

Elements of a metric spectral theory

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Dedicated to Margulis, with admiration

Abstract

This paper discusses a general method for spectral type theorems using metric spaces instead of vector spaces. Advantages of this approach are that it applies to genuinely non-linear situations and also to random versions. Metric analogs of operator norm, spectral radius, eigenvalue, linear functional, and weak convergence are suggested. Applications explained include generalizations of the mean ergodic theorem, the Wolff-Denjoy theorem and Thurston's spectral theorem for surface homeomorphisms.

1 Introduction

In one line of development of mathematics, considerations progressed from concrete functions, to vector spaces of functions, and then to abstract vector spaces. In parallel, the standard operations, such as derivatives and integrals, were generalized to the abstract notions of linear operators, linear functionals and scalar products. The study of the category of topological vector spaces and continuous linear maps is basically what is now called functional analysis. Dieudonné wrote that if one were to reduce the complicated history of functional analysis to a few keywords, the emphasis should fall on the evolution of two concepts: *spectral theory* and *duality* [Di81]. Needless to say, as most often is the case, the abstract general study does not supersede the more concrete considerations in every respect. In the context of analysis, one can compare the two different points of view in the excellent texts [L02] and [StS11].

The metric space axioms were born out of the same development, see the historical note in [Bo87] or [Di81]. In the present paper, I would like to argue for a another step: from normed vector spaces to metric spaces (and their generalizations), and bounded linear operators to semicontractions. This could be called *metric functional analysis*, or in view of the particular focus here, a *metric spectral theory*. Indeed we will in the metric setting discuss a spectral principle and duality in form of metric functionals. This is motivated by situations which are genuinely non-linear, but there is also an interest in the metric perspective even in the linear case. The latter can be exemplified by a well-known classical instance: for many questions in the study of groups of 2×2 real matrices, it is easier to employ their (associated) isometric action on the hyperbolic plane, which is indeed a metric and not a linear space, instead of the linear action on \mathbb{R}^2 . The isometric action of $\mathrm{PSL}_2(\mathbb{R})$ is by fractional linear transformations preserving the upper half plane. This generalizes to $n \times n$ matrices and the associated symmetric space.

Geometric group theory is a subject that has influenced the development of metric geometry during the last few decades. Gromov has been the leading person in this subject with his

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many and diverse contributions. Originally he found some inspiration from combinatorial group theory and the Mostow-Margulis rigidity theory (for example, the Gromov product appeared in Lyndon’s work, Mostow introduced the crucial notion of quasi-isometry, and Margulis noted that one can argue in terms of word metrics in this context of quasi-isometries and boundary maps).

There is another strand of metric geometry sometimes called the *Ribe program*, see Naor’s recent ICM plenary lecture [N18] for some history and appropriate references. Bourgain wrote already in 1986 [B86] in this context that: “the notions from local theory of normed spaces are determined by the metric structure of the space and thus have a purely metrical formulation. The next step consists in studying these metrical concepts in general metric spaces in an attempt to develop an analogue of the linear theory.” The present text suggests something similar, yet rather different. The properties of the Banach spaces and metric spaces studied in the Ribe program are rather subtle, in contrast, we are here much more basic and in particular motivated by understanding distance-preserving self-maps. This latter topic we see as a kind of metric spectral theory with consequences within several areas of mathematics: geometry, topology, group theory, ergodic theory, probability, complex analysis, operator theory, fixed point theory etc.

We consider metric spaces (X, d) , at times with the symmetry axiom removed, and the corresponding morphisms, here called semicontractions (in contrast to bi-Lipschitz maps in the context of Bourgain, Naor et al). A map f between two metric spaces is a *semicontraction* if distances are not increased, that is, for any two points x and y it holds that:

$$d(f(x), f(y)) \leq d(x, y).$$

Synonyms are 1-Lipschitz or non-expansive maps.

It is reasonable to wonder whether in such a general setting there could be anything worthwhile to uncover. One useful general statement is well-known: the *contraction mapping principle* which is a basic tool for finding solutions to equations. The abstract statement appeared in Banach’s thesis, but some version might have been used before (for the existence and uniqueness of solutions to certain ordinary differential equations). In this paper I will suggest a complement to this principle, which basically appeared in [Ka01], and that is applicable more generally than the contraction mapping principle since isometries are included.

The objective here is to discuss metric space analogs of the following linear concepts:

- linear functionals and weak topology
- operator norm and spectral radius
- eigenvalues and Lyapunov exponents,

and then show how these metric notions can be applied. At the center for applications is, as already indicated, a complement to the contraction mapping principle, namely a *spectral principle* [Ka01, GV12], its ergodic theoretic generalization [KaM99, KaL11, GK15], see also [G18], and a special type of metrics that could be called spectral metrics [T86, Ka14].

Here is an example: Let M be an oriented closed surface of genus $g \geq 2$. Let \mathcal{S} denote the isotopy classes of simple closed curves on M not isotopically trivial. For a Riemannian metric ρ on M , let $l_\rho(\beta)$ be the infimum of the length of curves isotopic to β . In a seminal preprint from 1976 [T88], Thurston could show the following consequence (the details are worked out in [FLP79, Théorème Spectrale]):

Theorem 1. ([T88, Theorem 5]) For any diffeomorphism f of M , there is a finite set $1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_K$ of algebraic integers such that for any $\alpha \in \mathcal{S}$ there is a λ_i such that for any Riemannian metric ρ ,

$$\lim_{n \rightarrow \infty} l_\rho(f^n \alpha)^{1/n} = \lambda_i.$$

The map f is isotopic to a pseudo-Anosov map iff $K = 1$ and $\lambda_1 > 1$.

