Noncommutative Ergodic Theorems

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To Robert J. Zimmer on the occasion of his sixtieth birthday

Abstract

We present recent results about the asymptotic behavior of ergodic products of isometries of a metric space X. If we assume that the displacement is integrable, then either there is a sublinear diffusion or there is, for almost every trajectory in X, a preferred direction at the boundary. We discuss the precise statement when X is a proper metric space ([KL1]) and compare it with classical ergodic theorems. Applications are given to ergodic theorems for nonintegrable functions, random walks on groups and Brownian motion on covering manifolds.

In this note, we survey some recent results about the asymptotic behavior of ergodic products of 1-Lipschitz mappings of a metric space (X,d). If the mappings are translations on the real line $(\mathbb{R},|\cdot|)$, then classical ergodic theorems apply, as we recall in Section 1. In more general settings, a suitable generalization of the convergence of averages is the ray approximation property: a typical orbit stay within a $O(\frac{1}{n})$ distance of some (random) geodesic ray ([Pa], [K3] and [KM], see Theorem 6 below). Most of this note is devoted to another generalization, valid in the case when the space $(X.d)$ is proper (see Theorem 7). It also says that there is a (random) direction followed by the typical trajectory, but now a direction is just a point in the

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metric compactification of (X,d). We discuss in Section 3 how Theorem 7 yields the ray approximation property when the space (X,d) is a CAT(0) metric space, and consequently Oseledets Theorem (following [K3]). We give in Section 4 some applications when the space (X,d) is a Gromov hyperbolic space. In particular, by choosing different metrics on $\mathbb{R}$ we directly show some known ergodic theorems for nonintegrable functions. We prove Theorem 7 in Section 5 and give applications to Random Walks in Section 6. Section 6 comes from [KL2], with slightly simpler proofs. The gist of our results is that for a random walk with first moment on a locally compact group with a proper metric, the Liouville property implies that the linear drift of the random walk, if any, completely comes from a character on the group (see Section 6 for precise statements). This is to be compared with the results of Guivarc'h ([G]) in the case of connected Lie groups. Since our result applies to discrete groups, it can, through discretization, be applied to Brownian motion on Riemannian covers of finite volume manifolds. We state in Section 7 the subsequent result from [KL3].

1 Classical Ergodic Theorems.

We consider a Lebesgue probability space $(\Omega, \mathcal{A}, \mathbb{P})$, an invertible bimeasurable transformation T of the space $(\Omega, \mathcal{A})$ that preserves the probability $\mathbb{P}$, a function $f : \Omega \to \mathbb{R}$, and we define $S_n(\omega) := \sum_{i=0}^{n-1} f(T^i\omega)$. This setting occurs in particular in Statistical Mechanics and in Mechanics, where Ω is the space of configurations, T the time 1 evolution and $\mathbb{P}$ is either the statistical distribution of states or the Liouville measure on the energy levels. The Ergodic Hypothesis led to assert that the ergodic averages

$$\frac{1}{n}S_n(\omega) = \frac{1}{n}\sum_{i=0}^{n-1} f(T^i\omega)$$

have some asymptotic regularity.

Around 1930, Koopman suggested that it might be useful to consider the operator U on functions f in $L^2(\Omega, \mathbb{P})$ defined by

$$(Uf)(\omega) = f(T\omega).$$

Since T is measure-preserving, the operator U is unitary. The ergodic average then becomes

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k f.$$

The system $(\Omega, \mathcal{A}, \mathbb{P}; T)$ is said to be ergodic if the only functions in L^2 which are invariant under the unitary operator U are the constant functions. In this text, for the sake of exposition, we assume that the system $(\Omega, \mathcal{A}, \mathbb{P}; T)$ is ergodic. Statements for nonergodic systems follow using the decomposition of the measure $\mathbb{P}$ into ergodic components. As an application of the Spectral Theorem, von Neumann indeed proved:

Theorem 1 *[von Neumann Ergodic Theorem, 1931] Assume that the transformation T is ergodic and that $\int f^2 d\mathbb{P} < \infty$, then*

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k f \to \int_\Omega f d\mathbb{P}$$

in L^2.

This prompted Birkhoff to prove an almost everywhere convergence theorem:

Theorem 2 *[Birkhoff Ergodic Theorem, 1931] Assume that the transformation T is ergodic, and that $\int \max(f, 0) d\mathbb{P} < \infty$, then for $\mathbb{P}$-almost every ω, as $n \to \infty$:*

$$\frac{1}{n} S_n(\omega) \ \to \ \int f d\mathbb{P}.$$

A variant of the ergodic theorem applies to *subadditive* sequences. A sequence S_n of real functions on Ω is said to be subadditive if, for $\mathbb{P}$-almost every ω, all natural integers n, m:

$$S_{n+m}(\omega) \ \leq \ S_m(\omega) + S_n(T^m \omega).$$

Theorem 3 *[Kingman Subadditive Ergodic Theorem, 1968] Assume that the transformation T is ergodic, and that $\int \max(S_1, 0) d\mathbb{P} < \infty$, then for $\mathbb{P}$-almost every ω, as $n \to \infty$:*

$$\frac{1}{n} S_n(\omega) \ \to \ \inf_n \frac{1}{n} \int S_n d\mathbb{P}.$$

Proofs of Theorems 2 and 3 often appeal to some combinatorics of the sequence $S_n(\omega)$ along individual orbits. The following technical Lemma was proven by the first author and Margulis:

Lemma 4 *[[KM], Proposition 4.2] Let S_n be a subadditive sequence on an ergodic dynamical system $(\Omega, \mathcal{A}, \mathbb{P}; T)$. Assume that $\int \max(S_1, 0) d\mathbb{P} < \infty$ and that $\alpha := \inf_n \frac{1}{n} \int S_n d\mathbb{P} > -\infty$. Then, for $\mathbb{P}$ a.e. ω, all $\varepsilon > 0$, there exist $K = K(\omega)$ and an infinite number of instants n such that:*

$$S_n(\omega) - S_{n-k}(T^k \omega) \ \geq \ (\alpha - \varepsilon)k \text{ for all } k, K \leq k \leq n.$$

In particular, it follows from subadditivity that $\liminf_k \frac{S_k(\omega)}{k} \geq \alpha$. Therefore Theorem 2 follows (in the case $\int f d\mathbb{P} > -\infty$) because in that case, both sequences S_k and $-S_k$ are subadditive and

$$\inf_n \frac{1}{n} \int S_n d\mathbb{P} = \sup_n \frac{1}{n} \int S_n d\mathbb{P} = \int f d\mathbb{P} = \alpha.$$

On the other hand, $\limsup_k \frac{S_k}{k}$ is a T-invariant function which, by subadditivity, is not bigger than $\limsup_p \frac{1}{pk} \sum_{j=0}^{p-1} S_k(T^{jk}\omega)$. Thus the constant $\limsup_k \frac{S_k}{k}$ is not bigger than $\frac{1}{k} \int S_k d\mathbb{P}$. Theorem 3 follows in the case when $\alpha > -\infty$. To treat the case $\alpha = -\infty$ in both theorems, it suffices to replace S_n by $\max(S_n, -nM)$, and to let M go to infinity, see [Kr] for details.

