to f. The cobordism $(T_2 - intF(T_1), F(M_1), M_2)$ is then an h-cobordism.

(8.2) <u>Corollary</u> : Two boundaries M^n_i of super stable thickening are h-cobordant if and only if they have the same tangential homotopy type $(n \geq 5)$.

Proof : Two manifolds which are CAT-h-cobordant always have the same CAT-tangential homotopy type. Conversely, let $g : M_1 \longrightarrow M_2$ be a CAT-tangential homotopy equivalence. Write $M_i = (V_i \times D^2)$, where V_i is a stable thickening. Thus $M_i = V \times I \cup V \times I$ and by general position one can assume that $g(V \times I) \subset V \times I$. One checks then that $g|V \times I$ is a stable tangential self-homotopy equivalence of $V \times I$ and thus M_1 is h-cobordant to M_2 by (8.1).

(8.3) Theorem : Let T_i^m be two super stable thickenings ($i = 1,2, m \geq 7$), with characteristic pairs (K_i, β). Then, ∂T_1 is CAT- homeomorphic to ∂T_2 if and only if there exists a homotopy equivalence $f : K_1 \longrightarrow K_2$ with $\beta_2 \circ f \sim \beta_1$ and $\tau(f) \in \mathrm{Ker}\Delta^V_{CAT}$.

Proof : Writing $\partial T_i = SOB(V_i, id)$, this follows easily from (7.1) .

III. INVARIANTS FOR (WMDH)-MANIFOLDS

9. The torsion invariant t(M)

Let M^n be a (WMDH)-manifold of dimension $n \geq 6$. Choose a stable pre-open book decomposition $M = V \times I \cup_\partial W$ for M. Denote by $\tau^M(W, V \times 0)$ the image in Wh(M) of the Whitehead torsion $\tau(W, V \times 0)$. Observe that $\tau^M(W, V \times 0) = (-1)^{n-1} \overline{\tau^M(W, V \times 0)}$. Indeed, $\tau(W, V \times 0) = \tau(W, K)$ where K is a (k-1)-dimensional complex over which V collapses (n = 2k or 2k + 1). By general position, the inclusions $K \hookrightarrow V \times 0$ and $K \hookrightarrow V \times 1$ are homotopic in $\partial(V \times I)$. Therefore, $\tau^M(W, V \times 0) = \tau^M(W, V \times 1)$ and the later is equal to $(-1)^{n-1} \overline{\tau^M(W, V \times 0)}$ by the duality formula [Mi, p. 394]. Thus, $\tau^M(W, V \times 0)$ defines a class in $H^{n-1}(\mathbb{Z}_2; Wh(M)) = \{ a \in Wh(M) \mid a = (-1)^{n-1}\bar{a} \} \ / \ \{ a + (-1)^{n-1}\bar{a} \}$.

(9.1) Lemma-Definition: The class of $\tau^M(W, V \times 0)$ in $H^{n-1}(\mathbb{Z}_2; Wh(M))$ is independent on the choice of the pre-open book decomposition of M. It thus defines an invariant of M which is denoted by $t(M) \in H^{n-1}(\mathbb{Z}_2; Wh(M))$.

Proof : Let $M = V' \times I \cup_\partial W'$ be another stable open book decomposition of M. By Proposition (2.2), we may suppose that $(V \times I, V \times 0) \subset ((\mathrm{int}V') \times I, (\mathrm{int}V') \times 0)$ and that V' is a σ-enlargement of V for some $\sigma \in Wh(\partial V)$. Therefore, $\tau^M(W, V \times 0)$ and $\tau^M(W', V' \times 0)$ are related by the following formula which is proven in [Ha 4, (4.3)] :

$$\tau^M(W', V' \times 0) = \tau^M(W, V \times 0) - [i_{\partial V, M}(\sigma) + (-1)^{n-1} \overline{i_{\partial V, M}(\sigma)}]$$

and thus $\tau^M(W,V \times 0)$ and $\tau^M(W',V' \times 0)$ represent the same class of $H^{n-1}(Z_2;Wh(M))$.

The use of the invariant $t(M)$ appears is the following result :

(9.2) <u>Theorem</u> : Let M^n be a (WMDH)-manifold, $n \geq 6$. Then $t(M) = 0$ if and only if M is homeomorphic to a stable open-book.

<u>Proof</u> : A stable open book admits a stable pre-open book decomposition $V \times I \cup_\partial W$ such that $(W, V \times 0, V \times 1)$ is an s-cobordism. Therefore, $t(M) = 0$ if $M = SOB(V,\alpha)$.

On the other hand, suppose that $t(M) = 0$. Let $M = V \times I \cup_\partial W$ be a stable pre-open book decomposition for M. Then, $\tau^M(W,V \times 0) = \sigma_0 + (-1)^{n-1}\sigma_0$ for some $\sigma_0 \in Wh(M)$. Observe that $\pi_1(\partial V) = \pi_1(M)$, so there exists an element $\sigma \in Wh(\partial V)$ with $i_{\partial V,M}(\sigma) = \sigma_0$. Let V_σ be a σ-enlargement of V.

The manifold M admits a stable pre-open book decomposition $M = V_\sigma \times I \cup_\partial W'$ (see [Ha 4, (4.2)]) and $\tau^M(W',V_\sigma \times 0) = 0$ by [Ha 4, (4.3)]. Thus $(W',V_\sigma \times 0, V_\sigma \times 1)$ is a s-cobordism, and $M = V_\sigma \times I \cup_\partial W'$ is a stable open book decomposition of M ($M = SOB(V_\sigma,\alpha)$ for some $\alpha \in Aut_{CAT}(V_\sigma \text{ rel}\partial)$).

In Section 10, we shall construct examples of (WMDH)-manifolds M with $t(M) \neq 0$. Moreover, these manifolds have the same tangential homotopy type as $SOB(V,id)$. Thus $t(M)$ is not an invariant of the tangential homotopy type. But the following two propositions show that, to some extent, $t(M)$ is as invariant

of the simple homotopy type and of the h-cobordism class of M. In Proposition (9.3) below, $w_1(M) \in H^1(M;Z_2)$ denotes the first Stiefel-Whitney class of M.

(9.3) <u>Proposition</u> : Let $f : M_0^n \longrightarrow M_1^n$ be a simple homotopy equivalence between (WMDH)-manifolds of dimension $n \geq 7$, such that $f^*(w_1(M_1)) = w_1(M_0)$. Then $t(M_1) = f_*(t(M_0))$, where $f_* : H^{n-1}(Z_2;Wh(M_0)) \longrightarrow H^{n-1}(Z_2;Wh(M_1))$ is the isomorphism induced by f.

<u>Remark</u> : Put $t(M_0) = 0$ in (9.3). It follows then from (9.2) that a manifold having the same (simple-and-w_1) homotopy type as a stable open book is a stable open book. This is proven directly in [Ha 4, (2.6) and (2.8)] without the condition on w_1. Observe that the hypothesis on w_1 in (9.3) is necessary to define the homomorphism f_*.

<u>Proof of (9.3)</u> : Let $M_1 = V_1 \times I \cup_\partial W_1$ be a stable pre-open book decomposition. Using [Wa 2, Theorem 12.1], one proves that $M_0 = V_0 \times I \cup_\partial W_0$ and that $f : M_0 \longrightarrow M_1$ is homotopic to a simple homotopy equivalence of 4-ads :

$$f : (M_0, V_0 \times I, W_0, \partial W_0) \longrightarrow (M_1, V_1 \times I, W_1, \partial W_1)$$

and $f|\partial W_0$ is a simple homotopy equivalence of 3-ads :

$$f|\partial W_0 : (\partial W_0, V_0 \times 0, \partial V_0 \times 0) \quad (\partial W_1, V_1 \times 0, \partial V_1 \times 0)$$

(see [Ha 4, proof of (2.6)] for details). Therefore, the equality $\tau^{M_1}(W_1,V_1 \times 0) = Wh(f)(\tau^{M_0}(W_0,V_0 \times 0))$ holds in $Wh(M_1)$, which implies that $f_*(t(M_0)) = t(M_1)$.

(9.4) Proposition : Let (R^{n+1}, M_0, M_1) be an h-cobordism between (WMDH)-manifolds M_1^n, $n \geq 6$. Let $r : R \longrightarrow M_0$ be a retraction by deformation. Then $(r|M_1)_*(t(M_1)) = t(M_0)$, where $(r|M_1)_* : H^{n-1}(Z_2;Wh(M_1))H^{n-1}(Z_2;Wh(M_0))$ is the isomorphism induced by $r|M_1$.

Proof : Let $M_0 = V_0 \times I \cup_\partial W_0$ be a stable pre-open book decomposition. The h-cobordism R is homeomorphic relative to M_0 to $(V_0 \times I \times I) \cup R_0$, where (R_0, W_0, W_1) is an h-cobordism. Also the retraction r can be deformed by a homotopy relative to M_0 so that $r|V_0 \times I \times I$ is the natural projection onto $V_0 \times I$. Call then $r_0 = r|R_0$. The decomposition $M_1 = V_0 \times I \cup_\partial W_1$ is a stable pre-open book decomposition. Thus we have :

$$\tau^{M_0}(W_0, V_0 \times 0) = (r|M_1)_*(\tau^{M_1}(W_1, V_0 \times 0) + i_{W_0,M_0}\tau(r_0|W_1)$$

By the duality formula [Mi, p.394], one has $i_{W_0,M_0}(\tau(r_0M_1)) = -[\sigma + (-1)^{n-1}\sigma]$, where $\sigma = \tau^0(R, M_0)$. Thus $\tau^{M_0}(W_0, V_0 \times 0)$ and $Wh(r\ M_1)(\tau^{M_1}(W_1, V_0 \times 0))$ represent the same class of $H^{n-1}(Z_2;Wh(M_0))$ which proves (9.4).

Remark : Proposition (9.4) in the case $t(M_0) = 0$ implies that a manifold which is h-cobordant to a stable open book is a stable open book. This result is also a direct consequence of (6.3).

10. Realization of elements of $H^{n-1}(Z_2;Wh(G))$ as invariant $t(M)$

Let G be a finitely presented group and $w : G \longrightarrow Z_2$ be a homomorphism. Recall the existence of the Rothenberg exact sequence [Sh] :

$$\cdots \longrightarrow L_n^s(G,w) \longrightarrow L_n^h(G,w) \xrightarrow{\gamma} H^n(Z_2;Wh(G)) \longrightarrow L_{n-1}^s(G,w) \longrightarrow \cdots$$

relating the Wall obstruction groups for surgery to a simple homotopy equivalence $L_n^s(G,w)$ and to a homotopy equivalence $L_n^h(G,w)$.

(10.1) Theorem : Let (G,w) be as above. Let V^{n-1} be a stable thickening $(n \geq 6)$ with an isomorphism $\zeta : \pi_1(V) \xrightarrow{\cong} G$ such that $w \circ \zeta$ is the orientation character of V. Let $\sigma \in H^{n-1}(Z_2;Wh(G))$ be an element in the image of $\gamma : L_{n+1}^h(G,w) \longrightarrow H^{n+1}(Z_2;Wh(G)) = H^{n-1}(Z_2;Wh(G))$. Then there exists a (WMDH)-manifold M^n with a tangential homotopy equivalence $f : M \longrightarrow SOB(V,id)$ such that $\zeta_* \circ f_*(t(M)) = \sigma$ (where $\pi_1(SOB(V,id))$ is identified with $\pi_1(V)$ via the inclusion $V \hookrightarrow SOB(V,id)$).

(10.2) Example : If G is finite abelian of odd order and n is odd, then :

- $H^{n-1}(Z_2;Wh(V)) \neq 0$ [Bs, Theorem 4.1 and Corollary 4.3]
- $: L_{n+1}^h(G) \xrightarrow{\gamma} H^{n-1}(Z_2;Wh(G))$ is onto. Indeed $L_{odd}^s(G) = 0$ [Bk 1].

Thus, this together with (10.1) produces (WMDH)-manifolds M with $t(M) \neq 0$.

Proof of (10.1) : An element $u \in L^h_{n+1}(G,w)$ can be represented by a normal map Φ of triads :

$$\Phi : (R, V \times I, W) \longrightarrow (V \times I \times I,\ V \times I \times 0,\ V \times I \times 1)$$

such that $\Phi|V \times I \cup \partial(V \times I) \times I$ is the natural inclusion and $\Phi|W: W \longrightarrow V \times I \times 1$ is a homotopy equivalence ([Wa 2, Theorem 10.4] or rather its version for homotopy equivalence mentioned in [Sh, p.301]). The definition of a normal map includes the data of a stable trivialization of $TR \oplus \Phi^*(\nu(V \times I \times I))$ where, in our case, $\nu(V \times I \times I)$ is the actual stable normal bundle to $V \times I \times I$, and TR denotes the tangent bundle of R. The element $\gamma(u) \in H^{n-1}(Z_2;Wh(G))$ can be interpreted as the class of the torsion $\tau(\Phi|W)$ [Sh, p. 313]. Now ∂R is a (WMDH)-manifold endowed with a stable pre-open book decomposition $R = V \times I \cup_\partial W$ and $\Phi|\partial R : \partial R \longrightarrow \partial(V \times I \times I) = SOB(V,id)$ is a homotopy equivalence. The fact that Φ is a normal map with respect to the actual stable normal bundle to $V \times I \times I$ implies that $f|\partial R$ is a tangential homotopy equivalence (i.e. $\Phi|\partial R$ can be covered by a map of the stable tangent bundles). The torsion $\Phi_*(\tau(W, V \times 0))$ is equal up to sign to $\tau(\Phi|W)$ and thus these two torsions represent the same class in $H^{n-1}(Z_2;Wh(G))$, the latter being a group of exponent 2. Hence, $t(\zeta_* \circ \Phi_*(t(\Phi|R)) = \gamma(u)$, which proves (10.1).

11. The virtual monodromy and the virtual detached page

Let us consider pairs $(\mathring{V}, \alpha)$ where :

1. $\mathring{V}^{n-1}$ is CAT-homeomorphic to the interior of a stable thickening .
2. $\alpha : \mathring{V} \longrightarrow \mathring{V}$ is a map with compact support (i.e.) there exists a compact $K \subset \mathring{V}$ such that $\alpha|\mathring{V} - K = id$).

We consider two such pairs (V_i, α_i) , $i = 1,2$, as equivalent if there exists a CAT-homeomorphism $\varphi : V_1 \longrightarrow V_2$ such that α_1 and $\varphi^{-1} \circ \alpha_2 \circ \varphi$ are homotopic among the maps with compact supports. An equivalence class of such pairs is called a virtual monodromy .

Let M^n be a (WMDH)-manifold, $n \geq 6$. Using (2.1), choose a stable pre-open book decomposition $M = V \times I \cup_\partial W$ of M. Being an h-cobordism, the cobordism $(W, V \times 0, V \times 1)$ admits a retraction by deformation $r : W \longrightarrow V \times 0$. The equivalence class of the pair $(intV, r')$ is a virtual monodromy, where $r' : intV \longrightarrow intV$ is defined by $r'(x) = r(x,1)$. The fact that r is well defined up to homotopy relative to $V \times 0 \cup_\partial V \times I$ and Proposition (2.2) make the class of $(intV, r')$ independent on the choice of the pre-open book decomposition and of the retraction r. We call this class the virtual monodromy of M and denote it by $\mu(M)$. Observe that $\mu(SOB(V,\alpha)) = [(intV, int\alpha)]$.

(11.1) Proposition : If M_0 and M_1 are two (WMDH)-manifolds of dimension $n \geq 6$ which are h-cobordant, then $\mu(M_0) = \mu(M_1)$.

