THE HILBERT METRIC AND GROMOV HYPERBOLICITY

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Abstract. We give some sufficient conditions for Hilbert's metric on convex domains D to be Gromov hyperbolic. The conditions involve an intersecting chords property, which we in turn relate to the Menger curvature of triples of boundary points and, in the case the boundary is smooth, to differential geometric curvature of ∂D. In particular, the intersecting chords property and hence Gromov hyperbolicity is established for bounded, convex C^2-domains in $\mathbb{R}^n$ with non-zero curvature.

We also give some necessary conditions for hyperbolicity: the boundary must be of class C^1 and may not contain a line segment. Furthermore we prove a statement about the asymptotic geometry of the Hilbert metric on arbitrary convex (i.e. not necessarily strictly convex) bounded domains, with an application to maps which do not increase Hilbert distance.

Introduction

Let D be a bounded convex domain in $\mathbb{R}^n$ and let h be the Hilbert metric, which is defined as follows. For any distinct points $x, y \in D$, let x' and y' be the intersections of the line through x and y with ∂D closest to x and y respectively. Then

$$h(x,y) = \log \frac{yx' \cdot xy'}{xx' \cdot yy'}$$

where zw denotes the Euclidean distance $\|z - w\|$ between two points. The expression $\frac{yx' \cdot xy'}{xx' \cdot yy'}$ is called the cross-ratio of four collinear points and is invariant under projective transformations. For the basic properties of the distance h we refer to [Bu55] or [dlH93].

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We will here give some sufficient conditions for the metric space (D, h) to be hyperbolic in the sense of Gromov. Namely, we show that a certain intersecting chords property implies Gromov hyperbolicity (Theorem 2.1). This intersecting chords property holds when the (Menger) curvature of any three points of the domain's boundary is uniformly bounded from both above and below in a certain way (Corollary 1.2). Domains with C^2 boundary of everywhere nonzero curvature satisfy this condition as will be proved in section 3. Beardon showed in [Be97] (see also [Be99]) that a weaker intersecting chords property holds for any bounded strictly convex domain and he used this to establish some weak hyperbolicity results for the Hilbert metric. In section 5 we prove a generalization of his results to any bounded convex domain.

It would also be interesting to understand what conditions on ∂D are necessary in order for (D, h) to be Gromov hyperbolic. For example, in section 4 we give an argument showing that ∂D must be of class C^1 and that it may not contain a line segment.

Some parts of the results in this paper might already be known: Y. Benoist told us that a convex domain with C^2 boundary is Gromov hyperbolic if the curvature of the boundary is everywhere nonzero. Benoist has also found examples of Gromov hyperbolic Hilbert geometries whose boundaries are C^1 but not C^2. In [Be99] it is mentioned that C. Bell has proved an intersecting chords theorem in an unpublished work. However, we have not found the present arguments or our main results in the literature. Furthermore, we have not found the simple and attractive Proposition 1.1 and Corollary 3.5 stated or discussed anywhere, although these facts are most likely known. They should belong to ancient Greek geometry and classical differential geometry respectively.

Since the Hilbert distance can be defined in analogy with Kobayashi's pseudo-distance on complex spaces [Ko84], we would like to mention that Balogh and Bonk proved in [BB00] that the Kobayashi metric on any bounded strictly pseudoconvex domain with C^2 boundary is Gromov hyperbolic.

Note that metric spaces of this type are CAT(0) only in exceptional cases (see [BH99] for the definition). Indeed, Kelly and Straus proved in [KS58] that if (D, h) is nonpositively curved in the sense of Busemann then D is an ellipsoid and hence (D, h) is the n-dimensional hyperbolic space. Compare this to the situation for Banach spaces: a Banach space is CAT(0) if and only if it is a Hilbert space. Another category of results is of the following type: if D has a large (infinite, cocompact, etc.) automorphism group and C^2-smooth boundary, then it is an ellipsoid, see the work of Socié-Méthou [SM00].

The Hilbert metric has found several applications, see [Bi57], [Li95] and [Me95] just to mention a few instances. Typically the idea is to apply the contraction mapping principle to maps which do not increase Hilbert distances (e.g. affine maps).

This work was mainly done during our stay at Bielefeld University. We are grateful to this university for its hospitality and for providing such excellent working conditions. Especially we wish to thank Professor H. Abels for inviting us there.