2 Noncommutative Ergodic Theorems.

Observe that Theorem 1 also holds true for any linear operator U of a Hilbert space assuming $\|U\| \leq 1$. One can take one step further and define for any $g \in \mathcal{H}$, $\phi(g) := Ug + f$. Then ϕ is an isometry (or merely 1-Lipschitz in the case $\|U\| \leq 1$).

Note that

$$\phi^n(0) = \sum_{k=0}^{n-1} U^k f.$$

Pazy proved in [Pa] that more generally for any map $\phi : \mathcal{H} \to \mathcal{H}$ such that $\|\phi(x) - \phi(y)\| \leq \|x - y\|$, it holds that there is a vector $v \in \mathcal{H}$ such that

$$\frac{1}{n} \phi^n(0) \to v$$

in norm. This can be reformulated as follows: There is a unit speed geodesic $\gamma(t) = tv / \|v\|$ in $\mathcal{H}$ such that

$$\frac{1}{n} \|\phi^n(0) - \gamma(n \|v\|)\| = \frac{1}{n} \|\phi^n(0) - nv\| \to 0. \tag{1}$$

We call this property *ray approximation*. It turns out that this generalization of the ergodic theorem still holds for more general group actions than the actions of $\mathbb{Z}$. Let G be a second countable, locally compact, Hausdorff topological semi-group, and consider $g : \Omega \to G$ a measurable map. We form

$$Z_n(\omega) := g(\omega)g(T\omega)...g(T^{n-1}\omega)$$

and we ask whether Z_n converges to infinity with some linear speed.

Assume G acts on a metric space (X, d) by 1-Lipschitz transformations. Then for a fixed $x_0 \in X$, we can define, for $g \in G$, $|g| := d(x_0, gx_0)$. Clearly, up to a bounded error, $|Z_n(\omega)|$ does not depend on our choice of x_0. We have:

Proposition 5 *Assume the transformation T is ergodic, and $\int |g|d\mathbb{P} < \infty$. Then there is a nonnegative number α such that for $\mathbb{P}$-almost every ω, as $n \to \infty$:*

$$\frac{1}{n}|Z_n(\omega)| \ \to \ \alpha.$$

The number α is given by

$$\alpha = \inf_n \frac{1}{n} \int |Z_n(\omega)| d\mathbb{P}. \tag{2}$$

Proof. It suffices to observe that the sequence $|Z_n(\omega)|$ satisfies the hypotheses of Theorem 3. Our hypothesis says that $\int Z_1 < \infty$. The subaditivity follows from the 1-Lipschitz property:

$$\begin{aligned} |Z_{n+m}(\omega)| &= d(x_0, g(\omega)...g(T^{n+m-1}\omega)x_0) \\ &\leq d(x_0, g(\omega)...g(T^{m-1}\omega)x_0) + \\ &\quad + d(g(\omega)...g(T^{m-1}\omega)x_0, g(\omega)...g(T^{n+m-1}\omega)x_0) \\ &\leq |Z_m(\omega)| + d(x_0, g(T^m\omega)...g(T^{n+m-1}\omega)x_0) \\ &= |Z_m(\omega)| + |Z_n(T^m\omega)|. \end{aligned}$$

Moreover we see that the limit α is given by $\inf_n \frac{1}{n} \int |Z_n(\omega)| d\mathbb{P}$. ■

When $\alpha > 0$, Proposition 5 says that the points $Z_n(\omega)x$ go to infinity with a definite linear speed. The question arises of the convergence in direction of the points $Z_n(\omega)x$. Given equation (1), we expect that an almost everywhere convergence theorem will say that $Z_n(\omega)x$ will stay at a sublinear distance of a geodesic. We present several results in that direction depending on different geometric hypotheses on the space X.

Assume X is a complete, Busemann nonpositively curved and uniformly convex (e.g. $CAT(0)$ or uniformly convex Banach space) metric space. Then,

Theorem 6 *[KM] Under these assumptions, there is a constant $\alpha \geq 0$ and, for $\mathbb{P}$-almost every ω, a geodesic ray γ_ω such that*

$$\frac{1}{n} d(Z_n(\omega)x_0, \gamma_\omega(n\alpha)) \to 0.$$

We outline the proof (see [KM], Section 5, for details). Let $a(n, \omega) = d(x_0, Z_n(\omega)x_0)$ for each n. Consider a triangle consisting of x_0, $Z_n(\omega)x_0$, and $Z_k(\omega)x_0$. Note that the side of this triangle have lengths $a(n, \omega)$, $a(k, \omega)$, and (at most) $a(n-k, T^k\omega)$. Given $\varepsilon > 0$ (and a.e. ω), for k large it holds that $a(k, \omega) \leq (\alpha + \varepsilon)k$. Assume now in addition to k being large that n and k are as in Lemma 4. This implies that the triangle is thin in the sense that $Z_k(\omega)x_0$ lies close to the geodesic segment $[x_0, Z_n(\omega)x_0]$, more precisely, the distance is at most $\delta(\varepsilon)a(k, \omega)$, where δ only depends on the geometry. Thanks to the geometric assumptions this $\delta(\varepsilon)$ tends to 0 as ε tends to 0. Selecting ε tending to 0 fast enough we can by selecting suitable n as in Lemma 4 and some simple geometric arguments obtain a limiting geodesic. Finally, one has essentially from the contruction that as $m \to \infty$, the points $Z_m(\omega)x_0$ lie at a sublinear distance from this geodesic ray.