<u>Proof</u> : Let (R,M_0,M_1) be an h-cobordism and let $M_0 = V_0 \times I \cup W_0$ be a stable pre-open book decomposition of M_0. The cobordism R is homeomorphic to $V \times I \times I \cup R_0$, where (R_0,W_0,W_1) is an h-cobordism. Thus, $M_1 = V_0 \times I \cup_\partial W_1$ is a stable pre-open book decomposition. Any retraction $r_0 : W_0 \longrightarrow V_0 \times 0$ extends to a retraction by deformation $\overline{r} : (R_0,W_1) \longrightarrow (V_0 \times 0 \times I, V_0 \times 0 \times 1)$. One has $\mu(M_0) = [(\text{int}V, r'_0)]$ and $\mu(M_1) = [(\text{int}V, r'_1)]$, where $r'_1 = (r_0|W_1)'$. By construction, r'_0 and r'_1 are homotopic (the h-cobordism R_0 being trivial over the boundary of W_0), which proves our proposition.

The following lemma shows that the virtual monodromy of a (WMDH)-manifold M contains a lot of information on the homotopy type of M.

(11.2) <u>Lemma</u> : Let M^n be a (WMDH)-manifold $(n \geq 6)$ with $\mu(M) = [(V,\alpha)]$. Let V^{n-1} be a(compact) stable thickening whose interior is homeomorphic to $\mathring{V}$. Then there exists a homeomorphism $h : \text{int}V \longrightarrow \mathring{V}$ such that M has the homotopy type of the Poincaré complex $X = (V \times I) \cup_\beta (V \times I)$, where $\beta : \partial(V \times I) \longrightarrow \partial(V \times I)$ is the map defined by

$$\beta(x) = \begin{cases} h^{-1} \circ \alpha \circ h(x) & \text{if } x \in \text{int}(V \times 0) \\ x & \text{oterwise} . \end{cases}$$

(observe that β is continuous since α has a compact support).

<u>Proof</u> : Let $M = V_0 \times I \cup_\partial W_0$ be a pre-open book decomposition of M. The stable thickening V is homeomorphic to a σ-enlargement

of V_0 for some $\sigma \in Wh(\partial V_0)$. Therefore M admits a stable pre-open book decomposition of the form $M = V \times I \cup_\partial W$ [Ha 4, (4.2)]. Let $r : W \longrightarrow V \times 0$ be a retraction by deformation. Choose a height function $s : (W, V \times 0, V \times 1) \longrightarrow (I,0,1)$ such that $s|\partial V \times I$ is the natural projection onto I. Then M has the homotopy type of $Y = (V \times I) \cup_\gamma (V \times I)$, where $\gamma : \partial(V \times I) \longrightarrow \partial(V \times I)$ is equal to r' on $V \times 1$ and to the identity elsewhere. Indeed, a homotopy equivalence $f : M \to Y$ can be defined by :

$$f(x) = \begin{cases} x & \text{if } x \in V \times I \\ (r(x), s(x)) , & \text{if } x \in W . \end{cases}$$

Now, by hypothesis, there exists a homeomorphism $h : \text{int}\,V \longrightarrow V$ such that r' is homotopic (with compact support) to $h^{-1} \circ \alpha \circ h$. This proves our lemma.

(11.3) <u>Proposition</u> : Let M^n be a (WMDH)-manifold, $n \geq 7$. Then M has the homotopy type of the boundary of a super stable thickening if and only if $\mu(M)$ is the class of $(\mathring{T}, \text{id})$ for some manifold $\mathring{T}$ homeomorphic to the interior of a stable thickening.

<u>Proof</u> : The "if" part is implied by Lemma (11.2). Conversely, let $f : M \longrightarrow (V \times I)_1 \cup_\partial (V \times I)_2 \cong \partial(V \times D^2)$ be a homotopy equivalence, where V is a stable thickening. By [Ca, Theorem 1 (ii)] , f can be deformed by a homotopy so that there exists a decomposition $M = W_1 \cup_\partial W_2$ and f is a homotopy equivalence of 4-ads : $(M, W_1, W_2, \partial W_1) \longrightarrow (\partial(V \times D^2), (V \times I)_1, (V \times I)_2, \partial(V \times I)_1)$. One may assume that a collar neighborhood $\partial W_1 \times I \subset W_2$ is a mapped

by f onto $\partial(V \times I)_1$. As $\pi_1(\partial W_i) \cong \pi_1(W_i)$, there is an h-cobordism $L \subset \partial W_i \times I \subset W_2$ over ∂W_i and such that $f|W'_1 : W'_1 \longrightarrow (V \times I)_1$ is a simple homotopy equivalence, where $W'_1 = W_1 \cup L$. We call the homotopy equivalence of 4-ads

$$f : (M, W'_1, W'_2, \partial W'_i) \longrightarrow (\partial(V \times D^2), (V \times I)_1, (V \times I)_2, \partial(V \times I)_i) \quad (\text{with } W'_2 = \overline{W_2 - \text{int}L})$$

a $(V \times I)_1$-simple splitting of f. As $n \geq 7$ and $\partial(V \times I) \cong V \times 0 \cup_\partial V \times 1$, one can apply the same argument to $f|\partial W'_i$ to obtain a $(V \times 0)$-simple splitting

$$f|\partial W'_i : (\partial W'_i, U, U', \partial U) \longrightarrow (\partial(V \times I), V \times 0, V \times I, \partial V \times 0).$$

The inclusion $U \subset W'_i$ is thus a simple homotopy equivalence and hence $W'_i = U \times I$ by the s-cobordism theorem, and $M = U \times I \cup_\partial W'_2$ is a stable pre-open book decomposition of M.

Now, $f|W'_1 = U \times I \longrightarrow V \times I$ can be deformed by a homotopy such that $f(x,t) = (f(x,0),t)$. Let $\rho : W'_2 \longrightarrow U \times 0$ be a retraction by deformation. The following diagram :

$$\begin{array}{ccccccc} & \overset{id}{\nearrow} & U \times 0 & \xrightarrow{\rho} & U \times 0 & \overset{f}{\searrow} & \\ U & & & & & & V \times 0 \\ & \underset{id}{\searrow} & U \times 1 & \xrightarrow{f} & V \times 1 & \underset{id}{\nearrow} & \end{array}$$

commutes up to homotopy. Therefore $\rho|U \times 1 : U \times 1 \longrightarrow U \times 0$ is homotopic to id_U and $\mu(M) = [\text{int}U, id]$.

Let M^n be a (WMDH)-manifold, $n \geq 6$, with $\mu(M) = (V,\alpha)$. The homeomorphism class of the manifold $\mathring{V}$ is called the <u>virtual detached page</u> of the manifold M. It is the homeomorphism class of the interior of a stable thickening, and any such class $[\mathring{V}]$ occurs as the virtual detached page of a (WMDH)-manifold (for instance, of SOB(V,id) for V such that $\text{int}V \simeq \mathring{V}$). By (11.1), the virtual detached page of a (WMDH)-manifold depends only

on its h-cobordism class.

The virtual detached page is an important invariant which permits us to break the classification up to h-cobordism of (WMDH)-manifold into two parts :

1) the classification of virtual detached pages, which is done below.

2) the classification of h-cobordism classes of (WMDH)-manifolds having a given virtual detached page. This is the object of Chapter IV (§ 13 to 19) .

The classification of virtual detached pages reduces to a homotopy-and-bundle problem as follows : let $\mathscr{V}_{CAT}(n)$ be the set of equivalence classes of pairs (K,f), where K is a complex having the homotopy type of a polyedron of dimension $\leq k-1$ ($n = 2k$ or $2k + 1$) and $f : K \longrightarrow BCAT$ is a continuous map ; two pairs (K_i, f_i), $i = 1,2$, are equivalent if there exists a homotopy equivalence $g : K_1 \longrightarrow K_2$ such that $f_2 \circ g$ is homotopic to f_1 .

(11.4) <u>Proposition</u> : For $n \geq 6$, the set of CAT-virtual detached pages of dimension $n - 1$ is in bijection with $\mathscr{V}_{CAT}(n)$.

<u>Proof</u> : Let $\mathring{V}$ represent a virtual detached page of dimension $n - 1$, i.e. $\mathring{V}$ is CAT-homeomorphic to intV where V is a stable thickening. The pair $(\mathring{V},V)$ defines a class in $\mathscr{V}_{CAT}(n)$, where $v : V \longrightarrow BCAT$ characterises the stable tangent bundle of V. This correspondance gives rise to required bijection by Proposition 5.1 of [Wa 1] and its proof.

12. <u>(WMDH)-manifolds which are not stable twisted doubles</u>.

In this section, we show how the torsion invariant $t(M)$ and the virtual detached page $\mathcal{V}(M)$ can be used to decide wether or not a (WMDH)-manifold is homeomorphic or h-cobordant to a stable twisted double.

Let $T^m = V \times I \times I$ be a super stable thickening (i.e. V is a stable thickening) ($m \geq 7$). Let $Aut^c_{CAT}(\partial T)$ be the group of CAT-concordance classes of CAT automorphisms of ∂T. Define a map $\theta : Aut^c_{CAT}(\partial T) \longrightarrow Wh(V)$ in the following way : let $\beta : \partial T \longrightarrow \partial T$ be a homeomorphism. On has $\partial T = SOB(V,id) = (V \times I) \cup W$, with the pair $(W, V \times 0)$ homeomorphic to $(V \times I, V \times 0)$. Thus $\beta : (V \times I) \cup W \longrightarrow \partial T$ is a stable pre-open book decomposition of ∂T. By (2.2), there is an ambient isotopy H_t with $H_0 = id$ so that :

1) $H_1(\beta(V \times 0)) \subset int(V\ 0)$

2) $H_1(\beta(x,t)) = (H_1(\beta(x,0)),t)$ for $(x,t) \in V \times I$

3) $V \times 0$ is a σ-enlargement of $H_1(\beta(V \times I)$ for some $Wh(H_1(\beta(\partial V \times 0))$.

We put $\theta(\beta) = i_{H_1(\beta(\partial V \times I)),V}(\sigma)$. One checks by general position that θ is well defined.

The image of θ can be described in terms of the homomorphism Δ^V_{CAT} of § 5 and of the set $E_{CAT}(V) \subset Wh(V)$ of torsions $\tau(f)$ of tangential self-homotopy equivalences $f : V \longrightarrow V$:

(12.1) <u>Lemma</u> : $Im\ \theta = E_{CAT}(V) \cap Ker\Delta^V_{CAT}$.

<u>Proof</u> : Clearly, $Im\ \theta \subset E_{CAT}(V)$, since the torsions belonging to $Im\ \theta$ are torsions of CAT-embeddings $V \hookrightarrow V$. Let β and σ as above. The stable open book decomposition $(V \times I \cup_\partial W)$ T has its monodromy equal to the identity, as well as the decomposition $T = V \times I \cup_\partial V \times I$. But if we compare these two stable open book decompositions using (6.4)1') and its proof, we get that the monodromy of $T = V \times I \cup V \times I$ is conjugate to $\Delta^{H_1(\beta(V \times 0))}_{CAT}(i_{H_1(\beta(\partial V \times 0)),H_1(\beta(V \times 0)}(\sigma))$. The latter is equal to $\Delta^V_{CAT}(\theta(\beta))$ by Formula (6.5) with $\alpha = id$. Thus $\theta(\beta) \in Ker\Delta^V_{CAT}$.

Now, let $\eta \in Ker\Delta^V_{CAT} \cap E_{CAT}(V)$. Let $f : V \longrightarrow V$ be a CAT-tangential homotopy equivalence with $\tau(f) = \eta$. By general position and [Wa 1, Proposition 5.1], there exists a CAT-embedding $F : V \longrightarrow intV$ homotopic to f. Thus V is a σ-enlargement of $F(V)$ for $\sigma \in Wh(\partial F(V))$ satisfying the equation $i_{F(V),F(V)}(\sigma) = i^{-1}_{F(V),V}(\eta)$. As $\eta \in Ker\Delta^V_{CAT}$ one has $i^{-1}_{F(V),V}(\eta) \in Ker\Delta^{F(V)}_{CAT}$ by (6.5). Therefore, the inclusion $F(V) \subset V$ can be extended in a homeomorphism $G : SOB(F(V);id) \to \longrightarrow SOB(V;id)$ by Proposition (6.4)1') and its proof. Of course the homomorphism $F : V \longrightarrow F(V)$ extends to a homeomorphism $H : SOB(V;id) \longrightarrow SOB(F(V);id)$. Clearly $\theta(G \circ H) = \eta$.

Belonging to $Ker\Delta^V_{CAT} \subset S_{n-1}(Wh(V))$, the torsion $\theta(\beta)$ determines a class in $H^{n-1}(Z_2;Wh(V))$. The following lemma is a direct consequence of the definitions :

(12.2) <u>Lemma</u> : Let $M^n = STD(V \times D^2,\beta)$, where V is a stable thickening ($n \geq 7$). Then $t(M)$ is the image if of $\theta(\beta)$ in

$H^{n-1}(\mathbb{Z}_2;Wh(M))$.

(The sentence "the image of ..." has the following meaning : by definition : $M = STD(V \times I, \ ) =$
$= (V \times I)_1 \amalg (V \times I)_2 \big/ \{\beta(x_2) = x_1, x_1 \in (V \times I)_1\}$;
one takes the image of $\theta(\beta)$ under the homomorphism induced by the inclusions and identification : $V \simeq (V \times 0)_1 \subset M$).

Let M^n be a CAT(WMDH)-manifold. Define $E_M =$
$= S_{n-1}(Wh(M) \cap [\bigcup_V i_{V,M}(E_{CAT}(V))]$ where the union is taken over all the CAT-submanifolds V of codimension 1 such that M admits a stable pre-open book decomposition $M = V \times I \cup W$. Let $\bar{E}_M$ be the image of E_M in $H^{n-1}(\mathbb{Z}_2;Wh(M))$.

(12.3) Theorem : Let M be a (WMDH)-manifold of dimension ≥ 7. Suppose that $\mathscr{V}(M) = [\mathring{V} \times \mathbb{R}]$, where $\mathring{V}$ is homeomorphic to the interior of a stable thickening. Then, M is h-cobordant to a stable twisted double if and only if $t(M) \in \bar{E}_M$.

The proof of (12.3) uses the following lemma :

(12.4) Lemma : Let (R,M,M') be an h-cobordism, with M and M' (WMDH)-manifolds of dimension ≥ 6. Let $r : R \longrightarrow M$ be a retraction by deformation. Then $(r|M')_*(E_{M'}) = E_M$.

Proof : Let $M' = V' \times I \cup_\partial W'$ be a stable pre-open book decomposition of M' . By chosing R trivial over $V' \times I$, one gets a stable pre-open book decomposition $M = V' \times I \cup_\partial W$ of M and r can be deformed by a homotopy so that $r|V' \times I$ is the inclusion of $V' \times I \subset M$. This argument, for any stable pre-open book decomposition of M' shows that $(r|M')_*(E_{M'}) \subset E_M$. The symmetric argument starting from an arbitrary stable pre-open book of M shows that $EM \subset (r|M')_*(E_{M'})$.

Proof of (12.3) : Let $M = STD(T \times I, \beta)$. One has $T = T_0 \times I$, where T_0 is a stable thickening. As $[\text{int}T] = \ (M) = [V \times \mathbb{R}]$ with V the interior of a stable thickening. Thus, by (12.2) and (12.1) one has $t(M) \in \bar{E}_M$. If now M' is h-cobordant to M. by an h-cobordism R having a retraction by deformation $r' : R \longrightarrow M'$, then $t(M') = (r'|M)_*(t(M))$ by (9.4). Thus $t(M') \in \bar{E}_{M'}$, by (12.4).

Conversely , suppose that $t(M) \in \bar{E}_M$. This implies that there exists a stable pre-open book decomposition $M = T \times I \cup_\partial W$ with $\tau^M(W, T \times 0) = \sigma + \nu + (-1)^{n-1}\bar{\nu}$, for $\sigma \in i_{V,M}(E_{CAT}(V)$ and $\nu \in Wh(M)$. As $\sigma = (-1)^{n-1}\bar{\sigma}$, one has $\tau^M(W \times 0) =$
$= -\sigma + (\nu + \sigma) + (-1)^{n-1}(\nu + \bar{\sigma})$. Let (R,M,M') be an h-cobordism trivial over $T \times I$ with source-torsion $\tau^0(R,M) = \sigma + \nu$. Then M' admits a stable pre-open book decomposition $M' = T \times I \cup W'$ and $\tau^{M'}(W', T \times 0) = (r'|M)_*(-\sigma)$, where $r' : R \longrightarrow M'$ is a retraction by deformation (see (9.4) and its proof). Therefore $(\tau^0(W', T \times 0) = i^{-1}_{T\times 0,M}(-\sigma)$. We will show that this implies that W' is homeomorphic to $T \times I$, hence $M' = STD(T \times I, \beta)$ for some $\beta \in Aut_{CAT}(\partial(T \times I))$. One uses again the fact that $T = V \times I$, where V is a stable thickening, which is a consequence of the condition on the virtual detached page.