1. Intersecting chords in convex domains

From elementary school we know that if c_1, c_2 are two intersecting chords in a circle, then $l_1 l_1' = l_2 l_2'$ where l_1, l_1' and l_2, l_2' denote the respective lengths of the segments into which the two chords are divided. (This follows immediately from the similarity of the associated triangles, see Fig. 1.)

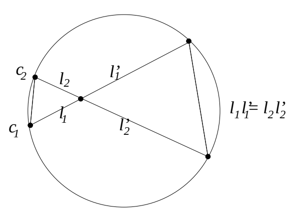

Figure 1
Intersecting chords in a circle

A generalization of this fact to any bounded strictly convex domain was given by Beardon in [Be97] by an elegant argument using the Hilbert metric. He proved that if D is such a domain then for each positive δ there is a positive number $M = M(D, \delta)$ such that for any intersecting chords c_1, c_2, each of length at least δ, one has

$$M^{-1} \leq \frac{l_1 l_1'}{l_2 l_2'} \leq M, \tag{1.1}$$

where l_1, l_1', and l_2, l_2' denote the respective lengths of the segments into which the two chords are divided.

We say that a domain satisfies the *intersecting chords property* (ICP) if (1.1) holds for *any* two intersecting chords c_1 and c_2. It is easy to see that ICP may fail for a general strictly convex domain (at a curvature zero point or a "corner").

We show in this section that ICP holds for domains that satisfy a certain (non-differentiable) curvature condition. Domains with C^2 boundary of nonvanishing curvature are proved to satisfy this condition in section 3.

1.1 Intersecting line segments and Menger curvature

This subsection clarifies the relation between the curvature of any triple of endpoints and the ratio considered above that two intersecting line segments define.

Three distinct points A, B and C in the plane, not all on a line, lie on a unique circle. Recall that the radius of this circle is

$$R(A,B,C) = \frac{c}{2\sin\gamma}, \tag{1.2}$$

where c is the length of a side of the triangle ABC and γ is the opposite angle. The reciprocal of R is called the *(Menger) curvature* of these three points and is denoted by $K(A,B,C)$.

Now consider two intersecting line segments as in Fig. 2.

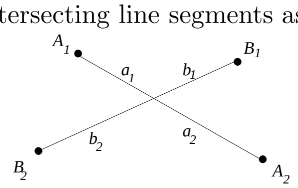

FIGURE 2
Intersecting line segments

PROPOSITION 1.1. *In the above notation, the following equality holds:*

$$\frac{a_1 a_2}{b_1 b_2} = \frac{K(A_1,B_1,B_2)K(A_2,B_1,B_2)}{K(B_1,A_1,A_2)K(B_2,A_1,A_2)}.$$

Proof. Let α_i be the angle between the line segments A_iB_j and B_1B_2, and let β_i be the angle between B_iA_j and A_1A_2, for $\{i,j\} = \{1,2\}$. By the sine law we have

$$\frac{a_1 a_2}{b_1 b_2} = \frac{\sin\alpha_1 \sin\alpha_2}{\sin\beta_2 \sin\beta_1} = \frac{2\sin\alpha_1}{|A_1B_1|}\frac{|A_2B_2|}{2\sin\beta_2}\frac{2\sin\alpha_2}{|A_2B_2|}\frac{|A_1B_1|}{2\sin\beta_1}$$
$$= \frac{K(A_1,B_1,B_2)K(A_2,B_1,B_2)}{K(B_1,A_1,A_2)K(B_2,A_1,A_2)}.$$

□

Corollary 1.2. *Let D be a bounded convex domain in $\mathbb{R}^n$. Assume that there is a constant $C > 0$ such that*

$$\frac{K(x,y,z)}{K(x',y',z')} \leq C$$

for any two triples of distinct points in ∂D all lying in the same 2-dimensional plane. Then D satisfies the intersecting chords property.

Proof. Any two intersecting chords define a plane and by Proposition 1.1 we have

$$\frac{a_1 a_2}{b_1 b_2} = \frac{K_{\alpha 1}K_{\alpha 2}}{K_{\beta 1}K_{\beta 2}} \leq C^2.$$

□

Remark 1.3. In view of this subsection it is clear that ICP implies restrictions on the curvature of the boundary, e.g. there cannot be any points of zero curvature. We were however not able to establish the converse of Corollary 1.2.