This note is devoted to the generalization of the ergodic theorem to groups of isometries of a metric space (X, d). We assume that the space (X, d) is *proper* (closed bounded subsets are compact) and we consider the *metric compactification* of X. Define, for $x \in X$ the function $\Phi_x(z)$ on X by:

$$\Phi_x(z) \ = \ d(x, z) - d(x, x_0).$$

The assignment $x \mapsto \Phi_x$ is continuous, injective and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of X. The *metric compactification* $\overline{X}$ of X is the closure of X for that topology. The *metric boundary* $\partial X := \overline{X} \setminus X$ is made of Lipschitz continuous functions h on X such that $h(x_0) = 0$. Elements of ∂X are called *horofunctions.* Our main result is the following

Theorem 7 *[Ergodic Theorem for isometries [KL1]] Let T be a measure preserving transformation of the Lebesgue probability space $(\Omega, \mathcal{A}, \mathbb{P})$, G a locally compact group acting by isometries on a proper space X and $g : \Omega \to$*

G a measurable map satisfying $\int |g(\omega)| d\mathbb{P}(\omega) < \infty$. Then, for $\mathbb{P}$-almost every ω, there is some $h_\omega \in \partial X$ such that:

$$\lim_{n\to\infty} -\frac{1}{n} h_\omega(Z_n(\omega)x_0) \;=\; \lim_{n\to\infty} \frac{1}{n} d(x_0, Z_n(\omega)x_0).$$

For the convenience of the reader, the proof of Theorem 7 is given in Section 5. We explain in Section 3 why the convergence in Theorem 7 is equivalent to the ray approximation under the $CAT(0)$ assumption. Note that by Theorem 7 the former convergence holds for all norms on $\mathbb{R}^d$, but that Theorem 7 does not apply to infinite dimensional Banach spaces. In this case, one can use Lemma 4 to prove a noncommutative ergodic theorem with linear functionals of norm 1, somewhat analogous to horofunctions. Namely,

Theorem 8 *[Ka] Let $Z_n(\omega)$ be an ergodic integrable cocycle of 1-Lipschitz self-maps of a reflexive Banach space. Then for $\mathbb{P}$-almost every ω there is a linear functional f_ω of norm 1 such that*

$$\lim_{n\to\infty} \frac{1}{n} f_\omega(Z_n(\omega)0) = \alpha.$$

On the other hand, Kohlberg-Neyman [KN] found a counterexample to the norm convergence, or more precisely to (1), for general Banach spaces.

3 Case when X is a $CAT(0)$ proper space.

When the space (X, d) is a proper $CAT(0)$ metric space, both Theorems 6 and 7 apply. Because it is a direct generalization of the important case when G is a linear group, it is often called the Oseledets Theorem. In this section we explain how to recover the ray approximation and other more familiar forms of Oseledets Theorem from Theorem 7. Many of the geometric ideas in this section go back to Kaimanovich's extension of Oseledets Theorem to more general semi-simple groups ([K3]).

A metric geodesic space (X, d) is called a $CAT(0)$ space if its geodesic triangles are thinner than in the Euclidean space. Namely, consider four points $A, B, C, D \in X$, D lying on a length minimizing geodesic going from B to C. Draw four points A', B', C', D' in the Euclidean plane with $AB = A'B', BD = B'D', DC = D'C', CA = C'A'$. The space is called

$CAT(0)$ if, for any such configuration $AD \leq A'D'$. Simply connected Riemannian spaces with nonpositive curvature, locally finite trees and Euclidean buildings are proper $CAT(0)$ spaces. If X is a $CAT(0)$ space, then the horofunctions $h \in \partial X$ are called Busemann functions, and for any $h \in \partial X$, there is a unique geodesic ray $\sigma_h(t), t \geq 0$ such that $\sigma_h(0) = x_0$ and $\lim_{t\to\infty} \Phi_{\sigma_h(t)} = h$. We have:

Corollary 9 *Assume moreover that X is a $CAT(0)$ space and that $\alpha > 0$. Then, for $\mathbb{P}$-almost every ω, as n goes to ∞,*

$$\lim_n \frac{1}{n} d\left(Z_n(\omega)x_0, \sigma_{h_\omega}(\alpha n)\right) \ = \ 0,$$

where h_ω is given by Theorem 7.

Proof. Consider a geodesic triangle $A = Z_n(\omega)x_0, B = x_0, C_t = \sigma_{h_\omega}(t)$, for t very large, and choose $D = \sigma_{h_\omega}(n\alpha)$. We want to estimate the distance AD. We have

$$\begin{aligned}
AB &= d(Z_n(\omega)x_0, x_0) = |Z_n(\omega)| =: n\alpha_n(\omega) \\
BC_t &= t,\ BD = n\alpha \text{ and} \\
C_tA &= t + \Phi_{\sigma_{h_\omega}(t)}(Z_n(\omega)x_0) =: t - n\beta_n(\omega) + o_n(t).
\end{aligned}$$

For almost every ω, we have

- $\lim_n \alpha_n(\omega) = \alpha$ by Theorem 3,
- $\lim_n \beta_n(\omega) = \lim_n -\frac{1}{n} h_\omega(Z_n(\omega)x_0) = \alpha$ by Theorem 7 and
- for a fixed n, $\lim_{t\to\infty} o_n(t) = h_\omega(Z_n(\omega)x_0) - h_{\sigma_{h_\omega}(t)}(Z_n(\omega)x_0) = 0$.

Construct the comparison figure $A'B'C'_tD'$, and let t go to ∞. The point E'_t of $B'C'_t$ at the same distance from C'_t than A' converges to the orthogonal projection E'_∞ of A' on $B'C'_t$ and satisfies $B'E'_t = n\beta_n - o_n(t)$. Therefore, $B'E'_\infty = n\beta_n$. We have:

$$(AE'_\infty)^2 = n^2(\alpha_n^2 - \beta_n^2), \ \ (D'E'_\infty)^2 = n^2(\beta_n - \alpha)^2,$$

and therefore, as $n \to \infty$:

$$\lim_n \frac{1}{n^2}(A'D')^2 \ = \ \lim_n \left((\alpha_n^2 - \beta_n^2) + (\beta_n - \alpha)^2\right) = 0.$$

■

Corollary 10 *With the same assumptions, we have, for $\mathbb{P}$-almost every ω, $Z_n(\omega)x_0$ converges to h_ω in $\overline{X}$.*

In particular, when $\alpha > 0$ and X is proper $CAT(0)$, the direction h_ω given by Theorem 7 is unique.