First, the torsion $\sigma^0 = i^{-1}_{V\times 0,M}(-\sigma)$ belongs to $E^0_{CAT}(V)$, the set of source-torsion $\tau^0(f) \in Wh(V)$ of CAT-tangential homotopy equivalences $f : V \longrightarrow V$. Indeed, $i^{-1}_{V\times 0,M}(\ ) \in E_{CAT}(V)$

by hypothesis and it is proven in [Ha 4, (5.1)] that $E_{CAT}(V) = -E^0_{CAT}(V)$. As $\partial W = \partial(T \times I)$, it follows from the definition of Δ^V_{CAT} that $\Delta^V_{CAT}(\sigma^0) = 0$ (see also (.)) . Thus $\sigma^0 \in E^0_{CAT}(V) \cap \mathrm{Ker}\Delta^V_{CAT}$. Observe that Lemma (12.1) can be rephrazed as follows :

(12.5) <u>Lemma</u> : Let V^{n-1} be a stable thickening, $n \geqslant 5$, and let W^{n+1} be an h-cobordism over $V \times I$. Then W is CAT-homeomorphic to $V \times D^2$ (not relative to $V \times I$) if and only if $\tau^0(W, V \times I) \in E^0_{CAT}(V) \cap \mathrm{Ker}\Delta^V_{CAT}$.

This achieves the proof of (12.3).

(12.6) <u>Remark</u> : The condition on $\mathscr{V}(M)$ in the statement of (12.3) is always satisfied when the dimension of M is odd. If M is of dimension 2k , this condition is equivalent to saying that M admits a handle decomposition whithout handles of index k - 1, k and k + 1 .

Let $\mathring{V}^{n-1}$ be the interior of a stable thickening, with n odd $\geqslant 6$ and $\pi_1(V) = C_p$, the cyclic group of prime order $p \geqslant 5$. Example (10.2) shows the existence of (WMDH)-manifolds M^n with $\mathscr{V}(M) = [\mathring{V}]$ with are not stable open books. The situation for the existence of a stable twisted structure depends more strongly on the virtual detached page. As an example, we give the following result, in which $L_{p,q}$ is the usual notation for the 3-dimensional lens space of type (p,q) [Co, § 27] (one has $\pi_1(L_{p,q}) = C_p$) .

(12.7) <u>Theorem</u> : Let n be an odd integer $\geqslant 9$. Then :

a) Let $\mathring{V}^{n-1}$ be the interior of a regular neighborhood of a 2-dimensional complex in R^{n-1}, with $\pi_1(V) \cong C_p$. Then every (WMDH)-manifold M^n with $\mathscr{V}(M) = [\mathring{V}]$ is CAT-homeomorphic to a stable twisted double.

b) There exists a (WMDH)-manifold M^n with $\mathscr{V}(M) = [L_{5,1} \times R^{n-4}]$ which is not PL-homeomorphic to a stable twisted double. But every (WMDH)-manifold M^n with $\mathscr{V}(M) = [L_{5,1} \times R^{n-4}]$ is h-cobordant to a stable twisted double.

c) A (WMDH)-manifold with M^n with $\mathscr{V}(M) = [L_{5,2} \times R^{n-4}]$ is h-cobordant to a stable twisted double if and only if it is homeomorphic to a stable open book.

(12.8) <u>Remark</u> : Theorem (12.7) c) together with Example (10.2) shows the existence of (WMDH)-manifolds which are not h-cobordant to a stable open twisted double.

<u>Proof of (12.7)</u> : Choose for each case a stable thickening V with $\mathrm{int}V \simeq \mathring{V}$. As V is parallelisable, a CAT-tangential homotopy equivalence of V is just a homotopy equivalence.

In Case a) one uses that $E_{CAT}(V) = Wh(V)$, [D - S, Theorem A] . (The result of [D - S, Theorem A] has been generalized in [Br] to any 2-complex with fundamental group π finited abelian with $SK_1(Z\pi) = 0$; thus, our result holds true in this case as well). Then case a) follows from Lemma (12.5).

In Case b), an example of a (WMDH)-manifold M with $\mathscr{V}(M) = [L_{5,1} \times R^{n-4}]$ which is not homeomorphic to a stable twisted double is given in [Ha 5, Theorem 3]. It is known that $Wh(V) = Wh(C_5) = \mathbb{Z}$ and $\bar{\sigma} = \sigma$ [Bs, 4.1 and 4.2]. Thus $H^{n-1}(\mathbb{Z}_2;Wh(C_5)) = \mathbb{Z}_2$. By [Ha 5, Lemma 3.1], $E_{CAT}(V)$ contains a generator of $Wh(V)$ and thus $\bar{E}_M = H^{n-1}(\mathbb{Z}_2;Wh(M))$ for any (WMDH)-manifold M with $\mathscr{V}(M) = [L_{5,1} \times R^{n-4}]$. Case b) then follows from (12.3).

To prove Case c), let V be a stable thickening with $\mathrm{int}V \simeq \mathring{V}$. There is a homotopy equivalence $f : V \longrightarrow V_0 = L_{5,2} \times D^{n-4}$, and any homotopy equivalence $h : V \to V$ is homotopic to $f \circ g \circ \varphi$, where φ is a homotopy inverse to f and g a self homotopy equivalence of V_0. The homomorphism $Wh(g)$ is either the identity or the multiplication by -1, since $Wh(V_0) \cong \mathbb{Z}$. On other hand, $\tau(g) = 0$ by [Ha 4, proof of (6.5)]. Then $\tau(h) = 0$ or $2 . \tau(f)$, thus $E_{CAT}(V) \subset 2Wh(V)$. As $H^{n-1}(\mathbb{Z}_2;Wh(-))$ is an abelian group of exponent 2, this implies that $\bar{E}_M = 0$ for any (WMDH)-manifolds M^n with $\mathscr{V}(M) = [L_{5,2} \times R^{n-4}]$. Case c) then follows from (12.3) and (9.2).

IV . (WMDH)-MANIFOLDS WITH A GIVEN VIRTUAL DETACHED PAGE

Let $\mathring{V}$ be the interior of a stable thickening of dimension $n - 1 \geq 5$. Let $\mathscr{M}_{CAT}(V)$ be the set of h-cobordism classes of CAT-(WMDH)-manifolds M with $\mathscr{V}(M) = [\mathring{V}]$ (this definition makes sense in view of Proposition (11.1)). The aim of the remaining sections is to determine $\mathscr{M}_{CAT}(V)$. All manifolds are from now on supposed to be orientable.

13. The group $\Gamma_{CAT}(V)$

Let V^{n-1} be an oriented stable thickening with $\mathrm{int}\,V \simeq \mathring{V}$. Let us consider pairs (M,γ) such that M^n is an oriented (WMDH)-manifold and $\gamma : V \times I \longrightarrow M$ is a codimension 0 oriented CAT-embedding which can be extended in a stable pre-open book decomposition $\bar{\gamma} : V \times I \cup_\partial W \xrightarrow{\cong} M$. This implies that $\mathscr{V}(M) = [\mathring{V}]$. Two pairs (M_0,γ_0) and (M_1,γ_1) as above will be considered as h-cobordant if there is an h-cobordism (R,M_0,M_1) and a CAT-embedding $\gamma : V \times I \times I \longrightarrow R$ such that $\gamma^{-1}(M_i) = V \times I \times \{i\}$ and $\gamma| V \times I \times \{i\} = \gamma_i$. The set of h-cobordism classes of such pairs is denoted by $\Gamma_{CAT}(V)$.

An interesting feature of $\Gamma_{CAT}(V)$ is that it admits a composition law, denoted by $\#$. Indeed, if (M_i,γ_i) represent two classes in $\Gamma_{CAT}(V)$, the class of $(M_1,\gamma_1) \# (M_2,\gamma_2)$ is by définition represented by the pair (M,γ) as follows:

$$M = M_1 - \mathrm{int}\gamma_1(V \times [0,\tfrac{1}{2}]) \amalg M_2 - \mathrm{int}\bar{\gamma}_2(V \times [0,\tfrac{1}{2}]) \Big/ \left\{ \begin{array}{l} \gamma_1(z) = \bar{\gamma}_2(z) \\ z \in (V \times [0,\tfrac{1}{2}]) \end{array} \right\}$$

where $\overline{\gamma}_1(x,t) = \gamma_1(x,1-t)$. As γ_1 preserves and $\overline{\gamma}_2$ reverses the orientations, the manifold M admits an orientation compatible with these of $M \cap M_1$. The embedding $\gamma : V \times I \longrightarrow M$ is obtained by :

$$\gamma(x,t) = \begin{cases} \gamma_2(x,t) & \text{if } 0 \leq t \leq 1/2 \\ \gamma_1(x,t) & \text{if } 1/2 \leq t \leq 1 . \end{cases}$$

One checks easily that (M,γ) represents a class in $\Gamma_{CAT}(V)$ depending only on the h-cobordism classes of (M_1,γ_1). Thus $\#$ is well defined.

(13.1) Theorem : If $n \geq 7$, the set $\Gamma_{CAT}(V)$ with the composition law $\#$ is an abelian group.

Proof : The associativity of $\#$ can be checked directly, but this is technically complicate (details like for [Ha 1, Lemma 2.2.3]). However, this point will be clear after Remark (15.6), so we leave it for the moment.

Let us prove that $\#$ is commutative. Let $(M,\gamma) = (M_1,\gamma_1) \# (M_2,\gamma_2)$ and $(M',\gamma') = (M_2,\gamma_2) \# (M_1,\gamma_1)$. Using Lemma (4.2), decompose V into a product $V = V_0 \times I$. The manifold $V_0 \times I \times 0 \cup V_0 \times I \times 1$ is contained in a submanifold of $\partial(V \times I)$ homeomorphic to $V \times S^1$. Any rotation of S^1 can be extended into a homeomorphism of $V_0 \times I \times I = V_0 \times D^2$. This permits us to construct an isotopy $\eta_u : V_0 \times I \times I \longrightarrow V_0 \times I \times I$ $(u \in I)$ such that $\eta_0 = id$ and $\eta_1(x,s,t) = (x,1-s,1-t)$. Thus the class of (M',γ') can be represented by (M'', γ'') where M'' is constructed as in the definition of $\#$, using the embeddings $\gamma_1 \circ \eta_1$ and

$\gamma_2 \circ \eta_1$. Observe that $M'' = M$ and $\gamma'' : V \times I \longrightarrow M$ is equal to $\gamma \circ \eta_1$. Therefore (M',γ') represents the same class of $\Gamma_{CAT}(V)$ then (M,γ).

The existence of a unit element for $\#$ and of inverse is proven in Lemma (12.2) and (12.4) below.

(12.2) Lemma : The class of (M_0,i), where $M_0 = SOB(V,id) = \partial(V \times I \times I)$ and $i : V \times I \xrightarrow{=} V \times I \times 0 \subset \partial(V \times I \times I)$ constitutes the unit element for the law $\#$ in $\Gamma_{CAT}(V)$.

Proof : Let (M,γ) represent a class in $\Gamma_{CAT}(V)$. The super stable thickening $V \times I \times I$ permits us to construct an h-cobordism between $(M,\gamma) \# (M_0,i)$ and (M,γ).

The main consequence of Lemma (12.2) is that the fact that (M,γ) represent the unit element of $\Gamma_{CAT}(V)$ is independent of γ. More precisely :

(12.3) Proposition : A pair (M,γ) represent the unit element of $\Gamma_{CAT}(V)$ if and only if M is the boundary of a super stable thickening.

Proof : If (M,γ) represents the unit element of $\Gamma_{CAT}(V)$, then there exists an h-cobordism (R,M,M_0) where M_0 is the boundary of the super stable thickening $V \times I \times I$ (Lemma 12.2)). Thus M is the boundary of the super stable thickening $V \times I \times I \cup R$.

On the other hand, if $M = \partial T$ with T a super stable thickening, then $\gamma(V \times I) \subset M$ has a collar neighborhood in

T of the form $\gamma(V \times I) \times I$. Pushing this collar neighborhood into the interior of T gives a decomposition $T = V \times I \times I \cup R$, with R an h-cobordism. The pushing permits us also to construct a concordance between the embeddings γ and i . Thus (M,γ) is h-cobordant to (M_0, i) and the result follows from (12.2).

(12.4) <u>Lemma</u> : The class of $(-m,\overline{\gamma})$ is an inverse for the class (M,γ), where $-M$ denotes the manifold M endowed with the opposite orientation.

<u>Proof</u> : Let $(N,\delta) = (M,\gamma) \# (-M,\overline{\gamma})$. Then $N = \partial(W \times I)$, where $W = M - \mathrm{int}\gamma(V \times I)$. As $W \times I$ is a super stable thickening, the result follows from (12.3).

14. <u>Relationship between</u> $\Gamma_{CAT}(V)$ <u>and</u> $\mathcal{M}_{CAT}(\mathring{V})$

In this section we study the obvious forgetfull map $\nu : \Gamma_{CAT}(V) \longrightarrow \mathcal{M}_{CAT}(V)$ which sends the class of (M,γ) onto the h-cobordism class of M.

Let $\mathrm{Emb}_{CAT}(V)$ be the group of concordance classes of orientation preserving CAT-embedding $V \hookrightarrow \mathrm{int}V$ which are homotopy equivalences. The group $\mathrm{Emb}_{CAT}(V)$ acts on the right on $\Gamma_{CAT}(V)$ by the formula $(M,\gamma)f = (M,\gamma \circ (f \times id))$. The fact that $(M,\gamma \circ (f \times id))$ represents a class of $\Gamma_{CAT}(V)$ can be easily shown, like for [Ha 4, (4.2)].

(14.1) <u>Proposition</u> : If $\dim V \geq 6$, the set $\mathcal{M}_{CAT}(\mathring{V})$ is the set of orbits of the above action of $\mathrm{Emb}_{CAT}(V)$ on $\Gamma_{CAT}(V)$.

<u>Proof</u> : The forgetfull map ν factors obviously through a map $\overline{\nu} : \Gamma_{CAT}(V) / \mathrm{Emb}_{CAT}(V) \longrightarrow \mathcal{M}_{CAT}(V)$. The map $\overline{\nu}$ is surjective because ν is surjective, which can be proved as follows : if $[M] \in \mathcal{M}_{CAT}(\mathring{V})$, then M has a stable pre-open book decomposition $M = T \times I \cup W$ with $\mathrm{int}T \simeq \mathring{V}$. Therefore V is a σ-enlargement of T (Lemma (7.3)) and thus M has a stable pre-open decomposition $M = V \times I \cup W'$ by [Ha 4, (4.2)]. (and then in $\mathrm{Im}(\nu)$). Thus, it is enough to prove that two pairs (M,γ) and (M',γ') such that M is h-cobordant to M' are in the same orbit of the action of $\mathrm{Emb}_{CAT}(V)$. As $\dim M \geq 5$, any h-cobordism from M to M' can be viewed as trivial over $\gamma(V \times I)$. Then, one can assume without loss of generality

that $M = M'$. The assertion then follows from Proposition (2.2).

An interesting feature about the action of $Emb_{CAT}(V)$ on $\Gamma_{CAT}(V)$ is the following one :

(14.2) Proposition : If $\dim V \geq 6$, the group $Emb_{CAT}(V)$ acts on $\Gamma_{CAT}(V)$ by automorphisms of groups, i.e. $(a \# b)f = af \# bf$ for $a,b \in \Gamma_{CAT}(V)$ and $f \in Emb_{CAT}(V)$.