This is analogous to a simple statement for linear transformations A in finite dimensions: given a vector v there is an associated exponent λ (absolute value of an eigenvalue), such that

$$\lim_{n \rightarrow \infty} \|A^n v\|^{1/n} = \lambda.$$

To spell out the analogy: diffeomorphism f instead of a linear transformation A , a length instead of a norm, and a curve α instead of a vector v . Below we will show how to get the top exponent, even for a random product of homeomorphisms, using our metric ideas and a lemma in Margulis' and my paper [KaM99]. This is a different approach than [Ka14]. To get all the exponents (without their algebraic nature) requires some additional arguments, see [H16].

One of the central notions in the present text is that of a Busemann function or metric functional. This notion appears implicitly in classical mathematics, with Poisson and Eisenstein, and is by now recognized by many people as a fundamental tool. In differential geometry, see the discussion in Yau's survey [Y11]. Busemann functions play a crucial role in the Cheeger-Gromoll splitting theorem for manifolds with non-negative Ricci curvature. The community of researchers of non-positive curvature also has frequently employed Busemann functions. For example, it has been noted by several people that the horofunction boundary (metric compactification) is the right notion when generalizing Patterson-Sullivan measures, see for example [CDST18] for a recent contribution. In my work with Ledrappier, we used this notion without knowing anything about the geometry of the Cayley graphs, in particular without any curvature assumption. Related to this, with a view towards another approach to Gromov's polynomial growth theorem, see [TY16]. There are many other instances one could mention, but still, it seems to me that the notion of a Busemann function remains a bit off the mainstream, instead of taking its natural place dual to geodesics.

A note on terminology. When I had a choice, or need, to introduce a word for a concept, I sometimes followed Serge Lang's saying that terminology should (ideally) be functorial with respect to the ideas. Hence I use the word *metric functional* for a variant of the the notion of horofunction usually employed and introduced by Gromov generalizing an older concept due to Busemann in turn extending a notion in complex analysis (and also from Martin boundary theory). While some people do not like this, I thought it could be useful to avoid confusion to have different terms for different concepts, even when, or precisely because, these are variants of each other. In addition to being functorial in the ideas, *metric functional* also sounds more basic and fundamental as a notion than *horofunction* does. Indeed, the present paper tries to argue for the analogy with the linear case and the basic importance of the metric concept of horofunctions or metric functionals.

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2 Functionals

Linear theory

For vector spaces E , *lines*

$$\gamma : \mathbb{R} \rightarrow E$$

are of course fundamental objects, as are their dual objects, the *linear functionals*

$$\phi : E \rightarrow \mathbb{R}.$$

In the case of normed vector spaces the existence of continuous linear functionals relies in general on Zorn's lemma via the Hahn-Banach theorem. It is an abstraction of integrals. The sublevel sets of ϕ define half-spaces. The description of these functionals is an important aspect of the theory, see for example the section entitled *The search for continuous linear functionals* in [Di81].

Metric theory

For metric spaces X , *geodesic lines*

$$\gamma : \mathbb{R} \rightarrow X$$

are fundamental. The map γ is here an isometric embedding. (Note that *geodesic* have two meanings: in differential geometry they are locally distance minimizing, while in metric geometry they are most often meant to be globally distance minimizing. The concepts coincide lifted to contractible universal covering spaces.) Now we will discuss what the analog of linear functionals should be, that is, some type of maps

$$h : X \rightarrow \mathbb{R}.$$

OBSERVATION 1: Let X be a real Hilbert space. Take a vector v with $\|v\| = 1$ and consider

$$\begin{aligned} \lim_{t \rightarrow \infty} \|tv - y\| - \|tv\| &= \lim_{t \rightarrow \infty} \sqrt{(tv - y, tv - y)} - t = \lim_{t \rightarrow \infty} \frac{(tv - y, tv - y) - t^2}{\sqrt{(tv - y, tv - y)} + t} = \\ &= \lim_{t \rightarrow \infty} \frac{t(-2(y, v) + (y, y)/t)}{t(\sqrt{1 - 2(y, v)/t + (y, y)/t^2} + 1)} = -(y, v). \end{aligned}$$

In this way one can recover the scalar product from the norm, differently than from the polarization identity.

In an *analytic continuation of ideas* as it were, one is then led to the next observation (which maybe is not how Busemann was thinking about this):

OBSERVATION 2: (Busemann) Let γ be a geodesic line (or just a ray $\gamma : \mathbb{R}_+ \rightarrow X$). Then the following limit exists:

$$h_\gamma(y) = \lim_{t \rightarrow \infty} d(\gamma(t), y) - d(\gamma(t), \gamma(0)).$$

The reason for the existence of the limit for each y is that the sequence in question is bounded from below and monotonically decreasing (thanks to the triangle inequality), see [BGS85, BrH99].

Example. The open unit disk of the complex plane admits the Poincaré metric, in its infinitesimal form

$$ds = \frac{2|dz|}{1-|z|^2}.$$

This gives a model for the hyperbolic plane and moreover it is fundamental in the way that every holomorphic self-map of the disk is a semicontraction in this metric, this is the content of the Schwarz-Pick lemma. The Busemann function associated to the ray from 0 to the boundary point ζ , in other words $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, is

$$h_\zeta(z) = \log \frac{|\zeta - z|^2}{1 - |z|^2}.$$

These functions appear (in disguise) in the Poisson integral representation formula and in Eisenstein series.

One can take one more step, which will be parallel to the construction of the Martin boundary in potential theory. This specific metric idea might have come from Gromov around 1980 (except that he considers another topology – an important point for us here).