1.2 Chords larger than δ

The following proposition provides a different approach to the result in [Be97] mentioned above.

Proposition 1.4. *Let D be a bounded convex domain in $\mathbb{R}^n$. Let δ be such that the length of any line segment contained in ∂D is bounded from above by some $\delta' < \delta$. Then there is a constant $C = C(D,\delta) > 0$ such that*

$$C(D,\delta) \leq K(x,y,z) \leq \frac{2}{\delta}, \tag{1.3}$$

whenever $x, y, z \in \partial D$ and $xy \geq \delta$.

Proof. The angle $\alpha(x,y,v) := \angle_y(xy, v)$ is continuous in $x, y \in \mathbb{R}^n$ and $v \in UT_y(\partial D)$, the unit tangent cone at y. The tangent cone at a boundary point y is the union of all hyperplanes containing y but which are disjoint from D. If $[x, y]$ does not lie in ∂D, then $0 < \alpha(x, y, v) < \pi$. The set

$$S = \{(x, y, v) \in \partial D \times \partial D \times UT_y(\partial D) : xy \geq \delta\}$$

is compact. Hence there is a constant $\alpha_0 > 0$ such that

$$\alpha_0 \leq \alpha(x, y, v) \leq \pi - \alpha_0 \tag{1.4}$$

for every $(x, y, v) \in S$. By the definition of the tangent cone and compactness there is an $\varepsilon > 0$ such that for any $y, z \in \partial D$, $0 < yz < \varepsilon$ there is an element $v \in UT_y(\partial D)$ for which

$$0 \leq \angle_y(yz, v) \leq \alpha_0/2. \tag{1.5}$$

The estimates (1.4) and (1.5) imply the existence of $C > 0$ and the other inequality in (1.3) is trivial. □

As an immediate consequence of Propositions 1.1 and 1.4 we have:

Corollary 1.5. (Cf.[Be99]) *Let D be a bounded convex domain such that any line segment in ∂D has length less than $\delta' < \delta$. Then the intersecting chords property holds for any two chords each of length greater than δ.*

2. Hyperbolicity of Hilbert's metric

Let (Y, d) be a metric space. Given two points $z, w \in Y$, let

$$(z|w)_y = \frac{1}{2}(d(z, y) + d(w, y) - d(z, w))$$

be their *Gromov product* relative to y. We think of y as a fixed base point. The metric space Y is *Gromov hyperbolic* (or *δ-hyperbolic*) if there is a constant $\delta \geq 0$ such that the inequality

$$(x|z)_y \geq \min\{(x|w)_y, (w|z)_y\} - \delta$$

holds for any four points x, y, z, w in Y. As is known, it is enough to show such an inequality for a fixed y (the δ changes by a factor of 2), see [BH99] for a proof of this and we also refer to this book for a general exposition of this important notion of hyperbolicity. By expanding the terms the above inequality is equivalent to

$$d(x, z) + d(y, w) \leq \max\{d(x, y) + d(z, w), d(y, z) + d(x, w)\} + 2\delta. \tag{2.1}$$

THEOREM 2.1. *Let D be a bounded convex domain in $\mathbb{R}^n$ satisfying the intersection chords property. Then the metric space (D,h) is Gromov hyperbolic.*

Proof. Suppose that the intersecting chords property holds with a constant M. Let y be a fixed reference point and consider any other three points x,z,w in D. Set $A(u,v)=h(u,v)+h(w,y)-h(u,w)-h(v,y)$ for any two points u,v. By (2.1) we need to show that there is a constant δ independent of x,z,w such that

$$\min\{A(x,z),A(z,x)\}\le 2\delta. \tag{2.2}$$

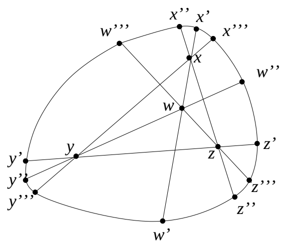

FIGURE 3
Four points

Using the definition of h and the notation in Fig. 3, we have (by rearranging the members of the product)

$$A(x,z)=\log\left(\frac{xz''\cdot zx''}{xx''\cdot zz''}\,\frac{wy''\cdot yw''}{ww''\cdot yy''}\,\frac{xx'\cdot ww'}{xw'\cdot wx'}\,\frac{zz'\cdot yy'}{zy'\cdot yz'}\right)=$$

$$=\log\left(\frac{xx'\cdot xz''}{xx''\cdot xw'}\,\frac{yy'\cdot yw''}{yy''\cdot yz'}\,\frac{zz'\cdot zx''}{zz''\cdot zy'}\,\frac{ww'\cdot wy''}{ww''\cdot wx'}\right).$$

