# Singular leaves of singular foliations

## Joint work with Leonid Ryvkin

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## Two well-known examples

Here are two well-known singular foliations

1. For $(M, \pi)$ a Poisson structure, *symplectic leaves* are a singular foliation.
2. For $\Gamma \rightrightarrows M$ a source-connected Lie groupoid, *end points of arrows* form a singular foliation

?

Natural answers :

1. A disjoint union of submanifolds of varying dimension ?
2. An integrable singular distribution ?
3. No need of definition, Lie algebroids give the generic example ?

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Let $M$ be a manifold, and $\mathfrak{X}(M)$ its sheaf of vector fields.

### Définition

A singular foliation $\mathcal{F}$ on a manifold $M$ is a sub-sheaf of $\mathfrak{X}(M)$ which is :

(i) stable under multiplication by $C^\infty(M)$,

(ii) stable under Lie bracket,

(iii) locally finitely generated over $C^\infty(M)$.

## Variations on that definition

### Définition

A singular foliation $\mathcal{F}$ on a manifold $M$ is a locally finitely generated sub-sheaf of the sheaf of Lie-Rinehart algebra $\mathfrak{X}(M)$.

Or the one used by non-commutative geometers (Androulidakis, Debord, Skandalis)

### Définition

A singular foliation $\mathcal{F}$ on a manifold $M$ is a sub $C^\infty(M)$-module of compactly supported vector fields $\mathfrak{X}_c(M)$ which are :

(i) stable under Lie bracket.

(ii) locally finitely generated over $C^\infty(M)$.

This makes no difference !

## Local picture

**Définition**

A singular foliation is said to be *locally real analytic* when around each point there exists a chart + generators $X_1, \ldots, X_r$ real anaytic in that chart.

Does not require the whole manifold to be real analytic ! These charts may not glue in real analytic manner.

## -oid examples

**Proposition**

For $(A, \rho, [\cdot, \cdot])$ a Lie algebroid, $\mathcal{F} = \rho(\Gamma(A))$ is a singular foliation.

Hence, quite a few singular foliations come from Lie algebroids :

1. Lie group actions,
2. Symplectic leaves,
3. regular foliations.

**Question (Androulidakis-Zambon)**

Are all singular foliations image through the anchor map of a Lie algebroid ?

Still open!

## Non "-oid" examples

### Question

Are vector fields on $\mathbb{R}^2$ vanishing at least quadratically at zero the image through the anchor map of a Lie agebroid over $\mathbb{R}^2$ ?

1. Vector fields on $M$ vanishing at order $\geqslant k$ at a given point is a singular foliation.
2. Polynomial vector fields on $\mathbb{C}^n$ tangent to a given affine variety.
3. Polynomial vector fields on $\mathbb{C}^n$ that "kill" some given polynomial functions $\varphi_1, \ldots, \varphi_k \in \mathbb{R}[x_1, \ldots, x_k]$

No Lie algebroid known in general! Probably no relevant one.

## Leaves

Let $\mathcal{F}$ be a singular foliation on $M$.

**Définition**

For all $m \in M$, the *tangent space of $\mathcal{F}$ at $m$, denoted $T_m\mathcal{F}$, is*

$$\{X_{|_m} | X \in \mathcal{F}\} \subset T_m M$$

.

**What is a leaf ?**

**Def 1**. $m, n$ in same leaf iff one can go from $m$ to $n$ through flows of vect. fi. in $\mathcal{F}$.

**Def 2**. Leaf = submanif. $L$ s.t. $T_m L = T_m\mathcal{F}$, $\forall m \in M$ + maximal among those.

## Leaves

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$\Longleftrightarrow$

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## Weinstein's splitting theorem

Poisson and Lie algebroids admits "transversal structures to leaves".

| If structure is | there is the | for a given leaf there is a transverse |
|---|---|---|
| Poisson | Weinstein's splitting Theorem | Poisson (vanishing at 0) |
| Lie algebroid | Zung's splitting theorem | Lie algebroid (with anchor 0 at 0) |

**Question**

What for singular foliation ?

## Transverse singular foliation

Let $\mathcal{F}$ be a singular foliation on a manifold of dimension $d$. Let $\ell$ be a point. Assume the leaf $L$ through $\mathcal{F}$ has dimension $k$.

1. Any submanifold $T_\ell$ transverse to $L$ at $\ell$ comes equipped with a natural singular foliation $\mathcal{T}_\ell$ near $\ell \in T \cap L$ ($\simeq$ Dirac reduction)
2. Any two such induced singular foliations are diffeomorphic (even for different $\ell$'s).
3. and locally, it is a trivial product.

