DOUBLE COVERS OF KUMMER SURFACES

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Abstract. Besides its construction as a quotient of an abelian surface, a Kummer surface can be obtained as the quotient of a K3 surface by a $\mathbb{Z}/2\mathbb{Z}$-action. In this paper, we classify all such K3 surfaces. Our classification is expressed in terms of period lattices and extends Morrison's criterion of K3 surfaces with a Shioda-Inose structure. Moreover, we list all the K3 surfaces associated to a general Kummer surface and provide very geometrical examples of this phenomenon.

1. Introduction

The definition of a Kummer surface unravels the natural link between K3 surfaces and abelian surfaces. Indeed, let A denote an abelian surface and let $A \xrightarrow{i} A$ be its involution automorphism, i.e. $i(a) = -a$. The quotient, $A/\{1, i\}$, is a singular surface with sixteen ordinary double points. Its minimum resolution, $\mathrm{Kum}(A)$, is a K3 surface called the Kummer surface associated to A. Closely related to Kummer surfaces, are the K3 surfaces with a Shioda-Inose structure ([19], [8]). A K3 surface X is said to have a Shioda-Inose structure if it admits a rational map of degree two into a Kummer surface, $X \overset{\tau}{\dashrightarrow} \mathrm{Kum}(A)$, which induces a Hodge isometry of the period lattices,

$$\mathrm{T}_X(2) \xrightarrow{\tau_*} \mathrm{T}_{\mathrm{Kum}(A)}.$$

Morrison proved that X admits a Shioda-Inose structure if and only if there exists a Hodge isometry, $T_X \simeq T_A$, between the period lattices of X and an abelian surface A ([8]).

However, in general, a rational map of degree two between a K3 surface and a Kummer surface, does not need to induce a Hodge isometry of period lattices.

The aim of this paper is to treat this more general situation, i.e. it classifies all the K3 surfaces that admit a rational map of degree two into a given Kummer surface. As a result, we prove in theorem 3.1 that if X is a K3 surface such that its period lattice admits an embedding into the period lattice of an abelian surface A, $T_X \hookrightarrow T_A$, with the property that

$$T_A/T_X \simeq (\mathbb{Z}/2)^\alpha \text{ with } 0 \leq \alpha \leq 4,$$

then there exists a rational map of degree two,

$$X \overset{\tau}{\dashrightarrow} \mathrm{Kum}(A).$$

Note that the case where $\alpha = 0$ recovers Morrison's criterion for K3 surfaces with a Shioda-Inose structure. Moreover, we give a complete list of all the K3 surfaces covering a general Kummer surface.

In the second part of this paper, we introduce the notion of an *even eight* on a K3 surface. We use this concept to construct explicit examples of the surfaces appearing in our list (theorem 6.4, theorem 7.3). Interestingly in these cases, the rational map $X \overset{\tau}{\dashrightarrow} \mathrm{Kum}(A)$

decomposes as

$$\begin{array}{ccc} X & \xrightarrow{\zeta} & \Omega \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Kum}(A) & \xrightarrow{\rho} & \mathbb{P}^2 \end{array}$$

where ρ is the double cover of $\mathbb{P}^2$ branched along a reducible sextic Σ decomposable as a (possibly degenerate) quartic Γ and a conic Δ and where φ and ζ are double covers branched along Δ and $\varphi^*(\Gamma)$ respectively (theorem 6.5, theorem 7.4).

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2. Lattices and involutions on K3 surfaces

The theory of K3 surfaces is intimately connected to the study of integral quadratic forms or *lattices*. In this section, we recall the basic properties of lattices and discuss their fruitful applications to involutions on K3 surfaces. A (non-degenerate) lattice S is a free $\mathbb{Z}$-module of finite rank together with a (non-degenerate) bilinear form, $b : S \times S \to \mathbb{Z}$. The lattice S is even if the associated quadratic form takes only even values. An embedding of lattices $M \hookrightarrow S$ is an homomorphism of $\mathbb{Z}$-modules preserving the bilinear forms. We say that the embedding is primitive if the quotient S/M is torsion free. The signature of S, (t_+, t_-) is the signature of the bilinear form b, over the real vector space $S \otimes \mathbb{R}$. The lattice $S(m)$ for $m \in \mathbb{Z}$ denotes the same free $\mathbb{Z}$-module as S, with the bilinear form $b_{mS}(x, y) = mb_S(x, y)$. If S is an non-degenerate even lattice, then the $\mathbb{Q}$-valued quadratic form on $S^* = \mathrm{Hom}(S, \mathbb{Z})$ induces a quadratic form on the finite abelian group $D_S = S^*/S$,

$$q_S : D_S \to \mathbb{Q}/2\mathbb{Z},$$

which is well defined mod $2\mathbb{Z}$ and a bilinear form

$$b_S : D_S \times D_S \to \mathbb{Q}/\mathbb{Z}.$$

The pair (D_S, q_S) is called the *discriminant form* of S, and $l(D_S)$ is the minimum number of generators of D_S. A non-degenerate lattice S is unimodular if $D_S \simeq \{0\}$. An embedding $S \hookrightarrow S'$ of even lattices with a finite cokernel is called an *overlattice*. Let $S \hookrightarrow S'$ be an overlattice of even non-degenerate lattices, then

$$|D_S| = [S' : S]^2 |D_{S'}|.$$

Consider the chain of embeddings $S \hookrightarrow S' \hookrightarrow S'^* \hookrightarrow S^*$ and let $H_{S'} = S'/S$ be the subgroup of D_S. Then the correspondence, $S' \leftrightarrow H_{S'}$, determines a bijection between even overlattices of S and isotropic subgroups of D_S.

A p-adic lattice and its discriminant form are defined in the same way as for an integral lattice, only replacing $\mathbb{Z}$ by $\mathbb{Z}_p$. Let S be a lattice over $\mathbb{Z}$, for every prime number p, q_p denotes the restriction of the discriminant form q_S to $(D_S)_p$, the p-component of D_S. There exists a unique p-adic lattice $K(q_p)$ whose discriminant form is isomorphic to $((D_S)_p, q_p)$ and whose rank is equal to $l((D_S)_p)$. The p-adic number $|K(q_p)|$ is then the determinant of a symmetric matrix associated to the lattice $K(q_p)$; it is therefore well defined mod $(\mathbb{Z}_p^*)^2$.

Theorem 2.1. [13] *There exists a primitive embedding of the even lattice S, with invariants (t_+, t_-, q_S) into the unimodular lattice L with signature (l_+, l_-) if and only if the following conditions are simultaneously satisfied:*

(1) $l_+ - l_- \equiv 0$ *(mod 8),*
(2) $l_- - t_- \geq 0$, $l_+ - t_+ \geq 0$, $l_+ + l_- - t_+ - t_- \geq l(D_S)$,
(3) $(-1)^{l_+ - t_+}|D_S| \equiv \pm|K(q_p)|(mod\ \mathbb{Z}_p^*)^2$
for all odd primes p for which $l_+ + l_- - t_+ - t_- = l((D_S)_p)$,
(4) $|D_S| \equiv \pm|K(q_2)|(mod\ \mathbb{Z}_2^*)^2$
if $l_+ + l_- - t_+ - t_- = l((D_S)_2)$ and $q_2 \ncong \frac{x^2}{\theta \cdot 2} \oplus q_2'$ for some $\theta \in \mathbb{Z}_2^*$, and some q_2'.

A K3 surface X is a smooth projective complex surface satisfying

$$K_X = 0 \quad \text{and} \quad \mathrm{H}^1(X, \mathcal{O}_X) = 0.$$

Let X be a K3 surface. The second cohomology group, $\mathrm{H}^2(X, \mathbb{Z})$, equipped with the cup product, is an even unimodular lattice of signature $(3, 19)$ and is isomorphic to $U^3 \oplus E_8^2(-1)$, where U is the hyperbolic plane and $E_8(-1)$ is the unique even negative definite lattice of rank eight. The period lattice of a surface X, denoted by T_X, is defined by

$$T_X = S_X^{\perp} \subset \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$$

where S_X is the Néron-Severi group of X. Recall that the lattice $\mathrm{H}^2(X, \mathbb{Z})$ admits a Hodge decomposition of weight two

$$\mathrm{H}^2(X, \mathbb{C}) \simeq \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}.$$

Similarly, the period lattice T_X inherits a Hodge decomposition of weight two denoted by

$$T_X \otimes \mathbb{C} \simeq T^{2,0} \oplus T^{1,1} \oplus T^{0,2}.$$

An isomorphism between two lattices that preserves their bilinear forms and their Hodge decomposition is called a Hodge isometry. K3 surfaces have the unusual property that they satisfy a "'Torelli theorem"', namely

Theorem 2.2. [16] *Let X and X' be two K3 surfaces and let $\phi : \mathrm{H}^2(X', \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})$ be a Hodge isometry. If ϕ preserves effective classes, then there exists a unique isomorphism $u : X \to X'$ such that $u^* = \phi$.*

Mukai improved this result for K3 surfaces of large picard number.

Theorem 2.3. [9] *Let X and X' be K3 surfaces and let $\phi : T_X' \to T_X$ be a Hodge isometry. If $\rho(X) \geq 12$, there exists an isomorphism $u : X \to X'$ such that ϕ is induced by u^*.*

Definition 2.4. Let X be a K3 surface, an involution $X \xrightarrow{\iota} X$ is called *a Nikulin involution* if it preserves any holomorphic two-form on X, i.e. $\iota^*(\omega) = \omega$ for all $\omega \in \mathrm{H}^{2,0}(X)$.

Theorem 2.5. [12] *Let $X \xrightarrow{\iota} X$ be a Nikulin involution on X. Then*
1) ι has eight isolated fixed points and the surface $X/\{1, \iota\}$ has eight ordinary double points,
2) the minimal resolution Y of $X/\{1, \iota\}$ is a K3 surface.

Let $\mathrm{Z} \xrightarrow{\epsilon} X$ be the blow-up of the eight fixed points of ι. The involution ι lifts to an involution $\hat{\iota}$ on Z whose fixed points consist of the eight exceptional curves. Thus the quotient

surface, $Z \xrightarrow{p} Z/\{1,\hat{\iota}\} = Y$, contains eight disjoint smooth rational curves, $C_1, \ldots, C_8$, which make up the branch locus of the map p. Consequently they satisfy

$$C_1 + \cdots + C_8 \in 2S_Y.$$

The commutative resolution diagram

$$\begin{array}{ccc} \mathrm{Z} & \xrightarrow{\epsilon} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle /\iota} \\ Y & \longrightarrow & X/\{1,\iota\} \end{array}$$

induces a rational map of degree two between K3 surfaces, $X \overset{\tau}{\dashrightarrow} Y$.

Definition 2.6. The *Nikulin lattice* is an even lattice N of rank eight generated by $\{c_i\}_{i=1}^8$ and $d = \frac{1}{2}\sum_{i=1}^8 c_i$, with the bilinear form $c_i \cdot c_j = -2\delta_{ij}$.

Conversely, any rational map of degree two between K3 surfaces, $X \overset{\tau}{\dashrightarrow} Y$, is obtained from a similar construction and therefore it implies the existence of a Nikulin involution on X.