Hence, by using $\frac{xx'}{xx''}\le M\frac{xz''}{xw'}$ and similar inequalities for the other fractions,

$$A(x,z)\le M'+2\log\left(\frac{xz''}{xw'}\,\frac{yw''}{yz'}\,\frac{zx''}{zy'}\,\frac{wy''}{wx'}\right).$$

Now, y is fixed and zy', wy'' are bounded from above and below respectively, so that

$$A(x,z)\le M''+2\log\left(\frac{xz''\cdot zx''}{xw'\cdot wx'}\right).$$

So (2.2) is equivalent to the boundedness of

$$\min\left\{\frac{xz''\cdot zx''}{xw'\cdot wx'},\frac{zx''\cdot xz''}{zw'''\cdot wz'''}\right\}.$$

from above. By symmetry we may assume without loss of generality that $zz''\leq xx''$. Now we have two cases:

Case 1: $xw\geq xx''$ or $zw\geq xx''$. If $xw\geq zw$ (so in particular $xw\geq xx''$), then

$$\begin{aligned}\frac{xz''\cdot zx''}{xw'\cdot wx'}&\leq\frac{(xz+zz'')(zx+xx'')}{(xw)^2}\\&\leq\frac{(xw+wz+zz'')(zw+wx+xx'')}{(xw)^2}\leq\frac{(3xw)^2}{(xw)^2}\leq 9\end{aligned}$$

When $zw\geq xw$, we estimate the other fraction instead (obtained by interchanging x and z) in the same way.

Case 2: $xw\leq xx''$ and $zw\leq xx''$. Considering chords at x we have

$$\frac{xz''\cdot zx''}{xw'\cdot wx'}\leq M\,\frac{xx'\cdot zx''}{xx''\cdot wx'}\leq M\frac{xx'}{wx'}\frac{(xw+wz+xx'')}{xx''}\leq 3M$$

since $xx''\cdot xz''\leq M(xx'\cdot xw')$. $\square$

Remark 2.2. Since the n-dimensional ball B^n obviously satisfies the assumption in Corollary 1.2 with $C=1$, Theorem 2.1 contains the standard fact that (B^n,h), which is Klein's model of the n-dimensional hyperbolic space, is Gromov hyperbolic.

Remark 2.3. The above proof does not appeal to compactness and therefore goes through in infinite dimensions provided that y lies on positive distance from the boundary. In particular, it proves that the unit ball in a Hilbert space with the Hilbert metric, which is the infinite dimensional hyperbolic space, is Gromov hyperbolic. Note however that ICP is not affinely invariant in infinite dimensions. (Kaimanovich brought this remark to our attention.)

3. Intersecting chords theorem for convex C^2-domains

Assume that D is a bounded, convex domain in $\mathbb{R}^n$ with C^2-smooth boundary. Let ρ be a C^2-defining function for D, that is, ρ is positive on points in D, negative outside $\overline{D}$ and zero on ∂D. Moreover the gradient $\nabla\rho =: \nu(x)$ is a unit vector field normal to ∂D directed inside D. The *curvature* (or *Weingarten*) *operator* $W_x\colon T_x\partial D \to T_x\partial D$ is by definition the directional derivative of ν in the direction v. The *second fundamental form* is the bilinear form II_x on $T_x\partial D$ given by

$$II_x(v,w) = (w, W_x(v)) = \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial x_i \partial x_j} v_i w_j.$$

The value $II_x(u,u) =: k_x(u)$ is called *the normal curvature* of ∂D at x in the direction of the unit tangent vector u. We will assume that the curvature of ∂D is everywhere nonzero, meaning that II is everywhere positive definite, so there is a constant $k_D > 0$ such that

$$k_D^{-1} \le k_x(u) \le k_D \tag{3.1}$$

for every $u \in UT_x\partial D$ and $x \in \partial D$.