Let (X, d) be a metric space (perhaps without the symmetric axiom for d satisfied, this point is being discussed in [W11] and [GV12]). Let

$$\Phi : X \rightarrow \mathbb{R}^X$$

be defined via

$$x \mapsto h_x(\cdot) := d(\cdot, x) - d(x_0, x).$$

This is a continuous injective map. The functions h and their limits are called *metric functionals*. In view of Observation 2 Busemann functions are examples of metric functionals and (easily seen) not being of the form h_x , with $x \in X$. Even though geodesics may not exist, metric functionals always exist. Note that like in the linear case functionals are normalized to be 0 at the origin: $h(x_0) = 0$.

Every horofunction (i.e. uniform limit on bounded subsets of functions h_x as x tends to infinity) is a metric functional and every Busemann function is a metric functional. On the other hand, in general it is a well-recognized fact that not every horofunction is a Busemann function (such spaces could perhaps be called non-reflexive) and also not every Busemann function is a horofunction, some artificial counterexamples showing this can be thought of:

Example. Take one ray $[0, \infty]$ that will be geodesic, then add an infinite number of points at distance 1 to the point 0 and distance 2 to each other. Then at each point n on the ray, connect it to one of the points around 0 with a geodesic segment of length $n - 1/2$. This way $h_\gamma(y) = \lim_{t \rightarrow \infty} d(\gamma(t), y) - d(\gamma(t), \gamma(0))$ still of course converge for each y but not uniformly. Hence the Busemann function h_γ is a metric functional but not a horofunction.

As already stated, to any geodesic ray from the origin there is an associated metric functional (Busemann function), compare this with the situation in the linear theory that the fundamental Hahn-Banach theorem addresses. In the metric category the theory of injective metric spaces considers when semicontractions (1-Lipschitz maps) defined on a subset can be extended, see [La13] and references therein. The real line is injective, which means that for any subset A of a metric space B and semicontraction $f : A \rightarrow \mathbb{R}$ there is an extension of f to $B \rightarrow \mathbb{R}$ without increasing the Lipschitz constant, for example

$$\bar{f}(b) := \sup_{a \in A} (f(a) - d(a, b))$$

or

$$\bar{f}(b) := \inf_{a \in A} (f(a) + d(a, b)),$$

It would require a lengthy effort to try to survey all the purposes horofunctions have served in the past. Two instances can be found in differential geometry, in non-negative curvature, the Cheeger-Gromoll theorem, and in non-positive curvature, the Burger-Schroeder-Adams-Ballmann theorem. In my experience, many people know of one or a few applications, but few have an overview of all the applications. Other applications are found below or in papers listed in the bibliography, for example let us mention a recent Furstenberg-type formula for the drift of random walks on groups [CLP17] in part building on [KaL06, KaL11]. It is also the case that the last two decades have seen identifications and understanding of horofunctions for various classes of metric spaces.

3 Weak convergence and weak compactness

Linear theory

One of the main uses for continuous linear functionals is to define weak topologies which have compactness properties even when the vector space is of infinite dimension (the Banach-Aloglu theorem), see [L02].

Metric theory

We will now discuss how the definition of metric functionals on a metric space will provide the metric space with a weak topology for which the closure is compact. There have been other more specific efforts to achieve this in special situations, maybe the first one for trees can be found in Margulis paper [Ma81], see also [CSW93] for another approach, [Mo06] for a discussion in non-positive curvature, and then [GV12] for the general method taken here.

Let X be a set. By a *hemi-metric* on X we mean be a function

$$d : X \times X \rightarrow \mathbb{R}$$

such that $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$ and $d(x, y) = 0$ if and only if $x = y$. (The latter axiom can be satisfied by passing to a quotient space.) In other words we do not insist that d is symmetric (one could symmetrize it), nor positive. For more discussion about such metrics, see [GV12, W11]. One way to proceed is to consider

$$D(x, y) := \max \{d(x, y), d(y, x)\}$$

which clearly is symmetric, but also positive, see [GV12], so an honest metric. One can take the topology on X from D .

For a weak topology there are a couple of alternative definitions, but we proceed as follows. As defined in the previous section, let

$$\Phi : X \rightarrow \mathbb{R}^X$$

defined via

$$x \mapsto h_x(\cdot) := d(\cdot, x) - d(x_0, x).$$

This is a continuous injective map. By the triangle inequality we note that

$$-d(x_0, y) \leq h_x(y) \leq d(y, x_0).$$

A consequence of this in view of Tychonoff's theorem is that with the pointwise (=product) topology the closure $\overline{\Phi(X)}$ is compact. In general this is not a compactification in the strict and standard sense that the space sits as an open dense subset in a compact Hausdorff space, but it is convenient to still call it a compactification, for a discussion about this terminology see [Si15, 6.5].

Example. This has by now been studied for a number of classes of metric spaces: non-positively curved spaces [BGS85, BrH99], Gromov hyperbolic spaces ([BrH99] or a more recent and closer to our consideration is [MT18]), Banach spaces [W07, Gu17, Gu18], Teichmüller spaces (see [Ka14] for references in particular to Walsh), Hilbert metrics [W11, W18, LN12], Roller boundary of CAT(0)-cube complex (Bader, Guralnick, Finkelshtein, unpublished), and symmetric spaces of noncompact type equipped with Finsler metrics [KL18].