Let $X \xrightarrow{\iota} X$ be a Nikulin involution. The two invariant lattices

$$T^\iota = \{x \in \mathrm{H}^2(X,\mathbb{Z}) | \iota^*(x) = x\} \quad \text{and} \quad S_\iota = \{x \in \mathrm{H}^2(X,\mathbb{Z}) | \iota^*(x) = -x\}$$

define the primitive embeddings

$$T_X \hookrightarrow T^\iota \hookrightarrow \mathrm{H}^2(X,\mathbb{Z}), \quad S_\iota \hookrightarrow S_X \hookrightarrow \mathrm{H}^2(X,\mathbb{Z})$$

and the isomorphisms

$$T^\iota \simeq U^3 \oplus E_8(-2) \quad \text{and} \quad S_\iota \simeq E_8(-2).$$

Let $X \overset{\tau}{\dashrightarrow} Y$ be the rational map associated to ι. The maps induced on the period lattices

$$T_Y \xrightarrow{\tau^*} T_X \quad \text{and} \quad T_X \xrightarrow{\tau_*} T_Y$$

preserve the Hodge decompositions and satisfy

$$\tau_*\tau^*(x) = 2x, \quad \tau^*\tau_*(x) = 2x, \quad \tau^*(x)\cdot\tau^*(y) = 2xy, \quad \tau_*(x)\cdot\tau_*(y) = 2xy.$$

The image of T_Y by τ^* is then a sublattice of T_X isomorphic to $T_Y(2)$ with two-elementary quotient, i.e.

$$T_X/T_Y(2) \simeq (\mathbb{Z}/2\mathbb{Z})^\beta \text{ for some } \beta \geq 0.$$

Lemma 2.7. [15] *Let X be a K3 surface and let $T_X \hookrightarrow T \simeq U^3 \oplus E_8(-2)$ be a primitive embedding of lattices. Then there exists a Nikulin involution ι of X such that for the corresponding rational map of degree two $X \overset{\tau}{\dashrightarrow} Y$*

$$\tau^*T_Y \simeq T_Y(2) \simeq 2(T^* \cap (T_X \otimes \mathbb{Q})) \subset T_X.$$

Finally, the following argument is due to Nikulin ([15]) and will be crucial for the proof of the theorem classifying the K3 surfaces covering a given Kummer surface. Let X be a K3 surface with a Nikulin involution ι, denote by T the period lattice T_X and by S, the invariant lattice S_ι. Let M be the primitive sublattice of $\mathrm{H}^2(X,\mathbb{Z})$ generated by $T \oplus S$. Then the subgroup $\Gamma = M/(T \oplus S) \subset D_T \oplus D_S$ is isotropic and satisfies

$$\Gamma \cap (D_T \oplus \{0\}) = 0 \quad \text{and} \quad \Gamma \cap (\{0\} \oplus D_S) = 0.$$

Let π_T and π_S be the projections in D_T and D_S respectively. Let $\mathfrak{H} = \pi_T(\Gamma) \subset D_T$. If $\alpha = l(\mathfrak{H}) \leq 4$, then the discriminant group of M is of the form $D_M \simeq D_T \oplus u(2)^{4-\alpha}$, where $u(2)$ denotes the discriminant form of the lattice $U(2)$. Note also that $\mathfrak{H}$ is not necessarily isotropic. The map $\xi = \pi_S \circ \pi_T^{-1}$ defines an inclusion of quadratic forms (i.e. an injection of groups preserving the bilinear form)

$$\xi : \mathfrak{H} \to (D_S, -q_S).$$

Moreover, Nikulin proved the equality

$$((S^{\perp})^* \cap (T \otimes \mathbb{Q}))/T = \mathfrak{H} \subset D_T.$$

3. Classification Theorem

In this section, we give necessary and sufficient conditions for a K3 surface to admit a rational map of degree two into a given Kummer surface. As a corollary, we show that there exist finitely many such K3 surfaces and give an exhaustive list of all the K3 surfaces covering a general Kummer surface.

Theorem 3.1. *Let* $\mathrm{Kum}(A)$ *be a Kummer surface. A K3 surface* X *admits a rational map of degree two,* $X \overset{\tau}{\dashrightarrow} \mathrm{Kum}(A)$*, if and only if there exists an embedding of period lattices* $\phi : T_X \hookrightarrow T_A$ *such that*

1. $\phi_{\otimes\mathbb{C}}$ *preserves the Hodge decomposition,*
2. $T_A/T_X \simeq (\mathbb{Z}/2\mathbb{Z})^{\alpha}$ *with* $0 \leq \alpha \leq 4$,
3. $|D_{T_X}| \equiv \pm|K(q_{T_X})_2|$ *mod* $(\mathbb{Z}_2^*)^2$*,*
   if $\alpha = \frac{\mathrm{rank}(T_X)+l((D_{T_X})_2)}{2} - 3$ *and* $(q_{T_X})_2 \oplus u^{4-\alpha}(2) \ncong \frac{x^2}{\theta\cdot 2} \oplus q_2'$ *for some* $\theta \in \mathbb{Z}_2^*$ *and some* q_2'*.*

Proof. Let Y denote the Kummer surface $\mathrm{Kum}(A)$ and suppose that X admits a rational map of degree two into Y, $X \overset{\tau}{\dashrightarrow} Y$. The covering involution $\iota : X \to X$ is then a Nikulin involution. Consider the induced Hodge isometry of lemma 2.7,

$$T_Y(2) \overset{\simeq}{\to} 2(T_X \otimes \mathbb{Q} \cap (T^{\iota})^*) \subset T_X.$$

Denote by $\tilde{\mathfrak{H}}$ the rational overlattice of T_X defined by $T_X \otimes \mathbb{Q} \cap (T^{\iota})^*$. Since $2\tilde{\mathfrak{H}} \subset T_X$, the quotient $\tilde{\mathfrak{H}}/\ T_X$ is two-elementary. Since the period lattice of $\mathrm{Kum}(A)$ is Hodge isometric to $T_A(2)$ ([11]), $T_A(4)$ is Hodge isometric to $T_Y(2)$. Thus there is a Hodge isometry, $T_A(4) \overset{\simeq}{\to} 2\tilde{\mathfrak{H}} \subset T_X$. The obvious identification of $T_A(4)$ with $2T_A$ shows that $T_A \simeq \tilde{\mathfrak{H}}$ and that the inclusion preserving the Hodge decomposition $T_X \hookrightarrow \tilde{\mathfrak{H}} \simeq T_A$ satisfies

$$T_A/T_X \simeq \tilde{\mathfrak{H}}/T_X \simeq (\mathbb{Z}/2\mathbb{Z})^{\alpha} \text{ for some } \alpha \geq 0.$$

Moreover, the lattice $\tilde{\mathfrak{H}}$ must be an integral overlattice of T_X, and conseqently the quotient $\tilde{\mathfrak{H}}/T_X$, is an isotropic subgroup of (D_{T_X}, q_{T_X}). Note that the group $\tilde{\mathfrak{H}}/T_X$ corresponds to the group $\mathfrak{H}$ introduced in the last part of the previous section. Therefore $\mathfrak{H}$ admits an inclusion of quadratic forms into $(D_{S_\iota}, -q_{|S_\iota})$, where $l(D_{S_\iota}) = 8$. Because $\mathfrak{H}$ is isotropic, we obtain the upper bound $\alpha \leq 4$. Finally, let M be the primitive closure of $T_X \oplus S_\iota$ in $\mathrm{H}^2(X, \mathbb{Z})$, then $D_M \simeq D_{T_X} \oplus u(2)^{4-\alpha}$. The lattice M admits a primitive embedding into the unimodular lattice in $\mathrm{H}^2(X, \mathbb{Z})$, so it satisfies the hypothesis of the theorem 2.1. In particular, it satisfies

$$|D_{T_X}| \equiv \pm|K(q_{T_X})_2| \text{ mod } (\mathbb{Z}_2^*)^2$$

if $\alpha = \frac{\text{rank}(T_X)+l(D_{T_X})_2}{2} - 3$ and $(q_{T_X})_2 \oplus u^{4-\alpha}(2) \ncong \frac{x^2}{\theta\cdot 2} \oplus q'_2$ for some $\theta \in \mathbb{Z}_2^*$.

Conversely, assume that X satisfies conditions (1), (2) and (3).

Claim: There exists a primitive embedding $T_X \hookrightarrow U^3 \oplus E_8(-2)$ such that

$$(T_X \otimes \mathbb{Q}) \cap (U^3 \oplus E_8(-2)^*) \simeq T_A.$$

Assuming the claim, by lemma 2.7, there exists a Nikulin involution on X such that the corresponding rational map of degree two $X \overset{\tau}{\dashrightarrow} \tilde{Y}$ satisfies

$$T_{\tilde{Y}}(2) \simeq 2(U^3 \oplus E_8(-2)^* \cap T_X \otimes \mathbb{Q}) \simeq 2T_A \subset T_X.$$

The Hodge isometry $T_{\tilde{Y}}(2) \simeq 2T_A$ implies that $T_{\tilde{Y}} \simeq T_A(2)$ and hence that $T_Y \simeq T_{\tilde{Y}}$. According to theorem 2.3, $\tilde{Y} \simeq Y$.

Proof of the claim: Let $\mathfrak{H} \simeq (\mathbb{Z}/2\mathbb{Z})^\alpha$ be the isotropic subgroup of D_{T_X} corresponding to the overlattice $T_X \subset T_A$. Since $\alpha \leq 4$, there exists an embedding of quadratic forms

$$\mathfrak{H} \overset{\xi}{\hookrightarrow} (D_S, -q_S),$$

where $S \simeq E_8(-2)$. Consider the subgroup

$$\Gamma_\xi = \{h + \xi(h) | h \in \mathfrak{H}\} \subset D_{T_X} \oplus D_S.$$

Clearly Γ_ξ is an isotropic subgroup of $D_{T_X} \oplus D_S$. Denote by M the overlattice it defines and observe that

$$\text{rank}(T_M) = \text{rank}(T_X) + 8 \text{ and } D_M \simeq q_{T_X} \oplus u(2)^{4-\alpha}.$$

In order for M to admit a primitive embedding into the unimodular lattice $\mathrm{H}^2(X, \mathbb{Z})$, it must satisfy the hypothesis of the theorem 2.1. In our setting it involves checking that

$$l((D_M)_p) + \text{rank}(T_X) \leq 14 \quad \text{for all } p \text{ prime},$$

since for any finite abelian group, $l(D) = \max_p l(D_p)$. If p is odd, then $(D_M)_p \simeq (D_{T_X})_p$ and the strict inequality $l((D_{T_X})_p) + \text{rank}(T_X) < 14$ always holds, as $l((D_{T_X})_p) \leq \text{rank}(T_X)$ and $2 \leq \text{rank}(T_X) \leq 5$. If p is 2, then $l((D_M)_2) = l((D_{T_X})_2) + 8 - 2\alpha$. The facts that $l((D_{T_X})_2) - 2\alpha \leq l((D_A)_2)$ (see the technical lemma 3.2 below), and that the lattice T_A embeds primitively into U^3 imply the inequalities

$$l((D_{T_X})_2)+8-2\alpha+\text{rank}(T_X) \leq l((D_{T_A})_2)+8+\text{rank}(T_A) \leq l(D_{T_A})+8+\text{rank}(T_A) \leq 6+8 = 14.$$

Thus the inequality $l((D_M)_p) + \text{rank}(T_X) \leq 14$ holds for all p. The assumption

$$|D_{T_X}| \equiv \pm |K(q_{T_X})_2| \bmod (\mathbb{Z}_2^*)^2$$

if $\alpha = \frac{\text{rank}(T_X)+l(D_{T_X})_2}{2} - 3$ and $(q_{T_X})_2 \oplus u^{4-\alpha}(2) \ncong \frac{x^2}{\theta\cdot 2} \oplus q'_2$ for some $\theta \in \mathbb{Z}_2^*$, is therefore enough to guarantee the existence of the primitive embedding of M into $\mathrm{H}^2(X, \mathbb{Z})$. Consider the embedding

$$T_X \oplus S \subset M \subset \mathrm{H}^2(X, \mathbb{Z}).$$

By construction, its restriction to T_X and to S are primitive. It follows that S is primitively embedded into S_X or equivalently T_X is primitively embedded into $S^\perp \simeq U^3 \oplus E_8(-2)$. Finally, from the last equality of section 2, we find that

$$\mathfrak{H} \simeq (T_X \otimes \mathbb{Q}) \cap (U^3 \oplus E_8(-2)^*)/T_X$$

and therefore

$$(T_X \otimes \mathbb{Q}) \cap (U^3 \oplus E_8(-2)^*) \simeq T_A.$$

□

Lemma 3.2. *Let D be a finite abelian 2-group with a bilinear form $b(x,y) \in \mathbb{Q}/\mathbb{Z}$ and a quadratic form $q(x) = b(x,x) \in \mathbb{Q}/2\mathbb{Z}$. Let $\mathfrak{H} \subset D$ be an isotropic subroup, then*

$$l(D) - 2l(\mathfrak{H}) \leq l(\mathfrak{H}^\perp/\mathfrak{H}).$$

Proof. It is enough to show that $l(D/\mathfrak{H}) \leq l(\mathfrak{H}^\perp)$. Indeed it would imply that

$$l(D) \leq l(D/\mathfrak{H}) + l(\mathfrak{H}) \leq l(\mathfrak{H}^\perp) + l(\mathfrak{H}) \leq l(\mathfrak{H}^\perp/\mathfrak{H}) + l(\mathfrak{H}) + l(\mathfrak{H})$$

where the first and last inequalities are obvious.

We assume first that $\mathfrak{H} \simeq \mathbb{Z}/2$ with generator h. There exist $y_1, \cdots, y_{l(D)}$ generators of D, where y_1 is of the form $\frac{1}{k}h$ for some $k \in \mathbb{Z}_{>0}$. Let $0 \leq l \leq l(D)$ be such that

$$y_1, \ldots, y_l \in \mathfrak{H}^\perp \text{ and } y_{l+1}, \ldots, y_{l(D)} \notin \mathfrak{H}^\perp.$$

Note that any element of order two $y \in D$ satisfies $b(y,x) = 0$ or $\frac{1}{2}$ for all $x \in D$. Indeed, if $b(y,x) \neq 0 \in \mathbb{Q}/\mathbb{Z}$, then $2b(x,y) = 0 \in \mathbb{Q}/\mathbb{Z}$, hence $b(x,y) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$. Thus

$$y_1, \ldots, y_l, y_{l+1} + y_{l+2}, \ldots, y_{l+1} + y_{l(D)} \in \mathfrak{H}^\perp$$

and $l(\mathfrak{H}^\perp) \geq l(D) - 1$. If $l(\mathfrak{H}^\perp) = l(D)$, clearly $l(\mathfrak{H}^\perp) \geq l(D/\mathfrak{H})$. If $l(\mathfrak{H}^\perp) = l(D) - 1$, then there exists a generator y_j of D with $b(h, y_j) = \frac{1}{2}$. Moreover y_j must be of order two, since if $2y_j \neq 0$, then $2y_j \in \mathfrak{H}^\perp$ and $l(D) = l(\mathfrak{H}^\perp)$. Hence

$$\frac{1}{2} = b(y_1, y_j) = b(\frac{1}{k}h, y_j) = \frac{1}{2k},$$

or equivalently $k = 1$ and $y_1 = h$. It follows that $l(D/\mathfrak{H}) \leq l(D) - 1 = l(\mathfrak{H}^\perp)$.