In this section we will establish:

Theorem 3.1. *Let D be a bounded convex domain in $\mathbb{R}^n$. Suppose that the boundary ∂D is smooth of class C^2 and the curvature of ∂D is everywhere nonzero. Then there is a constant $C > 0$ such that*

$$C^{-1} \le \frac{K(x,y,z)}{K(x',y',z')} \le C$$

for any two triples of distinct points in ∂D all lying in the same 2-dimensional plane.

In view of Corollary 1.2 and Theorem 2.1 this implies:

Corollary 3.2. *Let D be as above. Then D has the intersecting chords property and (D,h) is Gromov hyperbolic.*

3.1 The two dimensional case

For this subsection, let D be a convex, bounded domain in $\mathbb{R}^2$ with C^2-boundary curve ∂D. Assume in addition that the differential geometric curvature κ is positive (nonzero) at every point of ∂D.

LEMMA 3.3. *The Menger curvature $K(x,y,z)$ of three points extends to a continuous function on $\partial D \times \partial D \times \partial D$. The value $K(x,x,z)$ equals the curvature of the circle tangent to ∂D at x and passing through z, and the value $K(x,x,x)$ equals $\kappa(x)$.*

Proof. The continuity for three distinct points is clear. When three points converge to one point on the boundary, it is a standard fact that K converges to κ, see [Sp78, Ch. 1], or [BG88, p. 304 or p. 306]. When y_t converges to $x \neq z$, then $d(y_t,z) \to d(x,z)$ and $\sin\angle(y_t x, xz) \to \sin\angle(T_x\partial D, xz)$. This proves the continuity and it is clear that the limit circle is tangent to ∂D at x. $\square$

The idea of the proof of the following proposition was supplied to us by M. Bucher.

PROPOSITION 3.4. *Let (x,y,z) be a global minimum or maximum point for K on $\partial D \times \partial D \times \partial D$. Then ∂D contains the shortest circle arc connecting x,y and z.*

Proof. Recall the formula (1.2) and consider the circle in question through the three boundary points x,y,z with extremal, say maximal, radius. Denote by γ a shortest arc on this circle connecting these three points, and assume that x and z are the boundary points of γ.

In the case $x=y=z$ there is nothing to prove. Assume now that the three points are all distinct and consider first a potential boundary point w between γ and xz. By convexity of D it cannot lie inside the triangle xyz.

If γ is larger than a halfcircle, then note that (depending on which region w belongs to) either $R(x,w,y) > R(x,y,z)$ or $R(z,w,y) > R(x,y,z)$ (compare the angle at w with the one at either z or x). Therefore w cannot belong to ∂D. If γ is less than a half-circle, then, again by looking at the angles and using the formula for R, we have $R(x,w,z) > R(x,y,z)$, for any such w.

Secondly, note that a potential boundary point w outside the circle in the half-plane defined by the line through x and z containing y cannot belong to ∂D, because either $R(w,y,z)$ or $R(x,y,w)$ (depending on where w lies) is greater than $R(x,y,z)$. Hence the arc γ must coincide with an arc of ∂D.

In the case $x = y \neq z$, no point outside the circle can lie on ∂D, again by the assumption on the maximality of the radius. On the other hand, a point w between γ and xz cannot belong to ∂D because $R(x,w,z) > R(x,x,z)$, and again we have the desired conclusion.

The case of maximal curvature can be treated analogously. □

In view of the continuity of κ, the following immediate consequence of Proposition 3.4 is somewhat analogous to a mean value theorem.

COROLLARY 3.5. *Denote by κ_{min} and κ_{max} the minimum and the maximum, respectively, of the curvature of ∂D. Then*

$$\kappa_{min} \leq K(x,y,z) \leq \kappa_{max},$$

for any three boundary points x, y, z.