## Transverse singular foliation

Let $\mathcal{F}$ be a singular foliation on a manifold of dimension $d$. Let $\ell$ be a point. Assume the leaf $L$ through $\mathcal{F}$ has dimension $k$.

1. There is a splitting theorem.
2. There is a well-defined notion of *transverse singular foliation at a point*,
3. ...which is a germ of singular foliation near 0 on on open ball.
4. ...and which is the same all along the leaf.

## Transverse singular foliation

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## Transverse singular foliation

Let $\mathcal{F}$ be a singular foliation on a manifold of dimension $d$. Let $\ell$ be a point. Assume the leaf $L$ through $\mathcal{F}$ has dimension $k$.

1. There is a well-defined notion of transversal singular foliation to a leaf.
2. There is a splitting theorem : *Near $m$, $\mathcal{F}$ is the direct product of vector fields along $L$ and the transverse singular foliation.*

Of course, this is not true around the leaf. Hence the question:

**Question**

When is a                    neighborhood of a leaf isomorphic to the direct product of vector fields along the leaf with the transverse foliation to the leaf?

Formal means "formal along $L$".

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**Question**

When is a neighborhood of a leaf isomorphic to the direct product of vector fields along the leaf with the transverse foliation to the leaf?

1. Ask it for Poisson: there is no clear answer,
2. Ask it for Lie algebroid: there is no clear answer,
3. Ask it for regular foliation: there is a clear answer, yes, if the leaf is simply connected.

## Definition

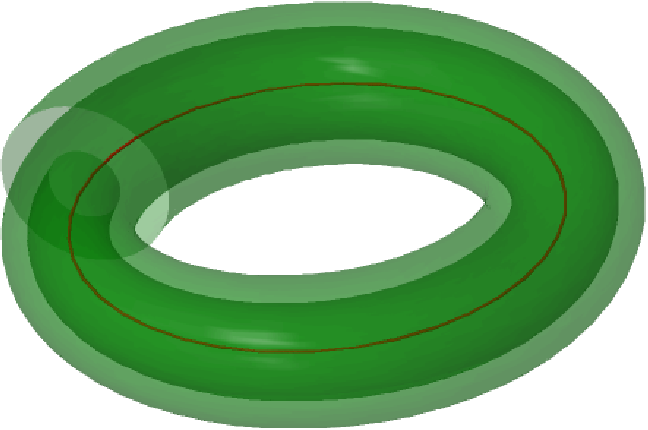
We say that a leaf $L$ is *formally trivial* if the formal jet $\widehat{\mathcal{F}}$ along $L$ is isomorphic to the direct product of $\mathfrak{X}(L)$ with the formal jet $\mathcal{T}$ of the transverse foliation.

"Formally trivial" could be defined for symplectic/Lie algebroid leaves.

1. When is a simply-connected symplectic leaf locally trivial? No idea.
2. When is a simply connected Lie algebroid leaf locally trivial? No idea.
3. When is a regular leaf locally trivial? always!
4. When is a simply connected singular leaf locally trivial? Quite often, as we will see.

## The non-simply connected case

Let us assume the leaf to be simply-connected to avoid the "self-eating snake".



## Main theorem

### Theorem

*(C.L.-G., Leonid Ryvkin) A simply-connected and locally closed leaf $L$ of a locally real analytic singular foliation $\mathcal{F}$ is formally trivial if and only if there exists a Lie algebroid section $TL \to A_L^{lin}$.*

Meaning:

1. (Androulidakis-Zambon) Every leaf $L$ comes with a transitive Lie algebroid defined by

$$\Gamma(A_L) = \frac{\mathcal{F}}{I_L \mathcal{F}}$$

with $I_L$ functions vanishing along $L$,

2. Divide by sections of $A_L$ coming from sections of $\mathcal{F}$ vanishing at order 2, the outcome is a transitive Lie algebroid over $L$ denoted $A_L^{lin}$.

## Transversally quatratic

Here an instance of a "non-oid" phenomena.

**Corollary**

(C.L.-G., Leonid Ryvkin) Every simply-connected, transversally quadratic and locally closed leaf $L$ of a locally real analytic singular foliation $\mathcal{F}$ is formally trivial.

**Proof.**

1. From the previous theorem.
2. Direct proof (to give the idea).

□

## The proof: several lemmas

1. Tubular neighborhoods $\simeq$ Euler-like vector fields.
2. For every flat bundle over a simply connected manifold $L$, $H^1(L, E) = 0$.
3. For every section $\sigma\colon \mathfrak{X}(M) \to \mathcal{F}_{proj}$ whose curvature is 0 up to order $n$, the bundle implicitly defined by

$$E_n := \frac{\text{v.f. vanishing at order } n+1 \text{ along } L}{\mathcal{I}_L^{n+1}\mathcal{F}}$$

is flat.