Let me try to introduce some terminology. We call $\overline{\Phi(X)}$ the *metric compactification* (the term was also coined for proper geodesic metric spaces by Rieffel in a paper on operator algebras and noncommutative geometry) and denote it by \overline{X} , even though this is a bit abusive, since the topology of X itself might be different. The closure that is usually considered starting from Gromov, see [BGS85, BrH99], is to take the topology of uniform convergence on bounded sets (note that uniform convergence on compact sets is in the present context equivalent to our pointwise convergence), and following [BrH99] we call this the *horofunction bordification*. For proper geodesic spaces the two notions coincide.

Example. A simple useful example is the following metric space, which I learnt from Uri Bader. Consider longer and longer finite closed intervals $[0, n]$ all glued to a point x_0 the point 0. This becomes a countable (metric) tree which is unbounded but contain no infinite geodesic ray. By virtue of being a tree it is CAT(0). It is easy to directly verify that there are no limits in terms of the topology of uniform convergence on bounded subsets. Alternatively, one can see this less directly since for CAT(0) spaces every horofunction is a Busemann function, but there are no (infinite) geodesic rays. So there are no horofunctions in the usual sense, the horofunction bordification is empty, no points are added. The metric compactification also does not add any new points, but new topology is such that every unbounded subsequence converges to h_{x_0} . This shows in particular that there are minor inaccuracies in [BrH99, 8.15 exercises] and [GV12, remark 14].

Some more terminology: we call as said above the elements in $\overline{\Phi(X)}$ *metric functionals*. We call *horofunctions* those that arise from unbounded subsequences via the strong topology, that of uniform convergence on bounded subsets. The metric functionals coming from geodesic rays, via Busemann's observation above are called *Busemann functions*. As observed above, not every Busemann function is a horofunction and vice versa.

In my opinion these examples show the need for a precise and new terminology, instead of just using the word *horofunction* for all these concepts, and let its precise definition depend on the context.

Moreover, we attempt to distinguish further between various classes of metric functionals. We have *finite metric functionals* and *metric functionals at infinity*. The latter are those functions which has $-\infty$ as its infimum. The former are hence those metric functionals that have an finite infimum. Busemann functions are always at infinity. The tree example above shows that even an unbounded sequence can converge to a finite metric functional. (What can easily be shown though is that every metric functional at infinity can only be reached via an unbounded sequence). An example of a metric functional at infinity that has finite infimum is the $h_{\infty,0} \equiv 0$ in the Hilbert space example in the next section.

One can have metrically improper metric functionals with infinite infimum. For the finite metric functionals we suggest moreover that the once coming from points $x \in X$, h_x are *internal (finite) metric functionals* and the complement of these are the *exotic (finite) metric functionals*. Examples of the latter are provided by the Hilbert space proposition in the next section (their existence is needed since we claim to obtain a compact space in which the Hilbert space sits). For related division of metric functionals in the context of Gromov hyperbolic spaces, see [MT18].

Example. Here is a simple illustration of how the notion of metric functionals interacts with Gromov hyperbolicity. Let h be a metric functional (Busemann function) defined by a sequence y_m belonging to a geodesic ray from x_0 . Assume that x_n is a sequence such that $h(x_n) < 0$ and $x_n \rightarrow \infty$. Then

$$2d(x_n, y_m) = d(x_n, x_0) + d(y_m, x_0) - d(x_n, y_m) > d(x_n, x_0)$$

for any n with m sufficiently large in view of $0 > h(x_n) = \lim_{m \rightarrow \infty} d(y_m, x_n) - d(y_m, x_0)$. So for each n we can find a sufficiently large m such this inequality holds, and along this subsequence $(x_n, y_m) \rightarrow \infty$ showing that the two sequences hence converge to one and the same point of the Gromov boundary. For more on metric functionals for (non-proper) Gromov hyperbolic spaces we refer to [MT18].

4 Examples: Banach spaces

Linear theory

The set of continuous linear functionals form a new normed vector space, called the *dual space*, with norm

$$\|f\| = \sup_{v \neq 0} \frac{|f(v)|}{\|v\|}.$$

Metric theory

The weak compactification and the horofunctions of Banach spaces introduces a new take on a part of classical functional analysis, especially as they have a similar role as continuous linear functionals. Two features stand out, first, the existence of these new functionals do not need any Hahn-Banach theorem which in general is based on Zorn's lemma, second, the horofunctions are always convex and sometimes linear. Horofunctions interpolate between the norm ($h_0(x) = \|x\|$) and linear functionals. More precise statements now follow.

Proposition 2. *Let E be a normed vector space. Every function $h \in \overline{E}$ is convex, that is, for any $x, y \in X$ one has*

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{2}h(x) + \frac{1}{2}h(y).$$

Proof. Note that for $z \in E$ one has

$$\begin{aligned} h_z((x+y)/2) &= \|(x+y)/2 - z\| - \|z\| = \frac{1}{2} \|x - z + y - z\| - \|z\| \\ &\leq \frac{1}{2} \|x - z\| + \frac{1}{2} \|y - z\| - \|z\| = \frac{1}{2} h_z(x) + \frac{1}{2} h_z(y). \end{aligned}$$

This inequality passes to any limit point of such h_z . □

Furthermore, as Busemann noticed in the context of geodesic spaces, any vector v gives rise to a horofunction via

$$h_{\infty v}(x) = \lim_{t \rightarrow \infty} \|x - tv\| - t \|v\|.$$

Often this is a norm one linear functional, it happens precisely when $v/\|v\|$ is a smooth point of the unit sphere [W07, Gu17, Gu18].

Note that in this case one has in addition to the convexity that $h_{\infty v}(\lambda x) = \lambda h_{\infty v}(x)$ for scalars λ , and so $h_{\infty v}$ is a homogeneous sublinear function. By the Hahn-Banach theorem we have a norm 1 linear functional ψ associated to unit vector v for which $\psi(v) = 1$ and such that $\psi \leq h_{\infty v}$.