3.2 The proof of Theorem 3.1

Assume that D is as in the theorem. To simplify the notation we will only discuss the 3-dimensional case. Each 2-dimensional plane section is Gromov hyperbolic by the above so we only need an overall bound for constants $\delta(S)$ when S runs through all the plane sections. The intersection of ∂D with a 2-dimensional plane gives rise to a smooth planar curve α, which we assume is parameterized by arclength. The constant δ of the hyperbolicity depends on the curvature of α. These curves could have an arbitrarily large curvature but we need only to bound from above (and hence from below) the ratio of the curvatures at different points of the curve. The curvature vector $\alpha''(t)$ of α at a point $x = \alpha(t)$ lies in this plane and is orthogonal to $\alpha'(t)$. Thus we need to bound the ratio $\frac{|\alpha''(t)|}{|\alpha''(s)|}$. It is a fact (Meusnier's lemma, see [Kl78, p. 43] that

$$k_x(\alpha'(t)) = |\alpha''(t)| \cos\theta(t),$$

where $k_x(\alpha'(t)) = II_x(\alpha'(t), \alpha'(t))$ is the normal curvature in the direction $\alpha'(t)$ and $\theta(t)$ is the angle between $\alpha''(t)$ and the normal of ∂D at x. In view of the assumption 3.1 and Corollary 3.5 we therefore need to bound the ratio $\frac{\cos\theta(s)}{\cos\theta(t)}$ independently of s, t and α. Near any point x the surface ∂D is the graph of a C^2 function $z = f(x,y)$ in suitable Cartesian coordinates. Hence any small plane section C_ε is given by the equation $f(x,y) = \varepsilon > 0$. Expressing θ in terms of f we arrive at the problem of bounding the ratio of the gradients $\frac{|\nabla f(p)|}{|\nabla f(q)|}$ along the section. By rotation in the xy-plane we may assume that the x- and y-axis are along the direction of principal curvature. By developing $f(x,y)$ into a Taylor's expansion around the origin, we obtain $f(x,y) = \frac{1}{2}(ax^2 + by^2) + r$, where r vanishes at $(0,0)$ together with all its derivatives up to second order, and where $a = f_{xx}(0,0), b = f_{yy}(0,0)$ are the principal curvatures. We conclude that $c < \frac{|\nabla f(x,y)|}{\sqrt{x^2+y^2}} < C$ near 0 for universal $c, C > 0$ and thus it remains to bound the ratio $\frac{x^2+y^2}{x'^2+y'^2}$ on C_ε. But this ratio is bounded in view of the estimate $\kappa^{-1}(x^2+y^2) < f(x,y) < \kappa(x^2+y^2)$ for some universal $\kappa > 0$ and of the fact that $f(x,y) = \varepsilon$ on C_ε. □

4. Consequences of Gromov hyperbolicity for the shape of the boundary

Proposition 4.1. *Let D be a bounded convex domain in $\mathbb{R}^n$ and let h be a Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is strictly convex, that is, it does not contain a line segment.*

This can be proven following the proof of N. Ivanov [Iv97] of Masur-Wolf's theorem [MW95] that the Teichmüller spaces (genus ≥ 2) are not Gromov hyperbolic. The proof makes use of Gromov's exponential divergence criterion, see [BH99, p. 412]. For another proof of the above proposition, see [SM00]

Theorem 4.2. *Let D be a bounded convex domain in $\mathbb{R}^n$ and let h be the Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is smooth of class C^1.*

Proof. 2-dimensional case. First, by the previous result, D is strictly convex. Let $y = f(x)$, $x \in (-a, a)$ be an equation of ∂D near some point. Then f is strictly convex and hence the one-sided derivatives $f'_-(x)$, $f'_+(x)$ exist and are strictly increasing on $(\varepsilon, \varepsilon)$, [RV73, §11].

We prove that $f'_-(0) = f'_+(0)$. Suppose not, then by choosing appropriate Cartesian coordinates we may assume that $f'_-(0) < 0$ and $f'_+(0) > 0$. For each sufficiently small ε construct an ideal triangle $\Delta = \Delta(\varepsilon)$ in D with one vertex 0 and two other vertices corresponding to the intersection of the line $y = \varepsilon$ with ∂D. We assert that the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero. Namely we show that the Hilbert distance between the point $P = (0, \varepsilon)$ and any point Q of the side $[0, B]$ tends to ∞. Let $f'_+(0) = \tan\alpha$, $0 < \alpha < \pi/2$. Let $x_1 < x_2$ be the points such that $f(x_1) = \varepsilon$ and $f'_+(0)x_2 = \varepsilon$. Then