If $|\mathfrak{H}| > 2$, then there exists a subgroup $\mathfrak{H}' \subset \mathfrak{H}$, such that $\mathfrak{H}/\mathfrak{H}' \simeq \mathbb{Z}/2$. Clearly $\mathfrak{H}/\mathfrak{H}'$ is an isotropic subgroup of $D_S/\mathfrak{H}'$. Since a minimal set of generators of $(\mathfrak{H}/\mathfrak{H}')^\perp$ define a set of independent elements of $\mathfrak{H}^\perp$, we get the inequality $l((\mathfrak{H}/\mathfrak{H}')^\perp) \leq l(\mathfrak{H}^\perp)$. By induction, we obtain

$$l(D/\mathfrak{H}) = l((D/\mathfrak{H}')/\mathfrak{H}/\mathfrak{H}') \leq l((\mathfrak{H}/\mathfrak{H}')^\perp) \leq l(\mathfrak{H}^\perp).$$

□

Let A be an abelian surface. If X is a K3 surface such that its period lattice T_X admits an embedding into T_A with $T_A/T_X \simeq (\mathbb{Z}/2\mathbb{Z})^\alpha$, then there is an embedding of $2T_A$ into T_X. The finite group $T_X/2T_A$ is a subspace of the $\mathbb{F}_2$-symplectic space $(T_A/2T_A, q_{T_A/2T_A})$. Conversely, any subspace U of $T_A/2T_A$ defines a sublattice of T_A, T_U, containing $2T_A$ and hence satisfying $T_A/T_U \simeq (\mathbb{Z}/2\mathbb{Z})^\alpha$. Moreover, the lattice T_U inherits a Hodge decomposition from the one on T_A and if T_U satisfies the condition (3) of the theorem 3.1, it determines a unique K3 surface. Note that any automorphism of T_A preserving its Hodge decomposition must be $\pm$ identity, [16], and therefore if U and U' are two subspaces of $T_A/2T_A$ such that there exists an orthogonal transformation $\phi \in O(T_A/2T_A)$ mapping U to U', then (if it exists) the lifting of ϕ to T_A cannot define a Hodge isometry between T_U and $T_{U'}$. Clearly, we have the following corollary.

Corollary 3.3. *Let* Kum(A) *be a Kummer surface. There exist finitely many isomorphism classes of K3 surfaces admitting a rational map of degree two into* Kum(A).

We say that a Kummer surface Y is general if it is associated to a general abelian surface A, that is to say A is the jacobian of a curve of genus two and its period lattice T_A is isomorphic to $U \oplus U \oplus \langle -2 \rangle$. In order to determine all the sublattices of T_A satisfying the conditions of 3.1, we classify all the subspaces of $(T_A/2T_A \simeq \mathbb{F}_2^5, q(x) = x_1x_2 + x_3x_4 + x_5^2)$. We

obtain in this way a short and exhaustive list of all K3 surfaces satisfying the conditions of theorem 3.1. In total, we find 373 K3 surfaces that are listed below. The number in the first column indicates the number of non-isomorphic K3 surfaces within a given type of period lattice. In other words, it represents the size of the orbit of a given subspace of $T_A/2T_A$ for the action of the orthogonal group $O(\mathbb{F}_2^5, q)$.

♯ K3's	T_X	D_{T_X}	q_{T_X}
1	$U \oplus U \oplus \langle -2\rangle$	$\mathbb{Z}/2\mathbb{Z}$	$\frac{1}{2}x^2$
15	$U(2) \oplus U \oplus \langle -2\rangle$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$x_1x_2 - \frac{1}{2}x_3^2$
10	$U \oplus U \oplus \langle -8\rangle$	$\mathbb{Z}/8\mathbb{Z}$	$-\frac{1}{8}x^2$
6	$U \oplus N$	$\mathbb{Z}/8\mathbb{Z}$	$\frac{3}{8}x^2$
20	$U(4) \oplus U \oplus \langle -2\rangle$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\frac{1}{2}x_1x_2 - \frac{1}{2}x_3^2$
15	$U(2) \oplus U(2) \oplus \langle -2\rangle$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 5}$	$x_1x_2 + x_3x_4 - \frac{1}{2}x_5^2$
45	$U(2) \oplus U \oplus \langle -8\rangle$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$x_1x_2 - \frac{1}{8}x_3^2$
60	$U \oplus \langle -2\rangle \oplus \langle 2\rangle \oplus \langle -8\rangle$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$-\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{8}x_3^2$
15	$U(2) \oplus N$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$x_1x_2 + \frac{3}{8}x_3^2$
15	$U(4) \oplus U(2) \oplus \langle -2\rangle$	$\mathbb{Z}/4\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$	$\frac{1}{2}x_1x_2 + x_3x_4 - \frac{1}{2}x_5^2$
60	$U(4) \oplus U \oplus \langle -8\rangle$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$
20	$U(4) \oplus N$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$\frac{1}{2}x_1x_2 - \frac{1}{2}x_1^2 + x_1x_3 + \frac{3}{8}x_3^2$
15	$U(2) \oplus U(2) \oplus \langle -8\rangle$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 4} \oplus \mathbb{Z}/8\mathbb{Z}$	$x_1x_2 + x_3x_4 - \frac{1}{8}x_5^2$
45	$\langle 2\rangle^2 \oplus \langle -2\rangle^2 \oplus \langle -8\rangle$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 4} \oplus \mathbb{Z}/8\mathbb{Z}$	$\frac{1}{2}(x_1^2 + x_2^2 - x_3^2 - x_4^2) - \frac{1}{8}x_5^2$
1	$U(4) \oplus U(4) \oplus \langle -2\rangle$	$\mathbb{Z}/4\mathbb{Z}^{\oplus 4} \oplus \mathbb{Z}/2\mathbb{Z}$	$\frac{1}{2}x_1x_2 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_5^2$
15	$U(4) \oplus U(2) \oplus \langle -8\rangle$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/8\mathbb{Z}$	$x_1x_2 + \frac{1}{2}x_3x_4 - \frac{1}{8}x_5^2$
15	$U(4) \oplus \langle 2\rangle \oplus \langle -2\rangle \oplus \langle -8\rangle$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/8\mathbb{Z}$	$\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + \frac{1}{2}x_3x_4 - \frac{1}{8}x_5^2$

The bilinear form of the lattices N is given by the symmetric integral matrix

$$N = \begin{pmatrix} 2 & 1 & 2 \\ 1 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

4. Even Eight

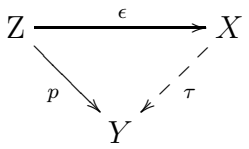
The remaining of this paper will be devoted to the geometrical description of the K3 surfaces appearing in the above list that have two-elementary period lattices. The study of such K3 surfaces is made easier by the existence on them of another involution.

Proposition 4.1. [13] Let X be a K3 surface. The period lattice T_X is two-elementary if and only if X admits an involution $\theta : X \to X$ such that $\theta^*_{S_X} = \mathrm{id}_{S_X}$ and $\theta^*_{T_X} = -\mathrm{id}_{T_X}$.

As a first step towards our geometrical description, we introduce the notion of an even eight. Secondly, in order to provide examples of different even eights on a Kummer surface, we give explicit generators for the Néron-Severi lattice of a general Kummer surface. We follow Naruki's notation in [10].

Definition 4.2. Let Y be a K3 surface, an even eight on Y is a set of eight disjoint smooth rational curves, $C_1, \cdots, C_8$, such that $C_1 + \cdots + C_8 \in 2S_Y$.

Recall, from section 2, that a rational map of degree two between K3 surfaces, $X \overset{\tau}{\dashrightarrow} Y$ factors as



where ϵ is the blow-up of X at eight points and p is a regular map of degree two, branched along eight disjoint smooth rational curves, hence an even eight. Conversely, given an even eight, $C_1, \cdots, C_8$, on a K3 surface Y, there is exists a double cover $Z \overset{p}{\to} Y$ branched along $\sum_{i=1}^{8} C_i$. Let $p^*(C_i) = 2E_i$, then each E_i is an exceptional divisor of the first kind. We may therefore blow down $\sum_{i=1}^{8} E_i$ to a smooth surface X, which is known to be a K3 surface. We recover, in this way, a rational map of degree two $X \overset{\tau}{\dashrightarrow} Y$ between the K3 surfaces X and Y (see [11]).

Let $Y \simeq \mathrm{Kum}(A)$ be a general Kummer surface and let C be the genus two curve of which A is the jacobian. The degree two map given by the linear system $|2C|$, $A \overset{|2C|}{\to} \mathbb{P}^3$, factors through the involution i, where $i(a) = -a$, and hence defines an embedding $A/\{1, i\} \hookrightarrow \mathbb{P}^3$. The image of this map, Y_0, is a quartic in $\mathbb{P}^3$ with sixteen nodes. Denote by L_0 the class of a hyperplane section of Y_0. Projecting from a node, Y_0 admits a regular map of degree two into the projective plane which induces the map $Y \overset{\phi}{\to} \mathbb{P}^2$, given by the linear system $|L - E_0|$, where L is a the pullback of L_0 on Y and E_0 is the exceptional curve resolving the center of projection. The branch locus of the map ϕ is a reducible plane sextic, which is the union of six lines, $l_1, \cdots, l_6$, all tangent to a conic W.

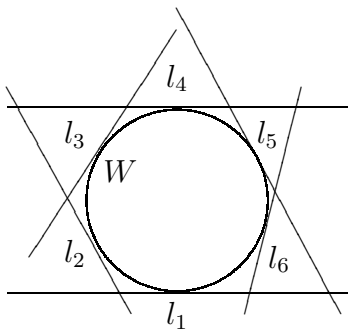


FIGURE 1.

Let $p_{ij} = l_i \cap l_j \in \mathbb{P}^2$, where $1 \leq i < j \leq 6$. Index the ten $(3,3)$-partitions of the set $\{1, 2, \ldots, 6\}$, by the pair (i, j) with $2 \leq i < j \leq 6$. Each pair (i, j) defines a plane conic l_{ij} passing through the sixtuplet $p_{1i}, p_{1j}, p_{ij}, p_{lm}, p_{ln}, p_{mn}$, where $\{l, m, n\}$ is the complement of $\{1, i, j\}$ in $\{1, 2, \ldots, 6\}$ and where $l \leq m \leq n$. The map ϕ factors as $Y \overset{\tilde{\phi}}{\longrightarrow} \tilde{\mathbb{P}}^2 \overset{\eta}{\longrightarrow} \mathbb{P}^2$ where η is the blowup of $\mathbb{P}^2$ at the p_{ij}'s and where $\tilde{\phi}$ is the double cover of $\tilde{\mathbb{P}}^2$ branched along the strict transform in $\tilde{\mathbb{P}}^2$ of the plane sextic of figure 1. Denote by $E_{ij} \subset Y$ the preimage of the exceptional curves of $\tilde{\mathbb{P}}^2$. The ramification of the map $\tilde{\phi}$ consists of the union of six disjoint smooth rational curves, $C_0 + C_{12} + C_{13} + C_{14} + C_{15} + C_{16}$. The preimage of the ten plane conics l_{ij} defines ten more smooth disjoint rational curves on Y, $C_{ij}, 2 \leq i < j \leq 6$. Finally,

note that $\phi(E_0) = W$. The sixteen curves $E_0, E_{ij} \quad 2 \leq i < j \leq 6$ are called the nodes of Y and the sixteen curves $C_0, C_{ij} \quad 2 \leq i < j \leq 6$ are called the tropes of Y. These two sets of smooth rational curves satisfy a beautiful configuration called the 16_6-configuration, i.e. each node intersects exactly six tropes and vice versa.
It is now possible to fully describe the Néron-Severi lattice S_Y of a general Kummer surface. Indeed, it is generated by the classes of E_0, E_{ij}, C_0, C_{ij} and L, with the relations:

(1) $C_0 = \frac{1}{2}(L - E_0 - \sum_{i=2}^{6} E_{1i})$,
(2) $C_{1j} = \frac{1}{2}(L - E_0 - \sum_{i \neq j} E_{ij})$, where $2 \leq j \leq 6$,
(3) $C_{jk} = \frac{1}{2}(L - E_{1j} - E_{1k} - E_{jk} - E_{lm} - E_{ln} - E_{mn})$ where $2 \leq i < j \leq 6$, and $\{l, m, n\}$ are as described above.