Proof. In the above triangle, the geodesic σ_n joining x_0 to $Z_n(\omega)x_0$ converges to σ_{h_ω}. Therefore all the accumulation points of $Z_n(\omega)x_0$ belong to the set seen from x_0 in the direction of h_ω. By the same proof, all the accumulation points of $Z_n(\omega)x_0$ belong to the set seen from $\sigma_{h_\omega}(K)$ in the direction of h_ω, for all K. As K goes to infinity, the intersection of those sets is reduced to the point h_ω. ∎

In the case when G is a linear group, Corollary 9 is closely related to the well known

Theorem 11 *[Oseledets Multiplicative Ergodic Theorem, [O], 1968] Let T be an ergodic transformation of the Probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and $A : \Omega \to GL(d, \mathbb{R})$ a measurable map such that $\int \max\{\ln ||A||, \ln ||A^{-1}||\} d\mathbb{P} < \infty$. Then there exist*

- *real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$*
- *integers $m_i, i = 1, \dots, k$ with $\sum_i m_i = d$, $\sum_i \lambda_i m_i = \int \ln |DetA| d\mathbb{P}$.*
- *for $\mathbb{P}$-almost every ω, a flag of subspaces of $\mathbb{R}^d$*

$$\{0\} = V_{k+1}(\omega) \subset V_k(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d$$

with, for all $i, 1 \leq i \leq k$, $DimV_i = \sum_{j \geq i} m_j$ and a vector v belongs to $V_i(\omega) \setminus V_{i+1}(\omega)$ if, and only if, as n goes to ∞,

$$\lim \frac{1}{n} \ln ||A(T^{n-1}\omega)A(T^{n-2}\omega) \dots A(\omega)v|| = \lambda_i.$$

Observe that, automatically, the V_i depend measurably of ω and are invariant in the sense that $A(\omega)V_i(\omega) = V_i(T\omega)$. The usual complete form of Oseledets Theorem follows by comparing the results of Theorem 11 for (T, A) and for $(T^{-1}, A^{-1} \circ T^{-1})$. Fix $\omega \in \Omega$, and let $e_i, i = 1, \dots, d$ be an orthogonal base of $\mathbb{R}^d$ such that $e_\ell \in V_i(\omega)$ as soon as $\ell \leq \sum_{j \geq i} m_j$. Write $\mu_1 \geq \cdots \geq \mu_d$ for the exponents λ_j, each counted with multiplicity m_j, and consider $A^{(n)}(\omega) := A(T^{n-1}\omega)A(T^{n-2}\omega) \dots A(\omega)$ in the base (e_i). To verify the statement of Theorem 11, it suffices to show that for all $\varepsilon > 0$ and for n large enough,

$$\big|A_{i,j}^{(n)}(\omega)\big| \leq e^{n(\mu_i+\varepsilon)} \text{ and } \big|\ln|DetA^{(n)}(\omega)| - \sum_j \mu_j\big| \leq \varepsilon.$$

With the notations of Section 2, consider the action by isometries of $GL(d,\mathbb{R})$ on the symmetric space $GL(d,\mathbb{R})/O(d,\mathbb{R})$ with origin $x_0 = O(d,\mathbb{R})$ and distance $|g| = \sqrt{\sum_{j=1}^d (\ln\tau_i)^2}$, where τ_i are the eigenvalues of gg^t. It is a $CAT(0)$ geodesic proper space. Set $g(\omega) = A^t(\omega)$. The moment hypothesis $\int |g| d\mathbb{P} < \infty$ is satisfied. We have $A^{(n)}(\omega) = (Z_n(\omega))^t$. If $\alpha = 0$, then the eigenvalues of $Z_n Z_n^t$ grow subexponentially and

$$\lim_n \frac{1}{n} \ln ||A^{(n)}(\omega)v|| = \frac{1}{2} \lim_n \frac{1}{n} \ln(||Z_n^t v||^2) = 0.$$

In this case $m_1 = d, \lambda_1 = 0$ and Theorem 11 holds. We may assume $\alpha > 0$, and apply Corollary 9.

Geodesics starting from the origin are of the form e^{tH}, where H is a nonzero symmetric matrix. Therefore, for $\mathbb{P}$-almost every ω, there is a nonzero symmetric matrix $H(\omega)$ such that $\frac{1}{n} d(exp(nH(\omega)), Z_n(\omega))$ goes to 0 as $n \to \infty$ (the constant α has been incorporated in H). In other words, $\frac{1}{n}\ln$ of the norm, and of the norm of the inverse, of the matrix $exp(-nH(\omega))(A^{(n)}(\omega))^t$ go to 0 as $n \to \infty$. We claim that this gives the conclusion of Theorem 11 with λ_i the eigenvalues of $H(\omega)$, m_i their respective multiplicities and V_i the sums of the eigenspaces corresponding to eigenvalues smaller than λ_i. Indeed, we write $exp(H(\omega)) = K(\omega)^t \Delta K(\omega)$ for K an orthogonal matrix and Δ a diagonal matrix with diagonal entries e^{μ_i}, and $A^{(n)}(\omega) = L_n(\omega)\Delta_n(\omega)K_n(\omega)$ a Cartan decomposition of $A^{(n)}$ with L_n, K_n orthogonal, Δ_n a diagonal matrix with nonincreasing diagonal entries $exp(n\delta_1^{(n)}(\omega)) \geq \cdots \geq exp(n\delta_d^{(n)}(\omega))$. The conclusion of Corollary 9 is therefore that, for $\mathbb{P}$-almost every ω, $\frac{1}{n}\ln$ of the norm, and of the norm of the inverse, of the matrix $\Delta_n(\omega)K_n(\omega)K^t(\omega)exp(-n\Delta)$ go to 0 as $n \to \infty$.

It follows that, for such an ω, $\big|\ln|DetA^{(n)}(\omega)| - \sum_j \mu_j\big|$ goes to 0 as n goes to ∞. Furthermore, for n large enough, the entries $k_{i,j}^{(n)}(\omega)$ of the matrix $K_n(\omega)K^t(\omega)$ satisfy:

$$\big|k_{i,j}^{(n)}(\omega)\big| \ \leq \ e^{n(\mu_j - \delta_i^{(n)} + \varepsilon)}.$$

We have

$$\|A^{(n)}(\omega)e_j\| \ = \ \|L_n(\omega)\Delta_n(\omega)K_n(\omega)e_j\| \ = \ \|\Delta_n(\omega)K_n(\omega)K^{-1}(\omega)f_j\|,$$

where f_i is the canonical base of $\mathbb{R}^d$. The components of this vector are $e^{n\delta_i^{(n)}(\omega)}k_{i,j}^{(n)}(\omega)$. Their absolute values are indeed smaller than $e^{n(\mu_j+\varepsilon)}$ for n large enough.

4 The case when X is a Gromov hyperbolic space (in particular $\mathbb{R}$).

Theorem 7 is due to Kaimanovich using an idea of Delzant when X is a Gromov hyperbolic geodesic space even without the condition that X is a proper space [K2]. As in the $CAT(0)$-case it is there formulated as Z_n lies on sublinear distance of a geodesic ray. From Theorem 7 one gets the following:

Corollary 12 *Assume moreover that X is a Gromov hyperbolic geodesic space and that $\alpha > 0$. Then, for $\mathbb{P}$-almost every ω, as n goes to ∞, there is a geodesic ray σ_ω such that*

$$\lim_n \frac{1}{n} d(Z_n(\omega)x_0, \sigma_{h_\omega}(\alpha n)) = 0.$$

Proof. Take h_ω given from Theorem 7. It is known, see [BH, p. 428], that for Gromov hyperbolic geodesic spaces it holds that there is a geodesic ray σ_ω such that $\sigma_\omega(x_0) = 0$ and

$$b_\omega(\cdot) = \lim_{t\to\infty} d(\cdot, \sigma_\omega(t)) - t$$

is a horofunction such that $|b_\omega(\cdot) - h_\omega(\cdot)| \leq C$ for some constant C. This b_ω therefore clearly satisfies the conclusion of Theorem 7.