$$PQ \geq \varepsilon \cos\alpha = f(x_1)\cos\alpha.$$

Let O, R be the intersection points of the line PQ with ∂D. We have therefore

$$QR \leq x_2 - x_1 = \frac{f(x_1)}{f'_+(0)} - x_1 = \frac{f(x_1) - f'_+(0)x_1}{f'_+(0)}$$

and hence combining the last two inequalities

$$\begin{aligned}\frac{PQ}{QR} &\geq \frac{f'_+(0)f(x_1)\cos\alpha}{f(x_1) - f'_+(0)x_1} \\ &= \frac{f'_+(0)\cos\alpha}{1 - f'_+(0)\frac{x_1}{f(x_1)}} \to \infty \text{ when } x_1 \to 0.\end{aligned}$$

It follows that

$$h(P,Q) = \ln\left(1 + \frac{PQ}{OP}\right)\left(1 + \frac{PQ}{QR}\right) \to \infty \text{ when } x_1 \to 0$$

and hence the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero.

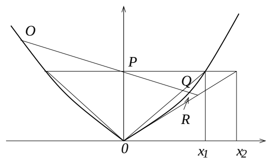

FIGURE 4
Hyperbolicity implies C^1

It remains to show that f' is continuous. By [RV73, §14] we have

$$\lim_{x\to x_0+} f'_+(x) = f'_+(x_0),$$

$$\lim_{x\to x_0-} f'_+(x) = f'_-(x_0).$$

From this we conclude that f'_+ is continuous at x_0 since $f'_+(x_0) = f'_-(x_0)$. But $f'(x_0) = f'_+(x_0)$ hence f' is also continuous at x_0.

n-dimensional case. Recall the known result that if f is a differentiable convex function defined on an open convex set S in $\mathbb{R}^{n+1}$, then it is C^1 on S, see for example [RV73]. Let D be a bounded convex domain in

$\mathbb{R}^{n+1}, n \geq 2$. It is enough to prove that ∂D is differentiable at any point. Given a point $p \in \partial D$, we can choose the coordinate axis of $\mathbb{R}^{n+1}$ so that the origin O of the coordinates is at p, all of D lies in the halfspace $x_0 \geq 0$ and in a neighbourhood of p the surface ∂D can be represented as the graph of a nonpositive convex function $x_0 = f(x_1, x_2, \ldots, x_n)$, $x = (x_1, x_2, \ldots, x_n)$, $f(0) = 0$. Considering the 2-dimensional sections in the planes $x_0, x_i, i = 1, \ldots, n$ we obtain that the partial derivatives of f at 0 exist and $f_{x_i}(0) = 0,\ i = 1, \ldots, n$. We have to prove that for each $\varepsilon > 0$ there is a neighbourhood U_ε of 0 such that $f(x) < \varepsilon|x|$ in this neighbourhood. But in view of $f_{x_i}(0) = 0,\ i = 1, \ldots, n$, we have $f(0, \ldots, 0, x_i, 0, \ldots, 0) < \varepsilon|x_i|$ for sufficiently small x_i and hence by convexity $f(x) < \varepsilon|x|$ for sufficiently small $|x|$. □

Remark 4.3. The following is announced in [B00] : If a strictly convex domain D is divisible, that is, if it admits a proper cocompact group of isometries Γ, then D is Gromov hyperbolic if and only if ∂D is C^1. Our Theorem 4.2 shows that that in the implication (Gromov hyperbolicity + divisibility $\Rightarrow C^1$) the condition of divisibility is superfluous.

5. Non-strictly convex domains

This section owes much of its existence to [Be97] and [Be99]. Using a different argument, we prove certain extensions to arbitrary convex bounded domains of some of the results obtained in those papers.