Proof : Let $f : V \hookrightarrow \text{int}T$ represent a class in $Emb_{CAT}(V)$. The stable thickening V is then an enlargement of f(V), i.e. $V = f(V) \cup U$, where $(U, \partial(f(V)), \partial V)$ is an h-cobordism.

Let (M_i, γ_i) , $i = 1,2$, be two pairs representing classes in $\Gamma_{CAT}(V)$ and write :

$$(M, \gamma) = (M_1, \gamma_1) \# (M_2, \gamma_2)$$

$$(M'_1, \gamma'_1) = (M_1, \gamma_1 \circ (f \times id)) \# (M_2, \gamma_2 \circ (f \times id))$$

for the representatives of the sum classes constructed as in the definition of the law $\#$.

Let L be a tubular neighborhood of $\gamma(\partial V \times I)$ in M (thus $L \cong \partial V \times I \times [-1,1]$) . By construction, one has : $M' = (M - \text{int}L) \cup_\partial T$, where T is obtained by gluing together two copies of $U \times I$ along $\partial(f(V)) \times I$. Observe that $T \cup_\partial L \cong \partial(U \times D^2)$ and both inclusions $T \subset U \times D^2$ and $L \subset U \times D^2$ are homotopy equivalences. This permits us to construct an h-cobordism from M to M' which is trivial over $M - \text{int}L$. Notice that the maps γ' and $\gamma \circ (f \times id)$ both factor through the same map from $V \times I$ into $M - \text{int}L = M' - \text{int}T$.

This proves Proposition (14.2).

(14.3) Corollary : If $\Gamma_{CAT}(V) \neq 0$, then $\mathscr{M}_{CAT}(V)$ contains more then one elements.

Proof : By (14.2) $0 \in \Gamma_{CAT}(V)$ is alone in its orbit. Thus, Corollary (14.3) follows from (14.1) .

(14.4) Corollary : If $\Gamma_{CAT}(V) = \mathbb{Z} \oplus A$, then $\mathscr{M}_{CAT}(V)$ is an infinite set.

Proof : By (14.2), the elements $(p,0) \in \mathbb{Z} \oplus A$, for p ranging among the positive prime integers, are all distinct. Corollary (14.4) then follows from (14.1).

15. The space $\mathscr{C}_{CAT}(V)$

In this section we construct a Kan Δ-set $\mathscr{C}_{CAT}(V)$ for every stable thickening V such that $\pi_i(\mathscr{C}_{CAT}(V)) = \Gamma_{CAT}(V \times D^i)$. Recall that a Δ-set is a semi-simplicial complex without degeneracies [R - S].

The notation Δ^k stands for any copy of the standard euclidean k-simplex in R^{k+1} with vertices $v_i = (0,\dots,0,1,0,\dots 0)$ with 1 occuring in the $(i + 1)$th place. A face of Δ^k is a simplex spanned by a proper subset of the v_i's. The i-th face Δ_i^{k-1} of Δ^k is the simplex spanned by v_j for $j \neq i$.

The word "manifold" in this section and in the next one means always "sub-manifold of R^∞" (otherwise we shall say "abstract manifold"). This precaution will insure us that the classes of k-simplexes in the definitions of our Δ-sets are actually sets. Also, if CAT = DIFF, it is suitable to consider manifolds with corners, like in [B - L - R].

(15.1) <u>Definition</u> : A k-simplex of $\mathscr{C}_{CAT}(V)$ is a 4-uple (W,N,f,α), where :

1) $(W, V \times \Delta^k, N)$ is an h-cobordism, possibly non-trivial on the boundary (but $\partial(W - \mathrm{int}[V \times \Delta^k] \cup N]$, $\partial(V \times \Delta^k), \partial N)$ is an h-cobordism).

2) $f : (W, V \times \Delta^k, N) \longrightarrow (V \times \Delta^k \times I, V \times \Delta^k \times 0, V \times \Delta^k \times 1)$ is a continuous map such that :

2a) $f|V \times \Delta^k = id$ and $f^{-1}(\partial(V \times \Delta^k \times I)) = \partial W$

2b) $f|f^{-1}(\partial V \times \Delta^k \times I)$ is a CAT-homeomorphism from $f^{-1}(\partial V \times \Delta^k \times I)$ onto $\partial V \times \Delta^k \times I$. Thus $f|f^{-1}(\partial V \times \Delta^k \times I) \cup V \times \Delta^k$ has an inverse $\varphi : \partial V \times \Delta^k \times I \cup V \times \Delta^k \times 0 \longrightarrow f^{-1}(\partial V \times \Delta^k \times I) \cup V \times \Delta^k$.

2c) For any face $\sigma < \Delta^k$, one has $f^{-1}(V \times \sigma \times I)$ is an h-cobordism W_σ from $V \times \sigma$ to N_σ (same condition as in 1) for the boundary) and $f_\sigma = f|W$ satisfies Conditions 2a) and 2b) in which Δ^k is replaced by σ and (W,N) by $(W\sigma, N\sigma)$.

3) A CAT-embedding $\alpha : V \times \Delta^k \times 0 \cup_\partial V \times \Delta^k \times I \cup V \times \Delta^k \times 1 \hookrightarrow \partial W$ such that :

3a) $\alpha\,|V \times \Delta^k \times 0 \cup \partial V \times \Delta^k \times I = \varphi$.

3b) $\alpha(V \times \Delta^k \times 1) = N$.

3c) For any face $\sigma < \Delta^k$, the CAT-embedding $\alpha_\sigma = \alpha|V \times \sigma \times 0 \cup \partial V \times \sigma \times I \cup V \times \sigma \times 1$ has its image in ∂W_σ and satisfies Condition 3a) and 3b) in which Δ^k is replaced by σ and (W,N) by (W_σ, N_σ).

The i-th face operator is defined by $\partial_i(W,N,f,\alpha) = (W_\sigma, N_\sigma, f_\sigma, \alpha_\sigma)$ where $\sigma = \Delta_i^{k-1}$. Thus $\mathscr{C}_{CAT}(V)$ is a Δ-set base point $*_k = (V \times \Delta^k \times I, V \times \Delta^k \times 1, id, id)$.

(15.2) <u>Lemma</u> : $\mathscr{C}_{CAT}(V)$ is a KanΔ-set.

<u>Proof</u> : Let $\Lambda_i = \bigcup_{j \neq i} \Delta_j^{k-1}$ be the i-th horn of Δ^k, viewed either as a subspace of Δ^k or as a Δ-set, depending on the context. Let $\mu : \Lambda_i \longrightarrow \mathscr{C}_{CAT}(V)$ be a Δ-map. We have to prove that Imμ is the sub-Δ-set of $\mathscr{C}_{CAT}(V)$ generated by $\{\partial_j\theta | i \neq j\}$ for a k-simplex θ of $\mathscr{C}_{CAT}(V)$. The union of

simplices of $\mathrm{Im}\mu$ constitutes a 4-uple $(W_1, N_1, f_1, \alpha_1)$, where $(W_1, V \times \Lambda_1, N_1)$ is an h-cobordism, $f_1 : (W_1, V \times \Lambda_1, N_1) \longrightarrow$
$\longrightarrow (V \times \Lambda_1 \times I, V \times \Lambda_1 \times 0, V \times \Lambda_1 \times 1)$ and
$\alpha_1 : (V \times \Lambda_1 \times 0 \cup \partial V \times \Lambda_1 \times I \cup V \times \Lambda_1 \times 1) \longrightarrow \partial W_1$
satisfy conditions analogous to conditions 1), 2) and 3) of Definition (15.1). In particular, $f^{-1}(V \times \partial\Lambda_1 \times I)$ is an h-cobordism W_∂ (non-trivial over the boundary) from $V \times \partial\Lambda_1$ to $N_\partial \subset \partial N$.

The natural inclusion $\Lambda_1 \subset \Delta^k$ can be extended to a homeomorphism h :

$$h : \Lambda_1 \times I \Big/ \{(\lambda, t) = (\lambda, 0) \,|\, \lambda \in \partial\Lambda_1\} \xrightarrow{\simeq} \Delta^k.$$

The required simplex $\theta = (W, N, f, \alpha)$ can be defined by

$$W = W_1 \times I \;\cup_h\; V \times \Delta^k$$

f is the composition $W = W_1 \times I \xrightarrow{f_1 \times id} (V \times \Lambda_1 \times I) \times I \xrightarrow{\zeta}$
$\longrightarrow V \times \Delta^k \times I$ where $\zeta\,((x, \lambda, t)s) = (x, h(\lambda, s), t)$ and
$\alpha = (\alpha_1 \times id) \circ \zeta^{-1}$.

(15.3) <u>Definition</u> : A Δ-set A is a structured Δ-set if, for each k, a composition law is defined on the set of k-simplices such that all the face operators are homomorphisms.

(15.4) <u>Lemma</u> : If $\dim V \geq 6$, the Δ-set $\mathscr{C}_{CAT}(V)$ is a structured Δ-set ; the induced composition law on $\pi_0(\mathscr{C}_{CAT}(V))$ is associative.

<u>Proof</u> : Let $a = (W_a, N_a, f_a, \alpha_a)$ and $b = (W_b, N_b, f_b, \alpha_b)$ be two k-simplices of $\mathscr{C}_{CAT}(V)$. Fix a decomposition $V = V_0 \times I$, using Lemma (4.2). Take $(V_0 \times I)_i$, $i = 1,2$, to be two copies of $V_0 \times I$. Define

$$S = (V_0 \times I)_1 \amalg (V_0 \times I)_2 \Big/ \{ (x,1)_1 = (x,0)_2,\ x \in V_0 \}$$

Choose a homeomorphism $\zeta : S \longrightarrow V_0 \times I$ such that $((x,0)_1) = (x,0)$ and $\zeta\,((x,1)_2) = (x,1)$. Denote by M_ζ the mapping cylinder of ζ . Define

$$W_0 = W_a \amalg W_b \Bigg/ \left\{ \begin{array}{l} \alpha_a(x,1,t,s) = \alpha_b(x,0,t,s)\ ; \\ \quad (x,1,t,s) \in V_0 \times 1 \times \Delta^k \times I \\ \qquad\qquad \subset \partial V \times \Delta^k \times I \\ \quad (x,1,t,s) \in V_0 \times 0 \times \Delta^k \times I \\ \qquad\qquad \subset \partial V \times \Delta^k \times I \end{array} \right\}$$

Then W_0 in an h-cobordism from S to $N = N_a \amalg N_b \Big/ \{ \alpha_a(x,1,t,s) = \alpha_b(x,0,t,s) \}$. Define $W = W_0 \cup_S M_\zeta \times \Delta^k$. There is an obvious homeomorphism $h : S \times \Delta^k \times I \cup_{S \times \Delta^k \times 0} M_\zeta \times \Delta^k \longrightarrow V \times \Delta^k \times I$. Define $f : W \longrightarrow V \times \Delta^k \times I$ as $f : h \circ (f_a \cup f_b \cup id_{M_\zeta})$ and

$$\alpha = (\alpha_a \cup \alpha_b \cup id_{M_\zeta}) \circ h^{-1} \,\Big|\, (V \times \Delta^k \times 0 \cup \partial V \times \Delta^k \times I \cup V \times \Delta^k \times 1) .$$

The 4-uple (W, N, f, α) is by definition the k-simplex ab. One checks that the face operators are homomorphisms.

Finally, if a,b,c are 0-simplices of $\mathscr{C}_{CAT}(V)$ the difference between (ab)c and (a(bc) is just a question of parametrisation of the factor I of $V_0 \times I$. Therefore one can easily construct a 1-simplex u of $\mathscr{C}_{CAT}(V)$ with $\partial_0(u) = (ab)c$ and $\partial_1(u) = a(bc)$. Thus the induced law on $\pi_0(\mathscr{C}_{CAT}(V))$ is associative.

(15.5) <u>Theorem</u> : Let V be a stable thickening of dimension ≥ 6. Then there is a bijection $\eta_k : \pi_k(\mathscr{C}_{CAT}(V)) \longrightarrow \Gamma_{CAT}(V \times D^k)$ which is an isomorphism of groups when $k \geq 1$. The bijection η_0 satisfies $\eta_0(ab) = \eta_0(a) \# \eta_0(b)$, where ab denotes the composition of a and b by the law of (15.4).

<u>Proof</u> : As $\mathscr{C}_{CAT}(V)$ is a Kan Δ-set, one can use the combinatorial definition of $\pi_k(\mathscr{C}_{CAT}(V))$ [Cu, § 2]. Recall that the elements of $\pi_k(\mathscr{C}_{CAT}(V))$ are thus equivalence classes of k-simplices σ such that $\partial_i(\sigma) = *_{k-1}$, where two such σ_1 and σ_2 are equivalent if there exists a $(k + 1)$-simplex a such that $\partial_0(a) = \sigma_1$, $\partial_1(a) = \sigma_2$ and $\partial_i(a) = *_k$ for $i \geq 2$.

Let $\sigma = (W,N,f,\alpha)$ be a k-simplex of $\mathscr{C}_{CAT}(V)$ with $\partial_i(\sigma) = *_{k-1}$. This implies that α can be extended in a CAT-homeomorphism $\bar{\alpha} : \partial(V \times \Delta^k \times I) \xrightarrow{\cong} \partial W$. ($\bar{\alpha}$ is defined to be the identity out of the domain of α). Construct à (WMDH)-manifold M by gluing W to $V \times \Delta^k \times I$ along their boundary, using $\bar{\alpha}$. There is an obvious inclusion $\gamma : V \times D^k \times I = V \times \Delta^k \times I \hookrightarrow M$. One checks and the pair (M,γ) represents an element of $\Gamma_{CAT}(V \times D^k)$. This gives rise to a well defined map

$$\eta_k : \pi_k(\mathscr{C}_{CAT}(V)) \longrightarrow \Gamma_{CAT}(V \times D^k) .$$

Let (M,γ) represent an element of $\Gamma_{CAT}(V \times \Delta^k) \cong \Gamma_{CAT}(V \times D^k)$. Call $W = M - \text{int}\gamma(V \times \Delta^k \times I)$. As $\dim V + k \geq 6$, the classification of h-cobordisms by their Whitehead torsion shows that there is an h-cobordism $(W_0, V \times \Delta^k, N)$ and a CAT-homeomorphism $h : (W_0, V \times \Delta^k \cup \partial(V \times \Delta^k) \times I) \xrightarrow{\cong} (W, \gamma(V \times \Delta^k \times 0 \cup \partial(V \times \Delta^k) \times I))$ such that $h|V \times \Delta^k \cup \partial(V \times \Delta^k) \times I) = \gamma$. Now the map $\gamma^{-1} \circ h \;|V \times \Delta^k \cup \partial(V \times \Delta^k) \times I$ can be extended to a map $f : W_0 \longrightarrow V \times \Delta^k \times I$ with $f^{-1}(V \times \Delta^k \times 1) = N$. Thus, the 4-uple $(W_0, N, f, h^{-1} \circ \gamma \mid V \times \Delta^k \times 0 \cup \partial V \times \Delta^k \times I \cup V \times \Delta^k \times 1)$ represents an element $\theta \in \pi_k(\mathscr{C}_{CAT}(V))$ and $\eta_k(\theta) = [(M,\gamma)]$. Therefore η_k is surjective. An analogous argument shows that if $\eta_k(\theta) = \eta_k(\theta')$, then θ is equivalent to θ'. Hence η_k is bijective.

It now we prove that $\eta_k(\theta_1 + \theta_2) = \eta_k(\theta_1) \# \eta_k(\theta_2)$, for $k \geq 1$. Choose a homeomorphism $\Delta^k = \Delta_0^{k-1} \times I$. As in the proof of (13.1), there exists an isotopy $h_t : V \times \Delta^{k-1} \times I \times I \longrightarrow V \times \Delta^{k-1} \times I \times I$ with $h_0 = \text{id}$ and $h_1(x,u,s,t) = (x,u,t,1-s)$. Therefore, the pairs (M_i,γ_i) and $(M_i, \gamma_i \circ h_1)$ represent the same element of $\mathscr{C}_{CAT}(V)$, where $[(M_i,\gamma_i)] = \eta_k(\theta_i)$. On the other hand, by taking the combinatorial definition of $\theta_1 + \theta_2$ [Cu, p.122] , it is clear that $\eta_k(\theta_1 + \theta_2)$ is equivalent to the representative on $(M_1, \gamma_1 \circ h_1) \# (M_2, \gamma_2 \circ h_1)$ constructed as in the definition of the law $\#$.