Now we use the notation and set-up in the proof of Corollary 9. Consider the triangle ABC_t. By δ-hyperbolicity D must lie at most δ away from either AB or AC_t. Call the closest point X. By the triangle inequality we must have that

$$\alpha n - \delta \leq XB \leq \alpha n + \delta.$$

If X lie on AB, then it is clear that $XA = o(n)$ and hence $AD = o(n)$. If X lie on AC_t, then

$$t - \alpha n - \delta \leq XC_t \leq t - \alpha n + \delta.$$

In view of that $b_\omega(Z_n(\omega)) \approx -\alpha n$ we again reach the conclusion that X, and hence also D, lie on sublinear distance from A. ■

Corollary 13 *With the same assumptions, we have that for $\mathbb{P}$-almost every ω, $Z_n(\omega)x_0$ converges to the point $[\sigma_\omega]$ in the hyperbolic boundary $\partial_{hyp}X$.*

Proof. Clearly, the Gromov product $(Z_n(\omega), \sigma_\omega(\alpha n)) \to \infty$ as $n \to \infty$ in view of the previous corollary. ■

In the case when $G = \mathbb{R}$ and $X = (\mathbb{R}, |\cdot|)$, Corollaries 12 and 13 yield Theorem 2. Indeed, in this case the drift is:

$$\alpha = \left| \int_\Omega f d\mathbb{P} \right|$$

and $\partial\mathbb{R} = \{h_+ = \Phi_{+\infty}(z) = -z,\ h_- = \Phi_{-\infty}(z) = z\}$. It follows from Corollary 13 that the index of h_ω is T invariant and is therefore almost everywhere constant. In other words, the existence of the h_ω with the required property amounts to the choice of the right sign:

$$h_\omega = \Phi_{sign\left\{\int_\Omega f(\omega)d\mathbb{P}(\omega)\right\}\infty}.$$

Then, Corollary 12 say exactly that if the function f is integrable, for $\mathbb{P}$-almost every ω

$$\frac{1}{n} S_n(\omega) \to \int_\Omega f(\omega) d\mathbb{P}(\omega).$$

The above observation is not a new proof of Theorem 2, because Theorem 2 is used in the proof of Theorem 7 (see section 5). We only want to illustrate the meaning of the metric boundary on the simplest example. Nevertheless, it turns out that modifying the translation invariant metric on $X = \mathbb{R}$ might have interesting consequences. The following discussion comes from [KMo], which in turn was inspired by [LL].

Let $D : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing function, $D(t) \to \infty$ such that $D(0) = 0$ and $D(t)/t \to 0$ monotonically. From the inequality

$$\frac{1}{t+s} D(t+s) \leq \frac{1}{t} D(t)$$

we get the following subadditivity property

$$D(t+s) \leq D(t) + \frac{s}{t} D(t) = D(t) + \frac{D(t)/t}{D(s)/s} D(s) \leq D(t) + D(s).$$

From all these properties of D, it follows that $(\mathbb{R}, D(|\cdot|))$ is a proper metric space, and clearly invariant under translations.

Now we determine $\partial\mathbb{R}$ with respect to this metric. Wlog we may assume that $x_n \to \infty$. We claim that for any z

$$h(z) = \lim_{n\to\infty} D(x_n - z) - D(x_n) = 0.$$

Assume not. Then for some $s > 0$ and an infinite sequence of $t \to \infty$ that $D(t+s) - D(t) > c > 0$ (wlog). For such s and t with t large so that $D(t)/t < c/s$, we have

$$\frac{D(t+s)}{t+s} \geq \frac{D(t)+c}{t+s} \geq \frac{D(t) + \frac{D(t)}{t}s}{t+s} = \frac{D(t)}{t}$$

but this contradicts that $D(t)/t$ is strictly decreasing. Hence $\partial\mathbb{R} = \{h \equiv 0\}$.

Applying Theorem 7 in this setting yields a result already obtained by Aaronson with a different argument.

Theorem 14 *[Aaronson [A]] Let $f : \Omega \to \mathbb{R}$ such that $\int_\Omega D(|f|)d\mu < \infty$. Then, for $\mathbb{P}$-almost every ω,*

$$\lim_{n\to\infty} \frac{1}{n} D\left(|S_n(\omega)|\right) = 0.$$

Proof. It was noted above that $\partial(\mathbb{R}, D(|\cdot|))$ only consisted of $h = 0$. The conclusion then follows from Theorem 7 since $h = 0$ forces $\alpha = 0$. ∎

One can relax the conditions on D: for one thing, one can remove having $D(0) = 0$. More interestingly, the condition that $D(t)/t$ decreases to 0 can be weakened in the following way.

Corollary 15 *[Aaronson-Weiss [A]] Let $d(t)$ be an increasing positive function, $d(t) \to \infty$, such that $d(t) = o(t)$, $d(t+s) \leq d(t)+d(s)$ and $\int_\Omega d(|f|)d\mu < \infty$ for some function $f : \Omega \to \mathbb{R}$. Then, for $\mathbb{P}$-almost every ω,*

$$\lim_{n\to\infty} \frac{1}{n} d\left(|S_n(\omega)|\right) = 0.$$

Proof. Define

$$D(t) = \sup\{d(ut)/u : u \geq 1\}.$$

Note that this satisfies all the assumptions made on D in Theorem 14. Moreover

$$d(t) \leq D(t) \leq 2d(t),$$

since if $D(t) = d(tu)/u$, set $n = [u] + 1$ and then $D(u) \leq d(nt)/u \leq nd(t)/u \leq 2d(t)$. See [A], page 66, for more details. This shows that Theorem 14 actually holds for d in place of D. ■

In particular, Corollary 15 applies to any metric $d(.,.)$ on $\mathbb{R}$ where balls grow superlinearly (where $d(t) := d(0,t)$). From this one obtains as a special case classical results like the one of Marcinkiewicz-Zygmund [MZ] and Sawyer ([S]):

Corollary 16 *Let $0 < p < 1$. If $f \in L^p$, then for $\mathbb{P}$-almost every ω*

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} S_n = 0.$$

Such moment conditions arise naturally in probability theory. These results are known to be best possible in certain ways (e.g. [S] and [A]). For the iid case the converse also holds ([MZ]). Another example

Corollary 17 *If f is log-integrable, then for $\mathbb{P}$-almost every ω*

$$\lim_{n\to\infty} |S_n|^{1/n} = 1.$$

One can modify the metric on any metric space X in the same way replacing $d(x,y)$ with $D(d(x,y))$, where $D(t)$ satisfies the assumptions for Theorem 14 or, more generally, the assumptions for Corollary 15. By estimating a subadditive by an additive cocycle in the obvious way,

$$a(n,\omega) \leq \sum_{k=0}^{n-1} a(1, T^k\omega),$$

Theorem 14 implies that

$$\frac{1}{n} D(d(Z_n x_0, x_0)) \to 0 \text{ a.e.}$$

under the condition that $D(d(g(\omega)x_0, x_0))$ is integrable.