The intersection pairing is given by:

(1) the E_0, E_{ij} are mutually orthogonal,
(2) $\langle L, L \rangle = 4, \langle L, E_0 \rangle = \langle L, E_{ij} \rangle = 0$,
(3) $\langle E_0, E_0 \rangle = \langle E_{ij}, E_{ij} \rangle = -2$,
(4) the C_0, C_{ij} are mutually orthogonal,
(5) $\langle L, C_0 \rangle = \langle L, C_{ij} \rangle = 2$.

The action on S_Y of the covering involution α of the map ϕ is given by:

$\alpha(C_0) = C_0$ $\quad$ $\alpha(C_{1j}) = C_{1j}$ $\quad$ $2 \leq j \leq 6$
$\alpha(E_{ij}) = E_{ij}$ $\quad$ $1 \leq i < j \leq 6$ $\quad$ $\alpha(L) = 3\mathrm{L} - 4E_0$
$\alpha(E_0) = 2L - 3E_0$ $\quad$ $\alpha(C_{ij}) = C_{ij} + L - 2E_0 \quad 2 \leq i < j \leq 6$.

Remark 4.3. Conversely, the minimal resolution of the double cover of $\mathbb{P}^2$ branched along the sextic of figure 1 is a Kummer surface (see [4] for a proof).

Consider the divisors

$e_1 := (L - E_0) - (E_{12} + E_{46})$,
$e_2 := 2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{46} + E_{56})$,
$e_3 := 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{36} + E_{45} + E_{46} + E_{56})$,
$e_4 := 4(L - E_0) - 2(E_{12} + E_{13} + E_{46}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{56})$,
$e_5 := 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) + (E_{24} + E_{25} + E_{34} + E_{36} + E_{45})$,
$e_6 := C_{23}, \quad e_7 := \alpha(C_{23}), \quad e_8 := E_{35}$.

Theorem 4.4. [10] *The eight* (-2)*-classes,* $e_1, \ldots, e_8$ *represent an even eight on* Y.

The geometric meaning of the (-2)-classes can be explained in terms of the double plane model of the general Kummer surface, $Y \xrightarrow{\phi} \mathbb{P}^2$. The class e_1 is represented by the proper inverse image of a line passing through p_{12} and p_{46}. The class of e_2 is represented by the proper inverse image of a conic passing through $p_{12}, p_{13}, p_{24}, p_{46}, p_{56}$. The class of e_3 is represented by the proper inverse image of a cubic passing through $p_{13}, p_{24}, p_{36}, p_{45}, p_{46}, p_{56}$ and having a double point at p_{12}. The class of e_4 is represented by the proper inverse image of a quartic passing through $p_{24}, p_{25}, p_{36}, p_{45}, p_{56}$ and having double points at p_{12}, p_{13}, p_{46}. The class of e_5 is represented by the proper inverse image of a quintic passing through $p_{24}, p_{25}, p_{34}, p_{36}, p_{45}$ and having double points at p_{13}, p_{46}, p_{56} and a triple point at p_{12}.

5. K3 surface with a Shioda-Inose structure

A particular case in the list of page 8 is given by the unique K3 surface X_0 that has its period lattice isomorphic to $U \oplus U \oplus \langle -2 \rangle$. In fact, Naruki showed that the map $X_0 \overset{\tau}{\dashrightarrow} Y$

is the rational map of degree two associated to the even eight of the theorem 4.4. Moreover, according to Morrison's criterion, X_0 has a Shioda-Inose structure ([8]).

Prior to describing the geometry of X_0, we will state some basics about elliptic fibrations on K3 surfaces (see [7] for details).

Definitions 5.1. (1) A smooth surface X is *elliptic* if it admits a regular map into a curve C, $X \xrightarrow{f} C$ where the general fiber is a smooth connected curve of genus one.
(2) When nonempty, the set of sections of f, $\mathrm{MW}_f(X)$, form a finitely generated group called the Mordell-Weil group, which can be identified with the set of $k(C)$-rational points of the generic fiber X_η.
(3) Let F denote the class of a fiber. The set $\{B \cdot F | B \in S_X\}$ is an ideal of $\mathbb{Z}$. The multisection index l of f is the positive generator of this ideal. An $l-$section is a divisor B on X for which $B \cdot F = l$.

We refer the reader to [7] for a table of Kodaira's classification of singular fibers on an elliptic fibration.

Theorem 5.2. [18] *Let $X \xrightarrow{f} \mathbb{P}^1$ be an elliptic K3 surface with a section S_0. Let A be the subgroup of S_X generated by the section S_0, the class of a fiber and the components of the reducible fibers that do not meet S_0. There is an short exact sequence of groups*

$$0 \to A \xrightarrow{\alpha} S_X \xrightarrow{\beta} \mathrm{MW}_f(X) \to 0$$

where β is the restriction to the generic fiber. Let Θ_i $(i = 1, \ldots, k)$ be all the singular fibers of f. Then

(1) $24 = \chi_{top}(X) = \sum_i \chi_{top}(\Theta_i)$,
(2) $\mathrm{rank}(\mathrm{MW}_f(X)) = \mathrm{rank}(S_X) - 2 - \sum_i (m(\Theta_i) - 1)$, *where $m(\Theta_i)$ denotes the number of irreducible components of Θ_i.*
(3) *When the rank of $\mathrm{MW}_f(X)$ is zero, then*

$$|D_{S_X}| = \frac{\prod_i s_i}{n^2}$$

where s_i is the number of simple components of the singular fiber Θ_i and n is the order of the finite group $\mathrm{MW}_f(X)$.

Pjateckiĭ-Šapiro and Šafarevič showed that on a K3 surface, if D is a non-zero effective nef divisor with $D^2 = 0$, then the system $|D|$ is of the form $|mD_0|, m \geq 1$ and the linear system $|D_0|$ defines a morphism $X \xrightarrow{|D_0|} \mathbb{P}^1$, with general fiber a smooth connected curve of genus one ([16]).

Proposition 5.3. Let Y be a general Kummer surface. Then there exists an elliptic fibration on Y whose singular fibers are of the type $6I_2 + I_5^* + I_1$ and whose Mordell-Weil group is finite of order two.

Proof. Consider the divisor class

$$D = 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45}).$$

Geometrically, D can be represented by the proper inverse image, under the map $Y \xrightarrow{\phi} \mathbb{P}^2$, of a quintic passing trough p_{24}, p_{25}, p_{36}, p_{45}, having double points at p_{13}, p_{46}, p_{56} and a triple point at p_{12}. Since $D \sim e_5 + E_{34}$ and $e_5 \cdot E_{34} = 2$, D must be a reduced nef effective divisor

with $D^2 = 0$. Let $Y \xrightarrow{f} \mathbb{P}^1$ be the elliptic fibration defined by the linear system $|D|$. The divisors

(1) $F_1 = e_5 + E_{34}$,
(2) $F_2 = e_4 + \underbrace{(L - E_0) - (E_{12} + E_{56})}_{a_1}$,
(3) $F_3 = e_3 + \underbrace{2(L - E_0) - (E_{12} + E_{13} + E_{46} + E_{56} + E_{25})}_{a_2}$,
(4) $F_4 = e_2 + \underbrace{3(L - E_0) - 2E_{12} - (E_{13} + E_{46} + E_{56} + E_{25} + E_{36} + E_{45})}_{a_3}$,
(5) $F_5 = e_1 + \underbrace{4(L - E_0) - 2(E_{12} + E_{13} + E_{56}) - (E_{46} + E_{24} + E_{25} + E_{36} + E_{45})}_{a_4}$,
(6) $F_6 = e_8 + \underbrace{5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{35})}_{a_5}$

represent six fibers of type I_2, as under the genericity assumption, the (-2)-cycles, $a_1, \dots a_5$ are represented by smooth rational curves. Their geometric meaning can be explained exactly in the same way as in the theorem [10]. The divisor

$$F_7 = e_6 + e_7 + 2(E_{23} + C_{12} + E_{26} + C_{16} + E_{16} + C_0) + E_{14} + E_{15}$$

defines a fiber of type I_5^*. Note also that $C_{14} \cdot D = C_{15} \cdot D = 1$, i.e. the curves C_{14} and C_{15} are sections of f. The fifteen components of the singular fibers, $e_1, \dots, e_8, C_{12}, E_{23}, E_{26}, C_{16}, E_{16}, C_0, E_{14}$ are independent over $\mathbb{Q}$. Consequently, by the theorem 5.2, there are no other reducible fibers and the Mordell-Weil group is finite of order two. □

We note that the curves $e_1, \cdots, e_8$ defining the even eight of theorem [10] are components of the singular fibers of this elliptic fibration. In fact, we see that the surface X_0 admits also an elliptic fibration, $X_0 \xrightarrow{|\tau^* D|} \mathbb{P}^1$, with singular fibers of type $I_2 + I_{10}^* + 6I_1$ and with a Mordell-Weil group of order two. The reducible fibers and the sections of this latter fibration form the diagram of curves on X_0, shown in figure 2. Nikulin proved that those curves are the only smooth rational curves lying on X_0 and that the group $\mathrm{Aut}(X_0)$ is generated by two commuting involutions θ and ι. The involution θ satisfies $\theta^*_{S_{X_0}} = \mathrm{id}_{S_{X_0}}$, $\theta^*_{T_{X_0}} = -\mathrm{id}_{T_{X_0}}$, while the involution ι is the Nikulin involution that gives the Shioda-Inose structure on X_0 ([14]).

Let $\mathcal{S}$ be a reducible plane sextic, decomposable as a cubic $\mathcal{C}$ with a cusp, a line $\mathcal{L}$ intersecting the cubic only at the cusp and a conic $\mathcal{Q}$ tangent to the line, (see figure 3).

Proposition 5.4. [3] The map $X_0 \dashrightarrow Y$ factors as

$$\begin{array}{ccc} X_0 & \xrightarrow{/\theta} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \Big| & & \Big\downarrow \pi_1 \\ \Big\downarrow /\iota & & \\ Y & \xrightarrow{\pi} & \mathbb{P}^2 \end{array}$$

where the map π_1 is the double cover of $\mathbb{P}^2$ branched along the conic $\mathcal{Q}$ and the map $X_0 \xrightarrow{/\theta} \mathbb{P}^1 \times \mathbb{P}^1$ is the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along $\pi_1^*(\mathcal{C} + \mathcal{L})$.

The map π is given by the divisor

$$\Delta = 9(L - E_0) - 5E_{12} - 4(E_{13} + E_{46} - 3E_{56} - 2(E_{24} + E_{36} + E_{45}) - (E_{25} + E_{34})$$

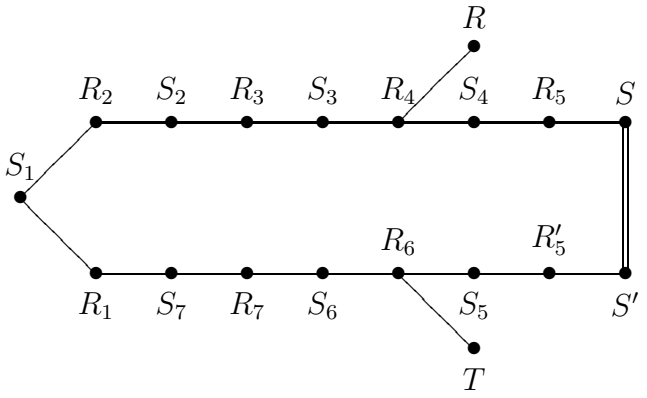


FIGURE 2.

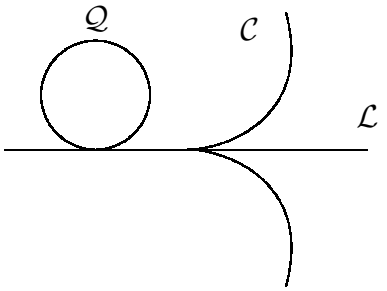


FIGURE 3.

and it contracts the e_i's. Conversely,

Proposition 5.5. [10] The canonical resolution of the double cover of $\mathbb{P}^2$ branched along the sextic $\mathcal{S}$ is a jacobian Kummer surface.

6. K3 Surfaces with $T_X \simeq U(2) \oplus U \oplus \langle -2 \rangle$

Building on the results of the previous section and using the same notation, we describe the geometry of the K3 surfaces, appearing in our list, that have their period lattices isomorphic to $U(2) \oplus U \oplus \langle -2 \rangle$.

In the proposition 5.3, we defined the curves $a_1, \cdots a_5$, which were components of the singular fibers of the elliptic fibration $Y \xrightarrow{f} \mathbb{P}^1$. Set $a_6 := e_6, \quad a_7 := e_7, \quad a_8 := E_{34}$.