The proof for $k = 0$ is the same, playing with the two factors I's of $V \times \Delta^k \times I = V_0 \times I \times \Delta^k \times I$ (one uses (4.2) to decompose V into $V_0 \times I$).

(15.6) <u>Remark</u> : Lemma (15.4) and the proof of (15.5) make clear that the law # of $\mathscr{C}_{CAT}(V)$ is associative (this was not established in details in the proof of (13.1)). Also (15.4) and (15.5) give the following consequences :

1) The composition law on $\pi_0(\mathscr{C}_{CAT}(V))$ of Lemma (15.4) endows $\pi_0(\mathscr{C}_{CAT}(V))$ with a structure of abelian group.

2) $\pi_1(\mathscr{C}_{CAT}(V))$ is abelian.

16. Fibrations involving $\mathscr{C}_{CAT}(V)$

Let $\mathscr{E}(V)$ be the Δ-set whose k-simplex are maps $g : V \times \Delta^k \longrightarrow V \times \Delta^k$ such that :

1) g restricted to a neighborhood of $\partial V \times \Delta^k$ is equal to the identity

2) for any face $\sigma < \Delta^k$, $g^{-1}(V \times \sigma) = V \times \sigma$.

Condition 2) permits us to define the face operators by $\partial_i(g) = g|\Delta_i$. If $V = V_0 \times I$, one can endow, for each k, the set of k-simplices of $\mathscr{E}(V)$ with a composition law in the following way : let $f_i : V_0 \times I \times \Delta^k \longrightarrow V_0 \times I \times \Delta^k$, $i = 1,2$, be two k-simplices of $\mathscr{E}(V)$. Put $f_i(x,t,s) = (f_i^1(x,t,s), f_i^2(x,t,s), f_i^3(x,t,s))$. Then $f = f_1\, f_2$ is defined by :

$$f(x,t,s) = \begin{cases} (f_1^1(x,2t,s), \tfrac{1}{2} f_1^2(x,2t,s), f_1^3(x,2t,s)) & \text{if } t \leqslant \tfrac{1}{2} \\ (f_2^1(x,2t-1,s), \tfrac{1}{2}(1 + f_2^2(x,2t-1,s)), f_2^3(x,2t-1,s)) & \text{if } \tfrac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

The face operators are homeomorphisms for this composition law, thus $\mathscr{E}(V)$ is a structured Δ-set. The fact that $\Delta^k \cong \Delta_i^{k-1} \times I$ makes $\mathscr{E}(V)$ a Kan Δ-set. Also, $\mathscr{C}_{CAT}(V)$ has a base point $*_k : V \times \Delta^k \xrightarrow{\text{id}} V \times D^k$.

One defines a Δ-map $m : \mathscr{C}_{CAT}(V) \longrightarrow \mathscr{E}(V)$ by associating to the k-simplex (W,N,f,α) of $\mathscr{C}_{CAT}(V)$ the k-simplex $g : V \times \Delta^k \longrightarrow V \times \Delta^k$ defined by $(g(x,t),1) = (f \circ \alpha(x,s,1),1)$. By a mild abuse language, we write $g = f \circ \alpha$. One checks that m is a map of structured Δ-set.

The map m might not be onto but it satisfies the Kan extension condition for simplices of dimension ≥ 1. Indeed, let $g : V \times \Delta^k \longrightarrow V \times \Delta^k$ be a k-simplex of $\mathscr{E}(V)$, $k \geq 1$, and $(W_\Lambda, N_\Lambda, f_\Lambda, \alpha_\Lambda)$ be a (k-1)-horn in $\mathscr{C}_{CAT}(V)$ such that $f \circ \alpha = g|V\times\Lambda$. As $\mathscr{C}_{CAT}(V)$ is a Kan Δ-set, $(W_\Lambda, N_\Lambda, f_\Lambda, \alpha_\Lambda)$ is a sub-horn of a k-simplex of $\mathscr{C}_{CAT}(V)$, say $(W_0, N_0, f_0, \alpha_0)$. Construct a new k-simplex (W_0, N_0, f, α_0) by in the following way : define f on N_0 by $f(\alpha_0, x, s, 1)) = (g(x,s),1)$.

This partial map f together with $id : V \times \Delta^k \longrightarrow V \times \Delta^k \times 0$ and f_Λ can be extended to $f : W_0 \longrightarrow V \times \Delta^k \times I$. Indeed, $V \times \Delta^k \cup W_\Lambda \cup N_0$ is a deformation retract of W_0 when $k \geq 1$. Clearly, $m(W_0, N_0, f, \alpha_0) = g$.

Define $\mathscr{F}_{CAT}(V) = m^{-1}(*)$, which is a pointed structured Kan Δ-set. One denotes by $i^m : \mathscr{F}_{CAT}(V) \rightarrow \mathscr{C}_{CAT}(V)$ the inclusion.

Let us now define an equivalence relation between two maps $f_i : (W_i, V \times \Delta^k, N_i) \longrightarrow (V \times \Delta^k \times I, V \times \Delta^k \times 0, V \times \Delta^k \times 1)$, $i = 1,2$, such that f_i are part of the k-simplices $(W_i, N_i, f_i, \alpha_i)$ of $\mathscr{C}_{CAT}(V)$. We say f_1 is equivalent to f_2 if :

a) $W_1 = W_2$ and for any face $\sigma < \Delta^k$, $(W_1)_\sigma = (W_2)_\sigma$ and $(N_1)_\sigma = (N_2)_\sigma$.

b) $f_1^{-1}(\partial V \times \Delta^k \times I) = f_2^{-1}(\partial V \times \Delta^k \times I) = L$ and $f_1|L = f_2|L$.

The definition permits us to define a Δ-set $\mathscr{H}_{CAT}(V)$ by describing a typical k-simplex as a triple (W,N,F), where

1) $(W, V \times \Delta^k, N)$ is an h-cobordism, as in (15.1) part.1).

2) F is an equivalence class of maps $f : (W, V \times \Delta^k, N) \longrightarrow (V \times \Delta^k \times I, V \times \Delta^k \times 0, V \times \Delta^k \times 1)$ satisfying 2a) and 2c) of (15.1).

3) For any face $\sigma < \Delta^k$, the inclusion $N_\partial \subset N$ is a simple homotopy equivalence.

The Δ-set $\mathscr{H}_{CAT}(V)$ is a Kan Δ-set ; the proof goes as for $\mathscr{C}_{CAT}(V)$. Also, $\mathscr{H}_{CAT}(V)$ has a base point $*_k = (V \times \Delta^k \times I, V \times \Delta^k \times 1, id)$.

(16.1) <u>Lemma</u> : If $V^n = V_0 \times I$, and $n \geq 5$, then :

1) $\mathscr{H}_{CAT}(V)$ is a structured Δ-set

2) There is a map of structured pointed Δ-set $t : \mathscr{C}_{CAT}(V) \rightarrow \mathscr{H}_{CAT}(V)$ which satisfies the Kan extension condition for the simplices of dimension ≥ 1.

<u>Proof</u> : The composition law has the same definition as the one for $\mathscr{C}_{CAT}(V)$ (Lemma (15.4)). The map t will send the k-simplex (W,N,f,α) to $(W,N,[f])$, where $[f]$ is the equivalence class of f. To prove that $(W,N,[f])$ is actually a k-simplex of $\mathscr{H}_{CAT}(V)$, it suffices to check Condition 3) of the definition of $\mathscr{H}_{CAT}(V)$. This condition is equivalent to the fact that for every face $\sigma < \Delta^k$, the map $f_\sigma : N_\sigma \longrightarrow V \times \sigma \times 1$ is a simple homotopy equivalence. The composed map :

$$V_0 \times I \times \sigma \times 1 = V \times \sigma \times 1 \xrightarrow{\alpha_\sigma} N_\sigma \xrightarrow{\sigma} V \times \sigma \times 1 = V_0 \times I \times \sigma \times 1$$

is equal to the identity on $V_0 \times 0 \times \sigma \times 1$. Therefore, this composed map is homotopic to the identity. As the whitehead torsion of the homeomorphism $\alpha_\sigma | V \times \sigma \times 1$ is zero, this implies that $f | N_\sigma$ is a simple homotopy equivalence. The map t is clearly a map of structured pointed Δ-sets.

Let us prove now that t satisfies the Kan extension condition for positive dimensional simplices. Let $\theta = (W,N,F)$ be a k-simplex of $\mathscr{H}_{CAT}(V)$ $(k \geq 1)$ and let $\Lambda = (W_\Lambda, N_\Lambda, F_\Lambda)$ be a $(k-1)$-horn of θ. Suppose that $\Lambda = t(\Lambda)$, for $\Lambda = (W_\Lambda, N_\Lambda, f_\Lambda, \alpha_\Lambda)$ a $(k-1)$-horn of $\mathscr{C}_{CAT}(V)$. Let $f : W \longrightarrow V \times \Delta^k \times I$ be a representative of F satisfying Condition 2) of Definition (15.1), with partial inverse $\varphi : V \times \Delta^k \times 0 \cup \partial V \times \Delta^k \times I \hookrightarrow \partial W$. The condition on the inclusions $N_\partial \subset N$ implies that $(N, N_\Lambda, N\text{-int } N_\Lambda)$ is a s-cobordism and therefore $N = N_\Lambda \times I$. (One uses here the hypothesis $k \geq 1$ which implies that N_Λ is not empty). Therefore, the embedding $\varphi \cup \alpha_\Lambda : V \times \Delta^k \times 0 \cup \partial V \times \Delta^k \times I \cup V \times \Lambda \times 1 \longrightarrow \partial W$ can be extended to an embedding $\alpha : V \times \Delta^k \times 0 \cup V \times \Delta^k \times I \cup$ $\cup V \times \Delta^k \times 1 \hookrightarrow \partial W$ satisfying Condition 3) of Definition (15.1). Now $f_\Lambda \cup \varphi^{-1}$ (or even $f_\Lambda \cup \varphi^{-1} \cup \alpha^{-1}|N$) can be extended to $f : W \longrightarrow V \times \Delta^k \times I$ such that $(W,N,F) = t(W,N,f,\alpha)$.

Let $\mathscr{A}$ be a structured Δ-set with base point $*$. Define $\circledast$ to be the sub-(structured Δ-set) of $\mathscr{A}$ generated by $*$. Observe that in the case of $\mathscr{A} = \mathscr{C}_{CAT}(V)$ or $\mathscr{H}_{CAT}(V)$, $\circledast$ is a Kan Δ-set having the same homotopy type as $*$. Also, $\circledast = *$ in the case $\mathscr{A} = \mathscr{E}(V)$.

Define $\mathscr{F}^t_{CAT}(V) = t^{-1}(\circledast)$, which is a structured pointed Kan Δ-set. Denote by $i^t : \mathscr{F}^t_{CAT}(V) \longrightarrow \mathscr{C}_{CAT}(V)$ the inclusion. As above, we show that the maps $m \circ i^t : \mathscr{F}^t_{CAT}(V) \longrightarrow \mathscr{E}(V)$ and $t \circ i^m : \mathscr{F}^m_{CAT}(V) \longrightarrow \mathscr{H}_{CAT}(V)$ satisfy the Kan extension condition for positive dimensional simplices. Thus, $\mathscr{F}_{CAT}(V) = \mathscr{F}^t_{CAT}(V) \cap \mathscr{F}^m_{CAT}(V) = (m \circ i^t)^{-1}(*) =$ $= (t \circ i^m)^{-1}(\circledast)$ is a structured pointed Kan Δ-set, with inclusions $j^t : \mathscr{F}_{CAT}(V) \hookrightarrow \mathscr{F}^t_{CAT}(V)$ and $j^m : \mathscr{F}_{CAT}(V) \longrightarrow \mathscr{F}^m_{CAT}(V)$.

A sequence $\mathscr{F} \xrightarrow{\ i\ } \mathscr{E} \xrightarrow{\ p\ } \mathscr{B}$ or maps of structured pointed Δ-sets is called a restricted structured fibration if i is injective, $i(\mathscr{F}) = p^{-1}(\circledast)$ and p satisfies the Kan extension condition for positive dimensional simplices. The results of this section can be summarized as follows :

(16.2) <u>Theorem</u> : There is a commutative diagram of maps of structured pointed Kan Δ-sets :

$$\begin{array}{ccccc}
\mathscr{F}_{CAT}(V) & \xrightarrow{\ j^t\ } & \mathscr{F}^t_{CAT}(V) & \xrightarrow{\ m^t\ } & \mathscr{E}(V) \\
\downarrow j^m & & \downarrow i^t & & \| \\
\mathscr{F}^m_{CAT}(V) & \xrightarrow{\ i^m\ } & \mathscr{C}_{CAT}(V) & \xrightarrow{\ m\ } & \mathscr{E}(V) \\
\downarrow & & \downarrow & & \\
\mathscr{H}_{CAT}(V) & = & \mathscr{H}_{CAT}(V) & &
\end{array}$$

such that each row and collumn is a restricted structured fibration.

17. Exact sequences . The braid diagram .

Let $\mathscr{F} \xrightarrow{i} \mathscr{E} \xrightarrow{p} \mathscr{B}$ be one of the restricted fibrations occuring in Theorem (16.2). As the sub-Δ-set $\circledast$ of $\mathscr{B}$ is a Kan Δ-set, the restricted fibration ($\mathscr{F}$, $\mathscr{E}$, $\mathscr{B}$,i,p) gives rise to a long exact sequence :

$$\ldots\pi_j(\mathscr{F}) \xrightarrow{i_j} \pi_j(\mathscr{E}) \xrightarrow{p_j} \pi_j(\mathscr{B}) \xrightarrow{\partial_j} \pi_{j-1}(\mathscr{F}) \xrightarrow{i_{j-1}} \ldots$$

$$\ldots \longrightarrow \pi_1(\mathscr{B}) \xrightarrow{\partial_1} \pi_0(\mathscr{F}) \xrightarrow{i_0} \pi_0(\mathscr{E}) \xrightarrow{p_0} \pi_0(\mathscr{B})$$

The map p_0 is not onto in general, since a restricted fibration does not satisfies the Kan extension property for the 0-simplices. The proof of the existence of the above long exact sequence is the same as for the case of an actual fibration (i.e. $\circledast = *$) . As the map i and p are map of structured Δ-sets, the maps i_0 and p_0 are homomorphisms for the induced composition laws on π_0 . We leave to the reader to check that ∂_1 is also a homomorphism.

The aim of this section is to interpret the long homotopy exact sequences of the restricted fibrations of (16.2).

Recall the definition of the set $\mathscr{S}^s_{CAT}(X \text{ rel}\partial)$ of homotopy CAT-structures on an oriented manifold pair $(X,\partial X)$. A class of $\mathscr{S}_{CAT}(X \text{ rel}\partial)$ is represented by a degree one simple homotopy equivalence $f : (U,\partial U) \longrightarrow (X,\partial X)$, where U is a CAT-manifold and $f|\partial U$ is a CAT-homeomorphism onto ∂X . Two such simple homotopy equivalences $f_i : (U_i, \partial U_i) \longrightarrow (X,\partial X)$ are equivalent if there exists a simple equivalence $F : (T,\partial T) \longrightarrow (X \times I, \partial(X \times I))$

such that $T = U_1 \cup U_2 \cup Z$, $f|U_1 = f_1$ and $f|Z$ is a CAT-homeomorphism onto $X \times I$. By omiting the word "simple" in the above definition, one gets a set $\mathscr{S}^h_{CAT}(X \text{ rel}\partial)$. Finally, denote by $[X,X]_{\text{rel}\partial}$ the set of homotopy classes of maps $f : X \longrightarrow X$ which are equal to the identity on a neighborhood of ∂X (homotopies relative to a neighborhood of the boundary). When $X = X_0 \times I$, such a map $f : X \longrightarrow X$ is a homotopy equivalence and thus $[X,X]_{\text{rel}\partial}$ is a group for the composition. This group is abelian, since the supports of two maps can be made disjoint by homotopies.