5 Proof of Theorem 7.

We begin by a few observations: firstly, we can extend by continuity the action of G to $\overline{X}$, and write, for $h \in \overline{X}, g \in G$:

$$g.h(z) \ = \ h(g^{-1}z) - h(g^{-1}x_0).$$

Define now the skew product action on $\overline{\Omega} := \Omega \times \overline{X}$ by:

$$\overline{T}(\omega, h) \ = (T\omega, g(\omega)^{-1}.h).$$

Observe that $\overline{T}^n(\omega, h) \ = (T^n\omega, (Z_n(\omega))^{-1}.h)$. Define the Furstenberg cocycle $\overline{F}(\omega, h)$ by $\overline{F}(\omega, h) := -h(g(\omega)x_0)$. We have:

$$\overline{F}_n(\omega, h) \ := \ \sum_{i=0}^{n-1} \overline{F}(\overline{T}^i(\omega, h)) \ = \ -h(Z_n(\omega)x_0). \tag{3}$$

Relation (3) is proven by induction on n. We have $\overline{F}_1(\omega, h) := -h(g(\omega)x_0) = -h(Z_1(\omega)x_0)$ and

$$\begin{aligned}
\overline{F}_n(\omega, h) \ &= \ \overline{F}_{n-1}(\omega, h) + \overline{F}(\overline{T}^{n-1}(\omega, h)) \\
&= \ -h(Z_{n-1}(\omega)x_0) - (Z_{n-1}(\omega))^{-1}.h(g(T^{n-1}\omega)x_0) \\
&= \ -h(Z_{n-1}(\omega)x_0) - h(Z_{n-1}(\omega)g(T^{n-1}(\omega))x_0) + h(Z_{n-1}(\omega)x_0) \\
&= \ -h(Z_n(\omega)x_0).
\end{aligned}$$

In particular, for any $\overline{T}$ invariant measure m on $\overline{\Omega}$ such that the projection on Ω is $\mathbb{P}$, we have $\int \overline{F} dm \leq \alpha$ because:

$$\int \overline{F}(\omega, h) dm(\omega, h) \ = \ \frac{1}{n} \int -h(Z_n(\omega)x_0) dm(\omega, h) \ \leq \frac{1}{n} \int |Z_n(\omega)| d\mathbb{P}(\omega).$$

There is nothing to prove if $\alpha = 0$. To prove Theorem 7 in the case $\alpha > 0$, it suffices to construct a $\overline{T}$ invariant measure m on $\overline{\Omega}$ such that the projection on Ω is $\mathbb{P}$ and such that $\int \overline{F}(\omega, h) dm(\omega, h) = \alpha$. Indeed, since α is the largest possible value of $\int \overline{F}$, we still have the same equality for almost every ergodic component of m. By the Ergodic Theorem 2, the set A of (ω, h) such that $-\frac{1}{n}h(Z_n(\omega)x_0) = \frac{1}{n}\overline{F}_n(\omega, h) \to \alpha$ as n goes to ∞ has full measure. Moreover, observe that if h is not a point in ∂X, $-\frac{1}{n}h_\gamma(Z_n(\omega)x_0)$ converges to $-\alpha$. Since $\alpha > 0$, this shows that $A \subset \partial X$. We get the conclusion of Theorem 7 by choosing for $\omega \mapsto h_\omega$ a measurable section of the set A.

We finally construct a measure m with those properties. We define a measure μ_n on $\overline{\Omega}$; for any measurable function Ξ on $\overline{\Omega}$ such that

$$\int \sup_{h \in \overline{X}} |\Xi(\omega, h)| \, d\mathbb{P}(\omega) < \infty,$$

we set:

$$\int_{\Omega\times\overline{X}} \Xi(\omega,h)d\mu_n(\omega,h) \;=\; \int_{\Omega} \Xi(\omega,\Phi_{Z_n(\omega)x_0})d\mathbb{P}(\omega).$$

The set of measures m on $\overline{\Omega}$ such that the projection on Ω is $\mathbb{P}$ is a convex compact subset of $L^\infty(\Omega,\mathcal{P}(\overline{X})) = (L^1(\Omega,C(\overline{X})))^*$ for the weak* topology. The mapping $m \mapsto (\overline{T})_* m$ is affine and continuous. We can take for m any weak* limit point of the sequence:

$$\eta_n \;=\; \frac{1}{n}\sum_{i=0}^{n-1}(\overline{T}^i)_*\mu_n.$$

The measure m is $\overline{T}$ invariant and, since

$$\|\overline{F}\|_{L^1(\Omega,C(\overline{X}))} = \int \sup_h \left|\overline{F}(\omega,h)\right| d\mathbb{P}(\omega) = \int \sup_h |h(g(\omega)x_0)| d\mathbb{P}(\omega) < +\infty,$$

we may write, using relation (3) and the formula (2):

$$\begin{aligned}
\int \overline{F} dm \;&=\; \lim_{k\to\infty}\frac{1}{n_k}\int\sum_{i=0}^{n_k-1}(\overline{F}\circ\overline{T}^i)d\mu_{n_k} \\
&=\; \lim_{k\to\infty}\frac{1}{n_k}\int \overline{F}_{n_k}(\omega,\Phi_{Z_{n_k}(\omega)x_0})d\mathbb{P}(\omega) \\
&=\; \lim_{k\to\infty}\frac{1}{n_k}\int (-\Phi_{Z_{n_k}(\omega)x_0}(Z_{n_k}(\omega)x_0))d\mathbb{P}(\omega) \\
&=\; \lim_{k\to\infty}\frac{1}{n_k}\int |Z_{n_k}(\omega)|d\mathbb{P}(\omega) \;=\alpha.
\end{aligned}$$

By the above discussion this achieves the proof of Theorem 7.