Lemma 6.1. *The a_i's form an even eight on Y. If $X \overset{\mu}{\dashrightarrow} Y$ is the associated rational map of degree two, then $T_X \simeq U(2) \oplus U \oplus \langle -2 \rangle$.*

Proof. A direct computation shows that

$$\sum_{i=1}^{8} a_i = 2C_{13} + 2E_{34} + 2 \cdot \{8(L - E_0) - (5E_{12} + 4E_{46} + 3E_{13} + 3E_{56} + E_{36} + E_{25} + E_{45})\} \in 2S_Y.$$

Recall that the rational double cover $X \overset{\mu}{\dashrightarrow} Y$ associated to the even eight $a_1, \cdots, a_8$ factors as

$$\begin{array}{ccc} Z & \xrightarrow{\epsilon} & X \\ {\scriptstyle p}\downarrow & \swarrow_{\mu} & \\ Y & & \end{array}$$

where $Z \overset{p}{\to} Y$ is a degree two map branched along the a_i's and $Z \overset{\epsilon}{\to} X$ contracts the eight exceptional curves $p^{-1}(a_i)$. Let $X \overset{i}{\to} X$ be the covering Nikulin involution. We will describe the proper inverse image under μ of the elliptic fibration $f : Y \overset{|D|}{\to} \mathbb{P}^1$ introduced in the proposition 5.3. Each of the curves $e_1, e_2, e_3, e_4, e_5, e_8$ meet the a_i's at exactly two points, thus their pullback under p are smooth rational curves, each meeting an exceptional curve at two points. After blowing down the exceptional curves, they become six disjoint nodal curves $f_1, f_2, f_3, f_4, f_5, f_8$. It follows from the properties of μ^* that the divisor μ^*D is a nonzero effective nef divisor with self-intersection 0. Thus the linear system $|\mu^*D|$ is of the form $|mD_0|, m \geq 1$, where D_0 is a smooth curve of genus one. Clearly the f_i's are singular fibers of the elliptic fibration defined by $|D_0|$, $X \overset{g}{\to} \mathbb{P}^1$. Moreover as $i(f_i) = f_i$, the involution i must act trivially on $\mathbb{P}^1$ under the map g.

Claim: The general member of the linear system $|mD_0|$ is a smooth curve of genus one.
Proof of the claim: Note that, since $mD_0 \cdot \mu^*C_{14} = \mu^*D \cdot \mu^*C_{14} = 2D \cdot C_{14} = 2$, we must have $m \leq 2$. If $m = 2$, then the preimage of the general member of $|D|$ would have to split into two disjoint smooth curves of genus one, $\mu^{-1}(D) = D_0 + D_1$, where $D_0 \overset{lin}{\sim} D_1$, and $i(D_0) = D_1$. The last assertion contradicts the assumption that i acts trivially on $\mathbb{P}^1$.
We deduce that the preimage of the I_1 fiber does not split into two I_1 fibers but instead it becomes a singular fiber of type I_2. We denote its two components by S and i^*S. Finally, the I_5^* fiber pulls back to a I_{10}^*-fiber whose components will be denoted by

$$\begin{aligned} E'_{14} + E'_{15} + 2(C'_0 + E'_{16} + C'_{16} + E'_{26} + C'_{12} + E'_{23} \\ + i^*C'_{12} + i^*E'_{26} + i^*C'_{16} + i^*E'_{16} + i^*C'_0) + i^*E'_{14} + i^*E'_{15}. \end{aligned}$$

The section C_{14} meets the a_i's at two points so it pulls back to a two-section C'_{14}. We conclude that X contains the diagram of (-2)-curves of figure 4.

Let M be the lattice of rank seventeen, generated by all the (-2) curves of figure 4. A direct computation shows that

$$D_M \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Claim: The lattice S_X is two-elementary, i.e. $D_{S_X} \simeq (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \geq 0$.
Assuming the claim and letting e be the index $[S_X : M]$, we get that

$$|D_M| = 2^5 = e^2 \cdot 2^a \Rightarrow e^2 = 2^{5-a} \Rightarrow a = 1, 3 \text{ or } 5.$$

If $a = 5$, then $e = 1$ and $M = S_X$, which contradicts the assumption that S_X is two-elementary. If $a = 1$, then by the classification of two-elementary lattices ([13]), T_X would be isomorphic to $U \oplus U \oplus \langle -2 \rangle$. We have mentioned in the previous section that a K3 surface with such a period lattice contains finitely many smooth rational curves forming the figure 2, which does not contain a subdiagram similar to the one in figure 4. We conclude that $a = 3$, or equivalently that $T_X \simeq U \oplus U(2) \oplus \langle -2 \rangle$.

Proof of the claim: In order to prove the claim, it is enough to show the existence of an involution $X \overset{\theta}{\to} X$ satisfying $\theta^*_{S_X} = \mathrm{id}_{S_X}$ and $\theta^*_{T_X} = -\mathrm{id}_{T_X}$ (see proposition 4.1). Observe

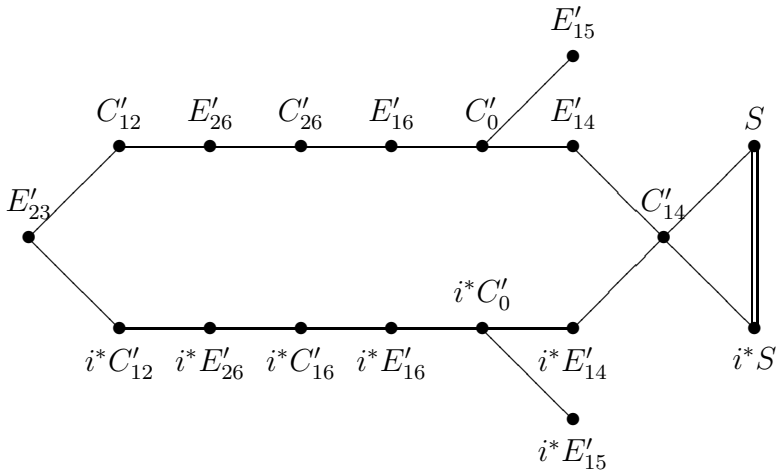


FIGURE 4.

that the curves C_{13} and C_{15} pull back to a curve of genus two W, and a curve of genus one E, respectively, both invariant under i.
The cycle

$$C'_{14}+E'_{14}+C'_0+E'_{16}+C'_{16}+E'_{26}+C'_{12}+E'_{23}+i^*C'_{12}+i^*E'_{26}+i^*C'_{16}+i^*E'_{16}+i^*C'_0+i^*E'_{14}$$

is disjoint from the genus one curve E. Consequently it defines the I_{14} singular fiber of the elliptic fibration given by the linear system $|E|$, which we will denote by $v : X \xrightarrow{|E|} \mathbb{P}^1$. The curves S, i^*S, E'_{15} and $i^*E'_{15}$ are sections of v. Fix $i^*E'_{15}$ as the zero section. Let $X \xrightarrow{\sigma} X$ be the map induced by the canonical involution on X_η, the generic fiber of v, i.e. $\sigma_{X_\eta}(x) = -x$ and let $X \xrightarrow{t_{E'_{15}}} X$ be the translation by the section E'_{15}. Set $\gamma := \sigma \circ t_{E'_{15}}$. The action of γ on the period lattice is given by

$$\gamma^*_{T_X} = \sigma^*_{T_X} \circ (t_{E'_{15}})^*_{T_X} = -\mathrm{id}_{T_X} \circ \mathrm{id}_{T_X} = -\mathrm{id}_{T_X}.$$

Moreover its action on the Néron-Severi lattice satisfies $\gamma^*_{S_X} = i^*_{S_X}$. Indeed, to show this, we note that the divisor $S + i^*S - E'_{15}$ is linearly equivalent to the divisor

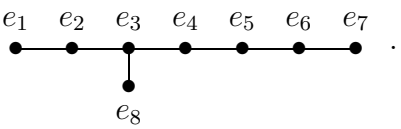
$$E'_{14}+2(C'_0+E'_{16}+C'_{16}+E'_{26}+C'_{12}+E'_{23}+i^*C'_{12}+i^*E'_{26}+i^*C'_{16}+i^*E'_{16}+i^*C'_0)+i^*E'_{14}+i^*E'_{15}$$

which belongs to the kernel of the map β of the theorem 5.2. In other words $-S+E'_{15} = i^*S$ in $\mathrm{MW}_{f_E}(X)$, or equivalently $\gamma^*S = i^*S$ and $\gamma^*E'_{15} = i^*E'_{15}$. It is now clear that $\gamma^*_M = i^*_M$ and consenquently that $\gamma^*_{S_X} = i^*_{S_X}$. Define the map $\theta = i \circ \gamma$. By construction θ satisfies $\theta^*_{T_X \oplus S_X} = -\mathrm{id}_{T_X} \oplus \mathrm{id}_{S_X}$, which implies, by the proposition 4.1, that S_X is two-elementary. □

Definition 6.2. Define the lattice $E_8^{\mathrm{twist}}(-1)$ to be the lattice $E_8(-1)$ with the following modification

$$b(e_i, e_7) = \left\{ \begin{array}{cc} -4 & \text{if } e_i = e_7 \\ 2 & \text{if } e_i = e_6 \\ 0 & \text{otherwise} \end{array} \right\}$$

where the label of the vertices of the dynkin diagram associated to $E_8(-1)$ is as follows



Note that $|D_{E_8^{\text{twist}}(-1)}| = 4$.

Proposition 6.3. Let Y be a general Kummer surface and N be the Nikulin lattice (see definition 2.6 in section 2) generated by the $a_i's$, then there is a primitive embedding $E_8^{\text{twist}}(-1) \hookrightarrow N^\perp \subset S_Y$.

Proof. The elements $E_{14}, C_{12}, E_{26}, C_{16}, E_{16}, C_0, E_{14}, 2C_{14} + a_5 + a_8$ generate a sublattice of $N^\perp$ isomorphic to $E_8^{\text{twist}}(-1)$. To prove that the embedding is primitive, it is enough to show that this set can be completed to a basis of $N^\perp$. Let T be the lattice generated by the elements $E_{14}, C_{12}, E_{26}, C_{16}, E_{16}, C_0, E_{14}, 2C_{14} + a_5 + a_8$ and $2E_{23} + a_6 + a_7$. T is a sublattice of $N^\perp$ of finite index e. By explicitly taking the determinant of a matrix representing the bilinear form of T, we find that $|D_T|$=4 and therefore, $4 = e^2|D_{N^\perp}|$. Either $e = 1$ or $N^\perp$ is unimodular, but according to Milnor's classification [6], there is no even unimodular lattice of rank nine. We conclude that $T = N^\perp$ and therefore the embedding is primitive. □

Theorem 6.4. *Let X be a K3 surface with $T_X \simeq U(2) \oplus U \oplus \langle -2 \rangle$. Then X has a Nikulin involution such that if $X \overset{\mu}{\dashrightarrow} Y$ is the rational quotient map,*

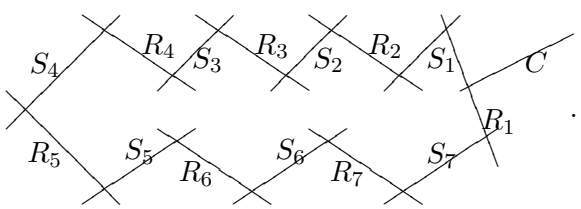
1) Y is a Kummer surface,

2) there is a primitive embedding $E_8^{\text{twist}}(-1) \hookrightarrow N^\perp$.