(17.1) <u>Proposition</u> : Let $V^{n-1} = V_0 \times I$, where V_0 is a compact CAT-manifold with non-empty boundary and $n \geqslant 6$. Then, for all $k \geqslant 0$, one has that :

a) $\pi_k(\mathscr{F}_{CAT}(V))$ is in bijection with $\mathscr{S}^s_{CAT}(V \times D^{k+1} \text{ rel}\partial)$

b) $\pi_k(\mathscr{F}^t_{CAT}(V))$ is isomorphic to $\text{Aut}^c_{CAT}(V \times D^k \text{ rel}\partial)$ (definition in § 5)

c) $\pi_k(\mathscr{E}(V))$ is isomorphic to $[V \times D^k, V \times D^k]_{\text{rel}\partial}$

d) In the homotopy exact sequence of the restricted fibration $\mathscr{F}_{CAT}(V) \xrightarrow{j^t} \mathscr{F}^t_{CAT}(V) \xrightarrow{m^t} \mathscr{E}(V)$ which, under the above bijection, becomes :

$$\xrightarrow{\partial} \mathscr{S}^s_{CAT}(V\times D^{k+1} \text{ rel}\partial) \xrightarrow{j^t_*} \text{Aut}_{CAT}(V\times D^k \text{ rel}\partial) \xrightarrow{m^t_*} [V\times D^k, V\times D^k]_{\text{rel}\partial}$$

$$\xrightarrow{\partial} \mathscr{S}^s_{CAT}(V\times D^k \text{ rel}\partial)$$

the homomorphisms j^t_* , m^t_* and ∂ have the following description :

d_1) j_*^t is the composed map $\mathcal{S}^s_{CAT}(V \times D^{k+1} \text{ rel}\partial) \cong$

$\cong \mathcal{S}^s_{CAT}(V \times D^k \times I \text{ rel}\partial) \xrightarrow{\delta} Aut_{CAT}(V \times D^k \text{ rel}\partial)$,

where δ is the map occuring in (5.2).

d_2) m_*^t is the forgetful homomorphism (an automorphism is seen as a map)

d_3) ∂ sends $f : V \times D^k \longrightarrow V \times D^k$ to the class determined by f in $\mathcal{S}^s_{CAT}(V \times D^k \text{ rel}\partial)$.

<u>Proof</u> : As $\mathcal{F}_{CAT}(V)$ is a Kan complex, an element of $\pi_k(\mathcal{F}_{CAT}(V))$ is represented by a k-simplex with all faces equal to the base point. Such an object is a k-uple (W,N,f,α) where $(W, V \times \Delta^k, N)$ is an s-cobordism and $f : W \longrightarrow V \times \Delta^k \times I$ satisfies $f|V \times \Delta^k \times 0 \cup \partial V \times \Delta^k \times I \cup N = \alpha^{-1}$. The conditions on the faces imply that f is a homeomorphism on the part of ∂W where α^{-1} is not defined. Thus $(f : (W,\partial W) \longrightarrow (V \times \Delta^k \times I, \partial(V \times \Delta^k \times I))$ represents a class in $\mathcal{S}^s_{CAT}(V \times \Delta^k \times I \text{ rel } \partial) =$ $= \mathcal{S}^s_{CAT}(V \times D^{k+1} \text{ rel}\partial)$. This defines a map $\gamma : \pi_k(\mathcal{F}_{CAT}(V)) \longrightarrow \mathcal{S}^s_{CAT}(V \times D^{k+1} \text{ rel } \partial)$. If $g : (X,\partial X) \longrightarrow (V \times \Delta^k \times I, \partial(V \times \Delta^k \times I))$ represents a class of $\mathcal{S}^s_{CAT}(V \times \Delta^k \times I, \text{ rel}\partial)$, then X is an s-cobordism from $g^{-1}(V \times \Delta^k \times 0)$. By the s-cobordism theorem, there exists a homeomorphism $\psi : V \times \Delta^k \times I \xrightarrow{\simeq} X$ extending g^{-1} over $V \times \Delta^k \times 0 \cup \partial(V \times \Delta^k) \times I$. Thus the 4-uple

$$(V \times \Delta^k \times I,\ V \times \Delta^k \times 1,\ g \circ \psi,\ \psi^{-1} \circ g^{-1}|V \times \Delta^k \times 0 \cup \partial V \times \Delta^k \times I \cup V \times \Delta^k \times 1)$$

represents an element θ of $\pi_k(\mathcal{F}_{CAT}(V))$ such that $\gamma(\theta)$ = class of (X,g). Thus γ is surjective. An analogous argument prooves the injectivity of γ .

b) An element of $\pi_k(\mathcal{F}^t_{CAT}(V))$ is represented by a 4-uple (W,N,f,α), where W is a s-cobordism and $f|W$ is an homeomorphism (since $t(W,N,f,\alpha) \in ®$). This determines a class in $\beta \in Aut^c_{CAT}(V \times \Delta^k \text{ rel}\partial)$ by the equation $(\beta(x),1) =$ $= (f \circ \alpha(x,1))$, $x \in V \times \Delta^k$. One checks that this correspondance defines an isomorphism $\pi_k(\mathcal{F}^t_{CAT}(V)) \cong$ $\cong Aut^c_{CAT}(V \times \Delta^k \text{ rel}\partial)$.

c) This is straighforward .

d_1) This follows immediatly from the proof of a) and of (5.2).

d_2) Straighforward from the proof of b) .

d_3) Let $g : V \times \Delta^k \longrightarrow V \times \Delta^k$ be a k-simplex of $\mathcal{E}(V)$ representing an element $[g] \in \pi_k(\mathcal{E}(V))$. Construct a k-simplex $\theta = (V \times \Delta^k \times I,\ V \times \Delta^k \times 1, f, \alpha)$ of $\mathcal{F}^t_{CAT}(V)$ such that : $\partial_i(\theta) = *$ for $i \geq 1$, $m^t(\theta) = g$ (i.e. $f|V \times \Delta^k \times 1 = g$) and $\alpha = id$. By definition of the boundary homomorphism in the homotopy sequence [Cu, p.125] one has $\partial([g]) = \partial_0(\theta)$. But the pair (W,f) makes $\partial_0(\theta)$ equivalent to g in $\mathcal{S}^s_{CAT}(V \times \Delta^{k-1} \times I \text{ rel}\partial) \cong$ $\cong \mathcal{S}^s_{CAT}(V \times \Delta^k \text{ rel}\partial)$.

(17.2) <u>Remark</u> : The space $\mathcal{F}^t_{CAT}(V)$ is actually homotopy equivalent to the Δ-group $\widetilde{CAT}(V)$ whose k-simplex are CAT-homeomorphisms from $V \times \Delta^k$ onto itself which are equal to the identity near the boundary [B-L-S] . Am homotopy equivalence $\widetilde{CAT}(V) \to \mathcal{F}^t_{CAT}(V)$ would send β to $(V \times \Delta^k \times I, V \times \Delta^k \times 1, id, \alpha_\beta)$, where $\alpha_\beta = \beta$ on $V \times \Delta^k \times 1$ and the identity on $\partial V \times \Delta^k \times I \cup$ $\cup V \times \Delta^k \times 0$.

(17.3) <u>Proposition</u> : Suppose that $V^{n-1} = V_0 \times I$ and $n \geq 6$. Then, for all $k \geq 0$, one has :

a) $\pi_k(\mathscr{F}^m_{CAT}(V))$ is in bijection with $\mathscr{S}^h_{CAT}(V \times D^{k+1}\ \mathrm{rel}\partial)$

b) $\pi_k(\mathscr{H}_{CAT}(V))$ is isomorphic to $H^{n-1-k}(Z_2;Wh(V))$

c) In the long exact sequence of the restricted fibration

$$\mathscr{F}_{CAT}(V) \xrightarrow{j^m} \mathscr{F}^m_{CAT}(V) \xrightarrow{t^m} \mathscr{H}_{CAT}(V)$$

which, under the identifications of a) and b) and of (17.1,a) becomes :

$$H^{n-k}(Z_2;Wh(V)) \xrightarrow{\partial} \mathscr{S}^s_{CAT}(V \times D^{k+1})\ \mathrm{rel}\partial) \xrightarrow{j^m_*} \mathscr{S}^h_{CAT}(V \times D^{k+1}\ \mathrm{rel}\partial)$$

$$\xrightarrow{t^m_*} H^{n-k-1}(Z_2;Wh(V)) \xrightarrow{\partial} \dots$$

the homomorphisms j^m_* , t^m_* and ∂ have the following interpretation :

c_1) j^m_* is the forgetful map $\mathscr{S}^s_{CAT} \longrightarrow \mathscr{S}^h_{CAT}$.

c_2) If (W,f) represents a class of $\mathscr{S}^h_{CAT}(V \times D^k\ \mathrm{rel}\partial)$, then the Whitehead torsion $\tau(f)$ of f represents a class $H^{n-k-1}(Z_2;Wh(V))$; the homomorphism t^m_* sends the class of (W,f) to the class of $\tau(f)$.

c_3) ∂ is the composed homomorphism

$$H^{n-k}(Z_2;Wh(V)) \longrightarrow H^{n+k+2}(Z_2;Wh(V)) \xrightarrow{\varepsilon} L^s_{n+k+1}(\pi_1(V)) \xrightarrow{u}$$

$$\longrightarrow \mathscr{S}^s_{CAT}(V \times D^{k+1}\ \mathrm{rel}\partial)$$

where ε is the boundary homomorphism in the Rothenberg exact sequence [Sh] and u is the action of the Wall surgery obstruction group $L^s_{n+k}(\pi_1(V))$ on $(V \times D^{k+1}, id)$ (see [Wa, Theorem 10.5] or § 5).

<u>Proof</u> :

a) The proof goes in the same way as for Part a) of (17.1) One uses that if (W,f) represents an element of $\mathscr{S}^h_{CAT}(V \times \Delta^k \times I\ \mathrm{rel}\partial)$, then W is an h-cobordism from $f^{-1}(V \times \Delta^k \times 0)$.

b) A class $[\theta]$ in $\pi_k(\mathscr{H}_{CAT}(V))$ is represent by a $\theta(W,N,F)$ such that $(W,V \times \Delta^k,N)$ is an h-cobordism trivial over the boundary (since all the faces are equal to the base point). If $k \geq 1$ or $V = V_0 \times I$, the duality formula [Mi, p.394] and Condition 3) of the Definition of $\mathscr{H}_{CAT}(V)$ imply that $\tau(W,V \times \Delta^k) = (-1)^{n-1+k}\ \tau(W,V \times \Delta^k)$. If $\theta' = (W',N',F')$ is another representative of $[\theta]$, there is a (k+1)-simplex $T = (W_0,N_0,F_0)$ with $\partial_0(T) = \theta$, $\partial_1(T) = \theta'$ and $\partial_i(T) = *$ for $i \geq 2$. By Condition 3) of the definition of $\mathscr{H}_{CAT}(V)$, the inclusions $N \subset N_0$ and $N' \subset N_0$ are both simple homotopy equivalences. By the s-cobordism theorem, one has $(N_0,N,N') = (N \times I, N \times 0, N \times 1)$. Thus W_0 is an h-cobordism (trivial over the boundary) between W and W' . The duality formula gives :

$$\tau^0(W,V \times \Delta^k) = \tau^0(W',V' \times \Delta^k) + i^{-1}_{V\times\Delta^k,W_0}\left[\tau(W_0,W) + \overline{+(-1)^{n-1+k}\tau(W_0,W)}\right]$$

This enables us to define a map $\gamma : \pi_k(\mathscr{H}_{CAT}(V)) \longrightarrow H^{n-1+k}(Z_2;Wh(V))$ by sending the class of (W,N,F) to the class of $\tau(W,V \times \Delta^k)$. By using the simplicial description of the sum in $\pi_k(\mathscr{H}_{CAT}(V))$

for $k \geq 1$ [Cu, p.122] or the sum in $\pi_0(\mathscr{H}_{CAT}(V))$ together with the sum formula for Whitehead torsion [Co, (23.1)] one proves that γ is a homomorphism. The surjectivity of γ is proven by using that any torsion $\nu \in Wh(V)$ occurs as $\tau^0(W_\nu, V \times \Delta^k)$ for an h-cobordism $(W_\nu, V_\nu \times \Delta^k, N_\nu)$ with $\partial W_\nu = V \times \Delta^k \cup \partial(V \times \Delta^k) \times I \cup N_\nu$. The duality formula guarantees Condition 3) of the definition of $\mathscr{H}_{CAT}(V)$ and $id : V \times \Delta^k \cup \partial(V \times \Delta^k) \times I \to V \times \Delta^k \times I$ can be extend to a map $f_\nu : W_\nu \longrightarrow V \times \Delta^k \times I$ so that the triple (W_ν, N_ν, f_ν) represents a class $\theta \in \pi_k(\mathscr{H}_{CAT}(V))$ with $\gamma(\theta) = [\nu]$. The proof of the injectivity of is analogous.

c_1) Follows directly from the proof of a) and of Part a) of (17.1).

c_2) Follows directly from the proof of a) and b) and the definition of t^m .

c_3) As in the proof of Part d_3) of (17.1), one uses the simplicial description of ∂ . The result follows then as for Proposition (5.2).

(17.4) Remark : Proposition (17.1) a) and (17.3) permits to put on $\mathscr{S}^e_{CAT}(V \times D^{k+1} \text{ rel}\partial)$ (e = s or h) an abelian group structure (or at least an associative composition law when k = 0). The composition law is constructed as the law on $\mathscr{E}_{CAT}(V)$. We leave to the reader to check that, in all cases, $\mathscr{S}^e_{CAT}(V \times \Delta^{k+1} \text{ rel}\partial)$ is an abelian group.

We study now the fibrations in which $\mathscr{E}_{CAT}(V)$ occurs. Recall that $\pi_k(\mathscr{E}_{CAT}(V)) \cong \Gamma_{CAT}(V \times D^k)$.

(17.5) Proposition : Let V^{n-1} be a stable thickening, $n \geq 6$. In the long exact sequence of the restricted fibration $\mathscr{F}^t_{CAT}(V) \xrightarrow{i^t} \mathscr{E}_{CAT}(V) \xrightarrow{t} \mathscr{H}_{CAT}(V)$ which, under the identifications of (15.5), (17.1) and (17.3) becomes :

$$\cdots \to H^{n+k}(Z_2; Wh(V)) \xrightarrow{\partial_k} Aut^c_{CAT}(V \times D^k \text{ rel}\partial) \xrightarrow{i^t_*} \Gamma_{CAT}(V \times D^k) \longrightarrow$$
$$\xrightarrow{t_*} H^{n+k-1}(Z_2; Wh(V))$$

the homomorphisms t_* , i^t_* and ∂_k have the following interpretation :

a) t_* sends the class of (M,β) in $\Gamma_{CAT}(V \times D^k)$ to $\beta_*^{-1}(t(M)) \in H^{n+k-1}(Z_2; Wh(V))$, where $t(M)$ is the torsion invariant of M (see § 9).

b) i^t_* sends the class of $\alpha \in Aut_{CAT}(V \times D^k)$ onto the class of $(SOB(V \times D^k, \alpha), i)$ where $i : V \times D^k \times I \to SOB(V \times D^k, \alpha)$ is the natural inclusion (see [Ha 4, (2.2)]) .

c) ∂_k is the homomorphism $\Delta^{V \times D^k}_{CAT}$ of § 5 .

Proof : All this follows directly from the way in which the various identifications are obtained (see the proofs of (15.5) (17.3,b and c_3)).