Proposition 3. *Let H be a real Hilbert space with scalar product (\cdot, \cdot) . The elements of \overline{H} are parametrized by $0 < r < \infty$ and vectors $v \in H$ with $\|v\| \leq 1$, and the element corresponding to $r = 0$, $v = 0$. When $\|v\| = 1$,*

$$h_{r,v}(y) = \|y - rv\| - r$$

and for general v

$$h_{r,v}(y) = \sqrt{\|y\|^2 - 2(y, rv) + r^2} - r.$$

In addition there is $h_0(y) := h_{0,0}(y) = \|y\|$ and the $r = \infty$ cases

$$h_{\infty,v}(y) = -(y, v)$$

where $v \in H$ with $\|v\| \leq 1$. A sequence (t_i, v_i) with $\|v_i\| = 1$ converges to $h_{r,v}$ iff $t_i \rightarrow r \in (0, \infty]$ and $v_i \rightarrow v$ in the standard weak topology, or to h_0 iff $t_i \rightarrow 0$.

Proof. In order to identify the closure we look at vectors $tv \in H$ where we have normalized so that $\|v\| = 1$. By weak compactness we may assume that a sequence $t_i v_i$ (or net) clusters at some radius r and some limit vector v in the weak topology with $\|v\| \leq 1$. In the case $r < \infty$ we clearly get the functions

$$h_{r,v}(y) = \sqrt{r^2(1 - \|v\|^2) + \|y - rv\|^2} - r,$$

which after developing the norms gives the functions in the proposition. Note that in case $t \rightarrow 0$ the function is just h_0 independently of v .

In the case $t_i \rightarrow \infty$ we have the following calculation

$$\begin{aligned} h_{\infty,v}(y) &= \lim_{i \rightarrow \infty} \sqrt{(t_i v_i - y, t_i v_i - y)} - t = \lim_{i \rightarrow \infty} \frac{(t_i v_i - y, t_i v_i - y) - t^2}{\sqrt{(t_i v_i - y, t_i v_i - y)} + t} = \\ &= \lim_{i \rightarrow \infty} \frac{t_i (-2(y, v) + (y, y)/t_i)}{t_i \left(\sqrt{1 - 2(y, v)/t_i + (y, y)/t_i^2} + 1 \right)} = -(y, v). \end{aligned}$$

It is rather immediate that the functions described are all distinct which means that for convergent sequences both t_i and v_i must converge (with the trivial exception of when $t_i \rightarrow 0$). \square

We have in this way compactified Hilbert spaces. To illustrate the relation with the (linear) weak topology consider an ON-basis $\{e_n\}$. It is a first example of the weak topology that $e_n \rightarrow 0$ weakly. Likewise does the sequence $\lambda_n e_n$ for any sequence of scalars $0 < \lambda_n < 1$. In

\overline{H} it is true that $e_n \rightarrow h_{1,0}$, but $\lambda_n e_n$ does not necessarily converge. On the other hand $n \cdot e_1$ does not converge weakly as $n \rightarrow \infty$ but $n \cdot e_1 \rightarrow h_{\infty, e_1}(\cdot) = -(\cdot, e_1)$ in \overline{H} .

For L^p spaces we refer to [W07, Gu17, Gu18, Gu19]. An interesting detail that Gutiérrez showed is that the function identically equal to zero is not a metric functional for ℓ^1 . He also observed how a famous fixed point free example of Alspach must fix a metric functional.

5 Basic spectral notions

Linear theory

Let E be a normed vector space and $A : E \rightarrow E$ a bounded (or continuous) linear map (operator). One defines the *operator norm*

$$\|A\| = \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}.$$

A basic notion is the *spectrum* and that it is a closed non-empty set of complex numbers. As Beurling and Gelfand observed its radius can be calculated by

$$\rho(A) = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}$$

called the *spectral radius* of A . (The existence of the limit comes from a simple fact, known as Fekete lemma, in view of the submultiplicative property of the norm, see [L02, 17.1]). One has the obvious inequality

$$\rho(A) \leq \|A\|.$$

In many important cases there is in fact an equality here, such as for normal operators which includes all unitary and self-adjoint operators.

For a given vector v one may ask for the existence of

$$\lim_{n \rightarrow \infty} \|A^n v\|^{1/n}.$$

Such considerations are called *local spectral theory*. In infinite dimensions this limit may not exist when the spectral theory fails. In finite dimensions the limit exists as is clear from the Jordan normal form. A counterexample can be given by the of ℓ^2 sequence and A is a combination of a shift and a diagonal operator, having two exponents each alternating in longer and longer stretches, making the behavior seem different for various periods of n . See for example [Sc91] for details.

When A^n is replaced by a random product of operators, an ergodic cocycle, then Oseledets multiplicative ergodic theorem asserts that these limits, called Lyapunov exponents, exist a.e.

Metric theory

Let (X, d) be a metric space and $f : X \rightarrow X$ a semicontraction (i.e. a 1-Lipschitz map). One defines the *minimal displacement*

$$d(f) = \inf_x d(x, f(x)).$$

Like in hyperbolic geometry, or for nonpositively curved spaces [BGS85], one can classify semicontractions of a metric space as follows:

- *elliptic* if $d(f) = 0$ and the infimum is attained, i.e. there is a fixed point
- *hyperbolic* if $d(f) > 0$ and the infimum is attained, or
- *parabolic* if the minimum is not attained.