Lemma 5.1. *Let D be a bounded convex domain in $\mathbb{R}^n$. Let $\{x_n\}, \{y_n\}$ be two sequences of points in D. Assume that $x_n \to \overline{x} \in \partial D$, $y_n \to \overline{y} \in \overline{D}$ and $[\overline{x}, \overline{y}] \nsubseteq \partial D$. Let x_n' and y_n' denote the endpoints of the chord through x_n and y_n as usual. Then x_n' converges to $\overline{x}$ and y_n' converges to the endpoint $\overline{y}'$ of the chord defined by $\overline{x}$ and $\overline{y}$ different from $\overline{x}$.*

Proof. Compare with Lemma 5.3. in [Be97]. Every limit point of chord endpoints must belong to the line through $\overline{x}$ and $\overline{y}$. In addition, in the case of x_n' for example, any limit point must lie on the halfline from $\overline{x}$ not containing $\overline{y}$. At the same time each limit point must belong to the boundary of D, and the statement follows since the line through $\overline{x}$ and $\overline{y}$ intersects ∂D only in $\overline{x}$ and $\overline{y}'$. □

THEOREM 5.2. *Let D be a bounded convex domain. Let $\{x_n\}$ and $\{z_n\}$ be two sequences of points in D. Assume that $x_n \to \overline{x} \in \partial D$, $z_n \to \overline{z} \in \partial D$ and $[\overline{x}, \overline{z}] \nsubseteq \partial D$. Then there is a constant $K = K(\bar{x}, \bar{z})$ such that for the Gromov product $(x_n|z_n)_y$ in Hilbert distances relative to some fixed point y in D we have*

$$\limsup_{n\to\infty} (x_n|z_n)_y \leq K.$$

Proof. By Lemma 5.1, the endpoints of the chords through x_n and z_n converge to $\overline{x}$ and $\overline{z}$.

Since $[\overline{x}, \overline{z}]$ is not contained in the boundary of D, there are small compact neighbourhoods $\overline{U}_x$ and $\overline{U}_z$ of $\overline{x}$ and $\overline{z}$ respectively, in ∂D, such that every chord with endpoints in $\overline{U}_x$ and $\overline{U}_z$ is contained in D. In particular the Euclidean midpoint of every such chord lies inside D and by compactness there is an upper bound K on $h(y, w)$, where w is the midpoint of such a chord.

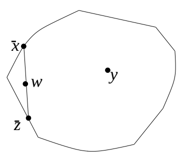

FIGURE 5
Partial hyperbolicity

Consider three points x, y, z and a point w on a (minimizing) geodesic segment $[x, z]$ in a (geodesic) metric space (Y, d). Then

$$\begin{aligned}(x|z)_y &= \frac{1}{2}(d(x,y) + d(z,y) - d(x,z)) \\ &= \frac{1}{2}(d(x,y) + d(z,y) - d(x,w) - d(w,z)) \\ &\leq \frac{1}{2}(d(y,w) + d(y,w)) = d(y,w)\end{aligned}$$

by the triangle inequality. It follows from this estimate and the above considerations that eventually

$$(x_n|z_n)_y \leq K.$$

□

Remark 5.3. The content of Theorem 5.2 is that (D,h) satisfies a weak notion of hyperbolicity. This property should be compared with Gromov hyperbolicity, especially with the fact that for Gromov hyperbolic spaces, two sequences converge to the same point of the boundary if and only if their Gromov product tends to infinity. Theorem 5.2 can be applied as in [Ka01, Theorem 8] to the study of random walks on the automorphism group of D, and it is also likely to be useful for analyzing commuting nonexpanding maps or isometries of (D,h).

Remark 5.4. We suggest that a similar statement might hold for the classical Teichmüller spaces and perhaps also for more general Kobayashi hyperbolic complex spaces. Hilbert geodesic rays from a point y that terminate on a line segment contained in the boundary may correspond to the Teichmüller geodesic rays defined by Jenkins-Strebel differentials that H. Masur considered when demonstrating the failure of CAT(0) for the Teichmüller space of Riemann surfaces of genus $g \geq 2$. The complement of the union of all line segments in the boundary ∂D may correspond to the uniquely ergodic foliation points on the Thurston boundary of Teichmüller space.

Using the arguments in [Ka01], see Proposition 5.1 of that paper, we obtain the following result as an application of Theorem 5.2:

Theorem 5.5. *Let D be a bounded convex domain and $\varphi : D \to D$ be a map which does not increase Hilbert distances. Then either the orbit $\{\varphi^n(y)\}_{n=1}^{\infty}$ is bounded or there is a limit point $\bar{y}$ of the orbit such that for any other limit point $\bar{x}$ of the orbit it holds that $[\bar{x},\bar{y}] \subset \partial D$.*

This theorem, which extends a theorem in [Be97], provides a general geometric explanation for a part of the main theorem in [Me01].

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