(17.5) Proposition : Let V^{n-1} be a stable thickening, $n \geq 6$. In the long exact of the fibration

$\mathscr{F}^m_{CAT}(V) \xrightarrow{i^m} \mathscr{E}_{CAT}(V) \xrightarrow{m} \mathscr{E}_{CAT}(V)$ which, under the identifications of (15.5), (17.1) and (17.3) becomes :

$$[V \times D^{k+1}, V \times D^{k+1}]_{rel\,\partial} \xrightarrow{\ \delta\ } \mathscr{S}_{CAT}(V \times D^{k+1}\ rel\,\partial) \xrightarrow{\ i_*^m\ }$$

$$\xrightarrow{\ i_*^m\ } \Gamma_{CAT}(V \times D^k) \xrightarrow{\ m_*\ } [\,V \times D^k,\ V \times D^k]_{rel\,\partial}$$

the homomorphisms δ , i_*^m and m_* have the following interpretation :

a) δ sends $f : V \times D^k \to V \times D^k$ to the class determined by f in $\mathscr{S}^h_{CAT}(V \times D^k\ rel\,\partial)$.

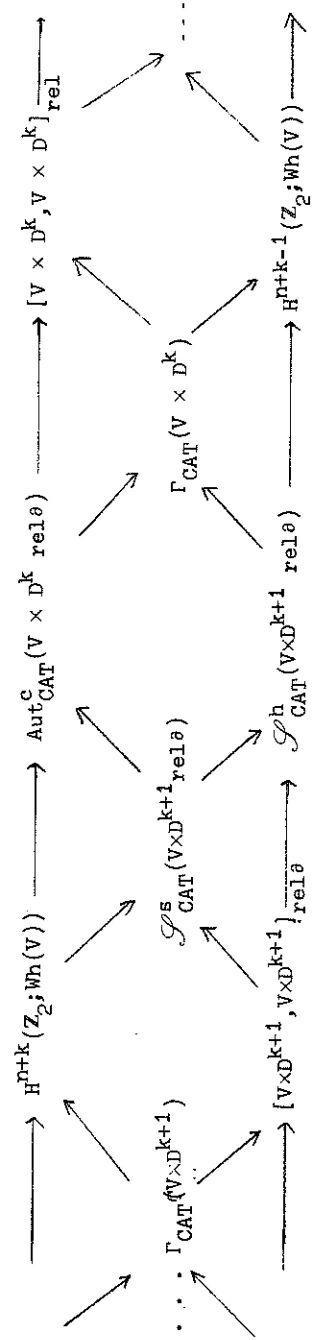
b) i_*^m sends the class of (W,f) to the class of the pair (M,i) where $M = V \times D^k \times I \cup_{f^{-1}|_{\partial W}} W$ and $i : V \times D^k \times I \longrightarrow M$ is the natural inclusion.

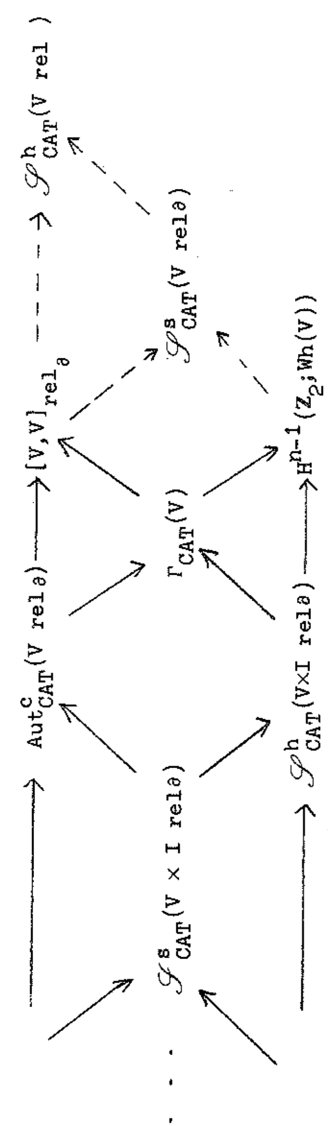
c) The pair $(\mathrm{int}(V \times D^k),\ m_*(M,\beta))$ represents the virtual monodromy $\mu(M)$ of M (see § 29 11).

Proof :

a) Is a consequence of Part d_3) of (17.1) and Part c_1) of (17.3) . Points b) and c) follows from the various identifications (proofs of (15.5), etc.).

(17.7) The braid diagram : Let V^{n-1} be a stable thickening, $n \geqslant 6$. From Theorem (16.2) and Propositions (17.1), (17.3), (17.5) and (17.6), we deduce the following commutative braid diagram of exact sequences of abelian groups :





The braid diagram .

The dotted homomorphisms at the right part of the braid diagram does not come from the fibrations but can be defined directly. As $V = V_0 \times I$, the sets $\mathcal{S}^e_{CAT}(V \text{ rel}\partial)$ are still groups. The dotted homomorphisms are defined as in (17.1) d_3), (17.3) c_1) and c_3) .

The braid diagram put a new light on several results of the previous chapters. The following comparisons are interestings :

- Exactness at $\Gamma_{CAT}(V \times D^k) \longleftrightarrow$ Theorem (9.2) and Corollary (11.3)
- Exactness at $\mathrm{Aut}^c_{CAT}(V \times D^k \text{ rel}\partial) \longleftrightarrow$ Theorem (7.4) .
- Exactness at $H^{n+1-k}(\mathbb{Z}_2; Wh(V)) \longleftrightarrow$ Theorem (10.1)

18. Applications of the exact sequences .

The first application is the use of the homomorphisms $h = \delta \circ m_* = \delta \circ t_* : \Gamma \longrightarrow \mathcal{S}^s_{CAT}(V \text{ rel}\partial)$.

(18.1) Proposition : Let (M,β) represent a class of $\Gamma_{CAT}(V)$, with $\dim V \geq 6$. Then $h(M,\beta) = 0$ if and only if M has the homotopy type of a stable open book with detached page homeomorphic to intV .

Proof : By (17.1, d_3) and (17.6, c) , the meaning of $h(M,\beta) = 0$ is that the virtual monodromy $\mu(M)$ of M is equal to the class of $(\mathrm{int}V,\alpha)$, where α is a CAT-homeomorphism. Therefore M has the homotopy type of $SOB(V,\alpha)$, using (11.2).

Conversely, let $[(M,\beta)] \in \Gamma_{CAT}(V)$ such that there exists a homotopy equivalence $f : M \longrightarrow SOB(T,\alpha) = T \times I \cup W$, with $\mathrm{int}T \simeq \mathrm{int}V$. As in the proof of (11.3), there exists a stable pre-open book decomposition $M = U \times I \cup W'$ and one can deform f by a homotopy so that $f(U \times 0) = V \times 0$, $f(x,t) = (f(x,0),t)$ for $(x,t) \in U \times I$ and $f(W') = W$. On the other hand, M has the stable pre-open book decomposition $M = \beta(V \times I) \cup_\partial X$. By Proposition (2.2), there is an ambient isotopy $H_t : M \longrightarrow M$ with $H_0 = \mathrm{id}$, $H_1(U \times 0) \subset \beta(V \times 0)$, $H_1(x,t) = \beta(\beta^{-1}(H_1(x,0),t)$ for $(x,t) \in U \times I$ and $\beta(V \times 0)$ is an enlargement of $H_1(U \times 0)$. Therefore, f can be deformed by a homotopy so that

$f(\beta(V \times 0)) = T \times 0, f(\beta(x,t)) = (f(\beta(x,0),t)$ and $f(X) = W$. With such an f, it is easy to check that if $(\overline{M},\gamma) = (M,\beta) \# (SOB(V,\alpha^{-1}),i)$, then $\overline{M}$ has the homotopy type of $SOB(V,id)$. By Proposition (11.3) one has $\mu(\overline{M}) = [intV, id]$. By (17.5, c) , one has $m_*(\overline{M},\gamma) = 0$ and thus $h(\overline{M},\gamma) = 0$. As $t_*(SOB(V,\alpha^{-1}),i) = 0$ (by (17.5, a) and (9.2)), it follows from the commutativity of the braid diagram that $h(M,\beta) = h(\overline{M},\gamma)$. Hence $h(M,\beta) = 0$.

(18.2) Corollary : Let G be a finitely presented group and $n \geq 7$ such that the homomorphism $L^h_{n+1}(G) \longrightarrow H^{n+1}(Z_2;Wh(G))$ of the Rothenberg exact sequence is surjective. Then any (WMDH)-manifold of dimension n with fundamental group isomorphic to G has the homotopy type of a stable open book.

Proof : Choose a stable pre-open book decomposition $M = \beta(V \times I) \cup_\partial W$. Then $(M,\beta) \in \Gamma_{CAT}(V)$. By the exactness of the Rothenberg exact sequence $L^h_{n+1}(G) \longrightarrow H^{n+1}(Z_2;Wh(G)) \longrightarrow L^s_n(G)$ and our hypothesis, one deduce using (17.2, c_3) and the braid diagram that $h(M,\beta) = 0$. The result then follows from (18.1).

(18.3) Remark and problems : Corollary (18.2) shows that the method of § 10 to construct (WMDH)-manifolds which are not homeomorphic to a stable open book (or, equivalently, not h-cobordant to a stable open book) always produces manifolds which are homotopy equivalent to a stable open book. The problem of finding a (WMDH)-manifold not homotopy equivalent to a stable open book (or to a stable twisted double) remains open.

Another problem is to decide whether or not Proposition (18.1) remains true if we drop the hypothesis on the virtual detached page.

The rest of this section is devoted to prove the following theorem :

(18.4) Theorem : Let V^{n-1} be a stable thickening, $n \geq 6$. If $\pi_1(V)$ is finite, then $\Gamma_{CAT}(V)$ is finitely generated.

Before proving Theorem (18.3), we need some preliminaries.

The group $[(X,Y);A;\varphi]_k$: Let A be a CW-complex and (X,Y) be a CW-pair. Let $\varphi : X \times I \times I^k \longrightarrow A$ be a map. Let us consider the set $[(X,Y);A;\varphi]_k$ of homotopy classes of maps $f : X \times I \times I^k \longrightarrow A$ such that $f|X \times \partial(I \times I^k) \cup Y \times I \times I^k = \varphi$ (homotopies relative to $X \times \partial(I \times I^k) \cup Y \times I \times I^k$).

Let F_1 and F_2 be two classes of $[(X,Y);A;\varphi]_k$. The class F_1 has a representative by a map $f_1 : X \times I \times I^k \longrightarrow A$ such that $f_1(x,s,t) = \varphi(x,s,t)$ for $s \geq \frac{1}{2}$. As for F_2, it admits a representative $f_2 : X \times I \times I^k \longrightarrow A$ such that $f_2(x,s,t) = \varphi(x,s,t)$ for $s \leq \frac{1}{2}$. Observe that the homotopy classes of f_1 relative to $X \times [0, \frac{1}{2}] \times I^k \cup Y \times I \times I^k \cup X \times \partial(I \times I^k)$ and the homotopy class of f_2 relative to $X \times [\frac{1}{2}, 1] \times I^k \cup Y \times I \times I^k \times X \cup \partial(I \times I^k)$ are well defined. Hence we can define $F = F_1 + F_2 \in [(X,Y);A;\varphi]_k$ as the class of the map f defined by :

$$f(x,s,t) = \begin{cases} f_1(x,s,t) & \text{for } s \leq \frac{1}{2} \\ f_2(x,s,t) & \text{for } s \geq \frac{1}{2} \end{cases}$$

(18.5) <u>Lemma</u> : The set $[(X,Y);A;\varphi]_k$ with the law "+" is an abelian group for all $k \geq 0$.

<u>Proof</u> : Checking the associativity is easy (reparametrization of I). The unit element is represented by φ. To find the inverse $-F$ of a class F , let represent F by a map f so that $f(x,s,t) = \varphi(x,s,t)$ for $s \geq \frac{1}{2}$. The map

$g : X \times [0,\frac{1}{2}] \times I^k \times 0 \cup Y \times I \times I^k \times I \cup X \times \partial(I^k \times I) \times I \cup X \times I \times I^k \times 1 \longrightarrow A$

defined by :

$$g(x,s,t,u) = \begin{cases} \varphi(x,s,t) & \text{for } u > 0 \\ f(x,s,t) & \text{for } u = 0 \end{cases}$$

can be extended to a map $G : X \times I \times I^k \times I \longrightarrow A$. Therefore, the class of the following map f' :

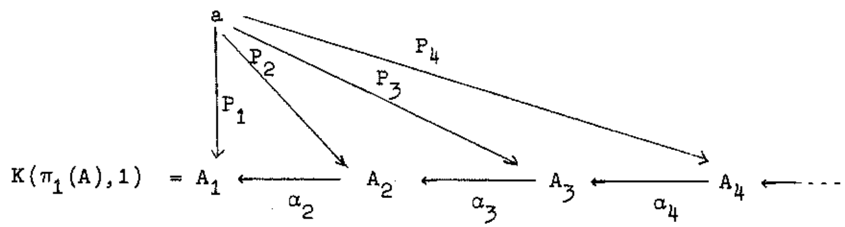
$$f'(x,s,t) = \begin{cases} \varphi(x,s,t) & \text{for } s \leq \frac{1}{2} \\ G(x,s,t,0) & \text{for } s \geq \frac{1}{2} \end{cases}$$

constitutes an inverse for the class F. The commutativity of the law "+" is checked as for the commutativity of the higher homotopy groups (observe that $[(I,\partial I);A;\varphi]_k$, where

$\varphi(z) = * \in A$ is isomorphic to $\pi_{k+2}(A;*)$.)

(18.6) <u>Lemma</u> : Let U^n be a compact PL-manifold with ∂U non-empty. Let A be a CW-complex such that $\pi_i(A)$ is a finitely generated (respectively finite) abelian group for $2 \leq i \leq n$. Let $\varphi : U \times I \times I^k \longrightarrow A$ be a map. Then $[(U,\partial U);A;\varphi]_k$ is finitely generated (respectively finite) for all $k \geq 0$.

<u>Proof</u> : Let us consider the Postnikov tower of A [Bau] :



As $\partial U \neq \emptyset$, the pair $(U \times I \times I^k, (U \times I \times I^k))$ is at least 2-connected, thus $[(U,\partial U);A_1;p_1 \circ \varphi]_k = 0$ for all $k \geq 0$. Let us take, as induction hypothesis, that $[(U,\partial U);A_{r-1};p_{r-1} \circ \varphi]_k$ is finitely generated (respectively finite). One has an exact sequence :

$$H^r(U \times I \times I^k, \partial(U \times I \times I^k);\pi_r(A)) \xrightarrow{\rho} [(U,\partial U);A_r;p_r \circ \varphi]_k \xrightarrow{(\alpha_r)_*} [(U,\partial U);A_{r-1}\ p_{r-1} \circ \varphi]_k$$

Indeed, by obstruction theory with local coefficients, the group $H^r(U \times I \times I^k, \partial(U \times I \times I^k);\pi_r(A))$ is in bijection with the homotopy classes of liftings of $p_{r-1} \circ \varphi$ into A_r [Bau, Theorem (4.3.7)]. Each obstruction can be realized by a lifting equal to $p_r \circ \varphi$ over $U \times [0,\frac{1}{2}] \times I^k$ (or over $U \times [\frac{1}{2},1] \times I^k$)

which permits us to show that ∂ is a homomorphism. Clearly, $\mathrm{Im}\,\partial = \ker(\alpha_r)_*$.

If $\pi_r(A)$ is finitely generated (respectively finite), the group $H^r(U \times I \times I^k, (U \times I \times I^k); \pi_r(A))$ is finitely generated (respectively finite). This proves by induction that $[(U,\partial U);A_r;p_r \circ \varphi]_k$ is finitely generated (respectively finite) for all r and k. But $[(U,\partial U);A;\varphi]_k = [(U,\partial U);A_r;p_r \circ \varphi]_k$ for $r > \dim(U \times I \times I^k)$, which proves Lemma (18.6).