Usually the parabolic maps are the more complicated. It might also be useful to divide semicontractions according to whether all orbits are bounded, all orbits are unbounded, and in the latter case whether all orbits tends to infinity. For example, a circle rotation is hyperbolic and bounded. In this general context let me again recommend [G18] for examples and a simpler proof of Calka's theorem, which asserts that for proper metric spaces unbounded orbits necessarily tend to infinity.

Another basic associated number is the *translation number* (or *drift* or *escape rate*)

$$\tau(f) = \lim_{n \rightarrow \infty} \frac{1}{n} d(x, f^n(x)).$$

Notice that this number is independent of x because by the 1-Lipschitz property any two orbits stay on bounded distance from each other. This number exists by the Fekete lemma in view of the subadditivity coming from the triangle inequality and the 1-Lipschitz property. It also has the tracial property: $\tau(fg) = \tau(gf)$ as is simple to see.

One has the obvious inequality

$$\tau(f) \leq d(f).$$

In important cases one has equality, especially under non-positive curvature: for isometries see [BGS85] and the most general version see [GV12]. In view of that holomorphic maps preserve Kobayashi pseudo-distances, one can study the corresponding invariants and ask when equality holds:

Problem. For holomorphic self-maps f , when do we have equality $\tau(f) = d(f)$ in the Kobayashi pseudo-distance?

This has recently been studied by Andrew Zimmer. This is analogous to operators when the spectral radius equals the norm.

The following fact is a spectral principle [Ka01] that is analogous to the discussion about the local spectral theory. Note that in contrast to the linear case it holds in all situations. The first statement can also be thought of as a weak spectral theorem or weak Jordan normal form. (For comparison, there is a stronger version in [GV12] for a restricted class of metric spaces.)

Theorem 4. (Metric spectral principle [Ka01]) *Given a semicontraction $f : (X, d) \rightarrow (X, d)$ with drift τ . Then there exists $h \in \overline{X}$ such that*

$$h(f^k(x_0)) \leq -\tau k$$

for all $k > 0$, and for any $x \in X$,

$$\lim_{k \rightarrow \infty} -\frac{1}{k} h(f^k(x)) = \tau.$$

Proof. Given a sequence $\epsilon_i \searrow 0$ we set $b_i(n) = d(x_0, f^n(x_0)) - (l - \epsilon_i)n$. Since these numbers are unbounded in n for each fixed i , we can find a subsequence such that $b_i(n_i) > b_i(m)$ for any $m < n_i$. We have for any $k \geq 1$ and i that

$$d(f^k(x_0), f^{n_i}(x_0)) - d(x_0, f^{n_i}(x_0))$$

$$\begin{aligned}
&\leq d(x_0, f^{n_i-k}x_0) - d(x_0, f^{n_i}x_0) \\
&= b_i(n_i - k) + (l - \epsilon_i)(n_i - k) - b_i(n_i) - (l - \epsilon_i)n_i \\
&\leq -(l - \epsilon_i)k.
\end{aligned}$$

By compactness, there is a limit point h of the sequence $d(\cdot, f^{n_i}(x_0)) - d(x_0, f^{n_i}(x_0))$ in \overline{X} . Passing to the limit in the above inequality gives

$$h(f^k(x_0)) \leq -lk$$

for all $k > 0$. Finally, the triangle inequality

$$d(x, f^k(x)) + d(f^k(x), z) \geq d(x, z)$$

implies that

$$h(f^k(x_0)) \geq -d(x_0, f^k(x_0)).$$

From this the second statement in the theorem follows in view of that changing x_0 to x only is a bounded change since f is 1-Lipschitz:

$$\begin{aligned}
\left| d(x_0, f^k(x)) - d(x_0, f^k(x_0)) \right| &\leq \max \left\{ d(f^k(x), f^k(x_0)), d(f^k(x_0), f^k(x)) \right\} \\
&\leq \max \{ d(x, x_0), d(x_0, x) \}.
\end{aligned}$$

□

Example. The classical instance of this is the Wolff-Denjoy theorem in complex analysis. This is thanks to Pick's version of the Schwarz lemma which asserts that every holomorphic map of the unit disk to itself is 1-Lipschitz with respect to the Poincaré metric ρ . It says that given a holomorphic self-map of the disk, either there is a fixed point or there is a point on the boundary circle which attracts every orbit. From basic hyperbolic geometry one can deduce this from our theorem. Wolff considered also horodisks, but may not have discussed lengths τ , which here equals $\inf_{z \in D} \rho(z, f(z))$, as follows for example from [GV12].

In the isometry case, in the same way, looking at times for which the orbit is closer to the origin than all future orbit points, one can show that there exists a metric functional h such that

$$h(f^{-n}x_0) \geq \tau_{f^{-1}} \cdot n$$

for all $n \geq 1$.

6 Application: Extensions of the mean ergodic theorem

In 1931 in response to a famous hypothesis in statistical mechanics, von Neumann used spectral theory to establish that for unitary operators U ,

$$\frac{1}{n} \sum_{k=0}^{n-1} U^k g \rightarrow P g$$

where P is the projection operator onto the U invariant elements in the Hilbert space in question. Carleman showed this independently at the same time (or before), and a nice proof of a more general statement (for U with $\|U\| \leq 1$) was found by F. Riesz inspired

by Carleman's method. Such convergence statement is known not to hold in general for all Banach spaces, in the sense that there is no strong convergence of the average. On the other hand, let $f(w) = Uw + v$, then we have

$$f^n(0) = \sum_{k=0}^{n-1} U^k v.$$

If $\|U\| \leq 1$, then f is semi-contractive and Theorem 4 applies, and it does so for *any* Banach space.