According to the theorem 3.1, we already know that X admits a rational map of degree two into a Kummer surface, so we need to show that the even eight associated to such a map is given by the a_i's. Before proving this, we will show, using very geometrical arguments, that any K3 surface with $T_X \simeq U(2) \oplus U \oplus \langle -2 \rangle$ must contain a diagram of smooth rational curves meeting as in figure 4. Let θ be the involution that satisfies $\theta^*_{T_X} = -\text{id}_{T_X}$ and $\theta^*_{S_X} = \text{id}_{S_X}$ which exists by proposition 4.1. Nikulin showed that

$$X^\theta := \{x \in X | \theta(x) = x\} = E \amalg \coprod_{i=1}^{7} R_i,$$

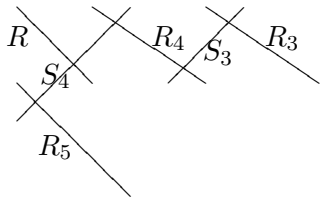
where E is a curve of genus one and the R_i's are disjoint rational curves. The system $|E|$ defines an elliptic fibration $X \overset{v}{\to} \mathbb{P}^1$, with a section C and exactly one reducible singular fiber $\mathfrak{F}$ of type I_{14}. Moreover θ acts non trivially on $\mathbb{P}^1$ under v ([14] p.1424-1430).
The R_i are components of $\mathfrak{F}$. Since θ preserves all the rational curves, any section of v must meet the singular fiber at a R_i component. Thus the surface X contains the following configuration of (-2)-curves



Consider the divisor

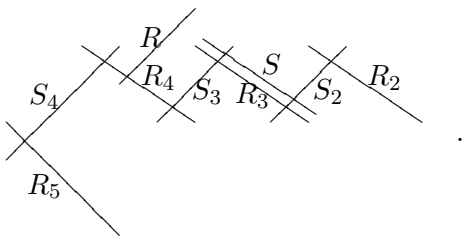
$$F := 2R_2 + 4S_1 + 6R_1 + 3C + 5S_7 + 4R_7 + 3S_6 + 2R_6 + S_5.$$

The system $|F|$ defines another elliptic fibration, $X \xrightarrow{q} \mathbb{P}^1$. By construction, q has a singular fiber of type II^*, a section R_5 and a 2-section S_2. Moreover, the curves S_4, R_4, S_3 and R_3 must be part of another reducible fiber of q, $\mathfrak{J}$. We will show by elimination that $\mathfrak{J}$ must be of type I_2^*. First, suppose that $\mathfrak{J}$ was of type I_b for $b \geq 4$:, then there would exist a rational smooth curve R meeting S_4 at exactly one point, p:



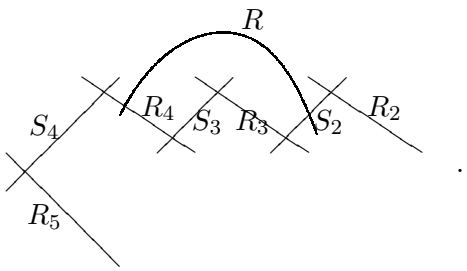
Since θ^* acts as the identity on S_X, $\theta(p)$ must also belong to $S_4 \cap R$. But R meets S_4 at exactly one point, so p must be fixed by θ. Either $p \in R_5 \cap S_4 \cap R$, which contradicts the fact that R_5 is a section of q, or $p \in R_4 \cap S_4 \cap R$, which contradicts the assumption that S_4, R_4, S_3 and R_3 are part of a fiber of type I_b. The fiber $\mathfrak{J}$ can neither be of type I_1^* : Again, suppose it is, then S_4 must be a component of multiplicity one (S_4 meets the section R_5). Denote by R the other component of multiplicity one of $\mathfrak{J}$ that meets R_4. Two cases are possible:

(1) R does not meet the curve S_2:



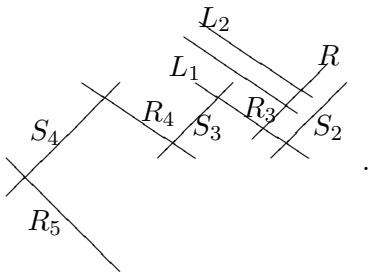
Then the sixth component of $\mathfrak{J}$, S would meet S_2 at exactly one point, p. Because $p \in S \cap S_2$, its image $\theta(p)$ belongs to $S \cap S_2$. So $\theta(p) = p$ and consequently $p \in R_3 \cap S_2$ or $p \in R_2 \cap S_2$. The first case contradicts the fact that $\mathfrak{J}$ is of type I_1^*, while in the second case, S would meet a component of another singular fiber, which is absurd.

(2) R does meet the curve S_2 :



Since S_2 is a 2-section of q and $S_2 \cdot R_3 = 1$, necessarily $R \cdot S_2 = 1$. Let $p \in R \cap S_2$, then $\theta(p) = p$. It follows that either $p \in R_3 \cap S_2$, which contradicts again the assumption that $\mathfrak{J}$ is of type I_1^*. Or $p \in R_2 \cap S_2 \cap R$, which implies again that R meets a component of another singular fiber.

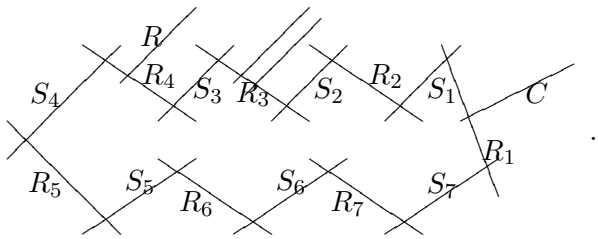
We exclude also the case where the fiber $\mathfrak{J}$ is of type I_3^* as follows. Suppose it is, then S_4 has to be a component of multiplicity one and R_4, S_3 and R_3 components of multiplicity two. Let R be the fourth component of multiplicity two of $\mathfrak{J}$ and L_1, L_2 the two components of multiplicity one intersecting it:



Since L_1 and L_2 do not meet any R_i's nor any S_i's, they must be part of a reducible fiber of the fibration defined by $|E|$. But this fibration is known to have a unique reducible singular fiber. The fiber $\mathfrak{J}$ cannot be of type I_b^*, with $b \geq 4$, as $\mathfrak{J}$ would have $b+5$ components. But by theorem 5.2, the number of components of $\mathfrak{J}$ cannot exceed 8, so $b \leq 3$. The same counting argument shows that $\mathfrak{J}$ cannot be of type II^*. The $\mathfrak{J}$ cannot be of type III^*. If it was, the group of sections of q would be finite of order n, where n would satisfy by the theorem 5.2

$$2^3 = |D_{S_X}| = \frac{1 \cdot 2}{n^2}$$

but no integer satisfies this equation. Finally $\mathfrak{J}$ cannot be of type IV^*. Indeed in this case one of the components with multiplicity one will have to be disjoint from every R_i and S_i. Such a component must be part of a reducible fiber of the fibration defined by $|E|$. But again, this fibration is known to have a unique reducible singular fiber. By elimination, we have shown that $\mathfrak{J}$ must be of type I_2^*. Moreover we have shown that X contains the following configuration of smooth rational curves

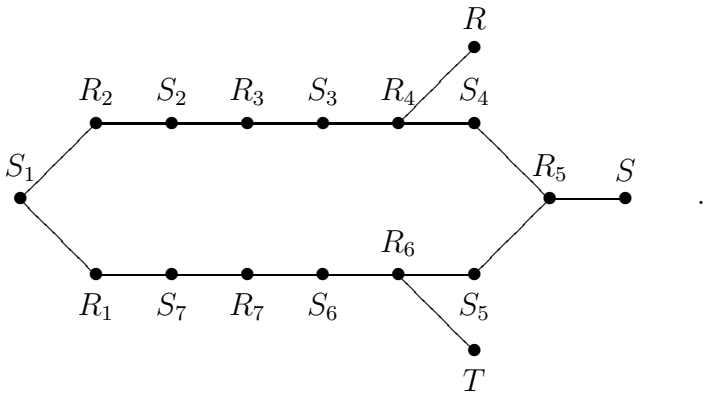


By symmetry, the divisor class

$$F' = 2R_7 + 2S_7 + 6R_1 + 3C + 5S_1 + 4R_2 + 3S_2 + 2R_3 + S_3$$

defines an elliptic fibration, $X \xrightarrow{h} \mathbb{P}^1$, for which the curves $R_6 + S_5 + R_5 + S_4$ are part of a reducible fiber of type I_2^*, $\mathfrak{H} := S + S_4 + 2R_5 + 2S_5 + 2R_6 + T + T'$. Since $R \cdot F' = 0$, the curve R is also part of a reducible fiber clearly disjoint from $\mathfrak{H}$. It follows from theorem 5.2 that R can only be part of a fiber with two components so generically it is of type I_2. We conclude that the R_i's, the S_i's and the smooth rational curves R, S, T define the diagram

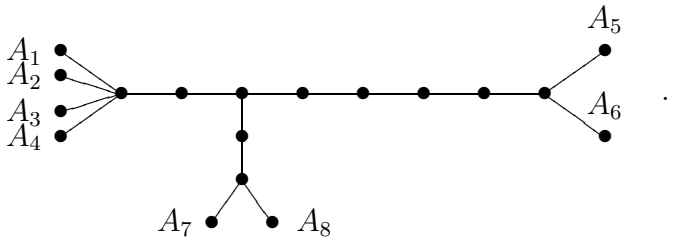
on X



The I_{10}^* configuration apparent in this diagram defines an elliptic fibration for which S is part of a I_2 fiber. By adding the other component of this fiber, we obtain a diagram of smooth rational curves meeting exactly as in the figure 4.

We can now prove theorem 6.4.

Proof. In order to explicitly construct the Nikulin involution i we apply the inverse process used in lemma 6.1 to construct the involution θ. The curves R, S and T, constructed above, are sections of the elliptic fibration $X \xrightarrow{v} \mathbb{P}^1$. Fix T as the zero section. Let σ be the map induced by inversion on the generic fiber and t_R be the translation by R. The map $\gamma = \sigma \circ t_R$ is an involution on X such that $\gamma^*_{T_X} = -\mathrm{id}_{T_X}$. Set $i := \theta \circ \gamma$. As the maps θ^* and γ^* commute on $\mathrm{H}^2(X, \mathbb{Z})$, i is an involution. Moreover it follows from the fact that $i^*_{T_X} = \mathrm{id}_{T_X}$ that i is a Nikulin involution. Its eight fixed points can be easily described. Indeed a point p is fixed by i if and only if $\gamma(p) = \theta(p)$. Since γ acts trivially on the fibers of v and θ acts non trivially on the same fibers, the fixed points of i must lie on the only two fibers fixed by θ, i.e E and $\mathfrak{F}$. The points lying on E satisfy $\gamma(p) = \theta(p) = p$ and therefore correspond to the four fixed points of γ restricted to E. The four remaining fixed points must lie on $\mathfrak{F}$. More precisely, two of them lie on S_1 and the other two lie on R_5. Indeed, since $S_j \cdot i^*(S_j) = 0$ for $j \neq 1$, a point p, lying on some S'_js is fixed by i if and only if $p \in S_1$ and $\theta(p) = p$. Thus the two fixed points of θ restricted to S_1 are fixed by i. Similarly, $R_j \cdot i^*(R_j) = 0$ for $j \neq 5$ implies that the two fixed points of γ restricted to R_5 are fixed by i. Consider $X \overset{\mu}{\dashrightarrow} Y$ the rational quotient map associated to i and denote by $A_1, \ldots, A_8$ the induced even eight. The A_i's and the image of the curves S_i, R_i, S, R and T under μ form the following diagram of rational curves on Y



The surface Y admits therefore an elliptic fibration with a section and whose singular fibers are of the type $6I_2 + I_5^* + I_1$. In the table of [17], we find that the Mordell Weil group of such a fibration must be of order two. It follows now from the proposition 5.3 that S_Y admits a primitive embedding of the *Kummer lattice* which implies in turn by [11] that Y is

a Kummer surface. The primitive embedding $E_8^{\text{twist}}(-1) \hookrightarrow N^\perp$ follows from the proposition 6.3. □

Let Σ be a reducible plane sextic, decomposable as an irreducible quartic Γ with an ordinary node and a cusp of type A_4, a line Λ_1 passing trough the two singular points of Γ and another line Λ_2 (see figure 5).

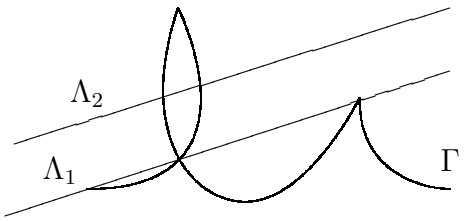


FIGURE 5.

Theorem 6.5. *The rational quotient map $X \overset{\mu}{\dashrightarrow} Y$ decomposes as*

$$\begin{array}{ccc} X & \xrightarrow{q_2} & Z \\ \downarrow \mu & & \downarrow q_1 \\ Y & \xrightarrow{q_0} & \mathbb{P}^2 \end{array}$$

where q_0 is the canonical resolution of the double cover of $\mathbb{P}^2$ branched along Σ. The maps q_1 and q_2 are the canonical resolutions of the double covers branched along Γ and $q_1^*(\Lambda_1 + \Lambda_2)$ respectively.