Proof of Theorem (18.4) : Let us consider the surgery exact sequence [Wa 2, § 10] :

$$L^h_{n+1}(\pi_1(V)) \xrightarrow{\gamma_3} \mathscr{S}^h_{CAT}(V\times I, \mathrm{rel}\,\partial) \xrightarrow{\gamma_2} [(V\times I, \partial(V\times I)); G/CAT] \longrightarrow$$
$$\xrightarrow{\gamma_1} L^h_n(\pi_1(V))$$

Observe that $[(V\times I,\partial(V\times I));G/CAT]$ is in bijection with $[(V,\partial V);G/CAT;\circ\,\varphi]_0$ with φ = constant map to the base point $* \in G/CAT$. Thus, the set $[(V\times I,\partial(V\times I));G/CAT]$ has an abelian group structure and, as in [Wa 2, Proposition 10.7], one checks that γ_1 is a homomorphism. Now $\mathscr{S}^h_{CAT}(V \times I\ \mathrm{rel}\,\partial) = \pi_0(\mathscr{F}^m_{CAT}(V))$ by (17.3,a) and thus $\mathscr{S}^h_{CAT}(V \times I\ \mathrm{rel}\,\partial)$ is an abelian group (see Remark (17.4)). One checks easily that γ_2 is a homomorphism (see(Wa 2, after Lemma 10.6]) and that α_3 is a homomorphism (using the usual geometric description of the sum in L^h_{n+1}). Now, $\pi_1(G/CAT)$ is finitely generated for all i. Therefore, $[(V\times I,\partial(V\times I));G/CAT]$ is finitely generated by Lemma (18.6). The surgery groups $L^h_{n+1}(\pi_1(V))$ are finitely generated when $\pi_1(V)$ is finite [Bs, § 11] and [Wa 2,p.240]. Thus $\mathscr{S}^h_{CAT}(V \times I_{\mathrm{rel}\,\partial})$ is finitely generated.

Now, write $V = V_0 \times I$. The group $[V,V]_{\mathrm{rel}\,\partial}$ is then isomorphic to $[(V_0,\partial V_0);V_0;\mathrm{id}]_0$. If $\pi_1(V)$ is finite, the universal covering $\tilde{V}$ of V is a finite simply connected complex and thus $\pi_i(V)$ is finitely generated for all i (it is clear that $H_i(\tilde{V})$ is finitely generated; then so are $\pi_i(\tilde{V}) = \pi_i(V)$ by the Serre-Hurewicz Theorem for the Serre class of finitely generated abelian groups). Therefore, $[V,V]_{\mathrm{rel}\,\partial}$ is finitely generated by (18.6). Using the exact sequence $\mathscr{S}^h_{CAT}(V \times I) \longrightarrow \Gamma_{CAT}(V) \longrightarrow [V,V]_{\mathrm{rel}}$ of the braid diagram, this proves Theorem (18.4).

19. Some computations :

Let C_p be the cyclic group of order p, with p odd. Denote by L_p^m a CAT-manifold quotient of the sphere S^m(m odd) by a free action of C_p (a fake lens space). In this section, we study the group $\Gamma_{CAT}(V)$ for $V = L_p^m \times D^r$. Recall that $\Gamma_{CAT}(V)$ is finitely generated by (18.4).

(19.1) Theorem : Let $V = L_p^m \times D^r$, with $r \geq m + 1 \geq 4$. Then :

$$\operatorname{rank} \Gamma_{CAT}(V) = \begin{cases} 0 & \text{if } r \text{ is even} \\ \frac{p-1}{2} & \text{if } r \equiv 1 \pmod 4 \\ \frac{p+1}{2} & \text{if } r \equiv 3 \pmod 4. \end{cases}$$

Proof : The homotopy groups $\pi_i(V) = \pi_i(S^m)$ are finite for $i > m$. Therefore, the group $[V,V]_{\mathrm{rel}\,\partial}$ is finite by (18.6) and and the proof of (18.4). Hence, by the braid diagram, one has $\operatorname{rank} \Gamma_{CAT}(V) = \operatorname{rank} \mathcal{S}^h_{CAT}(V \times I \ \mathrm{rel}\,\partial)$. The latter can be computed by the surgery exact sequence (see Proof of (18.4)). One has :

$$\pi_i(G/CAT) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } i \neq 4k \\ \mathbb{Q} & \text{if } i = 4k \end{cases}$$

Let us first consider the case r even. It follows by obstruction theory that :

$$\operatorname{rank}[(V\times I,\partial(V\times I));G/CAT] = \begin{cases} 1 & \text{if } r+m+1 \equiv 0 \pmod 4 \\ 0 & \text{if } r+m+1 \not\equiv 0 \pmod 4 \end{cases}$$

If $r + m + 1 \equiv 0 \pmod 4$, the image of the surgery obstruction map $[(V\times I,\partial(V\times I));G/CAT] \longrightarrow L^h_{4s}(C_p)$ has rank at least 1 (signature obstruction). Thus, if r is even, one has $\operatorname{rank} \mathcal{S}^h_{CAT}(V\times I \ \mathrm{rel}\,\partial) \leq \operatorname{rank} L^h_{m+r+2}(C_p)$. But $L^h_{odd}(C_p) = 0$[Bk 1], which shows that $\operatorname{rank} \Gamma_{CAT}(V) = 0$ when r is even.

Now, assume that $m \equiv 3 \pmod 4$. If $r \equiv 1 \pmod 4$, one has by obstruction theory that $\operatorname{rank}[(V\times I,\partial(V\times I));G/CAT] = 0 =$ $= \operatorname{rank}[V\times D^2,\partial(V\times D^2);G/CAT]$. Thus $\operatorname{rank} \mathcal{S}^h_{CAT}(V\times I \ \mathrm{rel}\,\partial) =$ $\operatorname{rank} L^h_2(C_p)$ by the surgery exact sequence. In [Bk 2, Theorem (1.2)], the rank of $L^h_2(C_p)$ is shown to be equal to $r_\infty - 1$, where r_∞ is the number of irreducible real representations of C_p. It is classical that $r_\infty = \frac{p+1}{2}$ [C - R , Theorem 42.8]. Thus $\operatorname{rank} \Gamma_{CAT}(V) = \frac{p-1}{2}$ when $m \equiv 3$ and $r \equiv 1 \pmod 4$.

If $m \equiv r \equiv 3 \pmod 4$, there is an infinite obstruction group $H^{r+1}(V\times I,\partial(V\times I);\pi_{r+1}(G/CAT)) \cong \mathbb{Z}$ in which lies the first obstruction to make a map $(V\times I,\partial(V\times I)) \longrightarrow G/CAT$ homotopic to a constant map. These obstructions are realized by normal maps id f : $L_p^m \times T^{r+1} \longrightarrow L_p^m \times D^r \times I$, where $f : (T^{r+1},\partial T) \longrightarrow (D^r \times I,\partial(D^r \times I))$ is a normal map equal to a CAT-homeomorphism on ∂T. As $L^h_{m+r+1}(C_p) = 0$ [Bk 1], one has an exact sequence :

$$0 \longrightarrow \mathbb{Q} \longrightarrow L^h_4(C_p) \otimes \mathbb{Q} \longrightarrow \mathcal{S}^h_{CAT}(V \times I \ \mathrm{rel}\,\partial) \otimes \mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow 0$$

$$\mathbb{Q} = [(V \times D^2,\partial(V \times D^2));G/CAT] \otimes \mathbb{Q}$$

and $\operatorname{rank} L_4(C_p) = r_\infty$ [Bk 2, Theorem (1.2)]. Thus $\operatorname{rank} \Gamma_{CAT}(V) = \frac{p+1}{2}$ if $r \equiv m \equiv 3 \pmod 4$.

The case $m \equiv 1 \pmod 4$ is similar.

We finish by a precise determination of the finite group $\Gamma_{PL}(V)$ in the case m = R and r even. We suppose in addition that L^3_p is equal to a homogeneous space S^3/C_p, for a discrete subgroup C_p of S^3.

(19.2) <u>Theorem</u> : Let $V = L^3_p \times D^r$, with L^3_p as above. If $8 \leqslant r \leqslant r+3 < 2p$, then :

$$\Gamma_{PL}(V) = \begin{cases} \pi_r(S^3) \oplus \pi_{r+3}(S^3) & \text{if } r \equiv 0 \pmod 4 \\ \text{a subgroup of } \pi_r(S^3) \oplus \pi_{r+3}(S^3) \text{ of} & \\ \text{index dividing } 2p \text{ if } r \equiv 2 \pmod 4 . & \end{cases}$$

<u>Proof</u> : Suppose first that $r \equiv 0 \pmod 4$. Using the classical fact that

$$\pi_i(G/PL) = \begin{cases} 0 & \text{if } i \text{ odd} \\ Z_2 & \text{if } i = 2 \pmod 4 \\ Z & \text{if } i = 0 \pmod 4 \end{cases}$$

One shows that the only non-zero obstruction group to compute $[(V \times I, \partial(V \times I)); G/PL]$ is the group $H^{r+4}(V\times I), \partial(V\times I); \pi_{r+4}(G/PL)) \cong Z$. Therefore, the surgery obstruction map $[(V\times I, \partial(V\times I)); G/PL] \longrightarrow L^h_4(C_p)$ is injective (signature obstruction). On the other hand, one has $L^h_{odd}(C_p) = 0$ [Bk 1], thus $\mathcal{S}^h_{PL}(V\times I \text{ rel}\partial) = 0$ by the surgery exact sequence. On the other hand, one proves by obstruction theory that $[(V,\partial V); G/PL] \cong Z$ and, as $L^h_{odd}(C_p) = 0$, the map $\mathcal{S}^h_{PL}(V \text{ rel}\partial) \longrightarrow [(V,\partial V); G/PL]$ is surjective. One gets the surgery exact sequence :

$$0 \longrightarrow [(V\times I, \partial(V\times I)); G/PL] \longrightarrow L^h_4(C_p) \longrightarrow \mathcal{S}^h_{PL}(V \text{ rel}\partial) \longrightarrow [(V,\partial V); G/PL] \longrightarrow 0$$

which proves that $\mathcal{S}^h_{PL}(V \text{ rel}\partial) = \tilde{L}^h_4(C_p) \oplus Z$, where $\tilde{L}^h_4(C_p)$ is the quotient of $L^h_4(C_p)$ by $L^h_4(1)$.

As $r \geqslant 5$, one has $V = V' \times I$, where V' is a stable thickening. From the braid diagram, one draws the following exact sequence :

$$\mathcal{S}^h_{PL}(V\times I \text{ rel}\partial) \longrightarrow \Gamma_{PL}(V) \longrightarrow [V,V]_{\text{rel}\partial} \longrightarrow \mathcal{S}^h_{PL}(V'\times I \text{ rel}\partial) \longrightarrow \Gamma_{PL}(V')$$
$$\longrightarrow \mathcal{S}^h_{PL}(V \text{ rel}\partial)$$

The torsion of $L^h_4(C_p)$ goes injectively into $H^4(Z_2; Wh(C_p))$ [Bk 2, proof of (1.6)] and therefore the torsion of $\mathcal{S}^h_{PL}(V \text{ rel}\partial)$ goes injectively into $\Gamma_{PL}(V')$, by the braid diagram. Thus the image of $[V,V]_{\text{rel}\partial}$ into $\mathcal{S}^h_{PL}(V \text{ rel}\partial)$ is torsion free. As $\pi_i(S^3)$ is finite for $i > 3$, the group $[V,V]_{\text{rel}\partial}$ is finite by (18.6) and the proof of (18.4). This shows that $\Gamma_{PL}(V) = [V,V]_{\text{rel}\partial}$.

When $r \equiv 2 \pmod 4$, one proves analogously that $\mathcal{S}^h_{PL}(V \times I \text{ rel}\partial) = 0$ (one seems to meet an obstruction in $H^{r+2}(V\times I, \partial(V\times I); \pi_{r+2}(G/PL)) = H^{r+2}(V\times I, \partial(V\times I); Z)$ in the computation of $[(V\times I, \partial(V\times I)); G/PL]$ but, by Poincaré duality, one has $H^{r+2}(V\times I, \partial(V\times I); Z) \cong H_2(L_p; Z) \cong H_2(C_p; Z) = 0$). However, the obstruction theory gives $[(V,\partial V); G/PL] = Z_2$ or Z_{2p}. Doing with these datas the same argument as for the case $r \equiv 0 \pmod 4$, we obtain that $\Gamma_{PL}(V)$ is isomorphic to a subgroup of $[V,V]_{\text{rel}\partial}$ of index dividing 2p.

It thus remains to prove that $[V,V]_{rel\partial} \cong \pi_r(S^3) \oplus \pi_{r+3}(S^3)$ for $r+3 < 2p$. The hypothesis implies that the p-torsion of $\pi_i(V) = \pi_i(S^3)$ vanishes for $i \leqslant r+3$ [Sp. p.515]. Thus there are only two non-zero obstruction groups relevant for the computation of $[V,V]_{rel\partial}$, namely $H^r(V,\partial V;\pi_r(S^3)) \cong \pi_r(S^3)$ and $H^{r+3}(V,\partial V;\pi_{r+3}(S^3)) \cong \pi_{r+3}(S^3)$. One gets an exact sequence :

$$\pi_{r+3}(S^3) \xrightarrow{\ \alpha\ } [V,V]_{rel} \xrightarrow{\ \beta\ } \pi_r(S^3)$$

where β is the first obstruction (always well defined) and α sends an element $u \in \pi_{r+3}(S^3)$ onto the map $f : V \longrightarrow V$ equal to the identity out of a small disk $D^{r+3} \subset \mathrm{int}V$ and sends D^{r+3} on to V accordingly to the class u. One checks easily that α and β are homomorphisms.

We first construct a section η of β. Choose a map $\varphi : D^r \longrightarrow D^r$ such that $\varphi_1|\partial D^r = id$ and $\varphi_1(t) = 0$ for $\|t\| \leqslant \frac{1}{2}$. Choose a map $\varphi_2 : (D^r, \partial D^r) \longrightarrow (S^r,*)$ of degree one such that $\varphi_2(t) = *$ for $\|t\| \geqslant \frac{1}{2}$. If $u : S^r \longrightarrow S^3$ represents a class in $\pi_r(S^3)$, we define a map $\varphi_u : S^3 \times D^r \longrightarrow S^3 \times D^r$ by :

$$\varphi_u(x,t) = (x \,.\, u \circ \varphi_2(t),\ \varphi_1(t))$$

where the dot stands for the multiplication in S^3. Clearly, if $z \in C_p$, one has $\varphi_u(z \cdot x,t) = z \cdot \varphi_u(x,t)$ and thus φ_u induces a map $\overline{\varphi}_u : V \longrightarrow V$ which, by definition, will represent $\eta(u)$. One checks that η is a homomorphism and that $\beta \circ \eta = id$.

It remains to prove that α is injective. Suppose that $\alpha(v) = id$. By definition of α, the class $\alpha(v)$ is represented by a map $\psi : V \longrightarrow V$ such that $\psi|V\text{-}\mathrm{int}D^{r+3} = id$, where $D^{r+3} \subset \mathrm{int}V$ is a small disk, and $\psi|D^{r+3}$ represents v. As $\alpha(v) = 0$, there exists a map $\xi : V\times I \longrightarrow V \times I$ such that $\xi(v,0) = \xi(v)$ and $\xi|V\times[\frac{1}{2},1] \cup \partial V\times I = id$. Let us try to homotopy ξ relative to $\partial(V \times I)$ to a map $\xi' : V\times I \to V\times I$ such that $\xi'|(V\text{-}\mathrm{int}D^{r+3}) \times I = id$. One finds a unique obstruction $u \in \pi_{r+1}(S^3)$. Let $f : V \times I \longrightarrow V \times I$ be a map satisfying $f|V \times[0,\frac{1}{2}] \cup \partial(V \times I) = id$ and representing the class $\eta(-u)$, where $\eta : \pi_{r+1}(S^3) \longrightarrow [V \times I, V \times I]_{rel\partial}$ is the section of β (in dimension r+1) constructed above. The map $\mu : V\times I \longrightarrow V\times I$ defined by :

$$\mu(x,t) = \begin{cases} \xi(x,t) & \text{for } t \leqslant \frac{1}{2} \\ f(x,t) & \text{for } t \geqslant \frac{1}{2} \end{cases}$$

is homotopic relative to $\partial(V \times I)$ to a map μ' such that $\mu'|(V\text{-}\mathrm{int}D^{r+3}) \times I = id$. Then, $\mu'|D^{r+3} \times I$ is a homotopy relative to ∂D^r from v to the natural inclusion $D^{r+3} \subset V$. This shows that $v = 0$.

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