In other words the theorem is weak enough to always hold. On the other hand when the situation is better, for example that we are studying transformation of a Hilbert space, then the weak convergence can be upgraded to stronger statement thanks to knowledge about the metric functionals. Here is an example:

Let U and f be as above acting on a real Hilbert space. Theorem 4 applied to f hands us a metric functional h , for which

$$\frac{1}{n} h \left(\sum_{k=0}^{n-1} U^k v \right) \rightarrow -\tau,$$

where as before τ is the growth rate in this case of the norm of the ergodic average. Either $\tau = 0$ and we have

$$\frac{1}{n} \sum_{k=0}^{n-1} U^k v \rightarrow 0,$$

or else we need to have that h is a metric functional at infinity (because h must be unbounded from below), see Proposition 3, in fact it must be of the form $h(x) = -(x, w)$ with $\|w\| = 1$ (since τ is the growth of the norm which h applied to the orbit matches). It is a well-known simple fact that if we have a sequence of points x_n in a Hilbert space and a vector w with norm $\|w\| \leq 1$, such that $(x_n, w) \rightarrow 1$ and $\|x_n\| \rightarrow 1$, then necessarily $x_n \rightarrow w$ and $\|w\| = 1$. In details for the current situation:

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k v - \tau w \right\|^2 = \left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k v \right\|^2 - 2 \left(\frac{1}{n} \sum_{k=0}^{n-1} U^k v, \tau w \right) + \|\tau w\|^2 \rightarrow \tau^2 - 2\tau^2 + \tau^2 = 0.$$

This finishes the proof of the classical mean ergodic theorem.

7 Spectral metrics

At the moment I do not see an appropriate axiomatization for the type of metrics that will be useful. Here is an informal description, precise definitions will follow in the particular situations studied later. We will have a group of transformations, with elements denoted f or g etc, of a space. This space has objects denoted α with some sort of length l , the set or subset of these objects should be invariant under the transformation and we define

$$d(f, g) = \log \sup_{\alpha} \frac{l(g^{-1}\alpha)}{l(f^{-1}\alpha)}.$$

The triangle inequality is automatic from the supremum, as is the invariance. The function d separates f and g if the set of α s is sufficiently extensive. On the other hand this "distance" is not necessarily symmetric. If desired it can be symmetrized in a couple of trivial ways.

Example. Define a hemi-metric between two linear operators A and B of a real Hilbert space H :

$$d(A, B) = \log \sup_{v \neq 0} \frac{\|B^t v\|}{\|A^t v\|}.$$

(Here t denotes the transpose.) Note that we may take the supremum over the vectors with have unit length, and also we see that there is the obvious connection to the operator norm:

$$d(I, A) = \log \|A^t\| = \log \|A\|,$$

where I denotes the identity operator.

Here is an example of classical and very useful metrics:

Example. Metrics on the Teichmüller space of a surface,

$$d(x, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_y(\alpha)}{l_x(\alpha)}$$

where x and y denote different equivalence classes of metrics (or complex structures) on a fixed surface, and \mathcal{S} is the set of non-trivial isotopy classes of simple closed curves, and l could denote various notions of length, depending on the choice the metric is asymmetric. See the next section for more details and applications.

Here is another possibility:

Example. Taken from [DKN18]. Given two intervals I, J and a C^1 -map $g : I \rightarrow J$ which is a diffeomorphism onto its image. The distortion coefficient is defined by

$$K(g; I) := \sup_{x, y \in I} \left| \log \left(\frac{g'(x)}{g'(y)} \right) \right|.$$

This is subadditive under composition and $K(g, I) = K(g^{-1}, g(I))$.

Other examples of such metrics include the Hilbert, Funk, and Thompson metrics on cones [LN12], Kobayashi pseudo-metric in the complex category, Hofer's metric on symplectomorphisms [Gr07]. and the Lipschitz metric on outer space.

8 Application: Surface homeomorphisms

Let Σ be a surface of finite type. Let \mathcal{S} be the set of non-trivial isotopy classes of simple closed curves on Σ . One denotes by $l_x(\alpha)$ the infimal length of curves in the class of α in the metric x . The metric x can be considered to be a point in the Teichmüller space \mathcal{T} of Σ and hence a hyperbolic metric, the length will be realized on a closed geodesic. Thurston introduced the following asymmetric metric on \mathcal{T}

$$L(x, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_y(\alpha)}{l_x(\alpha)}.$$

Thurston in a seminal work provided a sort-of Jordan normal form for mapping classes of diffeomorphisms of Σ and deduced from this the existence of Lyapunov exponents or eigenvalues as it were. A different approach was proposed in [Ka14]. In this section we will use the metrics

directly, without metric functionals explicitly. We will use a lemma in a paper by Margulis and me [KaM99], that was substantially sharpened in [GK15].

Let (Ω, ρ) be a measure space with $\rho(\Omega) = 1$ and let $T : \Omega \rightarrow \Omega$ be an ergodic measure preserving map. We consider a measurable map $\omega \mapsto f_\omega$ where f_ω are homeomorphisms of Σ (or more generally semi-contractions of \mathcal{T}). We assume the appropriate measurability and integrability assumptions. We form $Z_n(\omega) := f_\omega \circ f_{T\omega} \circ \dots \circ f_{T^{n-1}\omega}$. Let

$$a(n, \omega) = L(x_0, Z_n(\omega)x_0),$$

which is a subadditive (sub-)cocycle and by the subadditive ergodic theorem

$$a(n, \omega)/n$$

converges for a.e. ω to a constant which we denote by τ . Given a sequence of ϵ_i tending to 0, Proposition 4.2. in [KaM99] implies that a.e there is an infinite sequence of n_i and numbers K_i such that

$$a(n_i, \omega) - a(n_i - k, T^k \omega) \geq (\tau - \epsilon_i)k$$

for all $K_i \leq k \leq n_i$. Moreover we may assume that $(\tau - \epsilon_i)n_i \leq a(n_i, \omega) \leq (\tau + \epsilon_i)n_i$ for all i . We will now use a property of L established in [LRT12] (that was not used in [Ka14]). Namely there is a finite set of curves $\mu = \mu_{x_0}$ such that

$$L(x_0, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_y(\alpha)}{l_{x_0}(\alpha)} \asymp \log \max_{\alpha \in \mu} \frac{l_y(\alpha)}{l_{x_0}(\alpha)}$$

up to an additive error.