Proof. We have shown in the course of proving theorem 6.4 that the map μ can be thought as the rational double cover associated to the even eight defined by the a_i's. Therefore there exists on X a curve of genus two defined by $W = \mu^* C_{13}$ that satisfies

$$W \cap E = \{p_1, p_2, p_3, p_4\}, \quad W \cap R_5 = \{p_5, p_6\} \quad i(p_i) = p_i \quad i = 1, \dots 6,$$

where the p_i's are fixed points of the Nikulin involution. The quotient X/θ is a smooth rational surface. Applying Hurwitz formula we find that $K^2_{X/\theta} = -7$. Note that the image in X/θ of the nine disjoint curves S_i, for $i = 1 \cdots 7$, R and T have self intersection -1. We may therefore blow them down to a surface Z, whose canonical divisor has self-intersection 2. The map $X \overset{q_2}{\to} Z$ is ramified along the R_i's and E and it sends W to a smooth rational curve C, with $C^2 = 2$. Since the involution i commutes with θ on X, it induces an involution $\bar{i}$ on Z. Note that $\bar{i}$ has four fixed points on C, denoted by $q_2(p_1)$, $q_2(p_2)$, $q_2(p_3)$, $q_2(p_4)$, thus $\bar{i}$ restricted to C is the identity and the quotient map, $Z \to Z/\bar{i}$, is ramified along C. Using Hurwitz formula a second time, we find that $K^2_{Z/\bar{i}} = 6$. Moreover, under the composition, $X \overset{q_2}{\to} Z \overset{/\bar{i}}{\to} Z/\bar{i}$, the curves R_2, R_3 and R_5 map to rational curves R_2', R_3', R_5', with the intersection properties

$$R_5'^2 = R_2'^2 = -1, \quad R_3'^2 = -2, \quad R_5' \cdot R_2' = R_5' \cdot R_3' = 0, \quad R_2' \cdot R_3' = 1.$$

We may successively blow down R_5', R_2' and R_3' to obtain a rational surface whose canonical divisor has self intersection equal to 9, i.e. the projective plane. It is an easy exercise to verify that under the composition, $X \overset{q_2}{\to} Z \overset{q_1}{\to} \mathbb{P}^2$, the genus two curve W maps to a quartic

Γ with an ordinary node and a cusp of type A_4, the R_i's get contracted to a line Λ_1 passing through the singularities of Γ, and the genus one curve E maps to another line Λ_2. The following proposition completes the proof. □

Proposition 6.6. A surface Y is a general Kummer surface if and only if Y is the canonical resolution of a double cover of $\mathbb{P}^2$ branched along the reducible sextic $\Sigma = \Gamma + \Lambda_1 + \Lambda_2$.

Proof. Assume first that Y is a general Kummer surface. Recall that Y admits a double plane model $Y \xrightarrow{\phi} \mathbb{P}^2$ that factors as $Y \xrightarrow{\tilde{\phi}} \tilde{\mathbb{P}}^2 \xrightarrow{\epsilon} \mathbb{P}^2$ where $\tilde{\mathbb{P}}^2$ is the blowup of $\mathbb{P}^2$ at the fifteen singular points of the union of six lines (see figure 1). The image in $\tilde{\mathbb{P}}^2$ of the curves $a_1, \cdots a_8, C_{15}, E_{15}, C_0, E_{16}, E_{26}, C_{12}, E_{26},\ E_{14}\ C_{14}$ and C_{13} intersect as in the figure 6. The numbers in the parentheses denote the self-intersection.

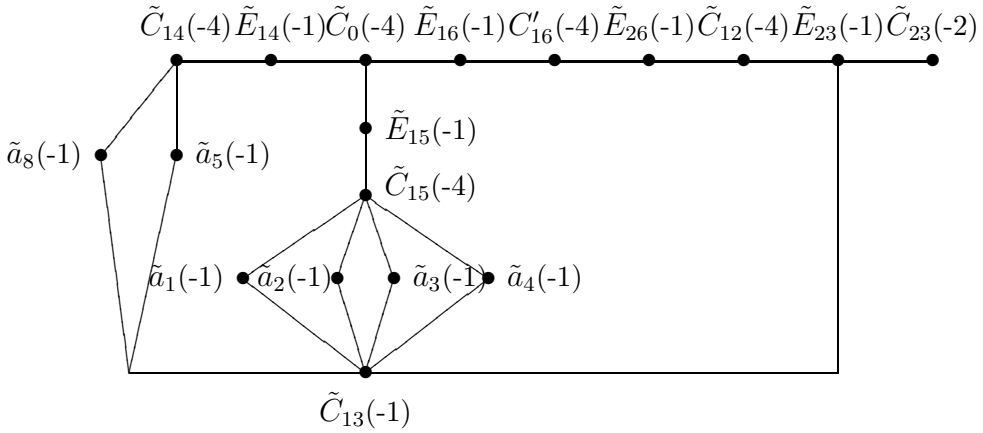
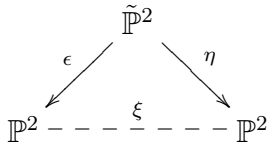


FIGURE 6.

The curve $\tilde{C}_{23}$ corresponds to the images $\tilde{\phi}(a_6) = \tilde{\phi}(a_7)$. We may blow down the curves $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_8, \tilde{E}_{14}, \tilde{E}_{16}, \tilde{E}_{15}, \tilde{E}_{26}, \tilde{E}_{23}, \tilde{C}_{14}, \tilde{C}_{23}, \tilde{C}_{12}$ and $\tilde{C}_{16}$. This sequence of blow-downs defines a Cremona transformation



which maps the six lines of figure 1 as follows

$$\xi(l_0) = \Lambda_1 \quad \xi(l_3) = \Gamma \quad \xi(l_5) = \Lambda_2$$

while the lines l_2, l_4 and l_6 get contracted to the singular points of Γ.

Conversely, let $Y_0 \xrightarrow{\eta_1} \mathbb{P}^2$ be the blow-up of the D_9 singularity of Σ (line passing through a cusp of type A_4) and denote by Υ_1 the exceptional divisor. By its recursive construction ([1] p.87), the canonical resolution of the double cover defined by Σ is the canonical resolution of the double cover defined by

$$\Sigma_1 = \eta_1^*\Sigma - 2\Upsilon_1 = \Gamma + 2\Upsilon_1 + \Lambda_1 + \Upsilon_1 + \Lambda_2 - 2\Upsilon_1 = \Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1$$

where we use the same symbol to denote a curve and its proper transform. Recursively, if $Y_1 \xrightarrow{\eta_2} Y_0$ is the blow-up of the point $\Gamma \cap \Upsilon_1$, then the canonical resolution of the double cover defined by Σ_1 is the canonical resolution of the double cover defined by

$$\Sigma_2 = \eta_2^* \Sigma_1 - 2\Upsilon_2 = \Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1 + \Upsilon_2.$$

Blow up the simple node of Γ and denote the exceptional divisor by Υ_3. We must now take the double cover branched along

$$\Sigma_3 = \Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1 + \Upsilon_2 + \Upsilon_3.$$

Since the singularities of Σ_3 are now all of type A_1, the exceptional divisors coming from their resolution will not be part of the branch locus along which we will take the double cover. The resolution of Σ_3 gives a graph identical to the one of figure 6 in $\tilde{\mathbb{P}^2}$ where $\Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1 + \Upsilon_2 + \Upsilon_3$ corresponds exactly to $\tilde{C}_{13} + \tilde{C}_0 + \tilde{C}_{15} + \tilde{C}_{16} + \tilde{C}_{12} + \tilde{C}_{14}$. □

7. K3 Surfaces with $T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle$

The last section of this paper is very similar to the two previous ones. It aims to describe the geometry of the K3 surfaces with period lattice isomorphic to $U(2) \oplus U(2) \oplus \langle -2 \rangle$.

Proposition 7.1. Let Y be a general Kummer surface. There exists an elliptic fibration on Y whose singular fibers are of the type $4I_2 + I_2^* + I_1^* + I_1$ and whose Mordell-Weil group is cyclic of order two.

Proof. Consider the divisor

$$B = 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{45} + E_{46} + E_{56}).$$

Geometrically, B can be represented by the proper inverse image, under the map $Y \xrightarrow{\phi} \mathbb{P}^2$, of a cubic passing trough p_{13}, p_{24}, p_{45}, p_{46}, p_{56} having a double point at p_{12}. By counting parameters, we immediately see that such a cubic exists. Generically, its proper transform is smooth and irreducible and satisfies $B^2 = 0$. Let $Y \xrightarrow{b} \mathbb{P}^1$ be the elliptic fibration defined by the linear system $|B|$.

The divisors

(1) $F_1 = \underbrace{(L - E_0) - (E_{12} + E_{45})}_{b_1} + 2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{46} + E_{56})$,

(2) $F_2 = \underbrace{2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{45} + E_{56})}_{b_2} + e_1$,

(3) $F_3 = \underbrace{3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{36} + E_{45} + E_{46} + E_{56})}_{b_3} + E_{36}$,

(4) $F_4 = \underbrace{E_{35}}_{b_4} + 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{45} + E_{56} + E_{46} + E_{35})$

define four fibers of type I_2. The divisor

$$F_5 = \underbrace{(L - E_0) - (E_{12} + E_{56})}_{b_5} + E_{14} + 2(C_0 + E_{14} + C_{14}) + E_{15} + E_{16}$$

defines a fiber of type I_2^*. The divisor

$$F_6 = e_6 + e_7 + 2(C_{12} + E_{23}) + E_{26} + E_{25}$$

defines a fiber of type I_1^*. Since $C_{15} \cdot B = C_{16} \cdot B = 1$, we see that the rational curves C_{15} and C_{16} are sections of b. Moreover, as $\sum_{i=1}^{6} \chi_{top}(F_i) = 23$, there must be one more singular fiber F_7 of type I_1. Finally, by theorem 5.2, we deduce that $\mathrm{MW}_b(X) \simeq \mathbb{Z}/2\mathbb{Z}$. □

Set $b_6 := e_6 \quad b_7 := e_7 \quad b_8 := E_{34}$.

Lemma 7.2. *The b_i's form an even eight on Y. If $X \overset{\nu}{\dashrightarrow} Y$ is the associated rational map of degree two, then $T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle$.*

Proof. A direct computation shows that

$$\sum_{i=1}^{8} b_i = 2C_{13} + 2E_{35} + 2E_{34} + 2\{4(L - E_0) - (3E_{12} + 2E_{45} + 2E_{56} + E_{13} + E_{24} + E_{46})\} \in 2S_Y.$$

We will apply the same strategy as in lemma 6.1 and analyze the fibration, $b_\nu : X \overset{\nu^* B}{\to} \mathbb{P}^1$. Let $X \overset{\iota}{\to} X$ be the associated Nikulin involution. Likewise in lemma 6.1, $\nu^* F_i$, for $i = 1, 2, 3, 4$, are nodal curves preserved by ι. Hence ι preserves the fibers of b_ν ($\nu^* B$ cannot split). As for the remaining singular fibers, $\nu^* F_5$ becomes of type I_4^*, $\nu^* F_6$ becomes of type I_2^* and $\nu^* F_7$ becomes of type I_2. Finally $\nu^* C_{15}$ and $\nu^* C_{16}$ become 2-sections.. We conclude that there exists an embedding of lattices of finite cokernel $U(2) \oplus D_8 \oplus D_6 \oplus A_1 \hookrightarrow S_X$. Since $D_{U(2) \oplus D_8 \oplus D_6 \oplus A_1} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 7}$, the overlattice S_X must be a two-elementary lattice. Note also that b_ν does not have a section. Indeed, if $C \cdot \nu^* D = 1$, then $\iota^* C$ also would satisfy

$$\iota^* C \cdot \iota^*(\nu^* D) = C \cdot \nu^* D = 1.$$

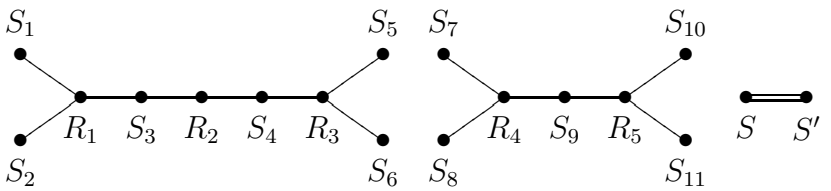
Let $\tilde{C} \in S_Y$ be such that $\iota^* C + C = \nu^* \tilde{C}$. Then $2\tilde{C} \cdot D = \nu^* \tilde{C} \cdot \nu^* D = 2$, thus $\tilde{C} \cdot D = 1$. So either $C + \iota^* C = \nu^* C_{15}$ or $C + \iota^* C = \nu^* C_{16}$. But we have seen that C_{15} and C_{16} do not split as they each meet the b_i's at two points. This proves that b_ν does not have a section. Let $J(X)$ be the jacobian fibration of b_ν (see [2] for the definition), then there is an embedding ([5])

$$T_X \hookrightarrow T_{J(X)}, \text{ with } T_{J(X)}/T_X \simeq \mathbb{Z}/2\mathbb{Z}.$$

Let a be the length of D_{S_X}, i.e. $D_{S_X} \simeq (\mathbb{Z}/2\mathbb{Z})^a$. If $a = 1$, then by the classification of two-elementary lattices ([13]), $T_X \simeq U \oplus U \oplus \langle -2 \rangle$ and X contains finitely many rational curves meeting as in figure 2 which do not define any $I_4^* + I_2^* + I_2$ configuration. If $a = 3$, then $T_{J(X)}$ has to be isomorphic to $U \oplus U \oplus \langle -2 \rangle$ and we get the same contradiction as in the previous case applied to the surface $J(X)$. We conclude that $T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle$. □

Theorem 7.3. *Let X be a $K3$ surface with $T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle$. Then the even eight associated to the rational map $X \overset{\nu}{\dashrightarrow} Y$ is given by the b_i's of lemma 7.2.*

Proof. From a purely lattice theoretical point of view, we deduce from lemma 7.2 that there exists an embedding $U(2) \oplus D_8 \oplus D_6 \oplus A_1 \hookrightarrow S_X$. Thus X admits an elliptic fibration $X \overset{b_\nu}{\to} \mathbb{P}^1$ with singular fibers of type $I_4^* + I_2^* + I_2$, i.e. it contains the following diagram of smooth rational curves