Observe that by putting together the discussions in sections 5 and 3, we obtain a proof of Oseledets Theorem 11. As proofs of Theorem 11 go, this one is in some sense rather close to the original one ([O]), with the somewhat simplifying use of the geometric ideas from [K3] and invariant measures as in [W].

6 Random Walks.

In this section we consider a probability ν on a group G and apply the preceeding analysis to the random walk $Z_n = g_0 g_1 \dots g_{n-1}$, where the g_i are independent with distribution ν. We assume:

- there is a proper left invariant metric d on G which generates the topology of G (when G is second countable locally compact, such a metric always exists, see [St]),

- $\int d(e,g)d\nu(g) < +\infty$ (we say that ν has a *first moment*) and

- the closed subgroup generated by the support of ν is the whole G (we say that ν is *non-degenerate*).

Then, there is a number $\ell(\nu) \geq 0$ such that, for almost every sequence $\{g_i\}$, $\lim_n \frac{1}{n} d(e, Z_n) = \ell(\nu)$. In the case when the group G is the group $SL_2(\mathbb{R})$ acting on the hyperbolic plane, $\ell(\nu)$ is twice the Lyapunov exponent of the independent product of matrices. In that case it is given by a formula involving the stationary measure on the circle, the Furstenberg-Khasminskii formula ([F1]; this appellation seems to be standard, cf. [Ar]). Seeing again the circle as the geometric boundary of the hyperbolic plane, we extend this formula to our general context:

Theorem 18 *[Furstenberg-Khasminskii formula for the linear drift, [KL2]]. Let (G,ν) verify all the above assumptions, and let $\overline{G}$ be the metric compactification of (G,d). Then there exists a measure μ on $\overline{G}$ with the following properties:*

- *μ is stationary for the action of G, i.e. μ satisfies $\mu = \int (g_*\mu) d\nu(g)$ and*

- *$\ell(\nu) = \int h(g^{-1}) d\mu(h) d\nu(g)$.*

Moreover, if $\ell(\nu) > 0$, then μ is supported on ∂G.

Proof. In the proof of Theorem 7, we constructed a measure m on $\Omega \times \overline{G}$. The measure μ can be seen as the projection on $\overline{G}$ of m, but it turns out that the measure μ can be directly constructed. Let $(\Omega^+, \mathcal{A}^+, \mathbb{P})$ be the space of sequences $\{g_0, g_1, \dots\}$ with product topology, σ-algebra and measure $\mathbb{P} = \nu^{\otimes \mathbb{N}}$. For $n \geq 0$, let ν_n be the distribution of $Z_n(\omega)$ in $\overline{G}$. In other words, define, for any continuous function f on $\overline{G}$:

$$\int f d\nu_n = \int f(g_0 g_1 \cdots g_{n-1}) d\nu(g_0) d\nu(g_1) \cdots d\nu(g_{n-1}), \quad \nu_0 = \delta_e.$$

We claim that any weak* limit μ of the measures $\frac{1}{n}\sum_{i=0}^{n-1} \nu_i$ satisfies the conclusions of Theorem 18. Clearly, the measure μ is stationary: for any continuous function f on $\overline{G}$, we have

$$\begin{array}{c} \int f(g.h)d\mu(h)d\nu(g) \\ = \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int f(gg_0g_1\cdots g_{i-1})d\nu(g_0)d\nu(g_1)\cdots d\nu(g_{i-1})d\nu(g) \\ = \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int f d\nu_{i+1} \\ = \int f d\mu + \lim_{k\to\infty}\frac{1}{n_k}[\int f d\nu_{n_k} - f(e)] \ = \ \int f d\mu. \end{array}$$

In the same way, we get:

$$\begin{array}{rc} & \int h(g^{-1})d\mu(h)d\nu(g) \\ = & \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int[d(Z_i(\omega),g^{-1})-d(Z_i(\omega),e)]d\mathbb{P}(\omega)d\nu(g) \\ = & \lim_{k\to\infty}\frac{1}{n_k}\int d(Z_{n_k},e)d\mathbb{P}(\omega) \ = \ \ell(\nu). \end{array}$$

This shows that the measure μ has the desired properties. Moreover, the measure $\mathbb{P}\times\mu$ on the space $\Omega^+\times\overline{G}$ is $\overline{T}$-invariant. There is a unique $\overline{T}$-invariant measure m on $\Omega\times\overline{G}$ that extends $\mathbb{P}\times\mu$. The measure m satisfies all the properties we needed in the proof of Theorem 7. In particular, if $\ell(\nu)$ is positive,

$$\mu(\partial G) \ = \ (\mathbb{P}\times\mu)(\Omega\times\partial G) \ = \ m(\Omega\times\partial G) \ = \ 1.$$

■

A bounded measurable $f : G \to \mathbb{R}$ is *ν-harmonic* if

$$f(g) = \int_G f(gh)d\nu(h)$$

for any $g \in G$. Constant functions are obviously ν-harmonic. If f is a bounded harmonic function, then $f(Z_n)$ is a bounded martingale and therefore converges almost surely. We say that (G,ν) satisfies the *Liouville property* (or (G,ν) is Liouville) if the constant functions are the only bounded ν-harmonic functions.

Corollary 19 *[KL2] Let G be a locally compact group with a left invariant proper metric and ν be a nondegenerate probability measure on G with first moment. Then, if (G,ν) is Liouville, there is a* 1*-Lipschitz homomorphism $T : G \to \mathbb{R}$ such that for almost every trajectory Z_n of the corresponding random walk, we have:*

$$\lim_{n\to\infty}\frac{1}{n}T(Z_n) \ = \ \int_G T(g)d\nu(g) \ = \ l(\nu).$$

Proof. The key observation is that if (G, ν) is Liouville and G acts continuously on a compact space Y, then every stationary measure μ is invariant. Indeed, for $f \in C(Y, \mathbb{R})$, the function $\varphi(g) := \int f d(g_* \mu)$ is harmonic and bounded, therefore constant. In particular, the measure μ from Theorem 18 is invariant, and if we set

$$T(g) := \int h(g^{-1}) \mu(dh),$$

The mapping T is Lipschitz continuous and is a group homomorphism because we have:

$$\begin{aligned} T(g'g) &= \int h(g^{-1}g'^{-1})\mu(dh) \\ &= \int (g'.h)(g^{-1})\mu(dh) + \int h(g'^{-1})\mu(dh) \\ &= \int h(g^{-1})(g'_*\mu)(dh) + T(g') \\ &= T(g) + T(g'), \end{aligned}$$

where we used the invariance of μ at the last line. Finally, by the Furstenberg Khasminskii formula, we have:

$$\ell(\nu) = \int T(g) d\mu(g).$$

■

A measure ν on G is called *symmetric* if it is invariant under the mapping $g \mapsto g^{-1}$. A measure is *centered* if every homomorphism of G into $\mathbb{R}$ is centered, meaning that the ν-weighted mean value of the image is 0. Every symmetric measure with first moment ν is centered, since for any homomorphism $T : G \to \mathbb{R}$, the mean value, which is

$$\int_G T(g) d\nu(g) = \int_G T(g^{-1}) d\nu(g) = -\int_G T(g) d\nu(g),$$

must hence equal 0. By simple contraposition from Corollary 19, we get:

Corollary 20 *[KL2] Let G be a locally compact group with a left invariant proper metric and ν be a nondegenerate centered probability measure on G with first moment. Then, if $l(\nu) > 0$, there exist nonconstant bounded ν-harmonic functions.*

Corollary 20 was known in particular for ν with finite support ([Va], [M]) or in the continuous case, for ν with compact support and density ([Al]).

One case when all probability measures on G are centered is when there is no group homomorphism from G to $\mathbb{R}$. We can apply Corollary 20 to a countable finitely generated group. Let S be a finite symmetric generator for G, and endow G with the left invariant metric $d(x,y) = |y^{-1}x|$ where $|z|$ is the shortest length of a S-word representing z. We say that G has subexponential growth if $\lim_n \frac{1}{n} \ln A_n = 0$, where A_n is the number of elements z of G with $|z| \leq n$. Such a group has automatically the Liouville property ([Av]). This yields:

Corollary 21 *[KL2] Let G be a finitely generated group with subexponential growth and $H^1(G,\mathbb{R}) = 0$. Then for any nondegenerate ν on G with first moment, we have $\ell(\nu) = 0$.*

Observe that conversely, if there exists a nontrivial group homomorphism T from a finitely generated group into $\mathbb{R}$, then there exists a nondegenerate probability ν on G, with first moment and $\ell(\nu) > 0$. Indeed, there exists M such that $[-M, M]$ contains all the images of the elements of the generating set S. We can choose ν carried by all the elements of S with images in $[0, M]$. Since T is nontrivial, $\int T(g)d\nu(g) > 0$. The measure ν is nondegenerate, has finite support and $\ell(\nu) > 0$ since for all $g \in G, |T(g)| \leq M|g|$.

7 Riemannian covers.

In this section we consider a complete connected Riemannian manifold (M, g) with bounded sectional curvatures. In particular, if d_M is the Riemannian distance on M, (M, d) is a proper space. Associated to the metric is the Laplace-Beltrami operator Δ. A function f is *harmonic* if $\Delta f = 0$. We say that M is *Liouville* if all bounded and harmonic functions are constant.

Associated to Δ is a diffusion process B_t called *Brownian motion*. Since the curvature is bounded and M is complete, the Brownian motion is defined for all time. For all $x \in M$, there is a probability $\mathbb{P}_x$ on $C(\mathbb{R}_+, M)$ such that the process B_t given by the t coordinate is a Markov process with generator Δ and $B_0 = x$. We can define

$$\ell_g := \limsup_{t\to\infty} \frac{1}{t} d_M(x_0, B_t),$$

for any $x_0 \in M$.

Theorem 22 *[KL3] Assume that (M,g) is a regular covering of a Riemannian manifold which has finite Riemannian volume and bounded sectional curvatures. Then M is Liouville if, and only if,*

$$\lim_{t\to\infty} \frac{1}{t} d(x_0, B_t) = 0 \ a.s..$$

The "if" part was proved by Kaimanovich, see [K1], and the converse is clear if the Brownian motion is recurrent on M. The proof of the new implication in Theorem 22 in the transient case uses the Furstenberg-Lyons-Sullivan discretization procedure. Let Γ be the covering group of isometries of M. This discretization consists in the construction of a probability measure ν on Γ, with the following properties:

- The restriction $f(\gamma) := F(\gamma x_0)$ is a one-to-one correspondence between bounded harmonic functions on M and bounded functions on Γ which satisfy
$$f(\gamma) = \sum_{g\in\Gamma} f(\gamma g)\nu(g)$$
.
- If $\gamma_1, \ldots, \gamma_n$ are chosen independent and with distribution ν, then $\lim_{n\to\infty} \frac{1}{n} d_M(x_0, \gamma_1 \ldots \gamma_n x_0)$ exists. It vanishes a.e. if, and only if, $\lim_{t\to\infty} \frac{1}{t} d_M(x_0, B_t) = 0$ a.s..
- In the case when the Brownian motion is transient, one can choose ν symmetric, i.e. such that for all γ in Γ, $\nu(\gamma^{-1}) = \nu(\gamma)$.

The first property goes back to Furstenberg ([F2]) and has been systematically developed by Lyons and Sullivan ([LS]) and Kaimanovich ([K5]). The second one was observed in certain situations by Guivarc'h ([G]) and Ballmann ([Ba]). Babillot observed that the modified construction of [BL] has the symmetry property. Given the above, proving Theorem 22 mostly reduces to Corollary 20, if we can show that hypotheses of Corollary 20 are satisfied. We endow Γ with the metric defined by the metric of M on the orbit Γx_0. This defines a left invariant and proper metric on Γ: bounded sets are finite, because they correspond to pieces of the orbit situated in a ball of finite volume. The measure ν is nondegenerate because its support is the whole Γ. It is shown in [KL3] that the measure ν has a first moment. The proof uses the details of the construction, but the idea is that the distribution of ν is given by choosing some random time and looking at the point γx_0 close to the trajectory of the Brownian motion at that time.

Since the curvature is bounded from below, if the expectation of the time is finite, the expectation of the distance of the Brownian point at that time is finite as well. It also follows that the rates of escape of the Brownian motion and of the Random walk are proportional. Therefore, if the manifold (M, g) is Liouville, then the Random walk (G, ν) is Liouville. By Corollary 20, $\lim_{n\to\infty} \frac{1}{n} d_M(x_0, \gamma_1 \dots \gamma_n x_0) = 0$ and therefore, $\lim_{t\to\infty} \frac{1}{t} d_M(x_0, B_t) = 0$ a.s..

There are many results about the Liouville property for Riemannian covers of a compact manifold. Theorem 22 implies that the corresponding statements hold for the rate of escape of the Brownian motion. Guivarc'h ([G]) showed that if the group Γ is not amenable, then (M, g) is not Liouville, whereas when Γ is polycyclic, (M, g) is Liouville (Kaimanovich [K2]). Lyons and Sullivan ([LS], see also [Er] for a simply connected example) have examples of amenable covers without the Liouville property.

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