Here $\mathcal{F}_{proj}$ = vector fields in $\mathcal{F}$ projectable on $L$ for some tubular neighborhood.

## The proof: step by step construction + vanishing obstructions

We construct by recursion:

1. A section $\sigma_n \colon \mathfrak{X}(M) \to \mathcal{F}_{proj}$,
2. An Euler-like vector field $E_n$,

such that

1. $\sigma_n$ takes values in vector fields "linear up to order $n$",
2. and therefore $\sigma_n$ is "flat up to order $n$".

Because "linearizable" + "flat at order 1" $\implies$ "flat".

## The proof: step by step construction + vanishing obstructions

## In fact, more general result

A transitive Lie algebroid admits a maximal solvable ideal: call semi-simple part its quotient. For $A_L$ we call it $A_L^s$.

**Theorem**

*(C.L.G, Leonid Ryvkin) Let L be a simply-connected locally closed leaf. If there exists a Lie algebroid section z from the semi-simple holonomy $A_L^s$ to the linear holonomy $A_L^{lin}$, then:*

1. *the normal bundle $\nu = \frac{TM|_L}{TL} \to L$ comes equipped with a flat $A_L^s$-connection,*
2. *there is a formal diffeomorphism*

$$\widehat{\mathcal{F}} \simeq A_L^s \,\widehat{\ltimes}\, \widehat{\mathcal{R}}_\nu$$

*where $\mathcal{R}_\nu$ is an $A_L^s$-invariant vertical singular foliation on $\nu$.*

## In fact, more general result

A transitive Lie algebroid admits a maximal solvable ideal: call semi-simple part its quotient. For $A_L$ we call it $A_L^s$.

**Theorem**

*Let L be a simply-connected locally closed leaf. If $A_L^{lin}$ satisfies Levi-Malcev theorem, then $\mathcal{F}$ also satisfies Levi-Malcev theorem (at least formally).*

$\Longrightarrow$ an old result by Dominique Cerveau (1978): for a singular foliation vanishing at a point $m$, there is a Lie algebra morphism from the semi-simple part of the isotropy Lie algebra at $m$ to $\widehat{\mathcal{F}}_m$.

## Universal Q-manifold

### Definition

(A bit wrong...) A Q-manifold over $M$ is a sequence $E_{-r}, \cdots, E_{-1}$ of vector bundles over $M$ + a degree $+1$ derivation $Q$ of $\Gamma(S(\oplus_{i=1}^{d} E^*_{-i}))$.

### Theorem

*(Vaintrob, Voronov)(A bit vague) Lie $\infty$-algebroids $\simeq$ Q-manifolds.*

Lie algebroid correspond to "only $E_{-1}$ is not zero".

### Theorem

*(C.L.G, Sylvain Lavau, Thomas Strobl) (Vague statement first) Behind almost any singular foliation there is a universal Q-manifold.*

"Universal" := in the category of Q-manifolds defining a sub-foliation of $\mathcal{F}$ + homotopy classes of Q-manifold morphisms.

## The universal Q-manifold: precise statement

**Theorem**

*(C.L.G, Sylvain Lavau, Thomas Strobl) Let $U$ be a relatively compact open subset of a locally real analytic singular foliation $\mathcal{F}$. Then:*

1. *$\mathcal{F}$ admits a projective resolution by sections of finitely many vector bundles $\Gamma(E_{-n-1}) \xrightarrow{d} \dots \xrightarrow{d} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F}$.*
2. *these vector bundles admit a Q-manifold structure whose linear part is d*
3. *Any two such Q-manifolds are homotopy equivalent.*

We call the homotopy classes of these $Q$-manifolds *the universal Q-manifold of a singular foliation* and denote it by $\mathbb{U}_{\mathcal{F}}$.

## The algebraic counterpart

Let $\mathcal{O}$ be a ring.

**Theorem**

*(Ruben Louis's PhD) There is an equivalence of category between:*

(ii) *Lie Rinehart algebras over $\mathcal{O}$*

(ii) *Lie $\infty$-algebroid structures on projective $\mathcal{O}$-resolutions, with homotopy classes of $\infty$-$\mathcal{O}$-morphisms as arrows.*

## What is the first return map for a singular leaf?

The holonomy of a leaf $L$ of a regular foliation $\mathcal{F}$ on a manifold $M$ is a group morphism:

$$Hol : \pi_1(L, \ell) \mapsto \mathrm{Diff}_\ell(T_\ell), \tag{1}$$

$\mathrm{Diff}_\ell(T_\ell)$ is the group of germs of diffeomorphisms of a transversal $T_\ell$ of $L$ at a point $\ell \in L$. This is classical.