We shall see that the multisection index of b_ν is two. Note that by construction b_ν admits a 2-section that we call R_6. Without loss of generality, $R_6 \cdot S_2 = R_6 \cdot S_6 = R_6 \cdot S_8 = R_6 \cdot S_{11} = 1$. Let $X \xrightarrow{\theta} X$ be the involution acting trivially on S_X. Nikulin showed that

$$X^\theta = \{x \in X | \theta(x) = x\}$$

consists of the union of seven disjoint smooth rational curves [14]. In fact, the curves $R_1, R_2, R_3, R_4, R_5, R_6$ belong to X^θ. Indeed, the equalities $R_1 \cdot S_1 = R_1 \cdot S_2 = R_1 \cdot S_3 = 1$ imply that there exist three points on R_1 fixed by θ, consequently $R_1 \subset X^\theta$. The same argument applies for R_3, R_4 R_5 and R_6. Assume that $R_2 \notin X^\theta$ and denote by p the point of $R_2 \cap S_3$. Because $R_2 \cdot S_3 = 1$, $\theta(p) = p$, so there exists $C \subset X^\theta$ such that $p \in C \cap R_2 \cap S_3$. If F denotes the class of a fiber of b_ν, then $C \cdot F \geq 4$. The curves S and S' cannot be fixed by θ as they don't meet S_1, S_5, S_7 and S_{10} which must intersect X^θ. It implies that $C \cdot (S + S') \leq 4$, so $C \cdot (S + S') = 4$. Similarly, if $q = R_2 \cap S_4$, then there exists $C' \subset X^\theta$, such that $C' \cdot F \geq 4$. If $C \neq C'$, then $C' \cdot (S + S') = 0$, which is absurd. If $C = C'$, then $C \cdot F \geq 8$, which contradicts $C \cdot (S + S') = 4$. So $R_2 \subset X^\theta$. Denote by R_7 the remaining curve of X^θ. Necessarily

$$(R_6 + R_7) \cdot S = (R_6 + R_7) \cdot S' = 2$$

and R_7 is also a 2-section satisfying

$$R_7 \cdot S_1 = R_7 \cdot S_5 = R_7 \cdot S_7 = R_7 \cdot S_{10} = 1.$$

Observe now that

$$D_1 = 2R_6 + S_2 + S_6 + S_8 + S_{11} \overset{lin}{\sim} 2R_7 + S_1 + S_5 + S_7 + S_{10} = D_2.$$

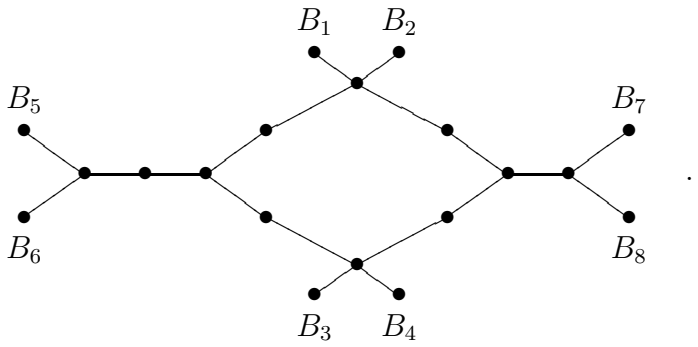
The divisors D_1 and D_2 are two I_0^* fibers of an elliptic fibration $X \xrightarrow{h} \mathbb{P}^1$. The curves S_3, R_2, S_4 and S_9 are components of the other singular fibers of h and R_1, R_3, R_4, R_5 are sections of h. Fix R_4 as the zero section. Let σ be the map induced by inversion of the generic fiber and t_{R_5} be the translation by R_5. The map $\gamma = \sigma \circ t_5$ is an involution on X such that $\gamma^*_{T_X} = -\mathrm{id}_{T_X}$. Set $\iota := \theta \circ \gamma$. As the maps θ^* and γ^* commute on $\mathrm{H}^2(X, \mathbb{Z})$ and $\gamma^* \circ \theta^*_{T_X} = \mathrm{id}_{T_X}$ we conclude that ι is a Nikulin involution. Note that

$$R_1 + S_3 + R_2 + S_4 + R_3 + S_3 + R_7 + S_1 \overset{lin}{\sim} R_4 + S_9 + R_5 + S_{11} + R_6 + S_8$$

as they both define two disjoint I_8 and I_6 fibers of some other fibration. Therefore $-R_1 + R_5 = R_3$ in $\mathrm{MW}_h(X)$ and consequently $\iota^* R_1 = \gamma^* R_1 = R_3$. Clearly $\iota^* R_4 = \gamma^* R_4 = R_5$. The involution γ^* acts trivially on the curves R_6, R_7, R_2 and S_9. Consequently so does ι and each of those curves contains two of the eight fixed points of ι. Finally, observe that

$$\iota^* S_1 = S_5, \quad \iota^* S_2 = S_6, \quad \iota^* S_3 = S_4, \quad \text{and} \quad \iota^* S_7 = S_{10} \quad \iota^* S_8 = \iota^* S_{11}.$$

Consider $X \overset{\nu}{\dashrightarrow} Y$ the rational quotient map by ι and denote by $B_1, \ldots B_8$ the induced even eight. Then Y contains the following diagram of -2-curves



By inspection of the diagram, we see that Y admits an elliptic fibration with a section and whose singular fibers are of the type $4I_2 + I_2^* + I_1^*$. In the table of [17], we find that the Mordell Weil group of such a fibration must be of order two. It follows now from theorem 6.5 that S_Y admits a primitive embedding of the *Kummer lattice* which implies by Nikulin's criterion [11] that Y is a Kummer surface. Finally, by proposition 7.1, the B_i's correspond to the b_i's. □

Let $\mathcal{X}$ be a reducible plane sextic, decomposable as three lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ meeting at a point, a line $\mathcal{L}_4$ and a conic $\mathcal{W}$ tangent to $\mathcal{L}_4$ (see figure 7).

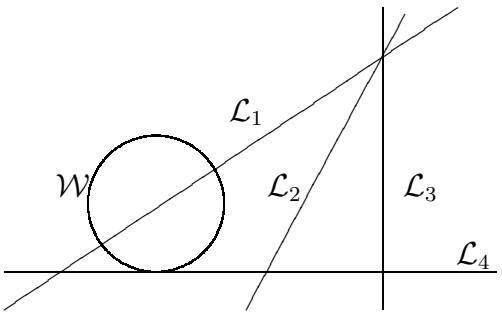


FIGURE 7.

Theorem 7.4. *The map $X \overset{\nu}{\dashrightarrow} Y$ decomposes as*

$$\begin{array}{ccc} X & \xrightarrow{\pi_2} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow \nu & & \downarrow \pi_1 \\ Y & \xrightarrow{\pi} & \mathbb{P}^2 \end{array}$$

where π is the canonical resolution of the double cover of $\mathbb{P}^2$ branched along $\mathcal{X}$. The maps π_1 and π_2 are the canonical resolutions of the double covers branched along $\mathcal{W}$ and $\pi_1^*(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4)$ respectively.

Proof. Let $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi_1} \mathbb{P}^2$ be the double cover branched along $\mathcal{W}$. Note that the line $\mathcal{L}_4$ splits under the cover and that $\pi_1^*(\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4)$ is a divisor of bidegree $(4,4)$ and consequently it is an even divisor. The canonical resolution of the double cover defined by this even divisor is a K3 surface X which admits a rational map $X \overset{\tilde{\nu}}{\dashrightarrow} Y$, where Y is a general Kummer

surface according to the proposition 7.5 below. Applying the same techniques as in theorem 6.5, we check that the corresponding even eight corresponds to the b_i's of lemma 7.2 and therefore $\tilde{\nu} = \nu$. □

Proposition 7.5. A surface Y is a general Kummer surface if and only if Y is the canonical resolution of the double cover of $\mathbb{P}^2$ branched along the reducible sextic $\mathcal{X}$. Moreover the double cover $Y \xrightarrow{\pi} \mathbb{P}^2$ contracts the b_i's of lemma 7.2.

Proof. We proceed exactly as in proposition 6.6. Consider the images of the curves $b_1, \dots b_8$, $C_{15}, E_{15}, C_0, E_{14}, C_{14}, C_{13}, E_{25}, E_{16}, E_{23}, C_{12}, E_{26}, C_{16}$ under the map $Y \xrightarrow{\tilde{\phi}} \tilde{\mathbb{P}}^2$. Blow down the fourteen curves $\tilde{b}_1, \dots, \tilde{b}_8, \tilde{E}_{14}, \tilde{E}_{15}, \tilde{E}_{23}, \tilde{E}_{26}, \tilde{E}_{25}, \tilde{E}_{16}, \tilde{C}_{12}, \tilde{C}_0$. Since $\tilde{\mathbb{P}}^2$ is the blow-up of $\mathbb{P}^2$ at fifteen points, the resulting surface is, the blow-up of $\mathbb{P}^2$ at a point, i.e. the Hirzenbruch surface $\mathbb{F}_1$. Recall that $\mathrm{Pic}(\mathbb{F}_1) = \mathbb{Z}H \oplus \mathbb{Z}F$, where $H^2 = 1$, $H \cdot F = 1$, and $F^2 = 0$. Under the sequence of blow-downs, $\tilde{C}_{15}$ and $\tilde{C}_{16}$ become linearly equivalent to H while $\tilde{C}_{14}$ and $\tilde{C}_{12}$ become linearly equivalent to F. Therefore the divisor $\tilde{C}_{14} - \tilde{C}_{15} \overset{lin}{\sim} E_p$, where is E_p is the exceptional divisor of $\mathbb{F}_1$ (we use the same symbol to denote the curves $\tilde{C}_{13}$, $\tilde{C}_{14}$, $\tilde{C}_{15}$, $\tilde{C}_{16}$ and their images in $\mathbb{F}_1$). By keeping track of the intersections all along the blow-downs, we see that, on $\mathbb{F}_1$, $\tilde{C}_{13} \cdot \tilde{C}_{14} = \tilde{C}_{13} \cdot \tilde{C}_{15} = 2$. If follows that $\tilde{C}_{13} \cdot E_p = 0$. Consider the contracting morphism $\mathbb{F}_1 \to \mathbb{P}^2$, under this map $\tilde{C}_{13}$ becomes a conic $\mathcal{W}$, the curves $\tilde{C}_{14}$, $\tilde{C}_{15}$, $\tilde{C}_{16}$ become three lines, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, meeting at a point and the curve $\tilde{C}_{12}$ becomes a line $\mathcal{L}_4$ tangent to $\mathcal{W}$. Again we have constructed a Cremona transformation

$$\begin{array}{ccc} & \tilde{\mathbb{P}}^2 & \\ {\scriptstyle\epsilon}\swarrow & & \searrow{\scriptstyle\eta} \\ \mathbb{P}^2 & \overset{\xi}{- - - - - - -} & \mathbb{P}^2 \end{array}$$

which maps the six lines of the figure 1 as follows

$$\xi(l_3) = \mathcal{W}, \quad \xi(l_2) = \mathcal{L}_4, \quad \xi(l_4) = \mathcal{L}_1, \quad \xi(l_5) = \mathcal{L}_2, \quad \xi(l_6) = \mathcal{L}_3.$$

and the line l_0 gets contracted to the point of intersection $\mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3$.

Conversely, let $Y_0 \xrightarrow{\eta_1} \mathbb{P}^2$ be the blow-up of the point $p \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3$ and denote by Υ_1 the exceptional divisor. By construction, the canonical resolution of the double cover defined by $\mathcal{X}$ is the canonical resolution of the double cover defined by

$$\mathcal{X}_1 = \eta^* \mathcal{X} - 2\Upsilon_1 = \mathcal{W} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \Upsilon_1$$

(again we use the same symbol to denote a curve and its proper transform). Since the singularities of $\mathcal{X}_1$ are now all of type A_1, the exceptional divisors coming from their resolution will not be part of the branch locus along which we will take the double cover. Moreover, the canonical resolution of $\mathcal{X}_1$ gives rise to a graph of rational curves in $\tilde{\mathbb{P}}^2$ meeting exactly as the curves $\tilde{b}_1, \cdots, \tilde{b}_8, \tilde{E}_{14}, \tilde{E}_{15}, \tilde{E}_{23}, \tilde{E}_{26}, \tilde{E}_{25}, \tilde{E}_{16}, \tilde{C}_{12}, \tilde{C}_0, \tilde{C}_{13}, \tilde{C}_{14}, \tilde{C}_{15}, \tilde{C}_{16}$. The divisor $\mathcal{W} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \Upsilon_1$ corresponds to $\tilde{C}_{13} + \tilde{C}_{14} + \tilde{C}_{15} + \tilde{C}_{16} + \tilde{C}_{12} + \tilde{C}_0$. It is clear now that the double cover along such a divisor is a general Kummer surface. □

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