Now by the pigeon-hole principle refine n_i such that there is one curve α_1 in μ which realizes the maximum for each $y = Z_{n_i}(\omega)x_0$, in other words

$$l_{Z_{n_i}x_0}(\alpha_1) \asymp \exp(n_i(\tau \pm \epsilon_i))$$

Given the way n_i were selected we have

$$-\log \sup_{\alpha \in \mathcal{S}} \frac{l_{Z_n x_0}(\alpha)}{l_{Z_k x_0}(\alpha)} \geq -a(n_i - k, T^k \omega) \geq (\tau - \epsilon_i)k - a(n_i, \omega)$$

(The first inequality is an equality in case the maps are isometries, and not merely semicontractions.) It follows, like in [Ka14], that

$$l_{Z_k x_0}(\alpha_1) \geq l_{Z_{n_i} x_0}(\alpha_1) e^{-a(n_i, \omega)} e^{(\tau - \epsilon_i)k}.$$

Since no length of a curve can grow faster $e^{\tau k}$ we get from this that

$$l_{Z_k x_0}(\alpha_1)^{1/k} \rightarrow e^\tau.$$

In other words, the top Lyapunov exponents exists in this sense. For the other exponents in the i.i.d case we refer to Horbez [H16] and in the general ergodic setting to a forthcoming joint paper with Horbez. The purpose of this section was to show a different technique to such results using spectral metrics and subadditive ergodic theory. For a similar statement instead with the complex notion of extremal length and using metric functionals, see [GK15].

9 Conclusion

9.1 A brief discussion of examples of metrics

The hyperbolic plane, recalled above, was discovered (rather late) as a consequence of the inquiries on the role of the parallel axiom in Euclidean geometry. At that time it was probably considered a curiosity but later it has turned out to be a basic example, connected to an enormous amount of mathematics. In particular it is often the first example in the following list of metric spaces (for references see [Gr07] or [Ka05, GK15]).

- L^2 -metrics: The fundamental group of a Riemannian manifold acts by isometry on the universal covering space. In geometric group theory it is of importance to have isometric actions on CAT(0) spaces, for example CAT(0)-cube complexes.
- Symmetric space type metric spaces: Extending the role of the hyperbolic plane for 2×2 matrices and the moduli of 2-dimensional tori, there are the Riemannian symmetric spaces. These have recently also been considered with Finsler metrics. Other extensions are Teichmüller space, Outer space, spaces of Riemannian metrics on which homeomorphisms or diffeomorphisms have induced isometric actions. Likewise for invertible bounded operators on spaces of positive operators.
- Hyperbolic metrics: The most important notion is Gromov hyperbolic spaces, appearing in infinite group theory (Cayley-Dehn see below), the curve complex (non-locally compact!) and similar complexes coming from topology and group theory, and for Hilbert and Kobayashi metrics in the next item.
- L^∞ -metrics. Again generalizing the hyperbolic plane and the positivity aspect of spaces of metrics, are cones and convex sets with metrics of Hilbert metric type. In complex analysis in one or several variables, we have pseudo-metrics of a similar type, generalizing the Poincaré metric, the maximal one being the Kobayashi pseudo-metric. The operator norm, Hofer's metric or Thurston's asymmetric metric are further examples. Roughly speaking these are the metrics referred to above as *spectral metrics*, and the natural maps in question in all these examples are semicontractions.
- L^1 -metrics: Cayley-Dehn graphs associated with groups and a generating set, the group itself acts on the graph by automorphisms, which amount to isometries with respect to the word metric.

9.2 Further directions

Horbez in [H16] extended [Ka14] to give all exponents in the i.i.d. case, thus in particular recovering Thurston's theorem (except for the algebraic nature of the exponents), and also implemented the same scheme for outer automorphisms group via an intricate study of the Culler-Vogtmann outer space, in particular its metric functionals. The paper by Gaubert-Vigeral [GV12], which in particular establishes, with another method, a strengthening of the metric spectral principle above in case the metrics admit a combing of non-positive curvature, contains further references to examples of hemi-metrics and semicontractions arising in areas such as game theory and optimal control. Other directions could be:

- Symplectomorphisms and Hofer's metric,

- Prove some statements for invertible linear transformations or compact operators using the asymmetric metric above,
- Diffeomorphisms of manifolds, there are several suggestions for spectral metrics here. See for instance the recent preprint [Na18] of Navas on distortion of 1-dimensional diffeomorphisms,
- The subject of Kalman filters via the metric approach of Bougerol and others, giving rise to semicontractions, see [Wo07].

In the works of Cheeger and collaborators on differentiability of functions on metric spaces, see [Ch99, Ch12], the notion of generalized linear function appears. In [Ch99] Cheeger connects this to Busemann functions, on the other hand he remarks in [Ch12] that non-constant such functions do not exist for most spaces. Perhaps it remains to investigate how metric functionals relate to this subject.

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