**Question**

How to define an analogue of the holonomy (1) for a singular leaf of a singular foliation?

Already two attempts:

1. Dazord (1985)
2. Androulidakis-Zambon.

Based on $\pi_1(L)$ and Androulidakis-Skandalis holonomy groupoid.
Our idea: use all $\pi_n(L)$.

## Our idea 1: what is $\pi_n(\mathcal{F})$?

### Definition

We call *n*-th homotopy group $\pi_n(\mathcal{F})$ the *n*-th homotopy group of any of its universal *Q*-manifold. In equation:

$$\pi_n(\mathcal{F}) := \pi_n(\mathbb{U}_{\mathcal{F}})$$

For instance,

1. $\pi_0(\mathcal{F})$ are diffeomorphisms of $M/\mathcal{F}$.
2. $\pi_1(\mathcal{F})$ is (more or less) the isotropy groups of Androulidakis-Skandalis isotropy holonomy groupoid.
3. $\pi_n(\mathcal{F})$ involves
   1. $\pi_n$ of all leaves.
   2. $\pi_n$ of all isotropy groups of the holonomy groupoid.
   3. $Tor^n(\mathcal{F}, \mathbb{R})$ ($\mathbb{R}$ is a module through evaluation)

## Our idea 1: what is $\pi_n(\mathcal{F})$?

### Definition

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3. $\pi_n(\mathcal{F})$ involves
    1. $\pi_n$ of $s$-fibers of the holonomy groupoid.
    2. $Tor^n(\mathcal{F}, \mathbb{R})$ ($\mathbb{R}$ is a module through evaluation)

A fundamental result is Claire Debord's statement that the holonomy algebroid $A_L$ is integrable.

## Our idea 2: what is generalized first return map

Let $L$ be a leaf, $\ell$ a point on that leaf. Assume "$L$ admits a complete Ehresmann $\mathcal{F}$-connection $(M_L, p, H)$." (= some assumptions that force leaves near $L$ to stay near $L$) (= no self-eating-snake!). Let $\mathcal{T}_l$ be the transverse singular foliation.

### Theorem

*(C.L.G., Leonid Ryvkin) For every $\ell \in L$, there exist canonical group morphisms*

$$Hol : \pi_n(L, \ell) \to \Gamma\left(\pi_{n-1}(\mathcal{T}_\ell)\right) \tag{2}$$

*such that for all $m \in p^{-1}(\ell)$ the sequence*

$$\dots \xrightarrow{Hol|_m} \pi_n(\mathcal{T}_\ell, m) \xrightarrow{i} \pi_n(\mathcal{F}|_{M_L}, m) \xrightarrow{P} \pi_n(L, \ell) \xrightarrow{Hol|_m} \pi_{n-1}(\mathcal{T}_\ell, m) \to \dots \tag{3}$$

*is exact.*

## Proof inspired by a similar result by Brahic-Zhu

The idea is

1. that the natural projection

$$\mathbb{U}_{\mathcal{F}} \longrightarrow TL$$

   is a surjective submersion of $Q$-manifolds with typical fiber $\mathbb{U}_{\mathcal{T}_\ell}$.
2. to remember that Olivier Brahic and Chenchang Zhu had given a long exact sequence of Lie algebroid fibrations
3. to extend Brahic-Zhu to Lie $\infty$-algebroid fibrations.

Then it paves the way to our construction.

!! Also, one has to compare with the same construction made with the holonomy groupoid of $\mathcal{F}$.

## Particular cases

Here is a list of examples:

1. The map $\pi_1(L) \to \pi_0(\mathcal{T}_\ell) = \mathit{Diff}(T_\ell/\mathcal{T}_\ell)$ is Dazord first return map.
2. The map $\pi_2(L) \to \pi_1(\mathcal{T}_\ell)$ highly linked to Crainic-Fernandes obstruction to integrability.
3. For $\mathcal{F}$ = vector fields on $TS^n$ tangent to the zero-section, the $n$-th holonomy is not trivial.

## Conclusions

Singular foliations do not behave as Lie algebroids or Poisson structure. At least not always.

Even if the answer to Androulidakis-Zambon's open question about the existence of a Lie algebroid was yes, this Lie algebroid might be arbitrary.

The important objects behind a singular foliations are rather:

1. Androulidakis-Skandalis holonomy groupoid, which is only leafwise Lie (Debord),
2. The universal Q-manifold.

## Open questions

There are natural questions.

For local structure.

1. Can we get semi-local convergence rather than formal?
2. What about non-simply connected leaves (and self-eating snakes)?

For first-return maps.

3. To understand the first return entirely, $\pi_n(L)$ is probably not enough. We need the whole homotopy of $L$, for instance $H^n(L)$